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NEURAL NETS AND NORMAL FORMS FROM FUZZY LOGIC POINT OF VIEW

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**Abstract**

The paper addresses the problem of efficient and adequate representation of functions using two soft computing techniques: fuzzy logic and neural networks. The principle approach to the construction of approximating formulas is discussed. We suggest the generalized definition of normal forms in predicate BL and ŁΠ logic and prove conditional equivalence between a formula and each of its normal form. Some mutual relations between normal forms will be also established.

**Key words:** Disjunctive, conjunctive and additive normal forms, BL-logic, Łukasiewicz logic, ŁΠ logic, fuzzy logic, fuzzy relation, approximation

# 1 Introduction

In the last twenty years, we witnessed the development of soft computing methodology for dealing with functions in two principal directions: functions are represented either qualitatively (fuzzy logic in broader sense), or given by their specific algorithmic representation (neural networks). By the qualitative representation we mean the set of fuzzy "IF-THEN" rules which can be translated into the formal language of fuzzy logic in narrow sense. Two special formulas of fuzzy predicate calculus, namely, generalized disjunctive and conjunctive normal forms are suggested in [14, 16] as candidates of formal representation of a set of "IF-THEN" rules. Also their suitability for representation of any continuous function on a compact set has been proved there. This fact, as well as a number of similar results (see e.g. [1, 2, 3, 9, 10, 12, 13, 15]) is known under the term "universal approximation". It is worth noticing that the formal logical proof of the universal approximation ability of generalized normal forms has been put forward only in [14] and partly (for particular disjunctive normal forms) also in [13].

In parallel, a sophisticated technique for approximation of functions by neural networks has been intensively developed. In this area, more attention has been paid to technical details than to the theoretical justification. However, observing neural networks with different architectures and reproducing their functional behaviour, the conjecture about their close connection to fuzzy logic comes to one's mind. The author has been personally inspired by A. DiNola and the paper [5]. He formulated the problem, whether the behaviour of neural networks can be represented by formulas of fuzzy logic? This paper is an attempt at giving the affirmative answer to this question.

We introduce three special forms of predicate fuzzy logic formulas which due to their similarity with the boolean prototypes are called normal forms. Two of them, namely the disjunctive and conjunctive normal forms are introduced in predicate BL-logic. The third one, so called *additive normal form* is introduced in predicate ŁΠ logic. It is worth noticing that the expressions for normal forms came from a formal translation of sets of fuzzy IF-THEN rules, which are widely used in applications of fuzzy logic for linguistic description of different kinds of dependencies.

BL-logic introduced by P. Hájek ([7]) has been chosen among other fuzzy logics because of its generality. In fact, each logic, based on a continuous $t$-norm, is a special case of BL-logic. The most known examples are Łukasiewicz logic, Gödel logic and product logic.

ŁΠ logic ([6]) enriches the language of BL and thus, its competence. This logic is obtained when the above mentioned three main logics are united. Hence, ŁΠ contains both fundamental arithmetical operations, namely product and sum (of course, confined to the interval of reals $[0, 1]$).

In this paper, conditional equivalence between a, so called, extensional formula and each of its normal form is established on a formal logical level (for detailed proofs see [14]). This substantiates theoretically that normal forms of fuzzy logic (as well as their linguistic representation in the form of "IF-THEN" rules) can be successfully used as universal approximating formulas. Moreover, we will show that a wide class of neural networks can be represented by formulas of predicate ŁΠ logic in the additive normal form. This fact is our answer to the above discussed problem. Furthermore, it has also practical significance, since it shows the meaning of parameters of neural networks and thus, helps to choose less exhaustive methods for their tuning.

# 2 The language of basic fuzzy predicate logic and ŁΠ logic

**BL logic.** The basic fuzzy predicate logic (also known as BL-predicate logic) has been introduced in [7]. In order to unify the terminology, we will adopt the notation used there.

