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REPRESENTATION OF LOGIC FORMULAS BY NORMAL FORMS

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Abstract

In this paper, we deal with the disjunctive and conjunctive normal forms in the frame of predicate BL-logic and prove theirs conditional equivalence to appropriate formulas. Our aim is to show approximation ability of special normal forms defined by means of reflexive binary predicate.

Key words Disjunctive and conjunctive normal forms, BL-logic, Fuzzy logic, Approximation

1 Introduction

In this paper, we deal with fuzzy logic formulas, which formalize linguistically expressed collections of fuzzy "IF-THEN" rules, namely disjunctive (DNF) and conjunctive (CNF) normal forms (see [6]). Both normal forms are regarded to be suitable for equivalent transformation of formulas of specific fuzzy logic theory. This transformation is called in [6] "logical approximation".

It is worth noticing that actually three different ways led us to the construction of normal forms in fuzzy logic. The first is the way of generalization of classical construction. On the second way, we have generalized logical formulas, which are used in the formalization of fuzzy "IF-THEN" rules. Finally, there are constructions of classical algebraic formulas used for interpolation or approximation of continuous functions and they have common structure which can be represented using logical means.

Let us remind that in classical logic, each formula can be transformed into unified normal form, either disjunctive or conjunctive, so that both normal forms are equivalent to the original formula. In fuzzy logic, the situation is different. Here, formulation of normal forms is no more equivalent to the original formula. Further, we will look at concrete conditions under which the considered normal forms will become equivalent with the original formula. This led us to the term "conditional equivalence".

We will consider Basic Fuzzy Logic (BL for short) introduced in [2]. BL-logic can be viewed as a basic logic for all fuzzy logics based on continuous t-norms.

Several authors deal with problems connected with the notion of disjunctive and conjunctive normal form, among them, e.g. I. Perfilieva, D. Mundici, V. Kreinovich etc. We can find an implicit definition of a disjunctive normal form in [1] arising from the constructive proof of McNaughton theorem. In [4], the explicit definition is given for Łukasiewicz logic. In [6], the conditional equivalence between any extensional formula and its normal forms has been proved formally. The extensionality has been defined w.r.t. similarity (the predicate characterized by axioms of reflexivity, symmetry and transitivity). In the present paper, we will concentrate ourselves on the extensionality w.r.t. arbitrary binary predicate.

In the sequel, we will use the definition of normal forms for predicate BL-logic in the sense of [6]. Moreover, we will look for such requirements which imply the information about the extensionality property of an arbitrary formula. The next subsection will be devoted to the study of the conditions under which DNF is equivalent to CNF, and both are equivalent to the initial formula.

The paper is organized as follows. At the beginning, the fuzzy predicate logic is introduced and the general definitions of disjunctive and conjunctive normal forms in predicate fuzzy logic are given. Section 3 is devoted to the basic notions and properties. In the following section, the extensionality property and its equivalence are studied. Finally, in Section 5 the conditional equivalence between an extensional formula and its normal forms is proved. Also, the relationship of normal forms to initial formula is shown there.

2 BL-Logic and BL-Normal Forms

We will consider BL-logic introduced by P. Hájek in [2]. This book is regarded as a fundamental one (see [1]). Therefore, we will follow the notation used in this book.

We will deal with some fixed language J of predicate BL-logic. Recall that it consists of a non-empty set of predicates, set of object constants, object variables, set of connectives $\{\neg, \&, \to, \vee, \wedge, \equiv\}$ and quantifiers, and it does not contain functional symbols.

The predicate BL-calculus (BL∀) contains a set of logical axioms on connectives (see 2.2.4 in [2]), quantifiers (see 5.1.7 in [2]) and the usual deduction rules (modus ponens and generalization rule).

A structure $\mathcal{M} = \langle M, (r_P)_{P\in J}, (m_c)_{c\in J}\rangle$ for the language J consists of a non-empty domain M, fuzzy relations $r_P : M^n \longrightarrow [0,1]$ assigned to each n-ary predicate symbol P, and designated elements $m_c \in M$ assigned to each object constant **c**.

