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1 Introduction

In this paper, we focus on two questions. The first one is, how a systematic theory of the specific linguistic expressions widely used in the applications of fuzzy logic can be developed. These expressions form a class of the so called evaluating linguistic expressions. These are all expressions such as "small, very small, roughly medium, more or less big", etc. and fuzzy numbers (also modified using linguistic hedges). Such a theory has been proposed in this paper. We argue that it fits well the meaning of the discussed expressions.

The second question is solution of fuzzy relation equations derived from the linguistic data consisting of the above expressions. We show that the pure evaluating linguistic expressions, i.e. those not containing fuzzy number, in most cases do not allow a solution. Thus, the only reasonable way is the use of fuzzy numbers.

2 Preliminaries

We will work with a set of truth values, which, in general, is a residuated lattice, i.e. an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

with four binary operations and two constants such that:

(i) $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with the ordering $\leq$ defined using the operations $\vee, \wedge$ as usual, and $\mathbf{0}, \mathbf{1}$ are its least and the greatest elements, respectively;

(ii) $\langle L, \otimes, \mathbf{1} \rangle$ is a commutative monoid, that is, $\otimes$ is a commutative and associative operation with the identity $a \otimes \mathbf{1} = a$;

(iii) the operation $\rightarrow$ is a residuation operation with respect to $\otimes$, i.e.

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c. \tag{2}$$

More specifically, we will suppose $\mathcal{L}$ to be one of the following.

A *BL-algebra* $\mathcal{L}_{BL}$, which is a residuated lattice fulfilling, moreover, the following conditions:

(a) *prelinearity*

$$(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1},$$

(b) *divisibility*

$$a \otimes (a \rightarrow b) = a \wedge b$$

for all $a, b \in L$. More specifically, we will suppose that $L = [0, 1]$ and $\otimes$ is some continuous t-norm.

The second possibility is that $\mathcal{L}$ is a Łukasiewicz MV-algebra

$$\mathcal{L}_{\text{L}} = \langle L, \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

where $L = [0, 1]$ and

$$\begin{aligned} a \otimes b &= 0 \vee (a + b - 1), && \text{(Łukasiewicz conjunction)} \\ a \oplus b &= 1 \wedge (a + b), && \text{(Łukasiewicz disjunction)} \\ \neg a &= 1 - a, && \text{(negation)} \end{aligned}$$

for all $a, b \in [0, 1]$. The Łukasiewicz implication is a residuation operation in $\mathcal{L}_{\text{L}}$ given by

$$a \rightarrow b = \neg a \oplus b = 1 \wedge (1 - a + b).$$

Let $\otimes$ be a t-norm. By $\rightarrow$, we denote the corresponding residuation operation. Furthermore, we put $z(a) = \bigwedge_{k=1}^{\infty} a^k$ for every $a \in [0, 1]$ where the power is taken with respect to $\otimes$.

A fuzzy set $A \underset{\sim}{\subset} V$ in the universe V is a function

$$A : V \longrightarrow L$$

where L is the support of one of the above mentioned algebras. By $\mathcal{F}(V)$ we denote the set of all fuzzy sets on V, i.e. $\mathcal{F}(V) = L^V$. A binary fuzzy relation in U and V is a fuzzy set

$$R \underset{\sim}{\subset} U \times V.$$

Let $A \underset{\sim}{\subset} U$, $B \underset{\sim}{\subset} V$ be fuzzy sets. Then the residuation operation between A and B is defined by

$$(A \ominus B)(u, v) = A(u) \rightarrow B(v), \qquad u \in U, v \in V.$$

We will also use the following symbol for a cut of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ to $[0, 1]$:

$$f^*(x) = \begin{cases} 1 & \text{if } 1 < f(x), \\ f(x) & \text{if } 0 \le f(x) \le 1, \\ 0 & \text{if } f(x) < 0. \end{cases}$$

The following function will also be used:

$$\delta_z(x) = \begin{cases} 1 & \text{if } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

3 Evaluating Linguistic Expressions

In the applications of fuzzy logic, an important role is played by the so called *evaluating linguistic expressions*, which are linguistic expressions characterizing a position on a bounded ordered scale (usually an interval of some numbers (real, rational, natural, etc.). Examples are "small, medium, big, very small, roughly large, extremely high", etc.

3.1 Linguistic characterization

The symbol $\langle \dots \rangle$ used below denotes a metavariable for the kind of a word given inside the angle brackets.

Definition 1
An evaluating linguistic expression is one of the following:

(i) *Simple evaluating linguistic expression* which is either of the linguistic expressions:

(a)

$$\langle \text{pure evaluating expression} \rangle := \langle \text{linguistic hedge} \rangle \langle \text{atomic evaluating linguistic expression} \rangle$$

where ⟨linguistic hedge⟩ is either *empty hedge* (i.e. no hedge is present), or an intensifying adverb with narrowing effect (*extremely, significantly, very*) or widening effect (*more or less, roughly, quite roughly, very roughly*). Furthermore,

$$\langle \text{atomic evaluating linguistic expression} \rangle$$

is any of the adjectives "small", "medium", or "big".

(b)

$$\langle \text{fuzzy number} \rangle := \langle \text{linguistic hedge} \rangle \langle \text{number} \rangle$$

where ⟨linguistic hedge⟩ is either "about" or an intensifying adverb with widening effect and ⟨number⟩ is a name of a real number (element taken from $\mathbb{R}$).

(ii) *Compound evaluating linguistic expression*
'$\mathcal{A}$ or $\mathcal{B}$' where $\mathcal{A}, \mathcal{B}$ are evaluating linguistic expressions.

Note that empty hedge is formally equal to stating that atomic evaluating expression is also simple one. However, it is advantageous to consider "empty hedge" as a specific linguistic hedge.

Example 1
Atomic evaluating linguistic expressions are *small, medium, big.* Fuzzy numbers are, e.g. *twenty five, the value z, roughly 100*, etc. Simple evaluating linguistic expressions are *very small, more or less medium, roughly big, about twenty five, approximately* x_0, etc. Compound evaluating linguistic expressions are *roughly small or medium* , etc.