We start with some fixed language $J$ of fuzzy predicate logic, which includes (besides others) the set of connectives $\{\neg, \&, \rightarrow, \vee, \wedge, \equiv\}$, truth constants $\bar{0}, \bar{1}$, and does not include functional symbols. By formula, we mean a formula of (fuzzy) predicate logic in the language $J$, including truth constants as atomic formulas.

A structure $\mathcal{M} = \langle M, (r_P)_{P \in J}, (m_c)_{c \in J} \rangle$ for the language $J$ consists of a non-empty domain $M$, fuzzy relations $r_P : M^n \longrightarrow [0,1]$ assigned to each $n$-ary predicate symbol $P$, and designated elements $m_c \in M$ assigned to each object constant $\mathbf{c}$.

The interpretation of the logical connectives $\wedge, \vee, \&, \rightarrow$ is determined by the respective operations of the BL-algebra $\mathcal{L}$ on $[0,1]$

$$\mathcal{L} = \langle [0,1], \wedge, \vee, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

where $*$ stands for some continuous $t$-norm. The other two connectives $\neg, \equiv$ are interpreted by the respective derived operations $\neg, \leftrightarrow$:

$$\neg a = a \rightarrow \mathbf{0},$$
$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

Note, that the notation of the connectives differs from the notation of the corresponding operations.

BL admits different interpretations of its logical connectives. That is why it is so general. However, three outstanding representatives of BL-logic are of the same importance: Łukasiewicz, Gødel and product logics. As logical systems they have additional (to BL) axioms characterizing the specificity of their connectives.

**ŁΠ logic.** The idea to enrich the language of BL and thus, its competence, appeared in [6]. The authors introduced ŁΠ logic as a union of three main logics: Łukasiewicz, Gødel and product.

The language $J^+$ of ŁΠ predicate logic includes all the connectives of three main logics mentioned above, and is considered here as an extension of $J$. The same connectives will be distinguished by one of the following subscripts: Ł, G or Π. For example, $\&_{\text{Ł}}$ is a symbol for the Łukasiewicz conjunction. Moreover, we will define the Łukasiewicz disjunction $\nabla$ as a short for the formula

$$p \nabla q = \neg_{\text{Ł}} p \rightarrow_{\text{Ł}} q.$$

$\nabla$ is interpreted by the operation of bounded sum $\oplus : [0,1] \times [0,1] \longrightarrow [0,1]$ given by

$$a \oplus b = \min(1, a + b).$$

**Remark 1**
To avoid confusion, we will below strictly indicate, which logic (language) we are dealing with.

# 3 Normal forms in BL and ŁΠ logic

In this section, we will introduce the disjunctive and conjunctive normal forms in predicate BL-logic and the so called additive normal form in the predicate ŁΠ logic. It is worth noticing that the expressions for normal forms came from a formal translation of sets of fuzzy IF-THEN rules, which are widely used in applications of fuzzy logic for linguistic description of different kinds of dependencies. Thus, the approximation property of normal forms w.r.t. other formulas is expected.

We will extend the languages $J$ and $J^+$ by a finite number of truth constants $\bar{d}$, for some $d \in [0,1]$.

**Definition 1**
Let $P_1, \dots, P_k$ be unary predicate symbols and $E_{i_1 \dots i_n}$, $1 \le i_j \le k$, $1 \le j \le n$, $n \ge 1$, be either truth constants or closed instances of some formula. The following formulas of fuzzy predicate logic are called the *disjunctive normal form*

$$\mathrm{DNF}(x_1, \dots, x_n) = \bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n) \mathbin{\&} E_{i_1 \dots i_n}) \tag{1}$$

and the *conjunctive normal form*

$$\mathrm{CNF}(x_1, \dots, x_n) = \bigwedge_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n) \to E_{i_1 \dots i_n}). \tag{2}$$

Analogously to this definition of normal forms in predicate logic, we introduce a new one in predicate ŁΠ logic.