Now we are able to introduce the disjunctive and conjunctive normal forms in the frame of BL-logic. The basic definitions of normal forms for Łukasiewicz logic are well established by I. Perfilieva in [4]. For additional properties, see [5]. The extension of this notation to predicate BL-logic and the next definition is taken from [6] with some reductions.

Definition 1
Let $P_1, \dots, P_k$ be unary predicate symbols and $E_{i_1\dots i_n}$, $1 \le i_j \le k$, $1 \le j \le n$, $n \ge 1$, be closed instances of some formula. The following formulas of fuzzy predicate logic are called the disjunctive normal form

$$\mathrm{DNF}(x_1, \dots, x_n) = \bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \,\&\cdots\&\, P_{i_n}(x_n) \,\&\, E_{i_1\dots i_n}) \tag{1}$$

and the conjunctive normal form

$$\mathrm{CNF}(x_1, \dots, x_n) = \bigwedge_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \,\&\cdots\&\, P_{i_n}(x_n) \to E_{i_1\dots i_n}). \tag{2}$$

We can see that this definition is too general for proving any properties, therefore further, we will specify all P_i and E_i in the previous definition.

3 Extensionality and Additional Properties

We will extend the language J by the binary predicate symbol R. Let T be a special theory over BL∀ which may include additional axioms for the considered predicate symbol R:

$(\forall x)R(x,x)$, (reflexivity)
$(\forall x,y)(R(x,y) \to R(y,x))$, (symmetry)
$(\forall x,y,z)((R(x,y) \,\&\, R(y,z)) \to R(x,z))$. (transitivity)

Note that R satisfying all three axioms is called similarity and is usually denoted by $\approx$. If R is reflexive and transitive then R is called quasiorder ($\preceq$).

Saying that R is reflexive(symmetric or transitive) we will mean that the respective axiom of reflexivity(symmetry or transitivity) belongs to the respective theory.

Definition 2
A formula φ with n variables is called extensional w.r.t. a binary predicate R if

$$T \vdash R(x_1,y_1) \,\&\cdots\&\, R(x_n,y_n) \to (P(x_1,\dots,x_n) \to P(y_1,\dots,y_n)). \tag{3}$$

Lemma 1
Let φ be a formula with n variables extensional w.r.t. a binary predicate R. If R is symmetric then

$$T \vdash R(x_1,y_1) \,\&\cdots\&\, R(x_n,y_n) \to (P(x_1,\dots,x_n) \equiv P(y_1,\dots,y_n)). \tag{4}$$

4 Determination of Extensionality

We will start with specification of normal forms. Then, DNF_φ-closure and CNF_φ-closure formulas will be presented in order to show that they become related with the initial formula. We will see that this relations are contingent on extensionality property of the initial formula.

We will deal with a theory T over BL$\forall$. The language $J(T)$ of the theory T is supposed to contain a finite number of object constants $C = \{\mathbf{c}_i \mid i = 1 \dots k\}$ and a binary predicate symbol R.

Suppose an arbitrary formula $\varphi(x_1, \dots, x_n)$. Let us specify $P_{i_j}(x)$ and $E_{i_1 \dots i_n}$ by $R(x, \mathbf{c}_{i_j})$ in CNF, $R(\mathbf{c}_{i_j}, x)$ in DNF and $\varphi(\mathbf{c}_{i_1} \dots \mathbf{c}_{i_n})$, respectively. This specifications will change the expressions for DNF and CNF of the forms (1), (2) into

$$\mathrm{DNF}_\varphi(x_1, \dots, x_n) = \bigvee_{i_1, \dots, i_n = 1}^{k} (R(\mathbf{c}_{i_1}, x_1) \mathbin{\&} \cdots \mathbin{\&} R(\mathbf{c}_{i_n}, x_n) \mathbin{\&} \varphi(\mathbf{c}_{i_1} \dots \mathbf{c}_{i_n})), \quad (5)$$

$$\mathrm{CNF}_\varphi(x_1, \dots, x_n) = \bigwedge_{i_1, \dots, i_n = 1}^{k} (R(x_1, \mathbf{c}_{i_1}) \mathbin{\&} \cdots \mathbin{\&} R(x_n\, \mathbf{c}_{i_n}) \to \varphi(\mathbf{c}_{i_1} \dots \mathbf{c}_{i_n})). \quad (6)$$

The next form of formulas are important for determining the extensionality property of initial formula.

