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Research report No. 40

2001

Submitted/to appear:

Soft Computing

Supported by:

Grant A1187901/99 of the GA AV ČR and project W-00-016 of the Cz-US scientific cooperation.

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Intensional Theory of Granular Computing*

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October 25, 2001

Abstract

This paper presents a new idea how the concept of granule can be modeled in formal fuzzy logic. The basic mathematical theory is fuzzy logic in narrow sense with evaluated syntax, which is based on Łukasiewicz algebra of truth values extended by the product. We have introduced the concepts of intension and extension of granules, internal and external operations on them, various kinds of relations, motion of granules, and others. The paper also outlines construction of some important examples of granules. At its end, we also introduce a more complex concept of association, which is formed of granules of various kinds.

Keywords: Fuzzy logic in narrow sense, evaluated syntax, intension, extension.

1 Introduction

This paper is an introductory paper, which presents a new idea how the concept of granule can be modeled and how the theory of computing with granules can be developed in fuzzy logic. From the formal point of view, granules form specific parts of fuzzy theories. A lot of operations with granules can be defined and relations among them can be studied. This opens an extensive area for research and, hopingly, puts forward an interesting way for utilization of formal logical means to modeling of phenomena, which till now have been out of reach of logic at all.

The term *granular computing* has been introduced by L. A. Zadeh. Let us quote him from [14]:

> *The granule is a clump of elements drawn together by indistinguishability, similarity, proximity or functionality.*

Example of granules (from Zadeh):

- *nose, cheeks, chin, forehead* (human face),
- *young, middle aged, old* (age),
- *hot, very hot, mild, very mild* (spiceness),
- *relevant, very relevant, moderately relevant* (relevance).

We argue that the granularity concept cannot be studied without considering *properties* of objects. These enable us to obtain the above mentioned clumps of objects forming the granules. However, the objects are encountered in some specific world (situation, context). On the formal level, we can speak about a a model of some formal theory. It is convenient to use the term "possible world" instead of "model". Then, from our point of view, granules in Zadeh's conception are possible interpretations of properties of objects in possible worlds and one property may be realized in a wide class of them. Hence, one property leads, in general, to various granules of the same character in various possible worlds.

Let us consider some other examples supporting our concept:

*This paper has been supported by the Grant A1187901/99 of the GA AV ČR and project W-00-016 of the Cz-US scientific cooperation.

- *The surface of Sun* — it has granular structure (granules are, in fact, large swirls of hot gas having a very complicated physical structure).
- *Nation* — this is a granule consisting of people, which can be distinguished by common language, common cultural background (education, training, etc.), common ancestors and relatives, common physiological features, etc.
- *Fuzzy number* — this is a typical granule, which is determined by the property "to be similar to the given number", for example "approximately 5", "roughly 200", etc.

How can we recognize that two granules are essentially the same? Consider, for example two granules on the surface of Sun, two nations, etc. From one sice, they are different phenomena. On the other hand, they have the same character, which means that they share the same properties. We encounter the substantial principle, on the basis of which a given property forms clumps of objects in various models. We argue that *not the elements themselves, but their properties determine the membership in the granule.* In other words, the elements forming a granule share the same properties, which make us possible to identify the granule in concern.

Essential characteristics of most granules is their unsharpness — they are delineated vaguely. This is apparent, e.g. from the photography of the Sun surface, but also from the other examples. Since classical predicate logic has no means to model the vagueness phenomenon, it is insufficient for a proper mathematical theory of granularity. We must find other tool how to characterize both vagueness of the property as well as vagueness of the clumps of objects having it (in some model). On the other hand, classical logic is an essential mathematical theory, which has been verified on many cases. We should have at disposal a mathematical theory preserving all the good from classical logic but generalize it in a nontrivial way.

The noble mathematical theory eligible to fulfil this task and, at the same time, already sufficiently developed, is fuzzy logic in narrow sense with evaluated syntax (FLn). This has been extensively presented in [9]. Its concept of evaluated formula seems to be suitable to formalize the vague properties. This, together with the model theory of FLn, provides proper means for the development of the theory of granularity and is general enough to capture a quite wide variety of granules.

In this paper, we will demonstrate some of the power of FLn. It may be stated that most concepts of fuzzy set theory, i.e. all the fundamental ones and also fuzzy IF-THEN rules, generalized modus ponens, fuzzy equality, fuzzy quantities, as well as (fuzzy) rough (fuzzy) sets, and some other ones can be explicated using FLn. The reason is that FLn is a formal logical theory generalizing classical logic and it is based on Łukasiewicz structure of truth values, possibly extended by some other continuous t-norms. This structure, however, stands beyond all the above mentioned concepts.

The theory of granularity is now being elaborated by many authors, e.g. using the theory of rough sets, information systems, and others. In this paper, we put forth the idea to develop it using very strong means of formal fuzzy logic. Since the paper is an introductory one, some ideas are quite fresh and thus, not fully elaborated so far. Therefore, it may be expected that some of the definitions may be modified in the future along with the development of the theory.

2 Preliminaries

As mentioned, we deal in this paper with fuzzy logic in narrow sense with evaluated syntax presented extensively in [9]. The set of truth values is supposed to form the Łukasiewicz MV-algebra

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle.$$

Our notation is due to [9], which is based on the notation proposed by J. R. Shoenfield in [11]. We will work with first-order fuzzy logic with functional symbols. Its language is denoted by J. In this paper, we will, moreover, suppose that J is many-sorted. However, because of great complexity of the formalism we will try to confine our explanation to one sort only whenever possible.

Formulas and terms are defined in a usual way. The connectives are $\boldsymbol{\vee}$ (disjunction), $\boldsymbol{\nabla}$ (Łukasiewicz disjunction), $\boldsymbol{\wedge}$ (conjunction), **&** (Łukasiewicz conjunction), $\boldsymbol{\Rightarrow}$ (implication) and $\boldsymbol{\neg}$ (negation). The set of all the well-formed formulas for the language J is denoted by F_J and the set of all the closed terms by M_J. Recall that a/A where $a \in L$ and $A \in F_J$ is an *evaluated formula*. Moreover, the language J is supposed to contain *logical constants* $\mathbf{a}$ being names of all the truth values $a \in L$. We write $\top, \bot$ instead of the logical constants $\mathbf{1}, \mathbf{0}$, respectively.

Let $A(x_1,\dots,x_n)$ be a formula and $t_1,\dots,t_n$ be terms substitutable into A for the variables $x_1,\dots,x_n$, respectively. Then $A_{x_1,\dots,x_n}[t_1,\dots,t_n]$ is an instance of A resulting from it when replacing all the free occurrences of the variables $x_1,\dots,x_n$ by the respective terms $t_1,\dots,t_n$.

A fuzzy theory T is a fuzzy set of formulas $T \underset{\sim}{\subseteq} F_J$ given by the triple $T = \langle \mathrm{LAx}, \mathrm{SAx}, R\rangle$ where $\mathrm{LAx} \underset{\sim}{\subseteq} F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \underset{\sim}{\subseteq} F_J$ is a fuzzy set of special axioms and R is a set of inference rules which includes the rules modus ponens (r_{MP}), generalization (r_G) and logical constant introduction (r_{LC}). If T is a fuzzy theory then its language is denoted by $J(T)$.

We will usually define a fuzzy theory only by the *fuzzy set of its special axioms*, i.e. we write

$$T = \{{}^a/A \mid \dots\} \tag{1}$$

understanding that $a > 0$ in (1) and A is a special axiom of T.