The "fuzzy number" is a linguistic characterization of some number. This means that every linguistic characterization of a number is understood imprecisely. We will take the form "about x_0" as canonical. It will play a similar role as "empty hedge" in the case of fuzzy number.

It is noticeable that atomic evaluating linguistic expressions usually form pairs of antonyms, i.e. the pairs

"nominal adjective — antonym".

Of course, there are a lot of other examples, e.g. "young — old", "ugly — nice", "stupid — clever", etc. The triple of expressions

⟨linguistic hedge⟩⟨nominal adjective⟩ —
⟨linguistic hedge⟩⟨middle member⟩ —
⟨linguistic hedge⟩⟨antonym⟩

is called the *evaluating trichotomy.* If all the linguistic hedges are empty then it is the *basic evaluating trichotomy.*

Definition 2
Let $\mathcal{A}$ be an evaluating linguistic expression. Then the linguistic expression

$$\langle \text{noun} \rangle \text{ is } \mathcal{A} \tag{3}$$

is an *evaluating predication.* If $\mathcal{A}$ is a simple evaluating linguistic expression then (3) is a a simple evaluating predication.

If $\mathcal{A}$ and $\mathcal{B}$ are evaluating predications then '$\mathcal{A}$ and $\mathcal{B}$' and '$\mathcal{A}$ or $\mathcal{B}$' are compound predications.

Example 2
Evaluating predications are, e.g. "*temperature is very high*" (here "high" is taken instead of "big"), "*pressure is not small*", "*frequency is small or medium*", "*cost is not small and not big*", "*income is roughly three million*", etc. Compound evaluating predication is, e.g. "*temperature is high and pressure is very high*".

The following definition characterizes a fragment $\mathcal{S}$ of natural language (English) employed in fuzzy logic.

Definition 3
The set $\mathcal{S}$ of the considered linguistic expressions consists of evaluating linguistic expressions, evaluating predications and the conditional clause

$$\mathsf{IF}\ \mathcal{A}\ \mathsf{THEN}\ \mathcal{B} \tag{4}$$

where $\mathcal{A}$, $\mathcal{B}$ are evaluating linguistic predications.

3.2 Semantics of evaluating linguistic expressions

The general model of the semantics of linguistic expressions is based on the distinction between their intension and extension in the sense introduced by R. Carnap ([2]).

Intension of a linguistic expression, sentence, or of a concept, can be identified with the property denoted by it. An intension may lead to different truth values in various possible worlds but it is invariant with respect to them.

Extension is a class of elements determined by an intension, which fall into the meaning of a linguistic expression in a given possible world. Thus, it depends on the particular context of use — it does not change when changing the possible world (context, time, place). The intension is assumed to keep the Frege's compositionality principle: a more complex intension is a function of simpler ones.

Expressions $\mathcal{A}$ of natural language are names of intensions. Let us remark that a lot of convincing arguments have been given to the statement (see, e.g., [8]) that the meaning of expressions of natural language cannot be identified with their extensions (the mentioned compositionality principle is broken).

Formalization of these ideas for the development of the semantics of evaluating linguistic expressions in fuzzy logic will follow the fundamental ideas considered in the intensional logic (cf. [8]).

Let W be a set of possible worlds. These can be understood as special parameters representing the particular state of affairs (or context of use). Moreover, let a set V be given, which represents objects. In the case of evaluating expressions, V will be a set of values of some features of objects, such as temperature, pressure, height, width, etc. It is reasonable to consider that $V = \mathbb{R}$.

Let us stress that on the basis of this idea, the real objects (e.g. a controlled system, situation, human being, etc.) can be mathematically represented by some vector of values of its features. More precisely, let o be such an object and let us distinguish its features $\varphi_1, \ldots, \varphi_n$. Each feature φ_i can attain values from some set V_i. Then a given object o is represented by a vector of values

$$o = \langle v_1, \ldots, v_n \rangle \in V_1 \times \cdots \times V_n.$$

As a special case, we may take $V_i = \mathbb{R}$, $i = 1, \ldots, n$. Then a real object is represented as a vector of real numbers. This is fully in accordance with the assumed practice, and also database systems where, e.g. a person is represented by a record being, in fact, a sequence of numbers (age, height, wage, etc.).

The *intension* is formally a function

$$A : W \longrightarrow \mathcal{F}(V). \tag{5}$$

The *extension* in the given possible world $w \in W$ is a fuzzy set

$$A(w) \underset{\sim}{\subset} V.$$

Note that this definition (taken from intensional logic) is in accordance with Carnap's idea that the extension must be recapturable from the intension. It is also clear that while extension changes, intension remains the same independently on the possible world.

For the semantics of the evaluating linguistic expressions we introduce the following definition.

Definition 4
The *intensional* space is given by:

(i) A set of possible worlds $W = \{\langle v_L, v_S, v_R \rangle\}$ where $v_L, v_S, v_R \in [0, \infty)$ and $v_L < v_S < v_R$.

(ii) An algebra of truth values is either a BL-algebra $\mathcal{L}_{BL}$ where $L = [0, 1]$ and $\otimes$ is a continuous t-norm, or it is a Łukasiewicz algebra $\mathcal{L}_{Ł}$.

(iii) A couple of linear functions $L, R : W \times \mathbb{R} \longrightarrow L$ defined in each possible world $w \in W$ by

$$L_w(x) = \left(\frac{v_S - x}{v_S - v_L}\right)^* \qquad \text{(left horizon)} \tag{6}$$

$$R_w(x) = \left(\frac{x - v_S}{v_R - v_S}\right)^* \qquad \text{(right horizon)} \tag{7}$$

and a *middle horizon* function

$$M_w(x) = \neg L_w(x) \wedge \neg R_w(x) = \left(\frac{x - v_L}{v_S - v_L}\right)^* \wedge \left(\frac{v_R - x}{v_R - v_S}\right)^* . \tag{8}$$

(iv) A set of linear functions defined in each possible world $w \in W$ by

$$\mathbf{B}_w = \{B_{w,x_0} \mid \\ B_{w,x_0}(x) = \left(\frac{x - x_0 + h_L}{h_L}\right)^* \wedge \left(\frac{x_0 - x + h_R}{h_R}\right)^* \text{ or } B_{w,x_0}(x) = \delta_{x_0}(x), \\ x_0 \in [v_L, v_R], 0 < h_L < x_0 - v_L, 0 < h_R < v_r - x_0\} \tag{9}$$