Definition 3
Consider an arbitrary formula φ with n variables. The CNF_φ-closure($\overline{\mathrm{CNF}_\varphi}$) and DNF_φ-closure($\overline{\mathrm{DNF}_\varphi}$) of formula φ w.r.t. R are defined by formulas

$$\overline{\mathrm{CNF}_\varphi}(x_1, \dots, x_n) = (\forall y_1, \dots, y_n)(R(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} R(x_n\, y_n) \to \varphi(y_1 \dots y_n)) \quad (7)$$

$$\overline{\mathrm{DNF}_\varphi}(x_1, \dots, x_n) = (\exists y_1, \dots, y_n)(R(y_1, x_1) \mathbin{\&} \cdots \mathbin{\&} R(y_n\, x_n) \mathbin{\&} \varphi(y_1 \dots y_n)) \quad (8)$$

Note that both formulas need not be closed w.r.t. all free variables. In the following lemma we will establish the relation between DNF_φ, CNF_φ and the corresponding closures for the same initial formula φ.

Lemma 2
Let T be a theory over BL$\forall$ and $\varphi(x_1, \dots, x_n)$ be an arbitrary formula. Moreover, let the language $J(T)$ be extended by $\mathbf{c}_1, \dots, \mathbf{c}_k$ as object constants and contains a binary predicate symbol R. Then

$$T \vdash \overline{\mathrm{CNF}_\varphi}(x_1, \dots, x_n) \to \mathrm{CNF}_\varphi(x_1, \dots, x_n) \quad (9)$$

$$T \vdash \mathrm{DNF}_\varphi(x_1, \dots, x_n) \to \overline{\mathrm{DNF}_\varphi}(x_1, \dots, x_n) \quad (10)$$

PROOF: For the simplicity, we will consider n=1. From the substitution axiom follows that

$$T \vdash (\forall y)(R(x, y) \to \varphi(y)) \to (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)),$$
$$T \vdash (R(\mathbf{c}_i, x) \mathbin{\&} \varphi(\mathbf{c}_i)) \to (\exists y)(R(y, x) \mathbin{\&} \varphi(y)),$$

for all i, which gives us

$$T \vdash (\forall y)(R(x, y) \to \varphi(y)) \to \bigwedge_{i=1}^{k} (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)),$$

$$T \vdash \bigvee_{i=1}^{k} (R(\mathbf{c}_i, x) \mathbin{\&} \varphi(\mathbf{c}_i)) \to (\exists y)(R(y, x) \mathbin{\&} \varphi(y)),$$

and hence

$$T \vdash \overline{\mathrm{CNF}_\varphi}(x) \to \mathrm{CNF}_\varphi(x),$$
$$T \vdash \mathrm{DNF}_\varphi(x) \to \overline{\mathrm{DNF}_\varphi}(x).$$

□

Next, we will take into account only extensional formulas and study their relationship with their DNF and CNF-closures. We will see that if a formula is extensional then it determines a relation to its closure.

Lemma 3
Let T be a theory over $BL\forall$. Moreover, let the language $J(T)$ contains a binary predicate symbol R and $\varphi(x_1,\ldots,x_n)$ be an arbitrary formula extensional w.r.t. *R. Then*

$$T \vdash \varphi(x_1,\ldots,x_n) \to \overline{\mathrm{CNF}_\varphi}(x_1,\ldots,x_n), \tag{11}$$
$$T \vdash \overline{\mathrm{DNF}_\varphi}(x_1,\ldots,x_n) \to \varphi(x_1,\ldots,x_n). \tag{12}$$

PROOF: Let $n=1$. To prove the lemma we have to show that $\varphi \to \overline{\mathrm{CNF}_\varphi}$ and $\overline{\mathrm{DNF}_\varphi} \to \varphi$. We start with two variants of extensionality axiom:

$$T \vdash R(x,y) \to (\varphi(x) \to \varphi(y)),$$
$$T \vdash (R(y,x) \mathbin{\&} \varphi(y)) \to \varphi(x).$$