The semantics is defined in by generalization of the classical semantics of predicate logic. The *structure* for the language J is

$$\mathcal{V} = \langle V, f_{\mathcal{V}}, \dots, P_{\mathcal{V}}, \dots, u_{\mathcal{V}}, \dots\rangle \tag{2}$$

where $f_{\mathcal{V}} : V^n \longrightarrow V$ are n-ary functions*) on V assigned to the functional symbols $f \in J$, $P_{\mathcal{V}} \underset{\sim}{\subseteq} V^n$ are n-ary fuzzy relations on V assigned to the predicate symbols $P \in J$ and $u_{\mathcal{V}} \in V$ are designated elements assigned to the object constants $\mathbf{u} \in J$. If the concrete symbols $f_{\mathcal{V}}, P_{\mathcal{V}}, u_{\mathcal{V}}, \dots$ are unimportant for the explanation then we will simplify (2) only to $\mathcal{V} = \langle V, \dots\rangle$.

If t is a term of the language J then $\mathcal{V}(t) = v \in V$ is an element being interpretation of t. To interpret the formula $A(x_1,\dots,x_n) \in F_J$, we must assign elements of V to its free variables. We either temporarily extend the language J into the language $J(\mathcal{V}) = J \cup \{\mathbf{v} \mid v \in V\}$ where $\mathbf{v}$ are new names for all the elements of V, i.e. if $\mathbf{v} \in J(\mathcal{V})$ is a name of $v \in V$ then $\mathcal{V}(\mathbf{v}) = v$ and then $\mathcal{V}(A_{x_1,\dots,x_n}[\mathbf{v}_1,\dots,\mathbf{v}_n]) \in L$ is an interpretation of A obtained after assignment of the elements $v_1,\dots,v_n \in V$ to the respective free variables $x_1,\dots,x_n$. The alternative way is to write this assignment explicitly, i.e. $\mathcal{V}(A(v_1/x_1,\dots,v_n/x_n)) \in L$ is an interpretation of A equivalent to the previous one.

We say that the structure $\mathcal{V}$ is a *model* of the fuzzy theory T and write $\mathcal{V} \models T$ if $\mathrm{SAx}(A) \le \mathcal{V}(A)$ holds for every formula $A \in F_{J(T)}$.

The concept of the provability degree as a generalization of the classical provability $T \vdash_a A$ and truth degree $T \models_a A$ can be introduced (for the precise definitions and a lot of properties of them — see [9]). Let us stress that the provability degree coincides with the truth due to the completeness theorem.

Theorem 1 (Completeness)

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A$$

holds for every formula $A \in F_J$ and every consistent fuzzy theory T.

The satisfaction fuzzy relation of $A(x_1,\dots,x_n)$ in the structure $\mathcal{V}$ is

$$A_{\mathcal{V}} = \left\{ {}^a/\langle v_1,\dots v_n\rangle \;\middle|\; a = \mathcal{V}(A[\mathbf{v}_1,\dots,\mathbf{v}_n]), v_1,\dots,v_n \in V \right\} \underset{\sim}{\subseteq} V^n. \tag{3}$$

Let T be a fuzzy theory and $\Gamma \underset{\sim}{\subseteq} J(T)$ be a fuzzy set of formulas. Then the extension of T by the special axioms from Γ is a fuzzy theory $T' = T \cup \Gamma$ given by the fuzzy set of special axioms $\mathrm{SAx}' = \mathrm{SAx} \cup \Gamma$. The extension T' is conservative if $T' \vdash_a A$ implies $T \vdash_a A$ for every formula $A \in J(T)$.

A formula A is *crisp* in a fuzzy theory T if

$$T \vdash A \lor \neg A. \tag{4}$$

Then in every model $\mathcal{V} \models T$ either $\mathcal{V}(A) = 1$ or $\mathcal{V}(A) = 0$. However, note that (4) *does not imply* that either $T \vdash A$ or $T \vdash \neg A$.

The fuzzy equivalence is a binary predicate $\approx$ such that the reflexivity, symmetry and transitivity with respect to **&** is provable in every fuzzy theory T. Fuzzy equality is a special case of fuzzy equivalence where the former should fulfil the standard equality axioms in the degree **1**. For some further discussion on fuzzy equivalence — see [6]. A special case of fuzzy equality (and, of course, of fuzzy equivalence) is the crisp equality predicate $=$, which except for the classical equality axioms is supposed to fulfil the crispness axiom (4).

*) The arity n is, of course, dependent on the given symbol. However, we will not explicitly stress this.

The following concept will play an essential role in this paper. Let $A(\mathbf{x})$ be a formula of the (many-sorted) language J, $\mathbf{x}$ be a sequence of free variables of the sorts from the corresponding type $\bar{\iota} = \langle \iota_1, \ldots, \iota_n \rangle$. Then the *multiformula* is a set of instances of the evaluated formulas

$$\mathbf{A}_{\langle \mathbf{x} \rangle} = \left\{ {}^{a_t}\!/A_{\mathbf{x}}[\mathbf{t}] \;\middle|\; \mathbf{t} \in \mathbf{M} \right\} \tag{5}$$

where $\mathbf{M} = M_{\iota_1} \times \cdots \times M_{\iota_n}$ is a set of sequences of closed terms with sorts according to the type of $\mathbf{x}$.

If $f : A \longrightarrow \mathbb{R}$ is a real function then f^* denotes its truncation to $[0, 1]$, i.e. it is the function

$$f^*(x) = \begin{cases} f(x) & \text{if } f(x) \in [0,1], \\ 1 & \text{if } f(x) > 1, \\ 0 & \text{if } f(x) < 0. \end{cases}$$

In FLn, various generalizations of most important theorems of classical logic hold, for example exquiv-alence, equality, closure, and many others (cf. [9]).

The proof of the following theorem can be found in [5].

Theorem 2
Let T be a consistent fuzzy theory, $T \vdash_a (\exists x)A$ and $v \notin J(T)$ be a constant. Then the fuzzy theory $T' = T \cup \{{}^{a}\!/A_x[v]\}$ is a conservative extension of T.

It is also easy to prove the classical theorem on extension of a fuzzy theory by a new predicate symbol.

The following lemma whose proof is analogous to the proof of Lemma 6.1 from [9]. will be applied in the theory of granular computing below.

Lemma 1
Let T be a consistent fuzzy theory and S a set of closed atomic formulas such that $T \vdash_a A$, $a > 0$, for every $A \in S$. Let the fuzzy set of special axioms of T fulfil for every formula B and every truth valuation $\mathcal{V} : F_{J(T)} \longrightarrow L$ the following condition: if A is an atomic subformula of B and $\mathrm{SAx}(A) \le \mathcal{V}(A)$ then $\mathrm{SAx}(B) \le \mathcal{V}(B)$. Then there is a model $\mathcal{V} \models T$ such that

$$\mathcal{V}(A) = a$$

for every $A \in S$.

We will close this section by the fuzzy logic version of the Kreig-Robinson's theorem on simultaneous consistency of theories, which is very important for our further explanation. It has been proved in [7].

Theorem 3
Let T and T' be consistent fuzzy theories. Then $T \cup T'$ is contradictory iff there is a closed formula $A \in F_{J(T)} \cap F_{J(T')}$ and $a, b \in L$ such that

$$T \vdash_a A \quad \text{and} \quad T' \vdash_b \neg A \quad \text{and} \quad a \otimes b > \mathbf{0}.$$

3 The Concept of Granule and Basic Operations

In this paper, we suppose to be given a many-sorted predicate language J with the sorts $\iota = 1, \ldots, p$, which for every sort ι contains a non-empty set $M_{J,\iota}$ of constants. If $p = 1$ then we simply write M_J and F_J. Sometimes, we will also introduce functional symbols in J. Then $M_{J,\iota}$ is supposed to contain all the closed terms of J constructed from the constants and these symbols.

