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Vilém Novák

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University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
Bráfova 7, 701 03 Ostrava 1, Czech Republic

tel.: +420-69-6160 233   fax: +420-69-6120 478
e-mail: Vilem.Novak@osu.cz

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Vilém Novák
University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22,
701 03 Ostrava 1, Czech Republic

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Abstract

This paper is a contribution to the development of fuzzy logic in narrow sense with evaluated syntax and connectives interpreted in Łukasiewicz algebra. The main results concern model theory of fuzzy logic (various kinds of submodels, chains of models) and generalization of the Kreig-Robinson's theorem on joint consistency of fuzzy theories as well as Kreig's interpolation theorem.

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1 Introduction

Fuzzy logic in the past ten years has profiled itself into a fuzzy logic in narrow sense (FLn) and that in broader sense. Informally, FLn is a special many-valued logic whose aim is to provide means, which can be used for modeling of various aspects of the vagueness phenomenon, the main characteristics of which is imperceptible change from its presence to non-presence. This can mathematically be captured by continuity. Therefore, it seems reasonable to consider structures of truth values based on the interval of real numbers $[0, 1]$ with continuous operations. A significant contribution to the metamathematics of fuzzy logic has been given by P. Hájek in [1]. According to him, we can distinguish three fundamental logics, namely Łukasiewicz, Gödel and product ones, all three of them being special cases of the so called BL-logic. The latter logic is based on a BL-algebra which is a residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle,$$

which, moreover fulfils the conditions of prelinearity ($(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}$) and divisibility ($a \otimes (a \rightarrow b) = a \wedge b$ for all $a, b \in L$).

All the mentioned logics are complete with respect to traditional syntax and generalized semantics (i.e. a formula of a given theory is provable iff it is true in the degree $\mathbf{1}$ in all its models).

A more radical departure from classical logic is FLn with evaluated syntax. In this logic, each formula is on syntactical level evaluated by some value $a \in L$. Then, we can introduce the concept of evaluated proof which is a sequence of evaluated formulas, the evaluation of the last one is at the same time *the value of the proof.* The supremum of the values of all the evaluated proofs of some formula A is the *provability degree* of A (or, equivalently, we can say that the given formula is a theorem in certain degree). It can be demonstrated that this approach is a generalization of the classical concept of provability.

This paper is a contribution to the theory of fuzzy logic in narrow sense with evaluated syntax. This logic has been extensively presented in [4]. The set of truth values is supposed to form the Łukasiewicz MV-algebra

$$\mathcal{L} = \langle [0, 1], \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

where

$$
\begin{aligned}
a \otimes b =& 0 \vee (a+b-1),\\
a \oplus b =& 1 \wedge (a+b),\\
\neg a =& 1-a,
\end{aligned}
$$

$a, b \in [0,1]$. Moreover, we define the residuation operation by $a \to b = 1 \wedge (1-a+b)$. On the semantical level, we work with the graded concept of model and so, each formula A is assigned its truth being infimum of all the truth values in all models. It has been proved that this logic is complete with respect to to the above introduced provability degree and the truth degree — both degrees coincide. Note that classical logic becomes isomorphic to this logic when confining $\mathcal{L}$ to the classical two-valued boolean algebra.

FLn with evaluated syntax possesses a lot of properties which are more or less apparent generalizations of the corresponding properties of classical logic. Their proofs are mostly non-trivial and more complex but, besides other outcomes, they enable us to see classical logic from different point of view.

This paper confirms such claims. Its results are two-fold: we contribute to model theory of FLn and prove a theorem on joint consistency of fuzzy theories. The latter is a generalization of the classical Kreig-Robinson's one.

The paper is organized as follows. In Section 2, we remind some definitions and results of FLn to help the reader to understand further explanation. In Section 3, we introduce several kinds relations among structures (mostly having no counterpart in classical logic) and prove some of their properties. Section 4 contains the proof of generalization of the mentioned Kreig-Robinson's theorem, and also generalization of the Kreig's interpolation theorem (cf. [8]).

2 Preliminaries

Our notation is due to [4], which is based on the notation proposed by J. R. Shoenfield in [8]. We will work in first-order (predicate) fuzzy logic.

Syntax. The language of FLn is denoted by J. Formulas and terms are defined in a usual way. The connectives are $\boldsymbol{\vee}$ (disjunction, interpreted by $\vee$), $\boldsymbol{\nabla}$ (Łukasiewicz disjunction, interpreted by $\oplus$), $\boldsymbol{\wedge}$ (conjunction, interpreted by $\vee$), **&** (Łukasiewicz conjunction, interpreted by $\otimes$), $\boldsymbol{\Rightarrow}$ (implication, interpreted by $\to$) and $\boldsymbol{\neg}$ (negation, interpreted by $\neg$). The set of all the well-formed formulas for the language J is denoted by F_J and the set of all the closed terms by M_J. Recall that a/A where $a \in L$ and $A \in F_J$ is an *evaluated formula*. It is important that the language J is supposed to contain *logical constants* $\mathbf{a}$ being names of all the truth values $a \in L$. Similarly as in classical logic, we use the symbol $\top$ instead of $\mathbf{1}$ and $\bot$ instead of $\mathbf{0}$.

Let $A(x_1, \ldots, x_n)$ be a formula and $t_1, \ldots, t_n$ be terms substitutable into A for the variables $x_1, \ldots, x_n$, respectively. Then $A_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$ is an instance of A resulting from it when replacing all the free occurrences of the variables $x_1, \ldots, x_n$ by the respective terms $t_1, \ldots, t_n$. If the variables $x_1, \ldots, x_n$ are clear from the context then we will simply write $A[t_1, \ldots, t_n]$.