From the first formula we obtain

$$T \vdash \varphi(x) \to (R(x,y) \to \varphi(y)),$$
$$T \vdash \varphi(x) \to (\forall y)(R(x,y) \to \varphi(y)),$$
$$T \vdash \varphi(x) \to \overline{\mathrm{CNF}_\varphi}(x).$$

Proof of the second implication is following

$$T \vdash (\forall y)((R(y,x) \mathbin{\&} \varphi(y)) \to \varphi(x))$$
$$T \vdash (\exists y)(R(y,x) \mathbin{\&} \varphi(y)) \to \varphi(x)$$
$$T \vdash \overline{\mathrm{DNF}_\varphi}(x) \to \varphi(x).$$

□

An inverse problem of characterization of extensionality is solved in the following theorem.

Theorem 1
Let T be a theory over $BL\forall$. Moreover, let the language $J(T)$ contains a binary predicate symbol R and $\varphi(x_1,\ldots,x_n)$ be an arbitrary formula. Then $\varphi(x_1,\ldots,x_n)$ is extensional w.r.t. *R if and only if*

$$T \vdash \overline{\mathrm{DNF}_\varphi}(x_1,\ldots,x_n) \to \varphi(x_1,\ldots,x_n) \quad \textit{or} \tag{13}$$
$$T \vdash \varphi(x_1,\ldots,x_n) \to \overline{\mathrm{CNF}_\varphi}(x_1,\ldots,x_n). \tag{14}$$

PROOF: Let $n=1$. The necessity part follows from Lemma 3. Therefore, we need to prove that formula $\varphi(x) \to \overline{\mathrm{CNF}_\varphi}$ implies extensionality of φ w.r.t. R.

$$T \vdash \varphi(x) \to \overline{\mathrm{CNF}_\varphi}(x),$$
$$T \vdash \varphi(x) \to (\forall y)(R(x,y) \to \varphi(y)),$$
$$T \vdash \varphi(x) \to (R(x,y) \to \varphi(y)),$$

which is just the extensionality property. To complete the proof, it is necessary to verify the analogous claim for the $\overline{\mathrm{DNF}_\varphi}$.

$$T \vdash \overline{\mathrm{DNF}_\varphi}(x) \to \varphi(x),$$
$$T \vdash (\exists y)(R(y,x) \mathbin{\&} \varphi(y)) \to \varphi(x),$$
$$T \vdash (R(y,x) \mathbin{\&} \varphi(y)) \to \varphi(x).$$

□

Two facts should be noticed when observing the above theorem. First, formulas (13) and (14) are equivalent. Second, formulas (13) and (14) fully characterize the extensionality property of the given initial formula.

5 Conditional Equivalence of Normal Forms

In this section, we will look for a sufficient condition for the equivalence between normal forms and initial formula. In the sequel, we will take into account only the special normal forms defined by (5) and (6).

Theorem 2
Let T be a theory over BL∀. Moreover, let the language $J(T)$ be extended by $\mathbf{c}_1, \ldots, \mathbf{c}_k$ as object constants and contains a binary predicate symbol R. Let $\varphi(x_1, \ldots, x_n)$ be extensional formula w.r.t. R. Let $P_{i_j}(x)$ and $E_{i_1 \ldots i_n}$ stand for $R(x, \mathbf{c}_{i_j})$ in CNF, $R(\mathbf{c}_{i_j}, x)$ in DNF and $\varphi(\mathbf{c}_{i_1} \ldots \mathbf{c}_{i_n})$ respectively. Then

$$T \vdash \mathrm{DNF}_\varphi(x_1, \ldots, x_n) \to \varphi(x_1, \ldots, x_n), \tag{15}$$

$$T \vdash \varphi(x_1, \ldots, x_n) \to \mathrm{CNF}_\varphi(x_1, \ldots, x_n). \tag{16}$$