(v) A class of *abstract hedges*

$$\mathbf{Hf} = \{\nu : [0, 1] \longrightarrow [0, 1] \mid \nu \in \mathbf{Hf}^{quad} \cup \mathbf{Hf}^{lin} \tag{10}$$

where

$$\mathbf{Hf}^{quad} = \{\nu^{quad}_{a,b,c} \mid a, b \in (-\infty, 1), c \in (0.5, 1], a < b < c\}$$

$$\nu^{quad}_{a,b,c}(y) = \begin{cases} 1, & c \leq y, \\ 1 - \frac{(c-y)^2}{(c-b)(c-a)}, & b \leq y < c, \\ \frac{(y-a)^2}{(b-a)(c-a)}, & a \leq y < b, \\ 0, & y < a \end{cases} \tag{11}$$

and

$$\mathbf{Hf}^{lin} = \{\nu^{lin}_{a,c} \mid a \in (-\infty, 1], c \in (0.5, 1], a < b < c\}$$

$$\nu^{lin}_{a,c}(y) = \begin{cases} 1, & c \leq y, \\ \frac{(y-a)}{(c-a)}, & a \leq y < c, \\ 0, & y < a. \end{cases} \tag{12}$$

Note that possible worlds are in this definition identified with intervals of real numbers. Of course, we can extend this definition to intervals in arbitrary ordered set. This is unnecessary for our explanation below. To simplify the notation, if $v \in \mathbb{R}$ and $w = \langle v_L, v_S, v_R \rangle$ is a possible world then we will often write $v \in w$ instead of $v \in [v_L, v_R]$.

Alternatively, we may introduce in (11) a simplified class

$$\nu^{quad}_{a,c}(y) = \begin{cases} 1, & c \leq y, \\ 1 - \frac{(c-y)^2}{(c-a)^2}, & \frac{a+c}{2} \leq y < c, \\ \frac{(y-a)^2}{(c-a)^2}, & a \leq y < \frac{a+c}{2}, \\ 0, & y < a. \end{cases}$$

If $a, b \in [0, 1)$ then the abstract hedge is *pure*, otherwise it is *modified.* The function ν can be also seen as a *deformation* of the horizon. To simplify notation, we will often write only $\nu \in \mathbf{Hf}$ understanding that it is, in fact, determined by the parameters a, b, c as in (10).

We will say that an abstract hedge $\nu_1 \in \mathbf{Hf}$ is sharper than $\nu_2 \in \mathbf{Hf}$, $\nu_1 < \nu_2$, if $\langle a_2, b_2, c_2\rangle < \langle a_1, b_1, c_1\rangle$.

Lemma 1
If $\nu_1 < \nu_2$ *then* $\nu_1(y) \leq \nu_2(y)$, $y \in [0, 1]$.

Definition 5
The following are special classes of intensions:

(i) Type *Small*

$$\mathbf{Sm} = \{\mathrm{Sm}_\nu : W \longrightarrow \mathcal{F}(\mathbb{R}) \mid \mathrm{Sm}_\nu(w) = \nu(L_w(x)), \nu \in \mathbf{Hf}\}$$

(ii) Type *Medium*

$$\mathbf{Me} = \{\mathrm{Me}_\nu : W \longrightarrow \mathcal{F}(\mathbb{R}) \mid \mathrm{Me}_\nu(w) = \nu(M_w(x)), \nu \in \mathbf{Hf}\}$$

(iii) Type *Big*

$$\mathbf{Bi} = \{\mathrm{Bi}_\nu : W \longrightarrow \mathcal{F}(\mathbb{R}) \mid \mathrm{Bi}_\nu(w) = \nu(R_w(x)), \nu \in \mathbf{Hf}\}$$

(iv) Type *fuzzy number*

$$\mathbf{Fn} = \{\mathrm{Fn}_{\nu,x_0} : W \longrightarrow P(\mathcal{F}(\mathbb{R})) \mid \\ \mathrm{Fn}_{\nu,x_0}(w)(x) = \{\nu(B_{w,x_0}(x)) \mid B_{w,x_0} \in \mathbf{B}_w\}, \nu \in \mathbf{Hf}\}\}$$

Note that this definition also includes crisp numbers as special case of the fuzzy ones. If x_0 is such a number then $\nu(B_{w,x_0}(x)) = 1$ for all *pure* $\nu \in \mathbf{Hf}$.

We will generally denote the class of intensions of the evaluating expressions by

$$\mathbf{Ev} = \mathbf{Sm} \cup \mathbf{Me} \cup \mathbf{Bi} \cup \mathbf{Fn}\,.$$

A special class are *pure evaluating expressions*, whose intensions are

$$\mathbf{Ev}^{pure} = \mathbf{Sm} \cup \mathbf{Me} \cup \mathbf{Bi}\,.$$

If $\mathrm{Ev} \in \mathbf{Ev}$ is an intension and $w \in W$ is a possible world then $\mathrm{Ev}(w)$ is the corresponding *extension* of Ev in w.

This model stems from the idea that the meaning of evaluating expressions is determined by some horizon, which we can see in the given possible world. This concept, which has been borrowed from P. Vopěnka [15], means that we can encounter in a given possible world a line the world before which is defined unsharply. This usharpness is mathematically captured by a simple linear functions $L(x), R(x)$ and $M(x)$ representing the truth degree uniformly diminishing when moving away from the position of observer. The meaning of the pure evaluating expressions is then obtain by specific deformation of the horizon.