A fuzzy theory T is a fuzzy set $T \underset{\sim}{\subset} F_J$ of formulas given by the triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, R \rangle$$

where $\mathrm{LAx} \underset{\sim}{\subset} F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \underset{\sim}{\subset} F_J$ is a fuzzy set of special axioms and R is a set of inference rules which includes the rules modus ponens (r_{MP}), generalization (r_G) and logical constant introduction (r_{LC}). If T is a fuzzy theory then its language is denoted by $J(T)$.

Given a fuzzy theory T and a formula A. If w_A is its proof with the value $\mathrm{Val}(w_a)$ then $T \vdash_a A$ means that A is provable in T (a theorem of T) in the degree

$$a = \bigvee \{\mathrm{Val}(w_a) \mid w_A \text{ is a proof of } A \in T\}.$$

If $a = \mathbf{1}$ then we simply write $T \vdash A$.

Since logical axioms are always given (for their detailed presentation see [4]) we will usually define a fuzzy theory only by the fuzzy set of its special axioms and thus, we will write

$$T = \{{}^{a}/A \mid \ldots\} \tag{1}$$

understanding that $a > 0$ in (1). Note that we may equivalently speak about a *set of evaluated formulas*, or about a *fuzzy set of formulas*. This double view is quite common and makes no harm to definiteness of the explanation.

Let T be a fuzzy theory and $\Gamma \underset{\sim}{\subset} J(T)$ be a fuzzy set of formulas. Then the extension of T by the special axioms from Γ is a fuzzy theory

$$T' = T \cup \Gamma$$

given by the fuzzy set of special axioms $\mathrm{SAx}' = \mathrm{SAx} \cup \Gamma$.

We will also use with the following symbols:

$$A^n := \underbrace{A \,\&\, A \,\&\, \cdots \,\&\, A}_{n-\text{times}}, \quad (n\text{-fold conjunction})$$

$$nA := \underbrace{A \nabla A \nabla \cdots \nabla A}_{n-\text{times}}. \quad (n\text{-fold disjunction})$$

The proofs of the following can be found in [4].

Theorem 1
Let T be a fuzzy theory, A be a formula and A' a formula obtained by some prenex operation. Then $T \vdash A \Leftrightarrow A'$.

It follows from this theorem that $T \vdash_a A$ iff $T \vdash_a A'$.

Theorem 2 (deduction theorem)
Let T be a fuzzy theory, A be a closed formula and $T' = T \cup \{{}^{\mathbf{1}}/A\}$. Then to every formula $B \in F_{J(T)}$ there is an n such that

$$T \vdash_a A^n \Rightarrow B \quad \textit{iff} \quad T' \vdash_a B$$

The symbols $<^*, >^*$ denote the bounded inequalities in $\mathcal{L}$

$$a <^* (>^*) b \quad \text{iff} \quad a < b \le \mathbf{1} \quad (\mathbf{1} \ge a > b) \text{ or } a = b = \mathbf{1}.$$

Given a fuzzy set $D \underset{\sim}{\subset} U$ then $\mathrm{Supp}(D) \subseteq U$ is its support, i.e. the set of all elements $u \in U$ with the membership degree $D(u) > \mathbf{0}$.

Theorem 3 (reduction for the consistency)
Let T be a fuzzy theory and $\Gamma \underset{\sim}{\subset} F_{J(T)}$ a fuzzy set of closed formulas. A fuzzy theory $T' = T \cup \Gamma$ is contradictory iff there are $m_1, \ldots, m_n$ and $A_1, \ldots, A_n \in \mathrm{Supp}(\Gamma)$ such that

$$T \vdash_c \neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}$$

and $a_1^{m_1} \otimes \cdots \otimes a_n^{m_n} > \mathbf{0}$ where $c >^* \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$, $a_1 = \Gamma(A_1), \ldots, a_n = \Gamma(A_n)$.

An extension T' of T is conservative if $T' \vdash_a A$ implies $T \vdash_a A$ for every $A \in F_{J(T)}$. In [4], Theorem 4.11, it has been proved that extension of T by new constants not belonging to $J(T)$ is conservative.

The proof of the following theorem can be found in [2].

Theorem 4
Let T be a consistent fuzzy theory, $T \vdash_a (\exists x)A$ and $v \notin J(T)$ be a constant. Then the fuzzy theory $T' = T \cup \{{}^{a}/A_x[v]\}$ is a conservative extension of T.

Semantics. Semantics of FLn is defined by generalization of the classical semantics of predicate logic. The *structure* for J is

$$\mathcal{V} = \langle V, f_\mathcal{V}, \dots, P_\mathcal{V}, \dots, u_\mathcal{V}, \dots \rangle$$

where $f_\mathcal{V} : V^n \longrightarrow V$ are n-ary[†)] functions on V assigned to the functional symbols $f \in J$, $P_\mathcal{V} \underset{\sim}{\subset} V^n$ are n-ary fuzzy relations on V assigned to the predicate symbols $P \in J$ and $u_\mathcal{V} \in V$ are designated elements assigned to the object constants $\mathbf{u} \in J$.

If t is a term of the language J then $\mathcal{V}(t) = v \in V$ is an element being interpretation of t in $\mathcal{V}$. To interpret the formula $A(x_1, \dots, x_n) \in F_J$, we must assign elements of V to its free variables. Therefore, we temporarily extend the language J into the language

$$J(\mathcal{V}) = J \cup \{\mathbf{v} \mid v \in V\}$$

where $\mathbf{v}$ are new names for all the elements of V, i.e. if $\mathbf{v} \in J(\mathcal{V})$ is a name of $v \in V$ then $\mathcal{V}(\mathbf{v}) = v$. Thus

$$\mathcal{V}(A_{x_1,\dots,x_n}[\mathbf{v}_1, \dots, \mathbf{v}_n]) \in L \tag{2}$$

is an interpretation of A (truth value) obtained after assignment of the elements $v_1, \dots, v_n \in V$ to the respective free variables $x_1, \dots, x_n$.