PROOF: Consider the case of one free variable ($n = 1$). From the extensionality axiom for φ (3) it follows that

$$\begin{aligned} &T \vdash R(x, \mathbf{c}_i) \to (\varphi(x) \to \varphi(\mathbf{c}_i)), \\ &T \vdash \varphi(x) \to (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)) \\ &T \vdash R(\mathbf{c}_i, x) \to (\varphi(\mathbf{c}_i) \to \varphi(x)), \\ &T \vdash (R(\mathbf{c}_i, x) \,\&\, \varphi(\mathbf{c}_i)) \to \varphi(x). \end{aligned}$$

Hence, using the properties of & and → (see 2.2.11 in [2]) we obtain

$$T \vdash \bigvee_{i=1}^{k} (R(\mathbf{c}_i, x) \,\&\, \varphi(\mathbf{c}_i)) \to \varphi(x),$$

$$T \vdash \varphi(x) \to \bigwedge_{i=1}^{k} (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)),$$

which gives us

$$T \vdash \mathrm{DNF}_\varphi(x) \to \varphi(x), \tag{17}$$

$$T \vdash \varphi(x) \to \mathrm{CNF}_\varphi(x). \tag{18}$$

□

In particular, Theorem 2 states that(without any additional conditions) $\mathrm{DNF}_\varphi \to \varphi$ and $\varphi \to \mathrm{CNF}_\varphi$. Further, also the next result holds.

Corollary 1
Under the assumptions of Theorem 2 it can be proved that

$$T \vdash (\forall x_1, \ldots, x_n)(\mathrm{DNF}_\varphi(x_1, \ldots, x_n) \to \mathrm{CNF}_\varphi(x_1, \ldots, x_n)). \tag{19}$$

The following corollary states that if two different variables are related then DNF_φ for the first variable implies CNF_φ for the other one.

Corollary 2
Let the theory T fulfill the assumptions of Theorem 2. Then

$$T \vdash (R(x_1, y_1) \,\&\, \ldots \,\&\, R(x_n, y_n)) \to (\mathrm{DNF}_\varphi(x_1, \ldots, x_n) \to \mathrm{CNF}_\varphi(y_1, \ldots, y_n)).$$

PROOF: For the simplicity, let us consider $n = 1$.

$$T \vdash R(x,y) \to (\varphi(x) \to \varphi(y)) \quad \text{(extensionality)}$$
$$T \vdash (R(x,y) \mathbin{\&} \varphi(x)) \to \varphi(y) \quad \text{(using axioms of BL for } \to)$$
$$T \vdash (R(x,y) \mathbin{\&} \varphi(x)) \to \mathrm{CNF}_\varphi(y) \quad \text{(by transitivity of } \to \text{ and (18))}$$
$$T \vdash R(x,y) \to (\varphi(x) \to \mathrm{CNF}_\varphi(y))$$
$$T \vdash \varphi(x) \to (R(x,y) \to \mathrm{CNF}_\varphi(y)) \quad \text{(by changing of assumptions)}$$
$$T \vdash \mathrm{DNF}_\varphi(x) \to (R(x,y) \to \mathrm{CNF}_\varphi(y)) \quad \text{(by transitivity of } \to \text{ and (17))}$$

and finally by changing of the assumptions we obtain

$$T \vdash R(x,y) \to (\mathrm{DNF}_\varphi(x) \to \mathrm{CNF}_\varphi(y)).$$

□

However, the one-way implication between normal forms and the initial formula is not satisfactory. The conditional equivalence is proved in the following theorem.