3.1 The concept of granule

As argued in Introduction, a granule is characterized by some specific property, a set of objects carrying it, and a characterization of relations among the given property and, possibly, some other properties of the objects. However, the objects are known only when a specific possible world is given. At the same time, the character of the granule remains the same independently on the possible world. This suggests an idea to replace concrete objects from possible worlds by some abstract ones. The latter, together with the (given) property assigned to them form the *intension* of the granule.

In FLn, the abstract objects can be represented by closed terms. Thus, we may suggest the following definition.

Definition 1
Let $H(\mathbf{x}) \in F_J$ be a formula where $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ is an n-tuple of variables (in general, of various sorts). Let $T_g \mathrel{\underset{\sim}{\subseteq}} F_J$ be a consistent fuzzy theory. Then

$$g = \langle \mathbf{H}, T_g \rangle \tag{6}$$

is a granule of the type $H(\mathbf{x})$ where $\mathbf{H}$ is a multiformula (5) (i.e. a set of evaluated formulas)

$$\mathbf{H} = \{{}^{h_{\mathbf{t}}}/H_{\mathbf{x}}[\mathbf{t}] \mid \mathbf{t} \in M_{J,\iota_1} \times \cdots \times M_{J,\iota_n}, T_g \vdash_{h_{\mathbf{t}}} H_{\mathbf{x}}[\mathbf{t}]\}. \tag{7}$$

The essential part of this definition is the multiformula $\mathbf{H}$. This is the holder of the main characteristics of the granule — the property as well as the objects carrying it. The former is represented by the formula $H(\mathbf{x})$ and the latter are abstract ones represented by n-tuples $\mathbf{t}$ of closed terms. Each set of abstract objects $\mathbf{t}$, which forms the granule, possesses the property H in a certain degree taken from the set of truth values L.

The fuzzy theory T_g characterizes the *structure* of the granule, i.e. it formally characterizes both the property H as well as its relation to other properties. We will call T_g the *schema* of the granule g. If g is a granule then we will sometimes write $J(g)$ for its language instead of $J(T_g)$.

For simplicity, we will in the sequel deal with granules of the type being a formula $H(x)$ with one variable x of some definite sort (provided that considering more sorts is unnecessary). Generalization to more sorts is in such cases left to the reader.

Depending on the type of $H(x)$, we will call the granule g as follows:

(i) g is of *atomic type* if $H(x)$ is an atomic formula,

(ii) g is of *compound type* if $H(x)$ is a compound formula.

3.2 Extension of granules

The extension of the granule is a fuzzy set of objects in some possible world (i.e. a model of T_g).

Definition 2
Let $g = \langle \mathbf{H}, T_g \rangle$ be a granule of the type $H(\mathbf{x})$ and $\mathcal{V} \models T_g$ be a model

$$\mathcal{V} = \langle \{V_\iota \mid \iota = 1, \ldots, p\}, \ldots \rangle$$

of the fuzzy theory T_g. Then the extension of g in $\mathcal{V}$ is the satisfaction fuzzy relation (3)

$$\mathcal{V}(g) = H_{\mathcal{V}} \mathrel{\underset{\sim}{\subseteq}} V_\iota^p. \tag{8}$$

Obviously, if the type of the granule g is a formula with one variable $H(x)$ then (8) becomes a fuzzy set

$$\mathcal{V}(g) = \left\{ {}^{\mathcal{V}(H(v/x))}/v \,\middle|\, v \in V \right\} \mathrel{\underset{\sim}{\subseteq}} V. \tag{9}$$

Let $\mathcal{V}(g)$ and $\mathcal{V}'(g)$ be two extensions of the same granule g. If

$$\mathcal{V}(g) \subseteq \mathcal{V}'(g)$$

then $\mathcal{V}(g)$ is *finer* than $\mathcal{V}'(g)$ (the latter is *coarser* than the former).

The extension $\mathcal{V}(g)$ of the granule $g = \langle \mathbf{H}, T_g \rangle$ is *limit* if

$$\mathcal{V}(H_x[t]) = \mathcal{V}(g)(v) = h_t \tag{10}$$

holds for every $v = \mathcal{V}(t) \in V$ and $t \in M_J$ where h_t is the provability degree of $T_g \vdash_{h_t} H_x[t]$, i.e. it is the evaluation in of $H_x[t]$ in the evaluated formula ${}^{h_t}/H_x[t] \in \mathbf{H}$.

Note that the limit extension need not exist. However, the following theorem holds.

Theorem 4
Let g be a granule of the atomic type $H(x)$. Let for every formula B containing instances $H_x[t_1], \ldots, H_x[t_m]$ of H as subformulas, the fuzzy set of special axioms SAx_g fulfil the following condition: if $\mathcal{V}$ is a truth evaluation such that $\mathcal{V}(H_x[t_i]) = h_{t_i}$, $i = 1, \ldots, m$, then $\mathrm{SAx}_g(B) \leq \mathcal{V}(B)$. Then g has a limit extension.

PROOF: Let $\mathcal{V} \models T_g$ be a model. Let us modify it into a truth valuation $\mathcal{V}'$ by putting $\mathcal{V}'(H_x[t_i]) = h_{t_i}$, $i = 1, \ldots, m$. This can be done because $H_x[t_i]$ are closed instances of the atomic formula. Then $\mathcal{V}'$ is a model of T_g.

Indeed, if B contains as subformula no instance $H_x[t]$ of H then $\mathcal{V}'(B) = \mathcal{V}(B) \geq \mathrm{SAx}_g(B)$. Otherwise, $\mathrm{SAx}_g(B) \leq \mathcal{V}'(B)$ by the assumption on SAx_g. This model gives the limit extension $\mathcal{V}'(g)$ in the sense of (10). □

The terms **t** determine, in a certain sense, the "precision" of the intension of the granule. This means that they are representatives of some "well selected" objects in arbitrary possible world. It should be our aim to select them so that each extension of the granule well characterized in the best possible way. However, so far it is not clear what criteria should be chosen and thus, we leave this as an open question.

3.3 Relations among granules

Definition 3
A granule $g' = \langle \mathbf{H}, T_{g'} \rangle$ is a variation of $g = \langle \mathbf{H}, T_g \rangle$ if $J(T_g) = J(T_{g'})$ and they have the same type $H(x)$. The variation is brighter (darker) if $h_t \leq h'_t$ ($h_t \geq h'_t$) for every $t \in M_J$. A variation g' which is both darker as well as brighter than g is identical variation of g.

Note that even if g' is identical variation of g, it may still happen that $T_g \neq T_{g'}$.

Lemma 2
Let $J(T_g) = J(T_{g'})$ and $g' = \langle \mathbf{H}, T_{g'} \rangle$, $g = \langle \mathbf{H}, T_g \rangle$ have the same type $H(x)$. If the fuzzy theories T_g and $T_{g'}$ are equivalent then the variations g and g' are identical. If $T_{g'}$ is an extension of T_g then g' is brighter than g.