The alternative way is to write this assignment explicitly, i.e.

$$\mathcal{V}(A(v_1/x_1, \dots, v_n/x_n)) \in L \tag{3}$$

is an interpretation of A equivalent to (2). In the sequel, we will mostly use the form (2) since it is syntactically more consistent. However, (3) turns out to be sometimes useful as well.

Let $A(x_1, \dots, x_n)$ be a formula and $\mathcal{V}$ a structure. The formula of the form (2) (alternatively, of the form (3)) is called $\mathcal{V}$*-instance* of A. The closure of A' is a $\mathcal{V}$-instance of A for any $\mathcal{V}$.

The satisfaction fuzzy relation of $A(x_1, \dots, x_n)$ in the structure $\mathcal{V}$ is

$$A_\mathcal{V} = \{{}^{a}/\langle v_1, \dots v_n\rangle \mid a = \mathcal{V}(A[\mathbf{v}_1, \dots, \mathbf{v}_n]), v_1, \dots, v_n \in V\} \underset{\sim}{\subset} V^n \tag{4}$$

where $\mathbf{v}_i$ are the temporary names of the corresponding elements $v_i \in V$, $i = 1, \dots, n$.

Given a fuzzy theory T. The structure $\mathcal{V}$ for $J(T)$ is a model of T, $\mathcal{V} \models T$, if $\mathrm{SAx}(A) \leq \mathcal{V}(A)$ holds for all A. Then we say that A is true in T, $T \models_a A$ if

$$a = \mathcal{V}(A) = \bigwedge\{\mathcal{V}(A) \mid \mathcal{V} \models T\}.$$

If $a = \mathbf{1}$ then we simply write $T \models A$.

A fuzzy theory T is consistent if $T \vdash_a A$ and $T \vdash_b \neg A$ implies $a \otimes b = \mathbf{0}$; otherwise it is inconsistent. It can be proved that $T = F_J$ for an inconsistent T (F_J is seen as a fuzzy set with all the membership degrees equal to $\mathbf{1}$).

The proof of the following can be found in [4].

Theorem 5 (Completeness theorem)

(a) A fuzzy theory T is consistent iff it has a model.

(b)

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A$$

holds for every formula $A \in F_J$ and every consistent fuzzy theory T.

A formula A is *crisp* in a fuzzy theory T if

$$T \vdash A \vee \neg A. \tag{5}$$

Then in every model $\mathcal{V} \models T$ either $\mathcal{V}(A) = 1$ or $\mathcal{V}(A) = 0$. However, note that (5) does not imply that either $T \vdash A$ or $T \vdash \neg A$. It may happen that both $T \vdash_0 A$ as well as $T \vdash_0 \neg A$ hold true because there may exist models $\mathcal{V}, \mathcal{V}'$ such that $\mathcal{V}(A) = 1$ and $\mathcal{V}(\neg A) = 0$, and at the same time $\mathcal{V}'(A) = 0$ and $\mathcal{V}'(A) = 1$. In the sequel, we will work only with the crisp equality predicate $=$ defined analogously as in classical logic and, moreover, every formula of the form $t = s$ for some terms t, s is supposed to fulfil also (5).

†) The arity n, of course, depends on the symbol in concern. However, we will not explicitly stress this.

3 Model theory in FLn

In [4], the concepts of submodel (substructure), elementary submodel, elementary equivalence, homomorphism and isomorphism of models have been defined. We will slightly modify the definition of substructure since a more subtle classification turned out to be important.

Definition 1
Let $\mathcal{V}$, $\mathcal{W}$ be two structures for the language J and let $\Gamma \subseteq F_J$.

(i) The $\mathcal{V}$ is a *weak substructure* of $\mathcal{W}$, in symbols

$$\mathcal{V} \subseteq \mathcal{W},$$

if $V \subseteq W$, $f_{\mathcal{V}} = f_{\mathcal{W}}|V^n$ holds for every n-ary function assigned to a functional symbol $f \in J$, $u_{\mathcal{V}} = u_{\mathcal{W}}$ for every object constant $\mathbf{u} \in J$ and

$$P_{\mathcal{V}} \leq P_{\mathcal{W}}|V^n \tag{6}$$

holds for the n-ary fuzzy relations $P_{\mathcal{V}}$ and $P_{\mathcal{W}}$ assigned to all the predicate symbols $P \in J$ in $\mathcal{V}$ and $\mathcal{W}$, respectively.

(ii) The $\mathcal{V}$ is a *substructure* of $\mathcal{W}$, in symbols

$$\mathcal{V} \subset \mathcal{W},$$

if the equality holds in (6) for all the predicate symbols $P \in J$.

(iii) The $\mathcal{V}$ is a *strong* Γ*-substructure* of $\mathcal{W}$ ($\mathcal{W}$ is a *strong* Γ*-extension* of $\mathcal{V}$), in symbols

$$\mathcal{V} \leq_\Gamma \mathcal{W},$$

if $V \subseteq W$ and

$$\mathcal{V}(A_{x_1,\dots,x_n}[\mathbf{v}_1,\dots,\mathbf{v}_n]) \leq \mathcal{W}(A_{x_1,\dots,x_n}[\mathbf{v}_1,\dots,\mathbf{v}_n]) \tag{7}$$

holds for every formula $A \in \Gamma$ and $v_1,\dots,v_n \in V$.

If $\mathcal{V} \leq_\Gamma \mathcal{W}$ for $\Gamma = F_J$ then $\mathcal{V}$ is a *strong substructure* of $\mathcal{W}$ ($\mathcal{W}$ is a strong extension of $\mathcal{V}$) and we write $\mathcal{V} \leq \mathcal{W}$.