Theorem 3
Let T be a theory over BL$\forall$. Moreover, let the language $J(T)$ be extended by $\mathbf{c}_1, \ldots, \mathbf{c}_k$ as object constants and contains a symmetric binary predicate R. Let $\varphi(x_1, \ldots, x_n)$ be extensional formula w.r.t. R. Let $P_{i_j}(x)$ be $R(x, \mathbf{c}_{i_j})$ in CNF, $R(\mathbf{c}_{i_j}, x)$ in DNF and $E_{i_1 \ldots i_n}$ stands for $\varphi(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_n})$. Moreover, let us define new n-ary predicate

$$C(x_1, \ldots, x_n) := \bigvee_{i_1, \ldots, i_n = 1}^{k} (R^2(\mathbf{c}_{i_1}, x_1) \mathbin{\&} \ldots \mathbin{\&} R^2(\mathbf{c}_{i_n}, x_n)), \tag{20}$$

Then

$$T \vdash C(x_1, \ldots, x_n) \to [\varphi(x_1, \ldots, x_n) \equiv \mathrm{DNF}_\varphi(x_1, \ldots, x_n)], \tag{21}$$
$$T \vdash C(x_1, \ldots, x_n) \to [\varphi(x_1, \ldots, x_n) \equiv \mathrm{CNF}_\varphi(x_1, \ldots, x_n)]. \tag{22}$$

PROOF: For simplicity let us consider $(n = 1)$. Let us start to prove (21). Using $(\varphi \to \varphi)$ and $(((\varphi_1 \to \psi_1) \mathbin{\&} (\varphi_2 \to \psi_2)) \to ((\varphi_1 \mathbin{\&} \varphi_2) \to (\psi_1 \mathbin{\&} \psi_2)))$ we obtain

$$T \vdash (R(\mathbf{c}_i, x) \to \varphi(\mathbf{c}_i)) \to (R^2(\mathbf{c}_i, x) \to (R(\mathbf{c}_i, x) \mathbin{\&} \varphi(\mathbf{c}_i))) \tag{23}$$
$$T \vdash R^2(\mathbf{c}_i, x) \to ((R(\mathbf{c}_i, x) \to \varphi(\mathbf{c}_i)) \to (R(\mathbf{c}_i, x) \mathbin{\&} \varphi(\mathbf{c}_i))) \tag{24}$$

by $((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)))$. The next formulas are the extensionality axiom and its modification.

$$T \vdash R(\mathbf{c}_i, x) \to (\varphi(\mathbf{c}_i) \equiv \varphi(x)),$$
$$T \vdash \varphi(x) \to (R(\mathbf{c}_i, x) \to \varphi(\mathbf{c}_i)).$$

From the last formula, (24) and using properties of $\to$ and $\vee$ we conclude

$$T \vdash R^2(\mathbf{c}_i, x) \to (\varphi(x) \to \bigvee_{i=1}^{k} (R(\mathbf{c}_i, x) \mathbin{\&} \varphi(\mathbf{c}_i)).$$

From the modification of the extensionality axiom and $(((\varphi_1 \to \psi) \wedge (\varphi_2 \to \psi)) \to ((\varphi_1 \vee \varphi_2) \to \psi))$ we get

$$T \vdash \bigvee_{i=1}^{k} (R(\mathbf{c}_i, x) \mathbin{\&} \varphi(\mathbf{c}_i)) \to \varphi(x)).$$

From the previous formulas it follows that

$$T \vdash R^2(\mathbf{c}_i, x) \to (\varphi(x) \equiv \bigvee_{i=1}^{k} (R(\mathbf{c}_i, x) \,\&\, \varphi(\mathbf{c}_i))$$

and finally

$$T \vdash \bigvee_{i=1}^{k} R^2(\mathbf{c}_i, x) \to (\varphi(x) \equiv \bigvee_{i=1}^{k} (R(\mathbf{c}_i, x) \,\&\, \varphi(\mathbf{c}_i)),$$

which allows us to state that

$$T \vdash \bigvee_{i=1}^{k} R^2(\mathbf{c}_i, x) \to (\varphi(x) \equiv \mathrm{DNF}_\varphi(x)).$$

The proof of (22) is analogous to the previous one.

$$T \vdash (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)) \to (R^2(x, \mathbf{c}_i) \to (R(x, \mathbf{c}_i) \,\&\, \varphi(\mathbf{c}_i))),$$
$$T \vdash R^2(x, \mathbf{c}_i) \to ((R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)) \to \varphi(x)).$$

