University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

Normal Forms in BL-algebra of Functions and Their Contribution to Universal Approximation

Irina Perfilieva

Research report No. 37

2001

Submitted/to appear:

Soft Computing

Supported by:

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
Bráfova 7, 701 03 Ostrava 1, Czech Republic

tel.: +420-69-6160 233 fax: +420-69-6120 478
e-mail: irina.perfilieva@osu.cz

NORMAL FORMS IN BL-ALGEBRA OF FUNCTIONS AND THEIR CONTRIBUTION TO UNIVERSAL APPROXIMATION

I. Perfilieva

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30.dubna 22, 701 03 Ostrava 1

Czech Republic

Abstract

This paper continues the investigation of approximating properties of generalized normal forms in fuzzy logic. The problem is formalized and solved algebraically. Normal forms are considered in two variants: infinite and finite. It is proved that infinite normal forms are universal representation formulas whereas finite normal forms are universal approximation formulas for extensional functions. The estimation of the quality of approximation is suggested.

Key words Disjunctive and conjunctive normal forms, BL-algebra, universal approximation, extensional functions

1 Introduction

This paper is the contribution to the study of BL-algebra of functions (which can be regarded as the generalization of boolean algebra of functions) and, at the same time, to the theory of representation of extensional functions by special formulas.

The latter general theory has many practical applications because it provides formal description of functions which can be taken as mathematical models and then used for different purposes. Let us briefly overview some principal results with respect to the theory of representation.

In boolean algebra of logical functions each function can be represented by disjunctive and conjunctive normal forms. In this connection we say that normal forms in boolean algebra play the role of universal representation formulas. In the algebra of many-valued logical functions, the Rosser-Turquette formula generalizes the disjunctive normal (boolean) form and also serves as universal representation formula. The other results concern the so called universal approximation. By this we mean that a class of approximating functions can be represented by one formula with a number of parameters such that each specification of parameters determines one approximating function of this class. From the theory of approximation, we know that e.g., Lagrange polynomials or Fourier series are examples of universal approximation formulas for continuous functions defined on bounded intervals. Fuzzy logic also contributed to this particular problem. It offers formulas of a first order calculus with fuzzy predicates which formalize a linguistic description given by a set of IF–THEN rules and thus, serve as universal approximation formulas for continuous functions defined on a compact set (see e.g.,[1, 2, 6, 9, 10]). Later, it has been shown that these formulas can be regarded as generalizations of boolean normal forms ([8, 10, 11, 12]). Moreover, in [12] this fact has been proved by formal logical means.

This paper continues the investigation of approximation properties of generalized normal forms in fuzzy logic. The problem is formalized and solved algebraically. To be more general, an arbitrary BL-algebra as an algebra of fuzzy logic operations has been chosen (Section 2). Normal forms are considered in two variants: infinite and finite (Sections 4, 5). It is proved that infinite normal forms are universal representation formulas whereas finite normal forms are universal approximation formulas for extensional functions (Section 3). The estimation of the quality of approximation is suggested.

2 BL-algebra

BL-algebra has been introduced in [5] as the algebra of logic operations which correspond to connectives of basic logic (BL). In the same sense as BL generalizes boolean logic we can say that BL-algebra generalizes boolean algebra. This appears in the extension of the set of boolean operations by two semigroup operations which constitute so called adjoined couple. The following definition summarizes definitions which have been introduced in [5].

Definition 1
A *BL-algebra* is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a lattice with $\mathbf{0}$ and $\mathbf{1}$ as the least and largest elements w.r.t. the lattice ordering,

(ii) $(L, *, \mathbf{1})$ is a commutative semigroup with the unit $\mathbf{1}$, such that multiplication $*$ is associative, commutative and $1 * x = x$ for all $x \in L$,

(iii) $*$ and $\rightarrow$ form an adjoint pair, i.e.
$z \leq (x \rightarrow y)$ iff $x * z \leq y$ for all $x, y, z \in L$,

(iv) and moreover, for all $x, y \in L$
$x * (x \rightarrow y) = x \wedge y$,
$(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$.

