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Abstract. The article presents early observations on possibilities of extension of the general resolution rule into fuzzy logic in narrow sense. Firstly the existing theory is summarized. Then it introduces possible resolution rule for general formulas of fuzzy logic (based on Łukasiewicz algebra).

keywords: Automated theorem proving, non-clausal deduction, general resolution, fuzzy logic.

1 Introduction

First-order logic covers many deductive systems and methods. In spite of high diversity of these systems, most of them have one essential fault: They are determined to narrow class of formulas, or they lead to highly ambiguous indeterminate proofs, that are not suitable for automated proving. The focus of the paper slip into a theory of resolution. The conventional view of resolution is closely related to clausal normal form of formula and to skolemization in predicate logic. It is not very difficult to obtain a clausal normal form in two-valued logic of any formula (particularly for non-quantified formulas).

In fuzzy logic, there is no direct way how a formula could be transformed into a normal form. One possibility is to find a normal form as approximation with some precision. The fact that we could avoid the translation may bring a new way to resolution in fuzzy logic, at least for its non-existential subclass.

The paper is based on the article presented by L. Bachmair and H. Ganzinger [Ba97] where the general resolution rule for two-valued logic is defined. Mathematical principles of fuzzy logic are mainly cited from the comprehensible book [No99] or from [Hj97] and [Mu95]. There is also work concerning to resolution theory with normal forms in fuzzy logic [Le95].

2 General resolution – brief overview

There is a paper concerning basics of generalized theory of resolution written by H. Ganzinger and L. Bachmair [Ba97]. They present deduction rules for general formulas and also resolution methods. Let's review these rules.

Definition 1. *General resolution*

$$\frac{F_1[G],\, F_2[G]}{F_1[G/\bot] \vee F_2[G/\top]} \tag{1}$$

where F_1 and F_2 are formulas – premises of first-order logic and G is an occurrence of a subformula of F_1 and F_2. The expression below the line means the resolvent of premises on G. Every occurrence of G is replaced by false in the first formula and by true in the second one. The formula F_1 is also called the positive, the formula F_2 the negative premise, and the subformula G the resolved subformula.

The proof of the soundness of this rule is similar to clausal resolution rule proof. Suppose an interpretation I in which both premises are valid. In I, G is either true or false. If G ($\neg G$) is true in I, so is $F[G/\top](F[G/\bot])$. From this point of view, it shows that the resolution rule is nothing more that assertion of the type: We have two formulas holding simultaneously in the interpretation I and they contain the same subformula. We can deduce that either the common subformula is true in I and then the truthfulness is assured by the first formula, or by the second formula in the opposite case. So it is possible to generate a new formula based on this fact. Nevertheless the above criterion for subformula is expressed by semantic means (interpretation), we can simulate this semantic item by syntactic counterparts – constants for true and false. Now we can have a look at the question how these facts influence the view of clausal resolution in Table 1. Consider the table showing various cases of resolution on similar clauses. The first and fourth row lead to true because premises have the same occurrences of atom b and it could not be obtained a reasonable resolvent. In the second row we took premises in wrong order. It should be mentioned that the selection of right order is also important question. There are several notions, which help to select the right order. They differ in efficiency and simplicity, but the enumeration of them lies out of the range of this paper and the reader can find some references in the cited technical report.

Premise+	Premise-	Resolvent	Simplified	Comments
$a \vee b$	$b \vee c$	$(a \vee \bot) \vee (\top \vee c)$	$\top$	no sense
$a \vee \neg b$	$b \vee c$	$(a \vee \top) \vee (\top \vee c)$	$\top$	redundant
$a \vee b$	$\neg b \vee c$	$(a \vee \bot) \vee (\bot \vee c)$	$a \vee c$	right resolution
$a \vee \neg b$	$\neg b \vee c$	$(a \vee \top) \vee (\bot \vee c)$	$\top$	no sense

Table 1. Clausal resolution in the frame of general resolution

Let us have a look into an example of a non-clausal refutation.

Example 1. General resolution with equivalence
Let us have two axioms $[a \wedge c] \leftrightarrow [b \wedge d]$ and $a \wedge c$ and we will prove $b \wedge d$ by the refutation of its negation.