**Definition 2**
Let $P_1, \dots, P_k$ be unary predicate symbols and $E_{i_1 \dots i_n}$, $1 \le i_j \le k$, $1 \le j \le n$, $n \ge 1$, be either truth constants or closed instances of some formula. The following formula is called the *additive normal form*

$$\mathrm{ANF}(x_1, \dots, x_n) = \mathop{\nabla}_{i_1,\dots,i_n=1}^{k} ((P_{i_1}(x_1) \mathbin{\&_{\text{Ł}}} \cdots \mathbin{\&_{\text{Ł}}} P_{i_n}(x_n)) \mathbin{\&_{\Pi}} E_{i_1 \dots i_n}). \tag{3}$$

Why are these formulas called the normal forms? First of all, the disjunctive normal form is represented by the disjunction of elementary conjunctions and thus, resembles the analogous boolean formula. Moreover, each elementary conjunction $P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n) \mathbin{\&} E_{i_1 \dots i_n}$ consists of the characterization of an area $P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n)$ combined with the description of a selected truth value of some formula $E_{i_1 \dots i_n}$ when its arguments lie inside the respective area. Second, the expression of the conjunctive normal form came from the formal translation of the fuzzy IF-THEN rules mentioned above. On the other side, this expression can be also generalized from the boolean conjunctive normal form written in the implicative form.

The normal forms are distinguished among the other formulas because they can be used as *universal approximators*. By this, we mean that any continuous fuzzy relation on a compact domain can be approximated by the fuzzy relation represented in one of the normal forms provided with specific interpretation of all included symbols (for different proofs regarding DNF see [1, 2, 9, 10] and regarding CNF see [15]).

# 4 Similarity and extensionality

Aiming at proving the approximation property of normal forms, we have to restrict formulas which can be approximately represented. Having on mind the result concerning a universal approximation cited above, we will introduce an analogy to the continuity property. This will be called "extensionality w.r.t similarity" which, for some special choices of similarity, is equivalent to Lipschitz continuity.

Let us further extend the language $J$ by the binary predicate symbol of similarity $\approx$. The corresponding atomic formula will be written as $x \approx y$ and has the meaning "$x$ is similar to $y$". The following axioms characterize similarity:

(S1) $(\forall x)(x \approx x)$,

(S2) $(\forall x, y)(x \approx y \to y \approx x)$,

(S3) $(\forall x, y, z)(x \approx y \mathbin{\&} y \approx z \to x \approx z)$.

Let $(x \approx^k y)$ be the abbreviation of $(x \approx y) \mathbin{\&} \cdots \mathbin{\&} (x \approx y)$, $k$ times, $k \geq 1$. It is easy to see that in the theory of similarity over BL∀ with special axioms (S1)–(S3), the same axioms with $\approx$ replaced by $(x \approx^k y)$ hold true.

The following axiom [1] characterizes the *extensionality* property for a predicate $P$ of arity $n$ w.r.t. $\approx$:

$$x_1 \approx y_1 \mathbin{\&} \cdots \mathbin{\&} x_n \approx y_n \to P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n).$$

Let us show that the extensionality is, in some sense, analogous to continuity. Let $\langle M, r \rangle$ be a model of the *similarity* axioms (over a BL-algebra). Then

$$r(a,a) = 1$$
$$r(a,b) = r(b,a)$$
$$r(a,b) * r(b,c) \leq r(a,c)$$

holds for all $a, b, c \in M$.

A fuzzy relation $s : M^n \longrightarrow [0,1]$ is *extensional* (w.r.t.a similarity $r$) if for each $u_1, \ldots, u_n$ and $v_1, \ldots, v_n \in M$

$$r(u_1, v_1) * \cdots * r(u_n, v_n) * s(u_1, \ldots, u_n) \leq s(v_1, \ldots, v_n).$$

It has been proved in [7] that a structure $\langle M, r, s \rangle$ is a model of the similarity axioms and extensionality iff $s$ is extensional w.r.t. $r$. Moreover, due to [4], in Łukasiewicz logic a fuzzy relation $s$ on $D$ is extensional w.r.t. $r$ iff it is Lipschitz continuous w.r.t. the pseudo-metric $d$, where $d(x,y) = 1 - r(x,y)$.