Another two operations of $\mathcal{L}$: unary $\neg$ and binary $\leftrightarrow$ can be defined by

$$\neg x = x \rightarrow \mathbf{0},$$
$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The following property will be widely used in the sequel:

$$x \leq y \quad \text{iff} \quad (x \rightarrow y) = \mathbf{1}.$$

Note that if a lattice $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is given, then BL-algebra is completely defined by the choice of multiplication operation $*$. In particular, $L = [0, 1]$ and $*$ is known as a t-norm (see examples below).
The following are examples of BL-algebras.

Example 1 (Boolean algebra for classical logic)

$$\mathcal{L}_B = \langle \{\mathbf{0}, \mathbf{1}\}, \vee, \wedge, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

where the multiplication $* = \wedge$ and $\rightarrow$ is the classical implication.

Example 2 (Gödel algebra)

$$\mathcal{L}_G = \langle [0, 1], \vee, \wedge, \rightarrow_G, 0, 1 \rangle$$

where the multiplication $* = \wedge$ and

$$x \rightarrow_G y = \begin{cases} 1 & \text{if} \quad x \leq y, \\ y & \text{if} \quad y < x. \end{cases} \tag{1}$$

Example 3 (Goguen algebra)

$$\mathcal{L}_P = \langle [0, 1], \vee, \wedge, \odot, \rightarrow_P, 0, 1 \rangle$$

where the multiplication $\odot = \cdot$ is the ordinary product of reals and

$$x \rightarrow_P y = \begin{cases} 1 & \text{if} \quad x \leq y, \\ \frac{y}{x} & \text{if} \quad y < x. \end{cases} \tag{2}$$

Example 4 (Łukasiewicz algebra)

$$\mathcal{L}_{\text{L}} = \langle [0,1], \vee, \wedge, \otimes, \to_{\text{L}}, 0, 1 \rangle \tag{3}$$

where

$$x \otimes y = 0 \vee (x + y - 1), \qquad \text{(Łukasiewicz conjunction)} \tag{4}$$

$$x \to_{\text{L}} y = 1 \wedge (1 - x + y). \qquad \text{(Łukasiewicz implication)} \tag{5}$$

3 BL-algebras of L-valued Functions and Extensional Functions

Let X be a nonempty set, $\mathcal{L}$ a BL-algebra on L, and P_L a set of all L-valued functions $f(x_1, \ldots, x_n)$, $n \geq 0$, which are defined on X and take values from L. To shorten the denotation we will write $\tilde{\mathbf{x}}^{(n)}$ instead of $(x_1, \ldots, x_n)$. Let us extend the operations from $\mathcal{L}$ on P_L so that for each $f(\tilde{\mathbf{x}}^{(n)}), g(\tilde{\mathbf{x}}^{(n)}) \in P_L$

$$(f \vee g)(\tilde{\mathbf{x}}^{(n)}) = f(\tilde{\mathbf{x}}^{(n)}) \vee g(\tilde{\mathbf{x}}^{(n)}), \qquad (f * g)(\tilde{\mathbf{x}}^{(n)}) = f(\tilde{\mathbf{x}}^{(n)}) * g(\tilde{\mathbf{x}}^{(n)}),$$
$$(f \wedge g)(\tilde{\mathbf{x}}^{(n)}) = f(\tilde{\mathbf{x}}^{(n)}) \wedge g(\tilde{\mathbf{x}}^{(n)}), \qquad (f \to g)(\tilde{\mathbf{x}}^{(n)}) = f(\tilde{\mathbf{x}}^{(n)}) \to g(\tilde{\mathbf{x}}^{(n)}).$$

Furthermore, let $\mathbb{U}$ and $\mathbb{O}$ be constant functions from P_L taking the values $\mathbf{1}$ and $\mathbf{0}$, respectively. Then

$$P_{\mathcal{L}} = \langle P_L, \vee, \wedge, *, \to, \mathbb{U}, \mathbb{O} \rangle$$

is a BL-algebra.

Note that

(i) in case $n = 0$, L-valued functions degenerate to constants (elements of L),

(ii) in case $n = 1$, L-valued functions are usually considered as membership functions of L-valued fuzzy sets,

(iii) in other cases, L-valued functions are identified with fuzzy relations.