1.	$[a \wedge c] \leftrightarrow [b \wedge d]$	(axiom)
2.	$a \wedge c$	(axiom)
3.	$\neg[b \wedge d]$	(axiom) – negated goal
4.	$[a \wedge \bot] \vee [a \wedge \top]$	(resolvent from (2), (2) on c)
4a.	a	(simplification of 4.)
5.	$[a \wedge \bot] \vee [[a \wedge \top] \leftrightarrow [b \wedge d]]$	((2), (1) on c)
5a.	$a \leftrightarrow [b \wedge d]$	(simplification of 5.)
6.	$\bot \vee [\top \leftrightarrow [b \wedge d]]$	((4), (5) on a)
6a.	$b \wedge d$	(simplification of 6.)
7.	$[\bot \wedge d] \vee [\top \wedge d]$	((6), (6) on b)
7a.	d	(simplification of 7.)
8.	$[b \wedge \bot] \vee [b \wedge \top]$	((6), (6) on d)
8a.	b	(simplification of 8.)
9.	$\bot \vee \neg[\top \wedge d]$	((8), (3) on b)
9a.	$\neg d$	(simplification of 9.)
10.	$\bot \vee \neg\top$	((7), (9) on d)
10a.	$\bot$	(refutation after simplification)

The simplification used above is also not an essential need, but it was performed only for lucidity. It is eventual to retain the resolvents unsimplified until it is completely empty of atoms and then to determine logical value of the resolvent.

3 Fuzzification of the rule

For the purposes of fuzzy extension the Modus ponens rule was considered as an inspiration. We will suppose that set of truth values is Łukasiewicz algebra. Therefore we will assume standard notions of conjunction, disjunction etc. to be bound with Łukasiewicz operators.

We will assume Łukasewicz algebra to be

$$\mathcal{L}_{\text{Ł}} = \langle [0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$$

where [0, 1] is the interval of reals between 0 and 1, which are the smallest and greatest elements respectively. Basic and additional operations are defined as follows:

$$a \otimes b = 0 \vee (a + b - 1) \tag{2}$$
$$a \to b = 1 \wedge (1 - a + b) \tag{3}$$
$$a \oplus b = 1 \wedge (a + b) \tag{4}$$
$$\neg a = 1 - a \tag{5}$$

The biresiduation operation $\leftrightarrow$ could be defined $a \leftrightarrow b =_{df} (a \to b) \wedge (b \to a)$, where $\wedge$ is infimum operation. The following properties of $\mathcal{L}_{\text{Ł}}$ will be used in the sequel:

$$a \otimes 1 = a \quad a \otimes 0 = 0 \quad a \oplus 1 = 1 \quad a \oplus 0 = a$$

$$a \to 1 = 1 \quad a \to 0 = \neg a \quad 1 \to a = a \quad 0 \to a = 1$$

Syntax and semantics of propositional fuzzy logic is following:

- propositional variables, logical constants – $\{\mathbf{a} \,|a \in [0,1]\}$. Instead of 0 we write $\bot$ and instead of 1 we write $\top$, connectives - **&** (Łukasiewicz conjunction), $\wedge$ (conjunction), $\boldsymbol{\nabla}$ (Łukasiewicz disjunction), $\vee$ (disjunction), $\Rightarrow$ (implication), $\Leftrightarrow$ (equivalence), $\neg$ (negation) and furthermore by F_J we denote set of all formulas of fuzzy logic in language J
- connectives have the following semantic interpretations: $\mathcal{D}(\mathbf{a}) = a$ for $a \in [0,1]$, $\mathcal{D}(A\,\&\,B) = \mathcal{D}(A) \otimes \mathcal{D}(B)$, $\mathcal{D}(A \wedge B) = \mathcal{D}(A) \wedge \mathcal{D}(B)$, $\mathcal{D}(A \nabla B) = \mathcal{D}(A) \oplus \mathcal{D}(B)$, $\mathcal{D}(A \vee B) = \mathcal{D}(A) \vee \mathcal{D}(B)$, $\mathcal{D}(A \Rightarrow B) = \mathcal{D}(A) \to \mathcal{D}(B)$, $\mathcal{D}(A \Leftrightarrow B) = \mathcal{D}(A) \leftrightarrow \mathcal{D}(B)$, $\mathcal{D}(\neg A) = \neg \mathcal{D}(A)$

Graded fuzzy propositional calculus assigns grade to every axiom, in which the formula is valid. It will be written as

$$a/A$$

where A is a formula and a is a syntactic evaluation. We will need to introduce several notions from fuzzy logic, in order to give the reader more exact definition of fuzzy theory.