(iv) Let $V \subseteq W$. The *expanded structure* is

$$\mathcal{W}_{\mathcal{V}} = \langle \mathcal{W}, \{\mathbf{v} \mid v \in V\}\rangle$$

where $\mathbf{v}$ are names for all the elements $v \in V$ taken in $\mathcal{W}_{\mathcal{V}}$ as new object constants, which, however, are interpreted by the same elements, i.e. $\mathcal{W}_{\mathcal{V}}(\mathbf{v}) = v$ for all $v \in V$.

(v) The $\mathcal{V}$ is an *elementary* Γ*-substructure* of $\mathcal{W}$ ($\mathcal{W}$ is an *elementary* Γ*-extension* of $\mathcal{V}$), in symbols $\mathcal{V} \prec_\Gamma \mathcal{W}$, if $\mathcal{V} \subseteq \mathcal{W}$ and

$$\mathcal{V}(A_{x_1,\dots,x_n}[\mathbf{v}_1,\dots,\mathbf{v}_n]) = \mathcal{W}(A_{x_1,\dots,x_n}[\mathbf{v}_1,\dots,\mathbf{v}_n]) \mid \qquad A(x_1,\dots,x_n) \in \Gamma \quad \text{and} \quad v_1,\dots,v_n \in V. \tag{8}$$

If $\Gamma = F_J$ then $\mathcal{V}$ is an *elementary substructure* of $\mathcal{W}$ ($\mathcal{W}$ is an *elementary extension* of $\mathcal{V}$) and write $\mathcal{V} \prec \mathcal{W}$.

Remark 1
Note that we may treat $\mathcal{V}$ and $\mathcal{V}_{\mathcal{V}}$ essentially in the same way. The difference between them is only formal since the names for the elements of V are in $\mathcal{V}$ used only temporarily for evaluation of the truth values of formulas while in $\mathcal{V}_{\mathcal{V}}$ they are explicitly added as special object constants.

Lemma 1
(a) If $\mathcal{V}_1 \subseteq \mathcal{V}_2$ ($\mathcal{V}_1 \subset \mathcal{V}_2$) and $\mathcal{V}_2 \subseteq \mathcal{V}_1$ ($\mathcal{V}_2 \subset \mathcal{V}_1$) then $\mathcal{V}_1 = \mathcal{V}_2$.

(b) If $\mathcal{V}_1 \leq \mathcal{V}_2$ then $\mathcal{V}_1 \subseteq \mathcal{V}_2$.

PROOF: This is an easy consequence of the previous definition. □

Note, however, that if $\mathcal{V}_1 \subseteq \mathcal{V}_2$ then there is no explicit relation between $\mathcal{V}_1(A)$ and $\mathcal{V}_2(A)$ for an arbitrary formula A.

Definition 2
Two structures $\mathcal{V}$ and $\mathcal{W}$ are *isomorphic*, $\mathcal{V} \cong \mathcal{W}$, if there is a bijection $g : V \longrightarrow W$ such that the following holds for all $v_1, \ldots, v_n \in V$:

(i) For each couple of functions $f_\mathcal{V}$ in $\mathcal{V}$ and $f_\mathcal{W}$ in $\mathcal{W}$ assigned to a functional symbol $f \in J$,

$$g(f_\mathcal{V}(v_1, \ldots, v_n)) = f_\mathcal{W}(g(v_1), \ldots, g(v_n)).$$

(ii) For each predicate symbol $P \in J$,

$$P_\mathcal{V}(v_1, \ldots, v_n) = P_\mathcal{W}(g(v_1), \ldots, g(v_n)). \tag{9}$$

(iii) For each couple of constants u in $\mathcal{V}$ and w in $\mathcal{W}$ assigned to a constant symbol $\mathbf{u} \in J$,

$$g(u) = u'.$$

The concepts of isomorphic as well as the elementary substructures can be generalized so that they may hold only in some degree (cf. [4]). Since we do not need this, we have simplified Definitions 1 and 2.

The Γ-*diagram* $D_\Gamma(\mathcal{V})$ of $\mathcal{V}$ is a fuzzy theory with the special axioms

$$\mathrm{SAx}_{D_\Gamma(\mathcal{V})} = \{ {}^{a}/A_{x_1,\ldots,x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n] \mid A \in \Gamma, a = \mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n]), \quad v_1, \ldots, v_n \in V\}. \tag{10}$$

If $\Gamma = F_J$ then we speak about *diagram* of $\mathcal{V}$ and write simply $D(\mathcal{V})$. Obviously, $\mathcal{V} \models D_\Gamma(\mathcal{V})$.

Lemma 2
Let $\mathcal{V}$, $\mathcal{W}$ be two structures for the language J and $V \subseteq W$. Then $\mathcal{V}$ is a Γ-substructure of $\mathcal{W}$ iff $\mathcal{W}_\mathcal{V} \models D_\Gamma(\mathcal{V})$.

PROOF: Let $A \in \Gamma$ and $v_1, \ldots, v_n \in V$. Then

$$\mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n]) \leq \mathcal{W}(A_{x_1,\ldots,x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n])$$

which means that $\mathcal{W}_\mathcal{V} \models D_\Gamma(\mathcal{V})$ since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are, at the same time, object constants of $\mathcal{W}_\mathcal{V}$. The converse is obvious. □

A set Γ of formulas is called *regular* if $x = y \in \Gamma$ or $x \neq y \in \Gamma$ and if $A' \in \Gamma$ is an instance of A then also $A \in \Gamma$.

Theorem 6
Let T be a (consistent) fuzzy theory, $\mathcal{V}$ a structure for $J(T)$ and $\Gamma \subseteq F_{J(T)}$ be a regular set of formulas. Then the following is equivalent.

(a) There is a Γ-extension of $\mathcal{V}$, $\mathcal{V} \leq_\Gamma \mathcal{W}$, being a model of T, i.e. $\mathcal{W} \models T$.