Till now, we spoke about extensionality for predicates. Having on mind that a formula with $n$ free variables defines an $n$-ary predicate we can formally consider the extensionality axiom for formulas too. It has been proved (P. Hájek, [7]) that if a theory $T$ over BL∀ contains the similarity axioms for $\approx$ and extensionality axioms for all its predicates then the extensionality for any formula of $T$ can be proved w.r.t. $(x \approx^k y)$, for some $k \geq 1$. Further we will use the term *extensional formula* when the extensionality axiom for this formula can be proved in the respective theory.

# 5 Normal forms based on similarity and what can be approximated by them

We will specify normal forms which will be used for approximation of extensional formulas. Of course, they should be connected with the approximated formula. How? Two parts will be distinguished in the constructions of normal forms. One part characterizes the universe and is independent on any represented formula, while the other part actually characterizes it.

Let $T$ be a consistent theory over BL∀ (or predicate LΠ logic) with the similarity predicate $\approx$ and the respective axioms. We suppose that the language $J(T)$ of the theory $T$ is extended by a finite number of truth constants as new atomic formulas, and stipulate that the evaluation of these constants is the same in each interpretation.

Let $\mathbf{c}_1, \ldots, \mathbf{c}_k$, $k \geq 1$, be object constants and $\overline{d}_{i_1 \ldots i_n}$, $1 \leq i_j \leq k$, $1 \leq j \leq n$, be truth constants in the language $J(T)$. Since this and furthermore, in expressions (1)–(3) of normal forms we will specify

$$P_i(x) \quad \text{is} \quad x \approx \mathbf{c}_i \quad \text{and} \quad E_{i_1 \ldots i_n} \quad \text{is} \quad \overline{d}_{i_1 \ldots i_n}.$$

The result will be demonstrated on the expression for DNF

$$\mathrm{DNF}(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_n = 1}^{k} ((x_1 \approx \mathbf{c}_{i_1}) \mathbin{\&} \cdots \mathbin{\&} (x_n \approx \mathbf{c}_{i_n}) \mathbin{\&} \overline{d}_{i_1 \ldots i_n}). \tag{4}$$

It can be easily verified that with such specification, predicates $P_i(x)$ are extensional w.r.t $\approx$. Thus, normal forms are extensional w.r.t. $\approx^l$, for some $l \geq 1$.

[1]This axiom has been introduced in [7] as axiom of congruence

**Lemma 1**
*Let $T$ fulfill the above written assumptions and $\langle M, r, c_1, \ldots, c_k \rangle$ be a model of $T$ where $r$ interprets $\approx$ and $c_1, \ldots, c_k$ are designated elements from $M$ which correspond to the constants $\mathbf{c}_1, \ldots, \mathbf{c}_k$. Then the fuzzy relations interpreting normal forms can be expressed by*

$$R_{\mathrm{DNF}}(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_n = 1}^{k} (r(c_{i_1}, x_1) * \cdots * r(c_{i_n}, x_n) * d_{i_1 \ldots i_n}), \tag{5}$$

$$R_{\mathrm{CNF}}(x_1, \ldots, x_n) = \bigwedge_{i_1, \ldots, i_n = 1}^{k} (r(c_{i_1}, x_1) * \cdots * r(c_{i_n}, x_n) \rightarrow d_{i_1 \ldots i_n}) \tag{6}$$

*in case $T$ is the theory over BL$\forall$ and*

$$R_{\mathrm{ANF}}(x_1,\dots,x_n) = \bigoplus_{i_1,\dots,i_n=1}^{k} ((r(c_{i_1},x_1)\otimes\cdots\otimes r(c_{i_n},x_n))\odot d_{i_1\dots i_n}) \tag{7}$$

*in case $T$ is the theory over predicate ŁΠ logic. Symbols $\otimes$, $\odot$ denote in (7) the operations of Łukasiewicz product and ordinary product, respectively.*

Let us now discuss the principal construction of normal forms based on similarity. In essence, they represent an aggregated description of an available local information about the formula. This local information is given by each elementary conjunction (or implication in case of CNF) and consists of the characterization of a domain (a neighborhood of a point $(c_{i_1},\dots,c_{i_n})$) combined with the description of a selected truth value of a formula when its arguments lie inside the respective domain.