Definition 2. *Inference rule*
An n-ary inference rule r in the graded logical system is a scheme

$$r : \frac{a_1/A_1, ..., \ a_n/A_n}{r^{evl}(a_{i_1}, ..., a_{i_n})/r^{syn}(A_{i_1}, ..., A_{i_n})} \tag{6}$$

using which the evaluated formulas $a_1/A_1, ..., \ a_n/A_n$ *are assigned the evaluated formula* $r^{evl}(a_1, ..., a_n)/r^{syn}(A_1, ..., A_n)$. *The syntactic operation* r^{syn} *is a partial n-ary operation on* F_J *and the evaluation operation* r^{evl} *is an n-ary lower semicontinous operation on L (i.e. it preserves arbitrary suprema in all variables).*

Definition 3. *Formal fuzzy theory*
A formal fuzzy theory T in the language J is a triple

$$T = \langle \text{LAx}, \text{SAx}, R \rangle$$

where $\text{LAx} \underset{\sim}{\subset} F_J$ *is a fuzzy set of logical axioms,* $\text{SAx} \underset{\sim}{\subset} F_J$ *is a fuzzy set of special axioms, and R is a set of sound inference rules.*

Definition 4. *Evaluated formal proof*
An evaluated formal proof of a formula A from the fuzzy set $X \underset{\sim}{\subset} F_J$ *is a finite sequence of evaluated formulas*

$$w := \ a_0/A_0, \ a_1/A_1, ..., \ a_n/A_n \tag{7}$$

such that $A_n := A$ *and for each* $i \leq n$*, either there exists an n-ary inference rule r such that*

$$a_i/A_i := \ r^{evl}(a_{i_1}, ..., a_{i_n})/r^{syn}(A_{i_1}, ..., A_{i_n}), \quad i_1, ..., i_n < n$$

or

$$a_i/A_i := X(A_i)/A_i$$

We will denote the value of the evaluated proof by $Val(w) = a_n$*, which is the value of the last member in (7).*

Definition 5. *Provability and truthfulness*
Let T *be a fuzzy theory and* $A \in F_J$ *a formula. We write* $T \vdash_a A$ *and say that the formula* A *is a theorem in the degree* a*, or provable in the degree* a *in the fuzzy theory* T*.*

$$T \vdash_a A \text{ iff } a = \bigvee\{Val(w)|\ w \text{ is a proof of } A \text{ from } \mathrm{LAx} \cup \mathrm{SAx}\} \tag{8}$$

We write $T \models_a A$ *and say that the formula* A *is true in the degree* a *in the fuzzy theory* T*.*

$$T \models_a A \text{ iff } a = \bigwedge\{\mathcal{D}(A) \,|\, \mathcal{D} \models T\}, \text{ where the condition } \mathcal{D} \models T \text{ holds}$$
$$\text{if for every } A \in \mathrm{LAx} : \mathrm{LAx}(A) \leq \mathcal{D}(A),\ A \in \mathrm{SAx} : \mathrm{SAx}(A) \leq \mathcal{D}(A) \tag{9}$$

The fuzzy modus ponens rule could be formulated:

Definition 6. *Fuzzy modus ponens*

$$r_{MP} : \frac{a/A,\ b/A \Rightarrow B}{a \otimes b/B} \tag{10}$$

where from premise A *holding in the degree* a *and premise* $A \Rightarrow B$ *holding in the degree* b *we infer* B *holding in the degree* $a \otimes b$*.*

In classical logic, r_{MP} could be viewed as a special case of resolution. The fuzzy resolution rule presented below is also able to simulate fuzzy r_{MP}. From this fact, the completeness of a system based on resolution can be deduced. It will only remain to prove the soundness. It is possible to introduce following notion of resolution, with respect to the modus ponens :

Definition 7. *General fuzzy resolution*

$$r_R : \frac{a/F_1[G],\ b/F_2[G]}{a \otimes b/F_1[G/\bot]\nabla F_2[G/\top]} \tag{11}$$

where F_1 *holding in the degree* a *and* F_2 *holding in the degree* b *are formulas – premises of fuzzy logic and* G *is an occurrence of a subformula in* F_1 *and* F_2*. The expression below the line means the resolvent of premises on* G *holding in the degree* $a \otimes b$*. Every occurrence of* G *is replaced by false in the first formula and by true in the second one.*

Before we give an example, let us compare the two-valued rule with the multi-valued rule. In the boolean case the correct behaviour is justified by validity of both premises. If we substitute true to the first one and false to the second one, at least one part of the resulting disjunction must be true, because there are no other alternatives of interpration for G.

The first (harder) problem in MV case lies in infinite (or finite, but high) amount of truth values for G. Every introduced connective has the property of monotonicity and it seems that every formula constructed from these ones will also be monotonous with respect to the interpretation of selected subformula. Since we can again restrict every resolvent of the resolution rule to mentioned two truth values true and false. And we again (as in boolean case) use logical constants $\bot, \top$ instead of truth value in the the resolvent expression.