(b) Let $B := \neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}$ be a formula where $A_1, \ldots, A_n \in \Gamma$ and $m_1, \ldots, m_n \in \mathbb{N}^+$ such that

$$T \vdash_b B.$$

Then

$$b \leq \mathcal{V}(\neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}).$$

PROOF: First note that $T \vdash (\neg A_1^{m_1} \boldsymbol{\nabla} \cdots \boldsymbol{\nabla} \neg A_n^{m_n}) \Leftrightarrow \neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n})$.

(a)⇒ (b): Let $\mathcal{V} \leq_\Gamma \mathcal{W}$ where $\mathcal{W} \models T$. Let $A_1, \ldots, A_n \in \Gamma$ and $T \vdash_b \neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n})$ for some $m_1, \ldots, m_n \in \mathbb{N}^+$. Then there are $v_1, \ldots, v_p$ such that

$$T \vdash_{b'} \neg((A_1')^{m_1} \mathbin{\&} \cdots \mathbin{\&} (A_n')^{m_n}, \qquad b' \geq b, \tag{11}$$

where $(A_i')^{m_i} := A_i^{m_i}[\mathbf{v}_1, \ldots, \mathbf{v}_p]$ is a $\mathcal{V}$-instance of $A_i^{m_i}$, $i = 1, \ldots, n$.

It follows from the assumption, the substitution axiom and monotonicity of $\otimes$ that

$$\mathcal{V}(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n}) = \mathcal{V}(A_1)^{m_1} \otimes \cdots \otimes \mathcal{V}(A_n)^{m_n} \leq \mathcal{V}(A_1')^{m_1} \otimes \cdots \otimes \mathcal{V}(A_n')^{m_n} \leq$$
$$\leq \mathcal{W}(A_1')^{m_1} \otimes \cdots \otimes \mathcal{W}(A_n')^{m_n} = \mathcal{W}((A_1')^{m_1} \mathbin{\&} \cdots \mathbin{\&} (A_n')^{m_n}).$$

Since $\mathcal{W} \models T$, we obtain

$$b \leq b' \leq \mathcal{W}(\neg((A_1')^{m_1} \mathbin{\&} \cdots \mathbin{\&} (A_n')^{m_n})) \leq \mathcal{V}(\neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n}))$$

by the properties of negation operation.

(b)⇒ (a): Let us extend conservatively T by constants for elements of V into the fuzzy theory T' and, furthermore, put

$$T'' = T' \cup D_\Gamma(\mathcal{V})$$

where $D_\Gamma(\mathcal{V})$ is the Γ-diagram (10) of $\mathcal{V}$. We will show that T'' is consistent.

Let T'' be contradictory. By Theorem 3, there are $A_1, \ldots, A_n \in \Gamma$ and $m_1, \ldots, m_n \in \mathbb{N}^+$ such that

$$T' \vdash_d \neg((A_1')^{m_1} \mathbin{\&} \cdots \mathbin{\&} (A_n')^{m_n}))$$

($(A_i')^{m_i}$ are as above) so that

$$d > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$$

where $a_i = \mathcal{V}(A_i')$ (axioms of $D_\Gamma(\mathcal{V})$). By the theorem on constants (Theorem 4.11 from [4])

$$T \vdash_d \neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n}).$$

¿From this and from the assumption, we obtain

$$d \leq \mathcal{V}(\neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n})) = \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}) < d$$

which is a contradiction.

Thus, T'' is consistent and it has a model $\mathcal{W}'' \models T''$. Moreover, since $\mathcal{W}'' \models D_\Gamma(\mathcal{V})$, we have $\mathcal{V}(A) \leq \mathcal{W}''(A)$ for every $A \in \Gamma$. This means that $\mathcal{V} \leq_\Gamma \mathcal{W}''$.

We must now show that there is an isomorphic structure W such that $\mathcal{W}_\mathcal{V} \models D_\Gamma(\mathcal{V})$. The problem is that there may be constants $\mathbf{v}_1, \mathbf{v}_2$ in $\mathcal{W}''$ assigned to only a single element $v \in V$. Thus, let $v_1, v_2 \in V$ be different elements. Since Γ is regular, $\mathcal{V}(\mathbf{v}_1 \neq \mathbf{v}_2) = \mathbf{1}$ and so, $D_\Gamma(\mathcal{V}) \vdash \mathbf{v}_1 \neq \mathbf{v}_2$. Therefore also $\mathcal{W}''(\mathbf{v}_1 \neq \mathbf{v}_2) = \mathbf{1}$ and consequently, we can consider an isomorphic structure $\mathcal{W}'$ instead of the original $\mathcal{W}''$. When considering its restriction $\mathcal{W}$ by names of elements from V, we realize that $\mathcal{W}' = \mathcal{W}_\mathcal{V}$. Since $\mathcal{W}_\mathcal{V} \models T''$, we have $\mathcal{W}_\mathcal{V} \models D_\Gamma(\mathcal{V})$ and thus, $\mathcal{W}$ is a Γ-extension of $\mathcal{V}$ such that $\mathcal{W} \models T$. □

Corollary 1
Let $\mathcal{V} \models T$. Then there is a strong extension $\mathcal{W}$, $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{W} \models T$.

PROOF: The existence of such structure follows from Theorem 6 by setting $\Gamma = F_{J(T)}$. Then for every formula $T \vdash_a A$ implies $a \leq \mathcal{V}(A)$ since $\mathcal{V} \models T$. □

Theorem 7
Let T be a fuzzy theory of the language J, $\mathcal{V}$ be a structure and $\Gamma \subseteq F_J$. Then there is a model $\mathcal{W} \models T$ which is an elementary Γ-extension of $\mathcal{V}$.