Remark 1 (to the notation)
We are working with several classes of functions, namely $\mathbf{Sm}, \mathbf{Me}, \mathbf{Bi}, \mathbf{Fn}$ which are altogether denoted by $\mathbf{Ev}$. Their members are functions from the set W of possible worlds to some set of fuzzy sets. Thus, if $\mathrm{Ev} \in \mathbf{Ev}$ is a function then $\mathrm{Ev}(w)$ is a fuzzy set in the given possible world w. Since $\mathrm{Ev}(w)$ is itself a function, then if $u \in w$ is some element (recall that the latter means $u \in [v_L, v_R]$), then its membership degree in $\mathrm{Ev}(w)$ is $\mathrm{Ev}(w)(u)$. To simplify the notation, we will better write $\mathrm{Ev}_w(u)$ in such a case. As a special case, we will write $\mathrm{Fn}_{\nu,x_0,w}$ instead of $\mathrm{Fn}_{\nu,x_0}(w)$

When writing an element of, say $\mathrm{Sm} \in \mathbf{Sm}$, then we know that Sm is determined also by some abstract hedge ν, i.e. we mean Sm_ν. However, if ν does not matter, we usually omit it from the subscript. This convention will be used also with other symbols.

Let $\mathrm{Ev}_1, \mathrm{Ev}_2$ be intensions of the same type containing the hedges ν_1, ν_2, respectively. Then Ev_1 is *sharper than* Ev_2, $\mathrm{Ev}_1 < \mathrm{Ev}_2$ if $\nu_1 < \nu_2$.

Lemma 2
Let $\mathrm{Ev}_1, \mathrm{Ev}_2 \in \mathbf{Ev}^{pure}$ *be of the same type and* $\mathrm{Ev}_1 < \mathrm{Ev}_2$. *Then*

$$\mathrm{Ev}_{1,w} \leq \mathrm{Ev}_{2,w}$$

for every possible world $w \in W$.

PROOF: This immediately follows from Lemma 1 since $\nu_1 < \nu_2$ by the definition. □

Lemma 3
In Łukasiewicz algebra $p \to \mathrm{Ev}_w \in \mathbf{Ev}$ *and* $p \otimes \mathrm{Ev}_w \in \mathbf{Ev}$ *holds for every* $p \in (0,1]$.

PROOF: This follows from rather tedious proof of the fact that $\mathrm{Ev}(w) + k \in \mathrm{Ev}$ for $k \in [0,1)$. □

Linguistic assignments

Definition 6
Let $\mathcal{A}$ be a a simple evaluating linguistic expression. Then its meaning is identified with its intension and, in general, it is a function

$$\mathrm{Int}(\mathcal{A}) \in \mathbf{Ev}$$

for which ν is a *pure hedge function.* More specifically,

$$\mathrm{Int}(\langle\text{linguistic hedge}\rangle\ \textit{small}) \in \mathbf{Sm},$$

$\mathrm{Int}(\langle\text{linguistic hedge}\rangle\ \textit{medium} \in \mathbf{Me}$, $\mathrm{Int}(\langle\text{linguistic hedge}\rangle\ \textit{big}) \in \mathbf{Bi}$ and

$$\mathrm{Int}(\langle\text{linguistic hedge}\rangle\langle\text{fuzzy number}\rangle) \in \mathbf{Fn}\,.$$

In more details, $\mathrm{Int}(\langle\text{linguistic hedge}\rangle x_0) = \mathrm{Fn}_{\nu,x_0}$.

Remark 2
If $\mathcal{A}$ is an evaluating linguistic predication (3) then we put its intension *equal* to the intension of the evaluating expression inside (3).

The extension of $\mathcal{A}$ in $w \in W$ is

$$\mathrm{Ext}_w(\mathcal{A}) = \mathrm{Int}(\mathcal{A})(w),$$

i.e. if $w = \langle v_L, v_S, v_R\rangle$ then it is a fuzzy set $\mathrm{Ext}_w(\mathcal{A}) = A \underset{\sim}{\subset} [v_L, v_R]$. The construction of the extension of the basic evaluating trichotomy is depicted on Fig. 1. We have marked in the figure also the distinguished

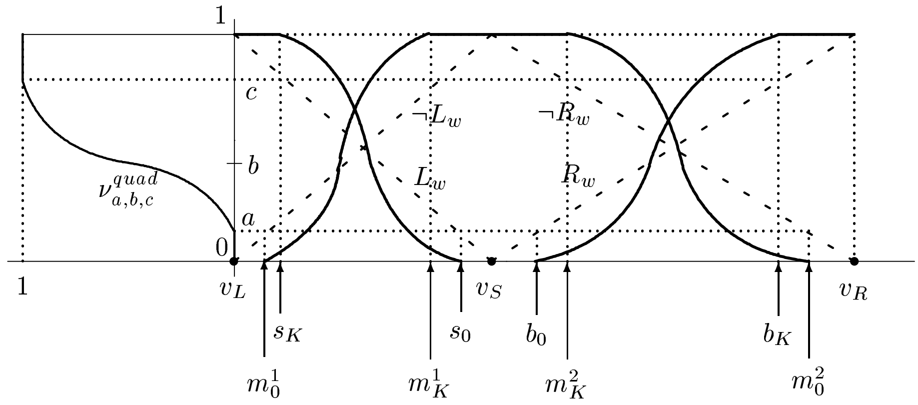

Figure 1: Schematic picture of the construction of the extension of evaluating linguistic expressions.

points defined in the given possible world for each intension. Let $\nu \in \mathbf{Hf}$ be given by the parameters a, b, c. Then we put:

$$\begin{aligned} s_K &= L_w^{-1}(c), & s_0 &= L_w^{-1}(a), && \text{(for small)} \\ m_K^1 &= (\neg L_w)^{-1}(c), & m_0^1 &= (\neg L_w)^{-1}(a), && \text{(for left part of medium)} \\ m_K^2 &= (\neg R_w)^{-1}(c), & m_0^2 &= (\neg R_w)^{-1}(a), && \text{(for right part of medium)} \\ b_K &= R_w^{-1}(c), & b_0 &= R_w^{-1}(a). && \text{(for big)} \end{aligned}$$

We will introduce the following terminology, which is useful for further explanation. The set $[v_L, s_K]$ is called the *kernel* of Sm_w and denoted by $\mathrm{Ker}(\mathrm{Sm}_w)$. Similarly, $\mathrm{Ker}(\mathrm{Me}_w) = [m_K^1, m_K^2]$, $\mathrm{Ker}(\mathrm{Bi}_w) = [b_K, v_R]$ are kernels of Me_w and Bi_w, respectively.