PROOF: This is a direct consequence of the definition of equivalence (and extension) of fuzzy theories. □

Recall that the elements of $\mathbf{H}$ in (6) are evaluated formulas. Thence, a function $f : \mathbf{H} \longrightarrow \mathbf{H}'$ means that $f({}^{h}/H)$ is some evaluated formula ${}^{h'}/H'$. On the other hand, to simplify the notation, and if the misunderstanding may not threaten from the context, we may also write $f(H)$ to mean the formula $f(H) = H'$ only, thus forgetting the evaluations. This abuse of notation is used in the following definition.

Definition 4
Let g, g' be granules of the types $H(x), H'(y)$, respectively. The homomorphism $f : g \longrightarrow g'$ is a function

$$f : \mathbf{H} \longrightarrow \mathbf{H}' \tag{11}$$

such that $f({}^{h_t}/H_x[t]) = {}^{h'_s}/H'_y[s]$ for each term $t \in M_J$ and some term $s \in M'_J$ and, moreover,

$$T_g \vdash_a H_x[t_1] \Rightarrow H_x[t_2] \quad \text{implies} \quad T_{g'} \vdash_b f(H_x[t_1]) \Rightarrow f(H_x[t_2]), \qquad a \leq b$$

for all terms $t_1, t_2 \in M_J$.

If f and f^{-1} are homomorphisms then f is an isomorphism.

Let $g_1 = \langle \mathbf{H}_1, T_{g_1} \rangle$ and $g_2 = \langle \mathbf{H}_2, T_{g_2} \rangle$ be granules. We say that they are *composable* if the fuzzy theory $T_{g_1} \cup T_{g_2}$ is consistent.

Note the following: let g_1, g_2 and at the same time g_2, g_3 be composable granules. Then g_1, g_3 need not be composable.

Let J be a language, M_J the set of all closed terms and $\Gamma_n \subseteq F_J$ a set of all formulas with n-free variables. For every formula $H(x_1, \ldots, x_n) \in \Gamma_n$, every consistent fuzzy theory $T_g \subsetapprox F_J$ determines a granule $\langle \mathbf{H}, T_g \rangle$. On the other hand, given a single fuzzy theory T_g, each $H(x_1, \ldots, x_n) \in \Gamma_n$ determines a granule $\langle \mathbf{H}, T_g \rangle$. Thus for every consistent $T_g \subsetapprox F_J$ we put

$$\mathfrak{G}_J(\Gamma_n, T_g) = \{\langle \mathbf{H}, T_g \rangle \mid H(x_1, \ldots, x_n) \in \Gamma_n\}. \tag{12}$$

Furthermore, we define

$$\mathfrak{G}_J(\Gamma_n) = \bigcup \{\mathfrak{G}_J(\Gamma_n, T_g) \mid T_g \subsetapprox F_J \text{ is consistent}\} \tag{13}$$

$$\mathfrak{G}_J(T_g) = \bigcup \{\mathfrak{G}_J(\Gamma_n, T_g) \mid n \in \mathbb{N}^+, \Gamma_n \subseteq F_J\} \tag{14}$$

($\mathbb{N}^+ = \mathbb{N} - \{0\}$). Finally, the set of all granules of the language J is

$$\mathfrak{G}_J = \bigcup\{\mathfrak{G}_J(\Gamma_n) \mid n \in \mathbb{N}^+\} \tag{15}$$

The granule $g = \langle \mathbf{H}, T_g\rangle$ is *saturated* if $\mathbf{H}$ consists of the evaluated formulas $\mathbf{1}/H_x[t]$ only, i.e. if

$$T_g \vdash H_x[t]$$

for every $t \in M)J$.

The granule g is negative if it is of the type $\neg H(x)$. A granule $g' = \langle \mathbf{H}', T_{g'}\rangle$ is a negation of $g = \langle \mathbf{H}, T_g\rangle$ if g is of the type $H(x)$, g' is of the type $\neg H(x)$ and

$$\neg a/\neg H_x[t] \in \mathbf{H}' \quad \text{iff} \quad a/H_x[t] \in \mathbf{H}.$$

We will write $\neg g$ for the negation of g.

A granule $g = \langle \mathbf{H}, T_g\rangle$ is *imaginary* if $\mathbf{0}/H_x[t] \in \mathbf{H}$ for all $t \in M_J$; otherwise it is *real*.

Lemma 3
A granule g is imaginary iff $\neg g$ is saturated. If T_g is a complete fuzzy theory then every $\mathfrak{G}_J(\Gamma_n, T_g)$ is closed with respect to negation of granules.

3.4 Operations on granules

3.4.1 Internal operation on granules

Internal operation is defined on specific granules in the language J with the crisp equality predicate $=$. This operation is based on extension of some operation defined on terms to formulas. Its motivation stems from the extension principle defined in fuzzy set theory, which can be explicated as a special case of our definition. We will define it for binary operation and formulas with two variables. Of course, arbitrary generalization is possible.

Let $H(x,y)$, $K(x,y)$, $N(x,y)$ be formulas and $t_1, t_2 \in M_J$ be closed terms. Let $\star \in J_g$ be a binary functional symbol[†)].

In general, of course, we can replace variables in the formulas by arbitrary terms but this does not lead to any connection of the above formulas. Such a connection must be established by a special axiom, which we will call the *axiom of extension:*

(Ext) $\mathbf{1}/((x = u_1 \star u_2) \Rightarrow ((y = v_1 \star v_2) \Rightarrow (H(u_1, v_1) \wedge K(u_2, v_2) \Rightarrow N(x,y))))$

where x, u_1, u_2, y, v_1, v_2 are variables.

Definition 5
Let $g_1 = \langle \mathbf{H}, T_{g_1}\rangle$ and $g_2 = \langle \mathbf{K}, T_{g_2}\rangle$ be composable granules of the types $H(t_1, y)$ and $K(t_2, y)$, respectively where t_1, t_2 are closed terms. Let $T_g = T_{g_1} \cup T_{g_2} \cup \{\text{Ext}\}$ be a new fuzzy theory and t_0 be a closed term such that $T_g \vdash t_0 = t_1 \star t_2$. Then the internal operation $\star$ on the granules g_1 and g_2 is the granule

$$g = g_1 \star g_2 = \langle \mathbf{N}, T_g\rangle \tag{16}$$

of the type $N(t_0, y)$ where

$$h_s/N_{x,y}[t_0, s] \in \mathbf{N} \quad \text{iff} \quad T_g \vdash_{h_s} N_{x,y}[t_0, s] \text{ and } T_g \vdash s = s_1 \star s_2 \tag{17}$$

where $s_1 \in M_J$ and $s_2 \in M_J$ are (closed) terms.

Theorem 5 (Extension principle)
Let g_1, g_2 be granules of the types $H(t_1, y)$ and $K(t_2, y)$, respectively and $g_1 \star g_2$ be the internal operation in the sense of Definition 5. Then for every $s \in M_J$,

$$T_g \vdash_{h_s} N(t_1 \star t_2, s) \qquad \textit{where} \qquad h_s \geq \bigvee\{h_{s_1} \wedge k_{s_2} \mid s_1, s_2 \in M_J, T_g \vdash s = s_1 \star s_2\}. \tag{18}$$

†) In the sequel, we will use infix notation, i.e. we will write $x \star y$ instead $\star(x,y)$.