The second problem consists in the truth value of the resolvent. The proposed interpretation by Łukasiewicz multiplication implies from the MP rule interpretation.

Example 2. Proof of d by r_R
Xa. steps represent application of simplification rules for $\bot$ and $\top$.

1. $^{1}/[a \Rightarrow b]$	(axiom with grade 1)
2. $^{0.9}/[(b \,\&\, c) \Rightarrow d]$	(axiom with grade 0.9)
3. $^{0.8}/c$	(axiom with grade 0.8)
4. $^{0.95}/a$	(axiom with grade 0.95)
5. $^{0.95 \otimes 1}/[\bot \nabla (\top \Rightarrow b)]$	(r_R on 4, 1)
5a. $^{0.95}/b$	(simplification for $\Rightarrow$, ∇)
6. $^{0.95 \otimes 0.9}/[\bot \nabla ((\top \,\&\, c) \Rightarrow d)]$	(r_R on 5a, 2)
6a. $^{0.85}/[c \Rightarrow d]$	
7. $^{0.8 \otimes 0.85}/[\bot \nabla (\top \Rightarrow d)]$	(r_R on 3, 6a)
7a. $^{0.65}/d$	

We have found a proof of d with the grade 0.65. Let us note with respect to (8) that this means only lower bound for the provability degree. The problem how to find the best provability degree is an open question. We suppose to find algorithms for this task in future.

Lemma 1. *Soundness of* r_R
The inference r_R *rule for propositonal fuzzy logic based on* $\mathcal{L}_L$ *is sound i.e. for every truth valuation* $\mathcal{D}$*,*

$$\mathcal{D}(r^{syn}(A_1, ..., A_n)) \geq r^{evl}(\mathcal{D}(A_1), ..., \mathcal{D}(A_n)) \tag{12}$$

holds true.

PROOF: For r_R we may rewrite the values of the left and right parts of equation (12):

$$\mathcal{D}(r^{syn}(A_1, ..., A_n)) = \mathcal{D}[\mathcal{D}(F_1[G/\bot]) \nabla \mathcal{D}(F_2[G/\top])]$$

$$r^{evl}(\mathcal{D}(A_1), ..., \mathcal{D}(A_n)) = \mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G])$$

It is sufficient to prove the equality for $\Rightarrow$ since all other connectives could be defined by it. By induction on the complexity of formula $|A|$, defined as the number of connectives, we can prove:

Let premises F_1 and F_2 be atomic formulas. Since they must contain the same subformula then $F_1 = F_2 = G$ and it holds

$$\mathcal{D}[\mathcal{D}(F_1[G/\bot]) \nabla \mathcal{D}(F_2[G/\top])] = D(\bot \nabla \top) = 0 \oplus 1 = 1 \geq \mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G])$$

Induction step: Let premises F_1 and F_2 be complex formulas and let A and B are subformulas of F_1, C and D are subformulas of F_2 and G is an atom where generally $F_1 = (A \Rightarrow B)$ and $F_2 = (C \Rightarrow D)$. The complexity of $|F_1| = |A| + 1$ or $|F_1| = |B| + 1$ and $|F_2| = |C| + 1$ or $|F_2| = |D| + 1$. Since they must contain the same subformula and for A, B, C, D the induction presupposition hold it remain to analyze the following cases:

1. $F_1 = A \Rightarrow G \quad F_2 = G \Rightarrow D$: $\mathcal{D}[\mathcal{D}(F_1[G/\bot]) \nabla \mathcal{D}(F_2[G/\top])] = \mathcal{D}([A \Rightarrow \bot] \nabla [\top \Rightarrow D]) = \mathcal{D}(\neg A \nabla D) = 1 \wedge (1 - a + d)$
   We have rewritten the expression into Łukasiewicz interpretation. Now we will try to rewrite the right side of the inequality, which has to be proven.
   $\mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G]) = \mathcal{D}(A \Rightarrow G) \otimes \mathcal{D}(G \Rightarrow D) = 0 \vee ((1 \wedge (1-a+g)) + (1 \wedge (1-g+d)) - 1) = 1 \wedge (1-a+d)$
   The left and right side of the equation (12) are equal and therefore

   $$\mathcal{D}[\mathcal{D}(F_1[G/\bot]) \nabla \mathcal{D}(F_2[G/\top])] \geq \mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G])$$

   for this case holds.
2. $F_1 = A \Rightarrow G \quad F_2 = C \Rightarrow G$: $\mathcal{D}[\mathcal{D}(F_1[G/\bot]) \nabla \mathcal{D}(F_2[G/\top])] = \mathcal{D}([A \Rightarrow \bot] \nabla [C \Rightarrow \top]) = 1 \geq \mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G])$
3. $F_1 = G \Rightarrow B \quad F_2 = G \Rightarrow D$: $\mathcal{D}[\mathcal{D}(F_1[G/\bot]) \nabla \mathcal{D}(F_2[G/\top])] = \mathcal{D}([\bot \Rightarrow B] \nabla [\top \Rightarrow D]) = 1 \geq \mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G])$