PROOF: Similarly as in the proof of Theorem 6, we will consider a fuzzy theory $T'' = T' \cup D(\mathcal{V})$ where T' is a conservative extension of T by constants for the elements of V and $D(\mathcal{V})$ is the diagram of $\mathcal{V}$.

The fuzzy theory T'' is consistent since otherwise there should exist formulas $A_1, \ldots, A_n$ and $m_1, \ldots, m_n \in \mathbb{N}^+$ such that $T' \vdash_d \neg((A_1')^{m_1} \,\&\, \cdots \,\&\, (A_n')^{m_n}))$ (A_i' are $\mathcal{V}$-instances of A_i) so that

$$d > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$$

where $a_i = \mathcal{V}_{\mathcal{V}}(A_i')$. Since $\mathcal{V}_{\mathcal{V}} \models T'$, we obtain

$$d \leq \mathcal{V}(\neg((A_1')^{m_1} \,\&\, \cdots \,\&\, (A_n')^{m_n}))) = \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}) < d$$

which is a contradiction. Hence, there is a model $\mathcal{W}'' \models T''$.

Let A' be a $\mathcal{V}$-instance of an atomic formula. Then $\mathcal{W}'(A') \geq \mathcal{V}(A')$ as well as $\mathcal{W}'(\neg A') \geq \mathcal{V}(\neg A')$ since A' and $\neg A'$ are axioms of the theory $D(\mathcal{V})$. Consequently

$$\mathcal{W}'(A') = \mathcal{V}(A').$$

If $A := B \Rightarrow C$ then $\mathcal{W}(A) = \mathcal{V}(A)$ by induction.

Let $A := (\forall x)B$. Then

$$\mathcal{V}(A) \leq \bigwedge_{w \in W} \mathcal{W}'(B_x[\mathbf{w}])$$

which means that

$$\mathcal{W}'(B_x[\mathbf{w}]) \geq \bigwedge_{v \in V} \mathcal{V}(B_x[\mathbf{v}]) \tag{12}$$

for every $w \in W - V$. By induction $\mathcal{W}(B_x[\mathbf{v}]) = \mathcal{V}(B_x[\mathbf{v}])$ for every $v \in V$. Consequently,

$$\mathcal{W}'(A) = \bigwedge_{w \in W} \mathcal{W}'(B_x[\mathbf{w}]) = \bigwedge_{v \in V} \mathcal{V}(B_x[\mathbf{v}]) \wedge \bigwedge_{w \in W - V} \mathcal{W}'(B_x[\mathbf{w}]) = \mathcal{V}(A)$$

by (12).

Replacing $\mathcal{W}'$ by an isomorphic structure $\mathcal{W}$, we obtain an elementary extension of $\mathcal{V}$. Finally, since $\Gamma \subseteq F_J$, $\mathcal{W}$ is an elementary Γ-extension. □

Definition 3
Let $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_\alpha \subset \ldots$ (or $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots \subseteq \mathcal{V}_\alpha \subseteq \ldots$) be a chain of structures, $\alpha < \xi$ for some ordinal number ξ. Put $V = \bigcup_{\alpha<\xi} V_\alpha$ and each $f_{\mathcal{V}} = \bigcup_{\alpha<\xi} f_{\mathcal{V}_\alpha}$. Furthermore, we extend $P_{\mathcal{V}_\alpha}$ into V by putting

$$P_{\mathcal{V}}(v_1, \ldots, v_n) = \bigvee_{\beta \leq \alpha < \xi} P_{\mathcal{V}_\alpha}(v_1, \ldots, v_n) \tag{13}$$

for each predicate symbol P and each sequence $v_1, \ldots, v_n \in V$, where β is the first ordinal such that $v_1, \ldots, v_n \in V_\beta$. Then the union of the above chain of structures is the structure

$$\mathcal{V} = \bigcup_{\alpha<\xi} \mathcal{V}_\alpha = \langle V, P_{\mathcal{V}}, \ldots, f_{\mathcal{V}}, \ldots, u_{\mathcal{V}}, \ldots \rangle.$$

Theorem 8
Let $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_\alpha \subset \ldots$, $\mathcal{V}_\alpha \models T$, be a chain of models of some fuzzy theory T, $\alpha < \xi$. Then $\mathcal{V} \models T$ where $\mathcal{V}$ is the union of this chain.

PROOF: By Theorem 1 we may confine only to formulas in prenex form. Let A be a formula and $T \vdash_a A$. We will show that $a \leq \mathcal{V}(A)$.

First, let $A := (\forall x_1)\cdots(\forall x_n)B(x_1,\ldots,x_n)$ for some formula B. Then

$$\mathcal{V}(A) = \bigwedge_{v_1,\ldots,v_n\in V} \mathcal{V}(B_{x_1,\ldots,x_n}[\mathbf{v}_1,\ldots,\mathbf{v}_n]) =$$

$$= \bigwedge_{\alpha<\xi}\ \bigwedge_{v_1,\ldots,v_n\in V_\alpha} \mathcal{V}_\alpha(B_{x_1,\ldots,x_n}[\mathbf{v}_1,\ldots,\mathbf{v}_n]) =$$

$$= \bigwedge_{\alpha<\xi} \mathcal{V}_\alpha((\forall x_1)\cdots(\forall x_n)B) \geq a$$

by the assumption on $\mathcal{V}_\alpha$.

It remains to show that if A does not contain quantifiers then $a \leq \mathcal{V}(A)$. This will be done by induction on the complexity of A.