The following two theorems give a partial answer to the problem of approximation ability of normal forms. In fact, they show that DNF, CNF and ANF can be respectively considered as lower, middle and upper approximate representations of the given formula.

**Theorem 1**
*Let $T$ be a theory over BL$\forall$ containing the similarity axioms for $\approx$ and extended by the truth constants $\overline{d}_{i_1\dots i_n}$. Let $\mathbf{c}_1,\dots,\mathbf{c}_k\in J(T)$ be object constants. Moreover, let $T$ contain the extensionality axiom for $\varphi(x_1,\dots,x_n)$ w.r.t. $\approx$ and*

$$T\vdash \varphi(\mathbf{c}_{i_1},\dots,\mathbf{c}_{i_n})\equiv \overline{d}_{i_1\dots i_n}. \tag{8}$$

*Then*

$$T\vdash \mathrm{DNF}(x_1,\dots,x_n)\rightarrow\varphi(x_1,\dots,x_n), \tag{9}$$
$$T\vdash \varphi(x_1,\dots,x_n)\rightarrow \mathrm{CNF}(x_1,\dots,x_n) \tag{10}$$

**Theorem 2**
*Let $T$ be a theory over ŁΠ predicate logic containing the similarity axioms for $\approx$ and extended by the truth constants $\overline{d}_{i_1\dots i_n}$. Let $\mathbf{c}_1,\dots,\mathbf{c}_k\in J(T)$ be object constants and $P_i(x)$ and $E_{i_1\dots i_n}$ stand for $x\approx\mathbf{c}_i$ and $\overline{d}_{i_1\dots i_n}$ respectively. Let the expressions for disjunctive (1) and conjunctive (2) normal forms be based on Łukasiewicz conjunction and implication, so that*

$$\mathrm{DNF}(x_1,\dots,x_n) = \bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1)\,\&_L\cdots\&_L\,P_{i_n}(x_n)\,\&_L\,E_{i_1\dots i_n})$$

*and*

$$\mathrm{CNF}(x_1,\dots,x_n) = \bigwedge_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1)\,\&_L\cdots\&_L\,P_{i_n}(x_n)\rightarrow{}_L E_{i_1\dots i_n}).$$

*Assume that for each $n$-tuple of indices $(j_1,\dots,j_n)$*

$$T\vdash \mathop{\nabla}_{\substack{i_1,\dots,i_n=1\\(i_1,\dots,i_n)\neq(j_1,\dots,j_n)}}^{k} (P_{i_1}(x_1)\,\&_L\cdots\&_L\,P_{i_n}(x_n))\equiv{}_L\neg_L(P_{j_1}(x_1)\,\&_L\cdots\&_L\,P_{j_n}(x_n)). \tag{11}$$

*Then the following holds true for the additive normal form given by (3):*

$$T\vdash \mathrm{DNF}(x_1,\dots,x_n)\rightarrow_L \mathrm{ANF}(x_1,\dots,x_n) \tag{12}$$
$$T\vdash \mathrm{ANF}(x_1,\dots,x_n)\rightarrow_L \mathrm{CNF}(x_1,\dots,x_n). \tag{13}$$

PROOF: Let us transform the assertion of the theorem into propositional ŁΠ logic. Suppose that $T$ is a theory of propositional ŁΠ logic and $a_1, \ldots, a_k, b_1, \ldots, b_k$ are propositional variables. Moreover, let $T$ prove the following (cf. (11)):

$$T \vdash \mathop{\nabla}_{\substack{i=1 \\ i\neq j}}^{k} a_i \equiv_{\text{Ł}} \neg_{\text{Ł}} a_j \qquad j = 1, \ldots, k. \tag{14}$$

Then (cf. (12)-(13))

$$T \vdash \bigvee_{i=1}^{k} (a_i \,\&_{\text{Ł}}\, b_i) \to_{\text{Ł}} \mathop{\nabla}_{i=1}^{k} (a_i \,\&_{\Pi}\, b_i) \tag{15}$$

$$T \vdash \mathop{\nabla}_{i=1}^{k} (a_i \,\&_{\Pi}\, b_i) \to_{\text{Ł}} \bigwedge_{i=1}^{k} (a_i \to_{\text{Ł}} b_i). \tag{16}$$

We will prove the last two formulas in theory $T$ ($\vdash \varphi$ means that formula $\varphi$ is provable in ŁΠ propositional calculus).