4. $F_1 = G \Rightarrow B \quad F_2 = C \Rightarrow G$: $\mathcal{D}[\mathcal{D}(F_1[G/\bot])\nabla\mathcal{D}(F_2[G/\top])] = \mathcal{D}([\bot \Rightarrow B]\nabla[C \Rightarrow \top]) = 1 \geq \mathcal{D}(F_1[G]) \otimes \mathcal{D}(F_2[G])$

By induction we have proven that the inequality holds and the r_R is sound. The induction of the case where only one of the premises has greater complexity is included in the above solved induction step. □

¿From this result we can conclude the completeness theorem. We will need two additional simplification rules for purposes of the proof:

Definition 8. *Simplification rules for* $\nabla, \Rightarrow$

$$r_{s\nabla} : \frac{a/\bot\nabla A}{a/A} \quad \textit{and} \quad r_{s\Rightarrow} : \frac{a/\top \Rightarrow A}{a/A}$$

The soundness of $r_{s\nabla}$ and $r_{s\Rightarrow}$ is straightforward.

Theorem 1 (Completeness for fuzzy logic with r_R, $r_{s\nabla}$, $r_{s\Rightarrow}$ instead of r_{MP}).
Formal fuzzy theory, where r_{MP} is replaced with r_R, $r_{s\nabla}$, $r_{s\Rightarrow}$, is complete i.e. for every A from the set of formulas $T \vdash_a A$ iff $T \models_a A$.

PROOF: The left to right implication (soundness of such formal theory) could be easily done from the soundness of the resolution rule. Conversely it is sufficient to prove that the rule r_{MP} can be replaced by r_R, $r_{s\nabla}$, $r_{s\Rightarrow}$. Indeed, let w be a proof:

$$w := a/A\{aproofw_a\}, b/A \Rightarrow B\{aproofw_{A\Rightarrow B}\}, a \otimes b/B\{r_{MP}\}$$

Then we can replace it by the proof:
$w := a/A\{aproofw_a\}, b/A \Rightarrow B\{aproofw_{A\Rightarrow B}\}, a \otimes b/\bot\nabla[\top \Rightarrow B]\{r_R\}$,
$a \otimes b/\top \Rightarrow B\{r_{s\nabla}, a \otimes b/B\{r_{s\Rightarrow}\}$

Using the last sequence we can easily make a proof with r_{MP} also with the proposed r_R and simplification rules. Since usual formal theory with r_{MP} is complete as it is proved in [No99], every fuzzy formal theory with these rules is also complete. □

4 Conclusions

In the above text, it was presented the formal system for fuzzy logic based on resolution. The advocacy of such system is simplified into a completeness theorem based on existing calculus. In future it supposed to improve formal correctness of proposed theorem. Another open question is related with the sufficiency of truth valuation of resolvent. The refutational proof with r_R is also interesting area of future research. We suppose to base it on existing notion of consistency threshold. The last (and perhaps the same important) objective lies in the implementation of the rule. With the existing application for two-valued logic [Ha99] it seems to be easy work.

References

[Ba97] Bachmair L., Ganzinger H.: "A theory of resolution", Technical report of the Max-Planck-Institut für Informatik, 1997

[No99] Novák V., Perfilieva I., Močkoř J.: "Mathematical principles of fuzzy logic", Kluwer Academic Publishers, 1999

[Hj97] Hájek P.: "Metamathematics of fuzzy logic", Kluwer Academic Publishers - Dordrecht, 2000

[Le95] Lehmke S.: "On resolution-based theorem proving in propositional fuzzy logic with 'bold' connectives", University of Dortmund, 1995

[Mu95] Mundici D., Cignoli R., D'Ottaviano I.M.L.: "Algebraic foundations of many-valued Reasoning", in preparation

[Ha99] Habiballa H.: "GERDS for Windows",
http://www.volny.cz/habiballa/thesis.exe