By property of $\to$ and $\vee$ we obtain

$$T \vdash R^2(x, \mathbf{c}_i) \to (\bigwedge_{i=1}^{k} (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)) \to \varphi(x))$$
$$T \vdash \varphi(x) \to (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)) \quad \text{(extensionality axiom)}$$
$$T \vdash \varphi(x) \to \bigwedge_{i=1}^{k} (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i))$$
$$T \vdash R^2(x, \mathbf{c}_i) \to (\bigwedge_{i=1}^{k} (R(x, \mathbf{c}_i) \to \varphi(\mathbf{c}_i)) \equiv \varphi(x))$$

and hence

$$T \vdash \bigvee_{i=1}^{k} R^2(x, \mathbf{c}_i) \to [\varphi(x) \equiv \mathrm{CNF}_\varphi(x)]$$

or equivalently

$$T \vdash \bigvee_{i=1}^{k} R^2(\mathbf{c}_i, x) \to [\varphi(x) \equiv \mathrm{CNF}_\varphi(x)].$$

□

Based on condition (20) described earlier, Theorem 3 states that the normal forms presented above become equivalent with the initial formula. But what does the condition actually mean? If all x_i are equal to the corresponding c_i then truth values of each of the normal forms coincide with the truth values of the original formula. Or, again from semantic but different point of view, the conditions say that equivalence of the normal forms and the initial formula depends first of all on the quality of the covering of the domain by the respective fuzzy sets. For example, we can regulate the truth degree of the condition with the number of used constants in the definition of normal forms.

Corollary 3
Let T be a theory fulfilling the assumptions of Theorem 3. Then

$$T \cup \{(\forall x_1, \ldots, x_n)(\bigvee_{i_1,\ldots,i_n=1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash$$
$$(\forall x_1, \ldots, x_n)(\mathrm{DNF}_\varphi(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)) \quad (25)$$

and

$$T \cup \{(\forall x_1, \dots, x_n)(\bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash$$
$$(\forall x_1, \dots, x_n)(\mathrm{CNF}_\varphi(x_1, \dots, x_n) \equiv \varphi(x_1, \dots, x_n)). \quad (26)$$

PROOF: Let us prove the first formula (25). We will start from conditional equivalence (21) formulated in Theorem 3, e.i.

$$T \vdash \bigvee_{i=1}^{k} R^2(\mathbf{c}_i, x) \to (\varphi(x) \equiv \mathrm{DNF}_\varphi(x)),$$
$$T \vdash (\forall x) \bigvee_{i=1}^{k} R^2(\mathbf{c}_i, x) \to (\forall x)(\varphi(x) \equiv \mathrm{DNF}_\varphi(x)),$$

and hence

$$T \cup \{(\forall x) \bigvee_{i=1}^{k} R(\mathbf{c}_i, x)\} \vdash (\forall x)(\varphi(x) \equiv \mathrm{DNF}_\varphi(x)).$$

The proof of (26) is analogous to the previous one. □

Corollary 4
Let T be a theory fulfilling the assumptions of Theorem 3. Then

$$T \cup \{(\forall x_1, \dots, x_n)(\bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash$$
$$(\forall x_1, \dots, x_n)(\mathrm{DNF}_\varphi(x_1, \dots, x_n) \equiv \mathrm{CNF}_\varphi(x_1, \dots, x_n)).$$

6 Conclusions

In this work, we have proposed two special formulas by means of which the extensionality of an arbitrary formula can be determined. Further, we have worked with a special class of the, so called, extensional formulas. The extensionality of a concrete formula is usually defined w.r.t. a similarity predicate, which gives us a continuity in a certain sense (see [3]). In our case, the extensionality is considered w.r.t. an arbitrary binary predicate.

We have found the special condition (20) under which normal forms (5) and (6) become equivalent with the initial formula. All requirements are shown in the section related to the conditional equivalence of normal forms. On the semantical level, the conditional equivalence means the approximation. The quality of the approximation can be extracted from a truth value of the respective condition.

The main contribution of this work is the generalization of the conditions of logical approximation, as given in [6], eliminating the properties of reflexivity (sometimes) and transitivity for the binary predicate R. The question of maximizing the truth degree of the equivalence is left unsolved in this paper and it is a topic for further study.

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