(a) The case when $A := \mathbf{a}$ is trivial. Therefore, let $A := P_{x_1,\ldots,x_n}[t_1,\ldots,t_n]$ be a closed atomic formula and $\mathcal{V}(t_i) = v_i \in V$, $i = 1,\ldots,n$. Then there is $\beta < \xi$ such that $v_1,\ldots,v_n \in V_\alpha$ for all $\beta \leq \alpha$. By (13), the assumed equality in (6) and the assumption on $\mathcal{V}_\alpha$ we have

$$a \leq \mathcal{V}_\alpha(A) = P_{\mathcal{V}_\alpha}(v_1,\ldots,v_n) =$$

$$= \bigvee_{\beta\leq\alpha<\xi} P_{\mathcal{V}_\alpha}(v_1,\ldots,v_n) = P_{\mathcal{V}}(v_1,\ldots,v_n) = \mathcal{V}(A).$$

(b) Let $A := B \Rightarrow C$ be a closed formula without quantifiers. By induction we may assume that $\mathcal{V}(B) = \mathcal{V}_\alpha(B)$ and $\mathcal{V}(C) = \mathcal{V}_\alpha(C)$ for all $\alpha \geq \beta$ and some $\beta < \xi$. From this and from the assumption on $\mathcal{V}_\alpha$ it follows that

$$a \leq \mathcal{V}_\alpha(A) = \mathcal{V}_\alpha(B) \to \mathcal{V}_\alpha(C) = \mathcal{V}(B) \to \mathcal{V}(C) = \mathcal{V}(A).$$

□

The following theorem has been proved in [4].

Theorem 9
Let $\mathcal{V}_1 \prec \mathcal{V}_2 \prec \ldots \prec \mathcal{V}_\alpha \prec \ldots$ be an elementary chain of models, $\alpha < \xi$ for some ordinal number ξ. Then

$$\mathcal{V} = \bigcup_{\alpha<\xi} \mathcal{V}_\alpha$$

is an elementary extension of each $\mathcal{V}_\alpha$, i.e., $\mathcal{V}_\alpha \prec \mathcal{V}$ for every α.

4 Joining fuzzy theories

In this section, we prove the main theorem of this paper. Though almost apparent in formulation, its proof is by no means trivial and required generalization of the technique originally developed by Kreig and Robinson.

Let T and T' be fuzzy theories. The union $T \cup T'$ is a fuzzy theory $\overline{T}$ in the language $J(\overline{T}) = J(T) \cup J(T')$ given by the fuzzy set of special axioms as follows. We extend the definition of $\mathrm{SAx} \underset{\sim}{\subsetneq} F_{J(T)}$ into $\mathrm{SAx} \underset{\sim}{\subsetneq} F_{J(\overline{T})}$ by putting $\mathrm{SAx}(A) = \mathbf{0}$ for every $A \in F_{J(T')-J(T)}$. Similarly we extend $\mathrm{SAx}' \underset{\sim}{\subsetneq} F_{J(T')}$ into $\mathrm{SAx}' \underset{\sim}{\subsetneq} F_{J(\overline{T})}$. Then the fuzzy theory $\overline{T} = T \cup T'$ is given by the fuzzy set of special axioms $\overline{\mathrm{SAx}} = \mathrm{SAx} \cup \mathrm{SAx}' \underset{\sim}{\subsetneq} F_{J(\overline{T})}$.

The following is a fuzzy logic version of the Kreig-Robinson's theorem on simultaneous consistency of theories.

Theorem 10
Let T and T' be consistent fuzzy theories. Then $T \cup T'$ is contradictory iff there is a closed formula $A \in F_{J(T)} \cap F_{J(T')}$ and $a, b \in L$ such that

$$T \vdash_a A \quad \textit{and} \quad T' \vdash_b \neg A \quad \textit{and} \quad a \otimes b > \mathbf{0}.$$

PROOF: Let the condition hold. Then there are a proof w_a in T and $w_{\neg A}$ in T' with the values $\mathrm{Val}(w_A) = a'$ and $\mathrm{Val}(w_{\neg A}) = b'$, respectively and the supremum of all a''s is a and that of b''s is b. Then there is a proof

$$w := \ldots\ a'/A \quad \{w_A\},\ b'/\neg A \quad \{w_{\neg A}\}, \ldots,\ a' \otimes b'/A \,\&\, \neg A.$$

Then $T \cup T' \vdash_c A \,\&\, \neg A$ where

$$c \geq \bigvee \{a' \otimes b' \mid \text{all proofs } w_A, w_{\neg A}\} = a \otimes b > \mathbf{0}.$$

This means that $T \cup T'$ is contradictory.

Vice-versa, let us denote $J = J(T) \cap J(T')$ and let

$$T \vdash_a A \quad \text{and} \quad T' \vdash_b \neg A \quad \text{implies} \quad a \otimes b = \mathbf{0} \tag{14}$$

for every formula $A \in F_J$.

Let us put

$$\Gamma = \left\{ {}^{c}/C \mid T' \vdash_c C, C \in F_J \text{ is closed} \right\}.$$

We will show that $T \cup \Gamma$ is a consistent fuzzy theory.

Let it be contradictory. Then there are formulas $A_1, \ldots, A_n \in \mathrm{Supp}(\Gamma)$ and $m_1, \ldots, m_n \in \mathbb{N}^+$ such that

$$T \vdash_d \neg(A_1^{m_1} \,\&\, \cdots \,\&\, A_n^{m_n}), \qquad d > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}) \tag{15}$$

where $T' \vdash_{a_i} A_i$, $i = 1, \ldots, n$. Let us denote $c = \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$. By the properties of fuzzy theories

$$T' \vdash_e A_1^{m_1} \,\&\, \cdots \,\&\, A_n^{m_n}, \qquad e \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n} = \neg c.$$

But then $d \otimes e > c \otimes \neg c = \mathbf{0}$ — a contradiction with (14).

Since $T \cup \Gamma$ is consistent, it has a model $\mathcal{V}_0$. Moreover, if $A \in F_J$ then its closure $A' \in \Gamma$ and $\mathcal{V}_0(A) = \mathcal{V}_0(A')$ by the properties of closure. Thus, we may consider the regular set F_J of formulas in the sequel.