PROOF: By the assumption, $T_g \vdash t_0 = t_1 \star t_2$. Since T_g is extension of both T_{g_1} and T_{g_2}, we have $T_g \vdash_{h'_{s_1}} H(t_1, s_1)$ and $T_g \vdash_{k'_{s_2}} K(t_2, s_2)$ where $h'_{s_1} \geq h_{s_1}$ as well as $k'_{s_2} \geq k_{s_2}$ (h_{s_1}, k_{s_1} are the corresponding provability degrees of $H(t_1, s_1)$ and $K(t_2, s_2)$ in $\mathbf{H}$ and $\mathbf{K}$, respectively). Then $T_g \vdash_{h'_{s_1} \wedge k'_{s_2}} H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2)$ can be proved.

Let $s \in M_J$ and $T_g \vdash s = s_1 \star s_2$ for some s_1, s_2. We consider the proof

$$\begin{aligned} w := {}& c/t_0 = t_1 \star t_2 \quad \{\textit{assumption}\},\ d/s = s_1 \star s_2 \quad \{\textit{assumption}\}, \\ & \ldots, \mathbf{1}/((t_0 = t_1 \star t_2) \Rightarrow ((s = s_1 \star s_2) \Rightarrow (H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2) \Rightarrow N(t_0, s)))) \\ & \{\textit{(Ext), substitution}\}, \ldots, c \otimes d/H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2) \Rightarrow N(t_0, s)) \quad \{r_{MP}\}, \\ & h/H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2) \quad \{\textit{a proof of } H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2)\}, \\ & c \otimes d \otimes h/N(t_0, s) \quad \{r_{MP}\} \end{aligned}$$

where c, d are values of some proofs of $t_0 = t_1 \star t_2$ and $s = s_1 \star s_2$, respectively, the suprema of both are equal to $\mathbf{1}$. Similarly, h is a value of some proof of $H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2)$, respectively, the supremum of which is equal to $h'_{s_1} \wedge k'_{s_2}$. Consequently, the supremum of the values of all the proofs w taken over all proofs of $t_0 = t_1 \star t_2$, $s = s_1 \star s_2$ and $H(t_1, s_1) \boldsymbol{\wedge} K(t_2, s_2)$ is equal to $h'_{s_1} \wedge k'_{s_2}$. Because, $h'_{s_1} \wedge k'_{s_2} \geq h_{s_1} \wedge k_{s_2}$, we conclude that

$$\begin{aligned} h_s \geq {}& \bigvee \{h'_{s_1} \wedge k'_{s_2} \mid s_1, s_2 \in M_J \text{ such that } T_g \vdash s = s_1 \star s_2\} \geq \\ & \geq \bigvee \{h_{s_1} \wedge k_{s_2} \mid s_1, s_2 \in M_J, T_g \vdash s = s_1 \star s_2\}. \end{aligned}$$

□

Note that (18) is, in fact, the *extension principle* introduced in fuzzy set theory by L. A. Zadeh. Here, it has been deduced on the basis of purely logical assumptions. The question raises, under which conditions the inequality in (18) can be replaced by equality (as is actually done in the extension principle).

The internal operation on granules is introduced for dealing with terms and thus, it is suitable for more specific kinds of granules. When considering a fuzzy theory T_g containing, e.g. the axioms of classical group, ring or field theory, then the following operations on granules take sense

$$g_1 + g_2, \quad g_1 \cdot g_2, \quad g_1/g_2,$$

etc. If the type $H(x, y)$ is the fuzzy equality/similarity then the extensions of the granules become fuzzy quantities (or fuzzy numbers) and the internal operation leads to the well known operations on them.

3.4.2 External operation on granules

While the internal operation on granules is based on functional symbols, the external one is based on logical connectives. We again suppose that the language J is unique for the granules in concern.

Definition 6
Let $g_1 = \langle \mathbf{H}_1, T_{g_1} \rangle$ and $g_2 = \langle \mathbf{H}_2, T_{g_2} \rangle$ be composable granules of the types $H_1(x)$, $H_2(y)$, respectively. Let $\Box$ be a (binary) logical connective. Then the external operation on g_1, g_2 is the granule

$$g_1 \Box g_2 = \langle \mathbf{H}, T_g \rangle \tag{19}$$

where $T_g \supseteq T_{g_1} \cup T_{g_2}$ and

$$h_{ts}/H_{x,y}[t, s] \in \mathbf{H} \quad \text{iff} \quad H_{x,y}[t, s] := H_{1,x}[t] \Box H_{2,y}[s] \text{ and } T_g \vdash_{h_{ts}} H_{x,y}[t, s]. \tag{20}$$

The fuzzy theory T_g is supposed to extend both T_{g_1}, T_{g_2} since when joining the granules, we may demand some more properties to be fulfilled, i.e. we may add some further axioms.

An example of the external operation may be the granule interpretation of the fuzzy IF-THEN rules are granules obtained by one of the external operations

$$g_1 \Rightarrow g_2 \quad \text{or} \quad g_1 \boldsymbol{\wedge} g_2.$$

The internal and external operations can be combined into a *complex operation*

$$g_1 \boxed{\star, \Box} g_2. \tag{21}$$

The definition of (21) is obvious.

3.5 Motion of granules

Let $\mathfrak{G}_J$ be a set of all granules over the language J. Let the set of all closed terms of J be M_J. It is possible to define distance of granules as follows.

Definition 7
Let $g_1 = \langle \mathbf{H}_1, T_{g_1}\rangle \in \mathfrak{G}$ and $g_2 = \langle \mathbf{H}_2, T_{g_2}\rangle \in \mathfrak{G}$ be composable granules. Then their distance $\rho : \mathfrak{G}^2 \longrightarrow [0,1]$ is given by the formula

$$\rho(g_1, g_2) = \neg \bigwedge \{c_t \mid T_{g_1} \cup T_{g_2} \vdash_{c_t} H_{1,x}[t] \Leftrightarrow H_{2,x}[t],\ t \in M_J\}. \tag{22}$$

The following follows from the properties of FLn and Łukasiewicz algebra.

Lemma 4
The distance $\rho : \mathfrak{G}_J(T_g) \longrightarrow L$ in (22) is a pseudometrics on the set $\mathfrak{G}_J(T_g)$.

Definition 8
Let $\Delta \subseteq \mathbb{R}$ be a set of real numbers.

(i) The function

$$\varphi : \Delta \longrightarrow \mathfrak{G}_J \tag{23}$$

is a granular time series. We will write

$$\{g_\tau\}_{\tau \in \Delta}.$$

(ii) Let $\Delta = [a, b] \subseteq \mathbb{R}$. The granular time series is continuous on Δ if to every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|\tau_1 - \tau_2| < \delta \text{ implies } \rho(\varphi(\tau_1), \varphi(\tau_1)) = \rho(g_{\tau_1}, g_{\tau_2}) < \varepsilon \tag{24}$$

for every $\tau_1, \tau_2 \in \Delta$, provided that the distance $\rho(g_{\tau_1}, g_{\tau_2})$ is defined on $\varphi(\delta)$.

Note that the definition of continuity of granular time series does not encompass only infinitesimal changes of the provability degrees $h_t(\tau)$ in the evaluated formulas $h_t/H_x[t](\tau) \in \mathbf{H}(\tau)$ but also "infinitesimal changes" of the formulas $H_x[t]$ themselves — in the sense that we take the formulas $H_x[t](\tau_1)$ and $H_x[t](\tau_2)$ to be "almost the same" if

$$T_g \vdash_d H_x[t](\tau_1) \Leftrightarrow H_x[t](\tau_2), \qquad d > 1 - \varepsilon.$$

This opens the potential for modeling of the development (growth).