Similarly, the sets $[s_0, v_R]$, $[v_L, m_0^1] \cup [m_0^2, v_R]$ and $[v_L, b_0]$ are *zero areas* and the remaining parts are *vagueness areas.*

Lemma 4
Let $\mathrm{Ev}_1, \mathrm{Ev}_2$ *be intensions of different types and* $\nu_1 = \nu_{a_1,b_1,c_1} \in \mathbf{Hf}$ *and* $\nu_2 = \nu_{a_2,b_2,c_2} \in \mathbf{Hf}$ *be their respective hedges. Then*

(a) $\mathrm{Ker}(\mathrm{Ev}_{1,w}) \cap \mathrm{Ker}(\mathrm{Ev}_{2,w}) = \emptyset$ *iff* $c_1 + c_2 > 1$.

(b) If $\mathrm{Ev}_1 < \mathrm{Ev}_2$ *then* $\mathrm{Ker}(\mathrm{Ev}_{1,w}) \subset \mathrm{Ker}(\mathrm{Ev}_{2,w})$.

PROOF: (a) easily implies from (6), (7) and (8) and the above definition.
(b) immediately from the definition of $<$. □

This lemma clarifies why we have put $c > 0.5$ in (10). If the kernels of neighboring extensions were overlapping then the intuitive meaning of them as interpretation of the meaning of evaluating linguistic expressions would become dubious.

We will now select several adverbs, namely

> *extremely (Ex), significantly (Si), very (Ve), more or less (ML), roughly (Ro), quite roughly (QR), very roughly (VR)*

as basic linguistic hedges (in the brackets are shorts used below).

Furthermore, we will choose numbers $a_0, b_0, c_0 \in [0, 1]$ and assign the abstract hedge $\nu_{a_0,b_0,c_0} \in \mathbf{Hf}^{quad}$ to the "empty hedge". Then we choose three abstract hedges $\nu_{Ex}, \nu_{Si}, \nu_{Ve} \in \mathbf{Hf}^{quad}$, for which

$$\nu_{Ex} < \nu_{Si} < \nu_{Ve} < \nu_{a_0,b_0,c_0}$$

Linguistic hedge	a	b	c
Extremely	0.5	0.75	0.95
Significantly	0.47	0.6	0.8
Very	0.35	0.58	0.83
empty	0.27	0.5	0.8
Rather	0.4	0.5	0.8
More or less	0.23	0.45	0.76
Roughly	0.2	0.4	0.7
Quite roughly	0.15	0.32	0.65
Very roughly	0.09	0.2	0.6

Table 1: Experimentally found values of parameters a, b, c of some linguistic hedges.

holds and four abstract hedges $\nu_{ML}, \nu_{Ro}, \nu_{QR}, \nu_{VR} \in \mathbf{Hf}^{quad}$, for which

$$\nu_{a_0,b_0,c_0} < \nu_{ML} < \nu_{Ro} < \nu_{QR} < \nu_{VR}.$$

Finally we assign these hedges to the above selected words and understand the former as the *meaning* of the latter. This procedure enables to construct the meaning of each evaluating expression, which is the intension and if given a possible world, also their extension. Let us remark that in fact, we should introduce also a special grammar which would prevent, e.g. combination of narrowing linguistic hedge with medium (e.g. "very medium" has no sense). We will not care for this problem in this paper.

Note that we can also define other kinds of modifiers not belonging to the above group, for example "rather". This should be assigned the abstract hedge $\nu_{a,b,c}$ with $a > a_0$, $b \leq b_0$ and $c < c_0$. Thus, the above theory encompasses a large class of linguistic hedges.

The experimentally found values of the parameters a, b, c of the discussed linguistic hedges are given on Table 1.

Shapes of the membership functions of extensions of the experimentally set pure evaluating expressions are depicted on Figure 2

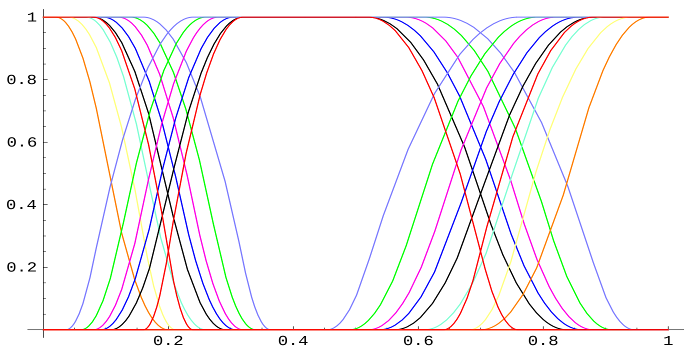

Figure 2: Membership functions of extensions of selected evaluating expressions from Table 1.

Let us also remark that linear abstract hedges could be used when a significant simplification (for the linguistic purposes non well suitable) is needed. However, from the linguistic point of view, they seem to be oversimplification of the problem. These hedges might be reasonable for the fuzzy numbers, which form a quite specific class of the linguistic expressions. Note that $\nu^{lin}_{0,1}$ gives extension equal to the original linear function (6), (7), (8) and (9).

4 Fuzzy Relation Equations

4.1 General solution

In this section, we will focus on fuzzy relation equations. Recall that this problem has been initiated by E. Sanchez ([14]) and studied further by many researchers (cf. Di Nola et. al. [3], S. Gottwald [4], I. Perfilieva and A. Tonis [11]). The initial situation is the following.

We are given fuzzy data, which is a finite set of tuples

$$\begin{array}{c}\langle A_{11}, \dots, A_{1n}, B_1 \rangle \\ \dots\dots\dots\dots\dots\dots\dots \\ \langle A_{m1}, \dots, A_{mn}, B_m \rangle\end{array} \tag{13}$$

$$ \tag{14}$$

where $A_{j,i} \mathrel{\underset{\sim}{\subset}} U_i$, $B_j \mathrel{\underset{\sim}{\subset}} V$, $j = 1, \dots, m$, $i = 1, \dots, n$ are fuzzy sets. We will often suppose that $U_i, V \subset \mathbb{R}$ are a compact sets.