We will expand $\mathcal{V}_0|J$ on the structure $\mathcal{W}_0'$ for the language $J(T')$. Let $A \in F_J$ and $T' \vdash_{a'} A$. Then $\mathcal{V}_0|J(A) = \mathcal{W}_0'(A) \geq a'$ (since $\mathcal{V}_0 \models T \cup \Gamma'$). Using Theorem 6, there is an F_J-extension of $\mathcal{W}_0'$ to a model $\mathcal{V}_0' \models T'$. Furthermore,

$$\mathcal{V}_0|J \leq \mathcal{V}_0'|J.$$

Since F_J contains all atomic formulas of the language J as well as their negations, using the same argumentation as in the proof of Theorem 7 we conclude that

$$\mathcal{V}_0|J \prec \mathcal{V}_0'|J.$$

Let us further consider $\mathcal{V}_0'|J$ and expand it to a structure $\mathcal{W}_1$ for the language $J(T)$. Now, using Theorem 6 we extend it to a model $\mathcal{V}_1 \models T$. Moreover, we take it as a model $\mathcal{V}_1 \models T \cup D(\mathcal{V}_0)$. This can be done since $\mathcal{V}_0|J = \mathcal{W}_1|J$ and thus, every formula $A \in F_J$ is an axiom of $D(\mathcal{V}_0)$ in the same degree as of $D_{F_J}(\mathcal{V}_0)$. But it follows from Theorem 7 that $\mathcal{V}_1$ is an elementary F_J-extension of $\mathcal{V}_0$.

Analogously we construct $\mathcal{V}_1'$ being elementary extension of $\mathcal{V}_0'$. When repeating this procedure we obtain the following elementary chains of models:

$$\mathcal{V}_0|J \prec \mathcal{V}_0'|J \prec \mathcal{V}_1|J \prec \mathcal{V}_1'|J \prec \cdots, \tag{16}$$

$$\mathcal{V}_0 \prec \mathcal{V}_1 \prec \cdots, \tag{17}$$

$$\mathcal{V}_0' \prec \mathcal{V}_1' \prec \cdots. \tag{18}$$

Let us now construct the union $\mathcal{V}$ of the chain (17) and the union $\mathcal{V}'$ of the chain (18). Since

$$V_0 \subseteq V_0' \subseteq V_1 \subseteq V_1' \subseteq \cdots,$$

the supports of $\mathcal{V}$ and $\mathcal{V}'$ are equal, $V = V'$. Since $\mathcal{V}_0 \models T$ and $\mathcal{V}_0' \models T'$ we have

$$\mathcal{V} \models T \quad \text{and} \quad \mathcal{V}' \models T' \tag{19}$$

by Theorem 9. Finally, since $V = V'$, we may construct a model $\mathcal{U}$ with the support V and the interpretation of symbols from J equal to $\mathcal{V}|J$, of symbols from $J(T) - J$ equal to $\mathcal{V}|(J(T) - J)$, and of symbols from $J(T') - J$ equal to $\mathcal{V}'|(J(T') - J)$. It follows from (19) that

$$\mathcal{U} \models T \cup T'.$$

□

Using Theorem 10 we can, analogously as in classical logic, prove also generalization of the classical Kreig's interpolation theorem.

Theorem 11 (Interpolation theorem)
Let T and T' be fuzzy theories, $A \in F_{J(T)}$ and $B \in F_{J(T')}$. Let

$$T \cup T' \vdash_a A \Rightarrow B$$

where $a > \mathbf{0}$. Then there is a closed formula $C \in F_{J(T)} \cap F_{J(T')}$ and $m \in \mathbb{N}^+$ such that

$$T \vdash_c A^m \Rightarrow C \quad \text{and} \quad T' \vdash_d C \Rightarrow mB$$

for some c and d such that $c \otimes d > \mathbf{0}$.

PROOF: We prove only the part for closed formulas. The rest proceeds analogously to the classical proof because the theorem on constants is valid in fuzzy logic, too.

Hence, let A, B be closed and let us consider the fuzzy theory

$$\bar{T} = (T \cup \{\mathbf{1}/A\}) \cup (T' \cup \{\mathbf{1}/\neg B\}).$$

This fuzzy theory is contradictory since $\bar{T} \vdash_{a'} A \Rightarrow B$ where $a' \geq a$, $\bar{T} \vdash A$, $\bar{T} \vdash \neg B$ and thus, $\bar{T} \vdash_{a''} B \,\&\, \neg B$ where $a'' > \mathbf{0}$. By Theorem 10, there is a closed formula $C \in F_{J(T)} \cap F_{J(T')}$ such that

$$T \vdash_c C \quad \text{and} \quad T' \vdash_d C \quad \text{and} \quad c \otimes d > \mathbf{0}.$$

By the deduction theorem there are $m', n' \geq 1$ such that

$$T \vdash_c A^{m'} \Rightarrow C \quad \text{and} \quad T' \vdash_d (\neg B)^{n'} \Rightarrow \neg C$$

where the right part implies $T' \vdash_d C \Rightarrow n'B$. Finally we take $m = \max(m', n')$. □

5 Conclusion

In this paper, we have focused on some properties of model theory of fuzzy logic in narrow sense with evaluated syntax and proved the theorem on joint consistency of fuzzy theories. It can be seen that when dealing with degrees, we can introduce a lot of special properties which have no counterpart in classical logic. This concerns especially various kinds of relations among models of fuzzy theories, which, moreover, can be generalized to hold only in some degree. As some other investigations indicate (cf. e.g. [5, 6]) the aim of which is to apply fuzzy logic in areas such as modeling of natural language semantics, such generalizations are not autotelic. Thus, we have opened a quite interesting and nontrivial area for further investigations.

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