4 Some Examples of Granules

We have already mentioned that the set of truth values of FLn, which is the Łukasiewicz algebra can be extended by arbitrary, in general n-ary operations, provided that they keep the equivalence (we say that they are logically fitting — for the details see [9]). Mathematically this means that the operations must be Lipschitz continuous. We will be careful in such extension, admitting it only when it turns out to be useful.

A significant strengthening of FLn can be obtained by introducing the ordinary product of reals as interpretation of an additional connective. This will extend its ability to serve in the intensional theory of granular computing and, in some sense closes FLn with respect to all continuous t-norms (due to the representation theorem — see, e.g., [2]).

We, therefore, introduce a new connective $\mathbin{\&}_P \in J$ and the following logical axioms:

Definition 9
Let A_1, A_2, B_1, B_2 be formulas. Then the following evaluated formulas are new logical axioms.

(P1) $\mathbf{1}/((A_1 \Leftrightarrow B_1) \mathbin{\&} (A_2 \Leftrightarrow B_2) \Rightarrow ((A_1 \mathbin{\&}_P A_2) \Leftrightarrow (B_1 \mathbin{\&}_P B_2)))$

(P2) Book-keeping axiom for product:
$\mathbf{1}/((\mathbf{a} \mathbin{\&}_P \mathbf{b}) \Leftrightarrow (\overline{\mathbf{a} \cdot \mathbf{b}}))$
where $\overline{\mathbf{a} \cdot \mathbf{b}}$ denotes the logical constant (atomic formula) for the truth value $a \cdot b$ when a and b are given.

(P3) $\mathbf{1}/(A \mathbin{\&_P} B \Rightarrow A)$

(P4) $\mathbf{1}/(A \mathbin{\&} B \Rightarrow A \mathbin{\&_P} B)$

This axiom scheme is probably redundant but we need not precise axiomatics in this paper. It can be demonstrated that the new FLn preserves the completeness property.

4.1 Fundamental granule

One of the concepts which plays fundamental role in our considerations on linguistic semantics, and also in nonstandard mathematization of the concept of infinity is that of *horizon* (cf. [13]). In FLn, we will model it within a special fuzzy theory. Therefore, we will consider a unary *horizon predicate* L modeling the idea of horizon running out from the observer. The truth of the fact that the given element is inside horizon diminishes with the increase of the distance from the observer. Hence, we need formalization of *distance* and also ordering of elements. The role of elements is taken on abstract level by closed terms.

Let us note that the motivation for introduction of this granule is the well known *sorites paradox* (heap) and its mathematization within FLn (cf. [9], Section 4.3.9).

We introduce the fuzzy theory T^{F} characterizing the properties of horizon. Its language $J(T^{\mathrm{F}})$ is the following:

$$J(T^{\mathrm{F}}) = \{D, \leq, =, L, \mathbf{u}_0, \mathbf{u}_1\} \tag{25}$$

where D is a binary predicate of distance, $\leq$ is a crisp binary ordering predicate and $=$ is the crisp equality predicate.

Let us now construct a fuzzy theory T^{F}. The special axioms (mostly taken in the degree **1**) are the following: three axioms for the distance

(D1) $\mathbf{1}/(D(x,x) \Leftrightarrow \bot)$,

(D2) $\mathbf{1}/(D(x,y) \Leftrightarrow D(y,x))$,

(D3) $\mathbf{1}/(D(x,z) \Rightarrow D(x,y) \nabla D(y,z))$.

Furthermore, we suppose that the crisp ordering $\leq$ fulfils the ordinary axioms of reflexivity, antisymmetry and transitivity (in the degree **1**) and crispness (4).

The following are axioms for the terms and horizon.

(H1) $\mathbf{1}/(\forall y)((\mathbf{u}_0 \leq y) \wedge (y \leq \mathbf{u}_1))$

The element $\mathbf{u}_0$ is the smallest and $\mathbf{u}_1$ is the largest one.

(H2) $\mathbf{1}/L_x[\mathbf{u}_0]$

(H3) $\mathbf{1}/\neg L_x[\mathbf{u}_1]$

(H4) $\mathbf{1}/(x \leq y) \Rightarrow (L(y) \Rightarrow L(x))$

In words: The farther lays an object, the less it is true that it lays inside the horizon.

(H5) $\mathbf{1}/((x \leq y) \mathbin{\&} (y \leq z) \mathbin{\&} (D(x,y) \Leftrightarrow D(y,z))) \Rightarrow ((L(x) \Rightarrow L(y))^2 \Leftrightarrow (L(x) \Rightarrow L(z)))$

In words: If the distance between x and y is the same as the distance between y and z then the degree of implication of $L(x) \Rightarrow L(y)$ is half of $L(x) \Rightarrow L(z)$.

(H6) $\mathbf{1}/(\forall x)(\forall z)(\exists y)(x \leq z \Rightarrow (x \leq y \mathbin{\&} y \leq z \mathbin{\&} (D(x,y) \Leftrightarrow D(y,z))))$

In words: For all x and z there is y, which is exactly between them.

It follows from (H1) and (H5) that

$$T^{\mathrm{F}} \vdash (D(\mathbf{u}_0, y) \Leftrightarrow D(y, \mathbf{u}_1)) \Rightarrow (L(y)^2 \Leftrightarrow \bot). \tag{26}$$

¿From (H6) we get

$$T^{\mathrm{F}} \vdash (\exists y)(\mathbf{u}_0 \leq y \mathbin{\&} y \leq \mathbf{u}_1 \mathbin{\&} (D(\mathbf{u}_0, y) \Leftrightarrow D(y, \mathbf{u}_1))). \tag{27}$$

Using Theorem 2 we can extend the theory T^{F} conservatively into the theory $T^{\mathrm{F}}_{0.5}$ by a constant $\mathbf{u}_{0.5}$ and from (26) and (27) we get

$$T^{\mathrm{F}}_{0.5} \vdash L(\mathbf{u}_{0.5})^2 \Leftrightarrow \bot$$

which, using the completeness theorem, gives

$$T^{\mathrm{F}}_{0.5} \vdash_{0.5} L(\mathbf{u}_{0.5}).$$

Similarly we can continue for the elements between $\mathbf{u}_0$ and $\mathbf{u}_{0.5}$, the latter and $\mathbf{u}_1$, etc.

This procedure enables us to extend the fuzzy theory T^{F} by constants $\mathbf{u}_b$ corresponding to all the dyadic numbers $d \in [0,1]$. This is the basis for the proof of the following theorem.

Theorem 6
Let $a \in [0,1]$. Then there is a conservative extension T^F_a by a constant $\mathbf{u}_a \notin J(T^F)$ such that

$$T^F_a \vdash_{\neg a} L(\mathbf{u}_a).$$

Corollary 1
The fuzzy theory T^{F} can be conservatively extended by constants $\{\mathbf{u}_a \mid a \in [0,1]\}$ into a fuzzy theory $T^{\mathrm{F}'}$ so that

$$T^{\mathrm{F}'} \vdash_{\neg a} L(\mathbf{u}_a), \qquad a \in [0,1].$$

On the basis of this corollary, we will assume that T^{F} is the above defined fuzzy theory and that its language $J(T^{\mathrm{F}})$ contains all the constants $\{\mathbf{u}_a \mid a \in [0,1]\}$. The *fundamental granule* characterizing horizon is thus the granule of the type $L(x)$

$$g = \langle \mathbf{L}, T^{\mathrm{F}} \rangle \tag{28}$$

where ${}^{\neg a}\!/L(\mathbf{u}_a) \in \mathbf{L}$, $a \in [0,1]$.