The imprecise information leading to fuzzy data is quite often given using linguistic expressions. Thus, the first information we are given takes the form

$$\begin{array}{c}\langle \mathcal{A}_{11}, \dots, \mathcal{A}_{1n}, \mathcal{B}_1 \rangle \\ \dots\dots\dots\dots\dots\dots\dots \\ \langle \mathcal{A}_{m1}, \dots, \mathcal{A}_{mn}, \mathcal{B}_m \rangle\end{array} \tag{15}$$

where $\mathcal{A}_{ji}, \mathcal{B}_j$ are evaluating linguistic predications. Then using the results of the previous section, (15) leads to the fuzzy data of the form

$$\begin{array}{c}\langle \mathrm{Ev}_{11,w_1}, \dots, \mathrm{Ev}_{1n,w_n}, \mathrm{Ev}_{1(n+1),w_{n+1}} \rangle \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \langle \mathrm{Ev}_{m1,w_1}, \dots, \mathrm{Ev}_{mn,w_n}, \mathrm{Ev}_{m(n+1),w_{n+1}} \rangle\end{array} \tag{16}$$

In (16), Ev_{ji,w_i}, $j = 1, \dots, m$, $i = 1, \dots, n+1$ are extensions of the linguistic predications from (15) in some possible worlds w_i. Clearly, the latter take the role of the universes $U_1, \dots, U_n, V$ considered above (these are compact sets by Definition 4).

The problem is to find a fuzzy relation $R \mathrel{\underset{\sim}{\subset}} U_1 \times \cdots \times U_n \times V$ such that

$$B_j = (A_{j1} \times \cdots \times A_{jn}) \circ R \tag{17}$$

holds for all $j = 1, \dots, m$ where the membership function of B_j in (17) is given by

$$B_j(v) = \bigvee_{\substack{u \in U_i \\ i=1,\dots,n}} ((A_{j1} \times \cdots \times A_{jn})(u_1, \dots, u_n) \otimes R(u, v)), \qquad v \in V \tag{18}$$

and $\otimes$ is a t-norm. If such relation exists, then we say that the system of fuzzy relation equations (17) is *solvable*. We will usually suppose that the composition in (18) is defined w.r.t. a continuous (left continuous) t-norm.

Let us denote

$$\hat{R} = \bigwedge_{j=1}^{m} ((A_{j1} \times \cdots \times A_{jn}) \ominus B_j). \tag{19}$$

Then the general solution is given in the following theorem.

Theorem 1
Given the data (13). Then the system (17) is solvable iff

$$B_k(v) = \bigvee_{\substack{u \in U_i \\ i=1,\dots,n}} ((A_{k1} \times \cdots \times A_{kn})(u_1, \dots, u_n) \otimes \hat{R}(u_1, \dots, u_n, v)), \qquad v \in V \tag{20}$$

holds for every $k = 1, \dots, m$. If (20) holds then $\hat{R}$ is the greatest solution of (17).

Note that it follows from this theorem that Cartesian product $(A_{j1} \times \cdots \times A_{jn})$ in (19) and (20) can be defined using arbitrary t-norm, or even by a more general operation. In the sequel, we will for simplicity take $n = 1$.

Given a couple of fuzzy data $\langle A, B\rangle$, we will put

$$X(v) := \{u \in U \mid A(u) \geq B(v)\} \subseteq U, \qquad v \in V. \tag{21}$$

In case that $A := \mathrm{Ev}_{1,w}$, $B := \mathrm{Ev}_{2,w}$ for some possible world and intensions $\mathrm{Ev}_1, \mathrm{Ev}_2$, we will denote (21) by $X_{E_1E_2}$, or, in more details, $X_{SS}, X_{SM}, \ldots$ if $\mathrm{Ev}_1 := \mathrm{Sm}$ and $\mathrm{Ev}_2 := \mathrm{Sm}$ or $\mathrm{Ev}_2 := \mathrm{Me}$, respectively.

The following theorem is proved in [11].

Theorem 2 (Perfilieva and Tonis)
Let all the membership functions of the fuzzy data (13) be continuous. Then the system (17) is solvable iff for each i, $i = 1, \ldots, m$ and every $v \in V$ there exists $u \in X_i(v)$ such that $z(B_j(v)) < B_i(v)$ implies

$$A_i(u) \to B_i(v) \leq A_j(u) \to B_j(v) \tag{22}$$

for each $j \neq i$ and $j = 1, \ldots, m$.

4.2 Pure evaluating expressions in fuzzy relations

In this subsection we will discuss the situation when the data are given by pure evaluating expressions.

Lemma 5
Let us consider a couple of data

$$\langle \mathrm{Ev}_{1,w}, \mathrm{Ev}_{2,w'}\rangle$$

for some possible worlds $w, w' \in W$ where $\mathrm{Ev}_1, \mathrm{Ev}_2$ are of the same type and $\mathrm{Ev}_1 < \mathrm{Ev}_2$. If $v \in \mathrm{Ker}(\mathrm{Ev}_{2,w'})$ then $X_{E_1E_2}(v) = \mathrm{Ker}(\mathrm{Ev}_{1,w})$.

PROOF: Immediately from the definition (21). □

Theorem 3
Let the fuzzy data (13) contain couples

$$\langle \mathrm{Ev}_{1,w}, \mathrm{Ev}_{2,w'}\rangle$$
$$\langle \mathrm{Ev}_{2,w}, \mathrm{Ev}_{3,w'}\rangle$$

where $\mathrm{Ev}_1, \mathrm{Ev}_2 \in \mathbf{Ev}^{pure}$, $\mathrm{Ev}_1 < \mathrm{Ev}_2$, are of the same type and $\mathrm{Ev}_3 \in \mathbf{Ev}$ is of the type different from the former. Then the system (17) is not solvable.

PROOF: We will use Theorem 2. We must show that there is $v \in w'$ such that for every $u \in X_{E_1E_2}(v)$, if $z(\mathrm{Ev}_{3,w'}(v)) < \mathrm{Ev}_{2,w'}(v)$ then

$$\mathrm{Ev}_{1,w}(u) \to \mathrm{Ev}_{2,w'}(v) > \mathrm{Ev}_{2,w}(u) \to \mathrm{Ev}_{3,w'}(v).$$

Let us take $v \in \mathrm{Ker}(\mathrm{Ev}_{2,w'}) - \mathrm{Ker}(\mathrm{Ev}_{1,w'})$. Then $z(\mathrm{Ev}_{3,w'}(v)) \leq \mathrm{Ev}_{3,w'}(v) < \mathrm{Ev}_{2,w'}(v)$ because $\mathrm{Ker}(\mathrm{Ev}_{2,w'}) \cap \mathrm{Ker}(\mathrm{Ev}_{3,w'}) = \emptyset$. By Lemma 5, $X_{E_1E_2}(v) = \mathrm{Ker}(\mathrm{Ev}_{1,w})$.