(i) proof of (15)

$$\vdash (a_i \,\&_{\text{Ł}}\, b_i) \to_{\text{Ł}} (a_i \,\&_{\Pi}\, b_i)$$

$$\vdash \mathop{\nabla}_{i} (a_i \,\&_{\text{Ł}}\, b_i) \to_{\text{Ł}} \mathop{\nabla}_{i} (a_i \,\&_{\Pi}\, b_i)$$

$$\vdash \bigvee_{i} (a_i \,\&_{\text{Ł}}\, b_i) \to_{\text{Ł}} \mathop{\nabla}_{i} (a_i \,\&_{\text{Ł}}\, b_i)$$

$$\vdash \bigvee_{i} (a_i \,\&_{\text{Ł}}\, b_i) \to_{\text{Ł}} \mathop{\nabla}_{i} (a_i \,\&_{\Pi}\, b_i)$$

(ii) proof of (16)

$$\vdash a_i \,\&_{\Pi}\, b_i \to_{\text{Ł}} a_i$$

$$\vdash \mathop{\nabla}_{i\neq j} (a_i \,\&_{\Pi}\, b_i) \to_{\text{Ł}} \mathop{\nabla}_{i\neq j} a_i$$

$$T \vdash \mathop{\nabla}_{i\neq j} a_i \equiv_{\text{Ł}} (\neg_{\text{Ł}} a_j) \qquad \text{assumption (14)}$$

$$T \vdash \mathop{\nabla}_{i} (a_i \,\&_{\Pi}\, b_i) \to_{\text{Ł}} (\neg_{\text{Ł}} a_j \nabla b_j)$$

$$T \vdash \mathop{\nabla}_{i} (a_i \,\&_{\Pi}\, b_i) \to_{\text{Ł}} (a_i \to_{\text{Ł}} b_i)$$

$$T \vdash \mathop{\nabla}_{i} (a_i \,\&_{\Pi}\, b_i) \to_{\text{Ł}} \bigwedge_{i} (a_i \to_{\text{Ł}} b_i)$$

□

Note that formula (11) characterizes a special covering of the universe by neighbourhoods of points $(c_{i_1}, \ldots, c_{i_n})$[2]. The semantical characterization of this covering as well as what it induces is described by the corollary below.

**Corollary 1**
*Let $T$ fulfill the assumptions of Theorem 2 and $\mathbf{M} = \langle M, r, c_1, \ldots, c_k \rangle$ be a model of $T$ where $r$ interprets $\approx$ and $c_1, \ldots, c_k$ are designated elements from $M$ interpreting the constants $\mathbf{c}_1, \ldots, \mathbf{c}_k$. Then the following*

[2] A formula with the same structure has been called JUST-ONE in [11].

*equation*

$$\bigoplus_{\substack{i_1,\dots,i_n=1\\(i_1,\dots,i_n)\neq(j_1,\dots,j_n)}}^{k} (r(c_{i_1},x_1)\otimes\cdots\otimes r(c_{i_n},x_n)) = 1-(r(c_{j_1},x_1)\otimes\cdots\otimes r(c_{j_n},x_n))$$

*is true in* $\mathbf{M}$ *for each* $n$*-tuple of indices* $(j_1,\dots,j_n)$. *This implies*

$$\bigoplus_{i_1,\dots,i_n=1}^{k} (r(c_{i_1},x_1)\otimes\cdots\otimes r(c_{i_n},x_n)) = \mathbf{1} \tag{17}$$

*and for all* $x_1,\dots,x_n\in M$

$$\begin{aligned} R_{\mathrm{DNF}}(x_1,\dots,x_n) \leq & R_{\mathrm{ANF}}(x_1,\dots,x_n),\\ R_{\mathrm{ANF}}(x_1,\dots,x_n) \leq & R_{\mathrm{CNF}}(x_1,\dots,x_n). \end{aligned}$$

The theorem below states that if the whole universe can be covered by a finite number of local domains (this is the meaning of our special condition, in other words, compactness) then the original extensional formula can be equivalently represented by each of its normal forms. Moreover, if we refine the covering, then the respective equivalence will be more precise.