4.2 Evaluating linguistic expressions as granules

The fundamental granule and its fuzzy theory can be used for definition of the granules corresponding to the meaning of the *evaluating linguistic expressions*, i.e. expressions such as "very small, more or less medium, roughly big", etc. which are considered in many applications of fuzzy logic. More about their linguistic theory can be found in [9], Chapter 6.

We will construct the fuzzy theory T^{EV} whose language $J(T^{\mathrm{EV}})$ is

$$J(T^{\mathrm{EV}}) = J(T^{\mathrm{F}}) \cup \{\eta\}$$

where η is a special unary functional symbol for order-reversing automorphism, and add to $J(T^{\mathrm{EV}})$ axioms for its behaviour (cf. [4]).

The linguistic hedges ("very, significantly, more or less, roughly", etc.) can be in our logical theory represented by special formulas. Instead of writing them explicitly, we will represent them (in this paper) by the quadratic function $\lhd_\alpha : [0,1]$ defined by

$$\lhd_\alpha(x) = (-x^2 + k_1 x - k_2)^* \tag{29}$$

where k_1, k_2 are suitable constants. Different subscripts α correspond to various settings of k_1, k_2 and thus, in our theory, $\lhd_\alpha$ interprets specific linguistic hedges including the empty one (i.e. the expressions "small, medium, big" are specific cases of the more complex containing the above mentioned hedges) . We will use special shorts for these formulas, namely $\blacktriangleleft_\alpha(A)$.

Finally we introduce the following shorts of formulas, which can also be understood as new predicate symbols:

$$M_\alpha Sm(x) := \blacktriangleleft_\alpha(L(x)) \tag{30}$$

$$M_\alpha Bi(x) := \blacktriangleleft_\alpha(L(\eta(x))) \tag{31}$$

$$M_\alpha Me(x) := \blacktriangleleft_\alpha(\neg L(x) \wedge \neg L(\eta(x))) \tag{32}$$

and using them, we complete T^{EV} by the following fuzzy sets of axioms

$$\left\{ {}^{\lhd_\alpha(0 \vee \frac{c-a}{c})}\!/M_\alpha Sm_x[\mathbf{u}_a] \mid a \in [0,1] \right\}, \tag{33}$$

$$\left\{ {}^{\lhd_\alpha(0 \vee \frac{a-c}{1-c})}\!/M_\alpha Bi_x[\mathbf{u}_a]) \mid a \in [0,1] \right\}, \tag{34}$$

$$\left\{ {}^{\lhd_\alpha(1 \wedge \frac{a}{c} \wedge \frac{1-a}{1-c})}\!/M_\alpha Me_x[\mathbf{u}_a] \middle| a \in [0,1] \right\} \tag{35}$$

where $\mathbf{u}_c$, $c \in (0,1)$, is a constant representing some central point.

Using Theorem 6 and the completeness theorem, we can prove:

Theorem 7

$$T^{EV} \vdash_{d(a)} M_\alpha Sm_x[\mathbf{u}_a], \qquad d(a) = \lhd_\alpha \left(0 \vee \frac{c-a}{c}\right) \tag{36}$$

$$T^{EV} \vdash_{d(a)} M_\alpha Bi_x[\mathbf{u}_a], \qquad d(a) = \lhd_\alpha \left(0 \vee \frac{a-c}{1-c}\right) \tag{37}$$

$$T^{EV} \vdash_{d(a)} M_\alpha Me_x[\mathbf{u}_a], \qquad d(a) = \lhd_\alpha \left(1 \wedge \frac{a}{c} \wedge \frac{1-a}{1-c}\right) \tag{38}$$

for all linguistic hedges M_α and $a \in [0,1]$.

Hence, the granules representing the meaning of evaluating linguistic expressions are the following.

(i) The granule for the meaning of all the linguistic expressions of the form "⟨linguistic hedge⟩ small" is

$$g_{M_\alpha Small} = \langle \mathbf{M}_\alpha \mathbf{Sm}, T^{\mathrm{EV}} \rangle \tag{39}$$

where, for all $a \in [0,1]$, $d(a)/M_\alpha Sm_x[\mathbf{u}_a] \in \mathbf{M}_\alpha \mathbf{Sm}$ and $d(a)$ is that from (36).

(ii) Similarly, the granule for the linguistic expressions "⟨linguistic hedge⟩ big" is

$$g_{M_\alpha Big} = \langle \mathbf{M}_\alpha \mathbf{Bi}, T^{\mathrm{EV}} \rangle \tag{40}$$

where, for all $a \in [0,1]$, $d(a)/M_\alpha Bi_x[\mathbf{u}_a] \in \mathbf{M}_\alpha \mathbf{Bi}$ and $d(a)$ is that from (37).

(iii) The granule for the linguistic expressions "⟨linguistic hedge⟩ medium" is

$$g_{M_\alpha Medium} = \langle \mathbf{M}_\alpha \mathbf{Me}, T^{\mathrm{EV}} \rangle \tag{41}$$

where, for all $a \in [0,1]$, $d(a)/M_\alpha Me_x[\mathbf{u}_a] \in \mathbf{M}_\alpha \mathbf{Me}$ and $d(a)$ is that from (38).

Let g_{EV} denote any of the granules (39)–(41) and, for short, let us write **EV** for its intension. Then the granules corresponding to fuzzy IF-THEN rules are

$$g_{IF-THEN} = \langle g_{EV,1} \Rightarrow g_{Ev,2}, T^{\mathrm{IT}} \rangle \tag{42}$$

where $T^{\mathrm{IT}} = T^{\mathrm{EV}} \cup \{\mathbf{EV}_1 \Rightarrow \mathbf{EV}_2\}$ and $\mathbf{EV}_1 \Rightarrow \mathbf{EV}_2$ contains all the evaluated formulas

$$(d_1(a) \to d_2(a))/(Ev_{1,x}[\mathbf{u}_{1,a}] \Rightarrow Ev_{2,x}[\mathbf{u}_{2,a}]) \in \mathbf{EV}_1 \Rightarrow \mathbf{EV}_2$$

(and, of course, also all the other instances for all the closed terms formed by $\mathbf{u}_a$ and the automorphism symbol η).

4.3 Fuzzy quantities

Fuzzy quantities are typical granules, which are determined by some fuzzy equivalence/equality relation. They can be thus seen as fuzzy sets of elements "fuzzy equal" to some central one.

Except for fuzzy equivalence, we may add other predicates and operations, such as ordering, group operations, etc. In this paper, we will assume only ordering. The fuzzy quantity will thus be constructed from two parts, separately left and right ones.

We will construct the fuzzy theory for fuzzy quantities T^{FQ} whose language is

$$J(T^{\mathrm{FQ}}) = \{\boldsymbol{\approx}, \leq\}$$

where $\leq$ is the crisp binary ordering predicate and $\boldsymbol{\approx}$ the binary predicate for fuzzy equivalence (or even fuzzy equality). In general, we may suppose that $J(T^{\mathrm{FQ}})$ contains more fuzzy equivalences.

By introducing analogous axioms as for the case of fundamental granule in Section 4.1 and by fixing a constant $\mathbf{w}$, we obtain the fuzzy quantity granule of the type $\mathbf{w} \boldsymbol{\approx} x$

$$g_w = \langle \mathbf{H}_w, T^{\mathrm{FQ}} \rangle$$

where

$$\mathbf{H}_w = \{\neg a/\mathbf{w} \boldsymbol{\approx} \mathbf{u}^l_a, \neg b/\mathbf{w} \boldsymbol{\approx} \mathbf{u}^r_b \mid a, b \in [0,1]\}. \tag{43}$$

The $\mathbf{u}^l_a$, $\mathbf{u}^r_b$ are couples of new constants conservatively extending the original fuzzy theory to obtain the fuzzy theory T^{FQ}.