Then for every $u \in X_{E_1E_2}(v)$, $\mathrm{Ev}_{1,w}(u) \to \mathrm{Ev}_{2,w'}(v) = 1$ and $\mathrm{Ev}_{2,w}(u) = 1$ because $\mathrm{Ker}(\mathrm{Ev}_{1,w}) = X_{E_1E_2}(v) \subset \mathrm{Ker}(\mathrm{Ev}_{2,w})$. However, $\mathrm{Ev}_{3,w'}(v) < 1$ because $\mathrm{Ker}(\mathrm{Ev}_{2,w'}) \cap \mathrm{Ker}(\mathrm{Ev}_{3,w'}) = \emptyset$. □

This theorem demonstrates that if we insist on the use of pure evaluating expressions then we are very limited provided we want the corresponding system of fuzzy relation equations to be solvable. In practice this means that for this case, pure linguistic expressions are inappropriate and thus, we should better take the fuzzy data to be fuzzy numbers.

Let us remark that this does not disqualify pure evaluating expressions from any use but only from the attempt to solve fuzzy relation equations with them. These expressions are, on the other hand, much more appropriate when deriving a conclusion using the logical deduction (cf., e.g., [1]).

4.3 Pure evaluating expressions in fuzzy IF-THEN rules

Let us remark that, in general, logical deduction leads to firing of one fuzzy IF-THEN rule. Let us analyze, what can be obtained by such a rule.

Recall from Definition 3 that fuzzy IF-THEN rules are linguistically a special kind of conditional clause. Its semantics will be defined as follows.

Definition 7
Let the fuzzy IF-THEN rule $\mathcal{R}$ have the form (4). Then its meaning*) is

$$\mathrm{Int}\,\mathcal{R} := \mathrm{Ev}_1 \Rightarrow \mathrm{Ev}_2$$

where $\Rightarrow$ is a connective interpreted by some implication operation $\rightarrow$. Its extension is defined in a couple of possible worlds $w, w' \in W$ by

$$\mathrm{Ext}_{\langle w,w'\rangle}(\mathcal{R}) := \mathrm{Ev}_{1,w} \rightarrow \mathrm{Ev}_{2,w'} \tag{23}$$

where (23) is a fuzzy relation defined pointwisely.

Let us now define a defuzzification operation DEE for the evaluating linguistic expressions as follows: let $\mathrm{Ev} \in \mathbf{Ev}$ be an evaluating expression and $w \in W$ a possible world. Then the defuzzification operation DEF is

$$\mathrm{DEE}(\mathrm{Ev}_w) = \begin{cases} \mathrm{LOM}(\mathrm{Sm}_w), & \text{if } \mathrm{Ev} \in \mathbf{Sm}, \\ \mathrm{COG}(\mathrm{Ev}_w), & \text{if } \mathrm{Ev} \in \mathbf{Me} \text{ or } \mathrm{Ev} \in \mathrm{Fn}, \\ \mathrm{FOM}(\mathrm{Bi}_w), & \text{if } \mathrm{Ev} \in \mathbf{Bi} \end{cases} \tag{24}$$

where LOM is the *Least of Maxima*, FOM is the *First of Maxima* and COG is the *Center of Gravity* method.

Let $w = \langle v_L, v_S, v_R\rangle$ be a possible world. Then

$$\mathrm{LOM}(\mathrm{Sm}_{\nu,w}) = cv_L + (1-c)v_S, \tag{25}$$
$$\mathrm{FOM}(\mathrm{Bi}_{\nu,w}) = cv_R + (1-c)v_S \tag{26}$$

where $c \in (0.5, 1]$ is the parameter of the corresponding linguistic hedge ν.

Let $\mathcal{R}$ be a fuzzy IF-THEN rule (4) with the extension (23) in some possible worlds w, w'. Then it determines a function f_R by

$$f_R(u) = \mathrm{DEE}(\mathrm{Ev}_{1,w}(u) \rightarrow \mathrm{Ev}_{2,w'}). \tag{27}$$

To distinguish more subtly various kinds of the function f_R dependingly on the used evaluating expressions in the rule (4), we will write $f_{S_1S_2}$ if $\mathcal{R} :=$ IF X is *Small* THEN Y is *Small*, $f_{S_1B_2}$ if $\mathcal{R} :=$ IF X is *Small* THEN Y is *Big*, etc.

A rather technical and routine proof gives us the following theorem.

Theorem 4
Let $w, w' \in W$ be possible worlds and $\mathcal{R}$ a fuzzy IF-THEN rule (E6). Then the function f_R from (27) is continuous. More specifically, f_{S_1,S_2} is nondecreasing, f_{B_1,B_2} is nondecreasing, f_{S_1,B_2} is nonincreasing, f_{B_1,S_2} is nonincreasing, f_{S_1,M_2} is nonincreasing until $v'_S \in w'$ and nondecreasing further and f_{B_1,M_2} is opposite.

Let us define a *pseudoinverse* of $\nu_{a,b,c}$ in (10) by

$$\nu^{(-1)}_{a,b,c}(z) = \begin{cases} \nu^{-1}(z) & \text{if } z \in (0,1), \\ c & \text{if } z = 1, \\ a & \text{if } z = 0. \end{cases}$$

*) In accordance with Remark 2, we semantically do not distinguish evaluating linguistic predications from the expressions inside them

Then, for example, the function $f_{S_1S_2}$ is given by the formula

$$f_{S_1S_2}(u) = \nu_2^{(-1)}(\mathrm{Sm}_{1,w}(u))v_L + (1 - \nu_2^{(-1)}(\mathrm{Sm}_{1,w}(u)))v_S$$

where $v_L, v_S \in w'$ and ν_2 is an abstract hedge inside Sm_2.