**Theorem 3**
*Let* $T$ *be a theory over BL*$\forall$ *containing the similarity axioms for* $\approx$ *and extended by the truth constants* $\overline{d}_{i_1\dots i_n}$. *Let* $\mathbf{c}_1,\dots,\mathbf{c}_k\in J(T)$ *be object constants. Moreover, let* $T$ *contain the extensionality axiom for* $\varphi(x_1,\dots,x_n)$ *w.r.t.* $\approx$ *and*

$$T\vdash\varphi(\mathbf{c}_{i_1},\dots,\mathbf{c}_{i_n})\equiv\overline{d}_{i_1\dots i_n}.$$

*In the expressions of normal forms, let* $P_i(x)$ *and* $E_{i_1\dots i_n}$ *stand for* $x\approx\mathbf{c}_i$ *and* $\overline{d}_{i_1\dots i_n}$ *respectively. Then for each of the three normal forms (denoted generally by* NF*) conditional equivalence*

$$T\cup\{(\forall x_1,\dots,x_n)(\bigvee_{i_1\dots i_n}(P_{i_1}(x_1)\,\&\cdots\&\,P_{i_n}(x_n))\}\vdash$$
$$(\forall x_1,\dots,x_n)(\mathrm{NF}(x_1,\dots,x_n)\equiv\varphi(x_1,\dots,x_n)) \tag{18}$$

*holds true.*

**Remark 2**
Precisely, the condition of equivalence (18) is given by the following formula $(\forall x_1,\dots,x_n)\bigvee_{i_1\dots i_n}(P_{i_1}(x_1)\,\&\cdots\&\,P_{i_n}(x_n))$ or after the specialization of $P_i$'s, by $(\forall x_1,\dots,x_n)\bigvee_{i_1\dots i_n}((x_1\approx\mathbf{c}_{i_1})\,\&\cdots\&(x_n\approx\mathbf{c}_{i_n}))$. This describes the "quality" of a covering of a universe by neighbourhoods of respective points $(c_{i_1},\dots,c_{i_n})$. On the semantic level, expressions (18) and (**??**) mean that in any model of $T$ where the above condition is evaluated by truth value $\mathbf{1}$, the equivalence between $\varphi$ and its normal forms is also evaluated by $\mathbf{1}$. Moreover, due to the used quantifiers, both evaluations are global. If we are interested in a local estimation of the "quality" of the above equivalences, actually in each point, we may use one intermediate formula in the given proof (see the Corollary below). In fact, a truth value of the antecedent in both formulas below gives a lower estimation of truth values of both following equivalences.

**Corollary 2**
*Let* $T$, $\varphi$ *and normal forms fulfill the assumptions of Theorem 3. Then*

$$T\vdash\bigvee_{i_1\dots i_n}((x_1\approx^2\mathbf{c}_{i_1})\,\&\cdots\&(x_n\approx^2\mathbf{c}_{i_n}))\to(\mathrm{NF}(x_1,\dots,x_n)\equiv\varphi(x_1,\dots,x_n)) \tag{19}$$

In analogy with the approximation theory, one may expect that conditions of equivalences (18) or (19) can be improved by the appropriate choice of the objects $\mathbf{c}_1,\dots,\mathbf{c}_N$. This is possible if we require additional properties of these objects. However, this special question should be left to the approximation theory and not to logic.

## 6 Logical representation of neural networks

Neural networks became famous especially as approximating tools. The functional behaviour of a neural network can be represented either by a certain formula or by an algorithm with the great number of parameters. Neural networks are thus special approximation methodologies which differ from the other ones by the way how the parameters are chosen. This is done according to some optimization criterion and thus, the relation to the original function is not visible.