4.4 Clusters based on extensions of granules

Cluster analysis is quite well developed discipline, the result of which is grouping of objects into clusters (in our case, fuzzy sets of objects) according to some concept of distance (or similarity). The data, however, are in most cases *not only tables of bare numbers* but some additional information about properties which led to them is known. We suggest an idea that clusters may be obtained on the basis of extension(s) of granules.

Let us consider some data consisting of p objects $o_1, \dots, o_p$, where each o_i-th object is given by an m-tuple of elements $u_{i,j} \in V_j$, $j = 1, \dots, m$, i.e.

object	$P_1(x_1)$	$\dots$	$P_m(x_m)$
o_1	u_{11}	$\dots$	u_{m1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
o_p	u_{p1}	$\dots$	u_{pm}

Each column of the data is, in practice, result of measuring of some characteristics of the objects o_i. The measurements are taken from sets V_j, $j = 1, \dots, m$, which may be understood as supports of some (m-sorted) possible world

$$\mathcal{V} = \langle \{V_j \mid j = 1, \dots, m\}, \dots \rangle.$$

This possible world should be a model of some fuzzy theory T. The problem is how to determine it. Since the columns represent certain characteristics of the objects, we may suppose that they are represented by some atomic formulas $P_1(x_1), \dots, P_m(x_m)$. These are the grounds for determination of T. Of course, to be able to do this, we must have some more information at disposal. However, this is often the case. One possible solution is the following.

The measurements may be (and in most case indeed are) the results of measuring of some physical characteristics such as length, strength, temperature, etc. Thus, they can be evaluated using the above described evaluating linguistic expressions†). Then the point of departure will the the fuzzy theory T^{EV}.

The problem of finding clusters of objects can thus be reduced to definition of some specific formula $C(P_1, \dots, P_m)$, finding model and then forming clusters of the above objects on the basis of this. To realize this procedure, the granules for linguistic expressions can be effectively used. Because of the lack of space, we will in detail present this idea in some other paper.

4.5 Pictures consisting of set of pixels as granules

This is a specific kind of granule in the language

$$J = \{P, \{\mathbf{u}_{ij} \mid i = 1, \dots, m, j = 1, \dots, n\}\}$$

where P is a unary predicate symbol representing a property "the given place shines" and $\mathbf{u}_{ij}$ is a set of terms representing places. Then the evaluated formula

$$h_{ij}/P_x[\mathbf{u}_{ij}]$$

is a pixel shining with the intensity $h_{ij} \in [0, 1]$. The fuzzy theory T_g may represent a given picture with the fuzzy set of axioms $\mathrm{SAx}_g = \mathbf{P}$ where

$$\mathbf{P} = \{h_{ij}/P_x[\mathbf{u}_{ij}] \mid i = 1, \dots, m, j = 1, \dots, n\}$$

and thus, the corresponding granule is simply

$$g = \langle \mathbf{P}, T_g \rangle.$$

Extensions $\mathcal{V}(g)$ of this granule can be understood as various versions of "distorted" picture. The limit extension is the "perfect" picture. This may be applied, e.g. for the pattern pre-recognition (cf. [10]), i.e. preliminary checking of the picture and to identify some substantial characteristics on the basis of which it can be decided whether the picture should be analyzed more carefully or be dismissed. More complicated pictures lead to associations — see the next section.

†)Note that we, in fact, evaluate them by terms of some linguistic variable.

5 Granules and Associations

Granules characterize one specific part of the world determined on the basis of one property — formally represented by the type of the granule $H(\mathbf{x})$. In the reality, however, granules form more complex phenomena, which we will call *associations*. Examples of associations are "picture, forest, Sun, society", and many others.

Definition 10
Given the granules $g_1, \ldots, g_m$. Let J be a language such that $J \supseteq J(g_1) \cup \cdots \cup J(g_m)$, M_J be its set of closed terms. The association formed by the granules $g_1 = \langle \mathbf{H}_1, T_{g_1} \rangle, \ldots, g_m = \langle \mathbf{H}_m, T_{g_m} \rangle$ is a couple

$$\mathfrak{A}(g_1, \ldots, g_m) = \langle \overline{\mathbf{H}}, T_g \rangle \tag{44}$$

where $T_g \supseteq T_{g_1} \cup \cdots \cup T_{g_m}$ is a consistent fuzzy theory of the language J and $\overline{\mathbf{H}}$ is a set of evaluated formulas of the form

$$h_t / \bigwedge_{\substack{i=1 \\ h_{i,t} \neq \mathbf{0}}}^{m} H_{i,x}[t] \tag{45}$$

where h_t is the provability degree of

$$T_g \vdash_{h_t} \bigwedge_{\substack{i=1 \\ h_{i,t} \neq \mathbf{0}}}^{m} H_{i,x}[t]$$

provided, that the conjunction is non-empty. Otherwise we set $h_t = \mathbf{0}$ and then (45) does not belong to $\overline{\mathbf{H}}$.

We will denote the formulas from (45) by

$$\overline{H}[t] := \bigwedge_{\substack{i=1 \\ h_{i,t} \neq \mathbf{0}}}^{m} H_{i,x}[t].$$

Note that $\overline{H}[t]$ is a conjunction of formulas selected from the list $H_{1,x}[t], \ldots, H_{m,x}[t]$ where $H_{i,x}[t]$ is a member of the conjunction only in the case when $h_{i,t} \neq \mathbf{0}$.

Analogously to the case of granules, we can introduce sets of associations in the fixed language J $As_J(m, \Gamma_n, T_g)$, $As_J(\Gamma_n, T_g)$, $As_J(m, \Gamma_n)$ and $As_J(m, T_g)$, $As_J(\Gamma_n)$, $As_J(T_g)$ and, finally, the set of all associations

$$As_J = \bigcup \{As_J(\Gamma_n, T_g) \mid m, n \in \mathbb{N}^+, \Gamma_n \subset F_J, T_g \subsetsim F_J \text{ is consistent}\}. \tag{46}$$

On the basis of that, we can introduce time series of associations

$$\{\mathfrak{A}_\tau\}_{\tau \in \Delta}$$

where $\mathfrak{A}_\tau \in As_J$.

It is possible to define distance between associations which is a pseudometrics on $As_J(T_g)$ in a way analogous to granules. On the basis of this the concept of continuous times series of associations can be introduced. This opens potential for modeling of growth of more complex systems using means of fuzzy logic (e.g. the development of face, etc.).

6 Conclusion

In this paper, we presented a new idea how the concept of granule, originally introduced by L. A. Zadeh, can be formally modeled and the theory of computing with granules can be developed in fuzzy logic. From the formal point of view, granules are specific parts of fuzzy theories of the fuzzy logic in narrow sense with evaluated syntax. This is based on Łukasiewicz algebra of truth values extended by the product. We have introduced the concepts intension and extension of granules, internal and external operations on them, various kinds of relations, motion of granules. The paper also presents construction how some important examples of granules can be constructed.

The paper should be considered as introductory one and so, we have only outlined the way how the theory can be developed. At any case, an extensive area for research is opened. We have demonstrated a way, how formal logical means can be utilized for modeling of phenomena, which till now have been out of reach of logic at all.

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