It follows from this theorem that by logical deduction we can obtain a piecewise continuous and monotonous function. An example of the behavior of the linguistic description (rule base) consisting of a set of monotonous rules such as

IF X is $ExSm$ THEN Y is $ExSm$
IF X is $SiSm$ THEN Y is $SiSm$
IF X is $VeSm$ THEN Y is $VeSm$
IF X is Sm THEN Y is Sm
IF X is $RoSm$ THEN Y is $RoSm$
etc.

is depicted on Figure 3

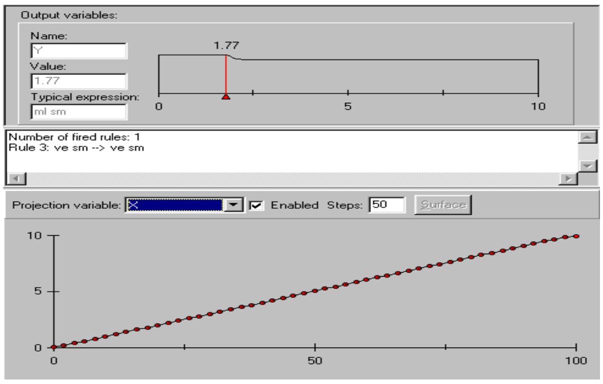

Figure 3: Example of the construction of linguistic expressions so that the monotonous linguistic description leads to almost linear course.

4.4 Fuzzy numbers in fuzzy relations

As discussed in the previous section, pure evaluating expressions lead to mostly unsolvable fuzzy relation equations. Therefore, we we will confine only to fuzzy numbers in this subsection.

By a simple computation we get the following lemma.

Lemma 6
Let the data (13) contain only fuzzy numbers, i.e. they are of the form

$$\langle \mathrm{Fn}_{\nu_1,x_0,w}, \mathrm{Fn}_{\nu_2,y_0,w'} \rangle \tag{28}$$

then the set (21) is

$$X_1(v) = [x_0 - h_{L,1}(1 - \nu_1^{(-1)}(\mathrm{Fn}_{\nu_2,y_0,w'})(v)), x_0 + h_{R,1}(1 - \nu_1^{(-1)}(\mathrm{Fn}_{\nu_2,y_0,w'})(v))].$$

Moreover, if $\mathrm{Fn}_{\nu_2,y_0,w'}(v) \geq \mathrm{Fn}_{\nu_2,y_0,w'}(v')$ *then* $X_1(v) \subseteq X_2(v')$.

Theorem 5
Let the fuzzy data (13) consist of fuzzy numbers of the form (28) and the corresponding system of fuzzy relations be solvable. Let

$$\langle A := \mathrm{Fn}_{\nu_1,x_0,w}, B := \mathrm{Fn}_{\nu_2,y_0,w'} \rangle$$

be a new data. If $A_j(x_0) = 0$ *for all* $j = 1, \ldots, m$ *then the new system is solvable.*

PROOF: By Theorem 2: Let $v = y_0$. Then $B(y_0) = 1$ and thus, $X(y_0) = \{u \in w \mid A(u) = 1\}$, i.e. $x_0 \in X(y_0)$. Let $z(B_j(y_0)) < 1 = B(y_0)$. Then there must exist $u \in X(y_0)$ such that

$$A(u) \to 1 \leq A_j(u) \to B_j(y_0). \tag{29}$$

By the assumption, $A_j(x_0) = 0$ and thus (29) is fulfilled. Since by Lemma 6, $X(v)$ forms a nested system, $x_0 \in X(v)$ for every $v \in w'$ and thus (29) is always fulfilled. □

To finish this discussion, we will present the following theorem, which first occurred in [6] and holds generally in BL-logic.

Theorem 6
Let the fuzzy data (13) consist of fuzzy numbers of the form (28). The solution of the corresponding system of fuzzy relations is the fuzzy relation

$$R(u, v) = \bigvee_{j=1}^{m} (\mathrm{Fn}_{\nu_j, x_{0j}, w}(u) \otimes \mathrm{Fn}_{\nu_j, y_{0j}, w'}(v)), \qquad u \in w, v \in w', \tag{30}$$

iff

$$\bigvee_{u \in w} (\mathrm{Fn}_{\nu_i, x_{0i}, w}(u) \otimes \mathrm{Fn}_{\nu_j, x_{0j}, w}(u)) \leq \bigwedge_{v \in w'} (\mathrm{Fn}_{\nu_j, y_{0i}, w}(v) \leftrightarrow \mathrm{Fn}_{\nu_j, y_{0j}, w'}(v))$$

holds for every $i, j = 1, \ldots, m$.

The intersection of fuzzy sets in the following corollary is taken w.r.t. the t-norm $\otimes$.

Corollary 1

(a) *If* $\mathrm{Fn}_{\nu_i, x_{0i}, w} \cap \mathrm{Fn}_{\nu_j, x_{0j}, w} = \emptyset$ *then the fuzzy relation* R *in (30) is a solution.*

(b) *If* $\mathrm{Ker}(\mathrm{Fn}_{\nu_i, y_{0i}, w'}) \cap \mathrm{Ker}(\mathrm{Fn}_{\nu_j, y_{0j}, w'}) = \emptyset$ *and the fuzzy relation* R *in (30) is a solution then* $\mathrm{Fn}_{\nu_i, x_{0i}, w} \cap \mathrm{Fn}_{\nu_j, x_{0j}, w} = \emptyset$.

This corollary demonstrates that if the data consist of fuzzy numbers then the system of fuzzy relation equations can be solved by a fuzzy relation in the form of Mamdani-Assilian formula (30) provided that the data can be seen as a fuzzy function — a set of couples of fuzzy sets such that the closer are first components, the more closer are the second ones. Note that this follows also from Theorem 2.

5 Conclusion

In this paper we have developed a systematic theory of the evaluating linguistic expressions and their semantics. In the second part, we have focused on fuzzy relation equations derived from the fuzzy data given by these expressions. We have demonstrated that pure expressions (i.e. the expressions containing the adjectives "small, medium, big") mostly cannot lead to solvable fuzzy relation equations. Thus, we should use fuzzy numbers. If the structure of the data corresponds to a function then the solution can even be in the form of the widely used Mamdani-Assilian formula.

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