In this section we will show that under natural assumptions, the behaviour of neural network can be represented by a formula of predicate ŁΠ logic in the additive normal form. This fact can help us to understand the meaning of parameters of neural networks and help to choose less exhaustive methods for their tuning.

Following [8], a single layer network with $n$ inputs and one output can be represented by a function $F : [0,1]^n \longrightarrow [0,1]$ in the following (algebraic) form:

$$F(x_1,\dots,x_n) = \sum_{i=1}^{k} \alpha_i \phi\left(\sum_{j=1}^{n} w_{ij} x_j + b_i\right) \tag{20}$$

where $\phi : \mathbb{R} \longrightarrow [0,1]$ is a non-decreasing continuous function (referred to as activation function), $k$ is a number of nodes in a hidden layer, and $\alpha_i, w_{ij}, b_i$ are real constants (parameters).

Suppose that an activation function $\phi$ is linear and rewrite the expression (20) into

$$F(x_1,\dots,x_n) = \sum_{i=1}^{k} \alpha_i \left(\sum_{j=1}^{n} w_{ij} \phi(x_j) + \phi(b_i)\right). \tag{21}$$

Recall once again formula (7)

$$R_{\mathrm{ANF}}(x_1,\dots,x_n) = \bigoplus_{i_1,\dots,i_n=1}^{k} ((r(c_{i_1},x_1) \otimes \cdots \otimes r(c_{i_n},x_n)) \odot d_{i_1\dots i_n})$$

which describes the fuzzy relation represented by ANF and compare it with (21).

Suppose that in a model $\langle M, r, c_1, \dots, c_k\rangle$ where the ANF is interpreted by $R_{\mathrm{ANF}}$, condition (17) holds true. Let us remark on this place that the similarity relation $r$ can take also linear form. Then $\oplus$ can be replaced by $\sum$ and we obtain

$$R_{\mathrm{ANF}}(x_1,\dots,x_n) = \sum_{i_1,\dots,i_n=1}^{k} (d_{i_1\dots i_n} \odot (r(c_{i_1},x_1) \otimes \cdots \otimes r(c_{i_n},x_n))) \tag{22}$$

(recall that $\odot$ is ordinary product). Finally replace $(r(c_{i_1},x_1) \otimes \cdots \otimes r(c_{i_n},x_n))$ by its algebraic representation

$$max(0, (r(c_{i_1},x_1) + \cdots + r(c_{i_n},x_n)) - (n-1)).$$

We can see that formulas (20) and (7) can be transformed one into another by a suitable choice of parameters. This confirms that a single layer network with $n$ inputs and one output can be represented by a formula of predicate ŁΠ logic in the additive normal form. Moreover, Theorem 2 and its corollary theoretically substantiate the approximation ability of neural networks.

## 7 Conclusions

In this paper the concepts of disjunctive, conjunctive and additive normal forms have been introduced as special formulas of predicate BL and ŁΠ logic. The expressions for normal forms stem from a formal translation of sets of IF-THEN rules using which various kinds of dependencies are linguistically described in the applications of fuzzy logic.

We have specified the basic predicates in normal forms by particular cases of similarity predicates. On the basis of this, we have established the approximation property of normal forms. We have shown that the approximation means the conditional equivalence on the syntactical level. Thus, we have justified that in the algebra of logical operations on $[0, 1]$ determined by a continuous $t$-norm, any extensional fuzzy relation on a compact domain can be approximated by the fuzzy relation represented in one of the normal forms. The precision of the approximation can be extracted from a truth value of the respective condition.

We dealt also with the problem, whether the behaviour of neural networks can be represented by formulas of fuzzy logic (posed by A. DiNola). We have solved it for a wide class of one layer neural networks. Namely, we have shown that their behaviour can be represented by formulas of predicate ŁΠ logic in the additive normal form. This fact can help us to understand the meaning of parameters of neural networks and to choose less exhaustive methods for their tuning.

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