We will be concerned with the subclass of L-valued functions formed by so called extensional functions. The reason comes from the fact that extensional functions have properties similar to continuity and therefore, they can be represented (precisely or approximately) by special formulas over $P_{\mathcal{L}}$ (see [11, 12]). But before we give the definition of extensional functions we will introduce the similarity relation on X which helps to describe a neighborhood of a point.

Definition 2
A binary fuzzy relation E on X given by L-valued function $E(x, y)$ is called a *similarity* if for each $x, y, z \in X$ the following properties hold true

$$E(x, y) = \mathbf{1}, \quad \text{(reflexivity)}$$
$$E(x, y) = E(y, x), \quad \text{(symmetry)}$$
$$E(x, y) * E(y, z) \leq E(x, z). \quad \text{(transitivity)}$$

In our text we will formally distinguish between the similarity relation (denoted by E) and the (membership) function representing it (denoted by $E(x, y)$), although they are closely connected. The value $E(x, y)$ can be interpreted as the *degree of similarity* of x and y or the degree which characterizes that x belongs to a neighborhood of y.

Let us consider some examples of similarity. Note that each crisp equivalence on X is also a similarity on X in any BL-algebra $P_{\mathcal{L}}$. If $L = [0, 1]$ and the multiplication $*$ is a continuous Archimedian t-norm with continuous generator $g : [0, 1] \longrightarrow [0, \infty]$, then for any pseudo-metric $d : X^2 \longrightarrow [0, \infty]$ on X the fuzzy relation E_d on X given by $E_d(x, y) = g^{(-1)}(d(x, y))$ is a similarity (see [7]). For example, if $*$ is the Łukasiewicz conjunction with generator $1 - x$ then $E_d(x, y) = \max(0, 1 - d(x, y))$ defines the similarity on X in BL-algebra $P_{\mathcal{L}_{\text{L}}}$.

The following definition of extensional function is taken from [5].

Definition 3
An L-valued function $f(x_1,\dots,x_n)$ is *extensional* w.r.t. a similarity relation E on X if for all $x_1,\dots,x_n,y_1,\dots,y_n \in X$

$$E(x_1,y_1) * \dots * E(x_n,y_n) * f(x_1,\dots,x_n) \leq f(y_1,\dots,y_n). \tag{6}$$

It is easy to see that if $f(x,y) = E(x,y)$ or $f(x,y) = E(x,y) * a$ where a is an arbitrary element of L then $f(x,y)$ is extensional w.r.t. similarity E. A general characterization of extensional functions can be given in the BL-algebras where the operation $*$ is a continuous Archimedean t-norm. This characterization is based on the property which is analogous to Lipschitz continuity.

Theorem 1
Let $L = [0,1]$ and multiplication $*$ be a continuous Archimedean t-norm with the continuous generator $g : [0,1] \longrightarrow [0,\infty]$. Let $d : X \longrightarrow [0,\infty]$ be a pseudo-metric on X and the fuzzy relation E_d on X given by $E_d(x,y) = g^{(-1)}(d(x,y))$ be a similarity. Then any extensional function $f(x_1,\dots,x_n)$, w.r.t. the similarity E_d fulfils the inequality

$$|g(f(x_1,\dots,x_n)) - g(f(y_1,\dots,y_n))| \leq \min(g(0), d(x_1,y_1) + \dots + d(x_n,y_n)) \tag{7}$$

where $n \geq 2$ or (in case $n = 1$)

$$|g(f(x)) - g(f(y))| \leq d(x,y). \tag{8}$$

PROOF: Let us recall that continuous generator g of $*$ is strictly decreasing function such that $g(1) = 0$. Moreover,

$$a * b = g^{-1}(\min g(0), g(a) + g(b)). \tag{9}$$

A pseudoinverse function $g^{(-1)} : [0,\infty] \longrightarrow [0,1]$ is defined as follows:

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if} \quad x \in [0, g(0)], \\ 0, & \text{otherwise.} \end{cases}$$

Without lost of generality it is sufficient to consider the case $n = 2$. By (6),

$$E(x_1,y_1) * E(x_2,y_2) * f(x_1,x_2) \leq f(y_1,y_2)$$

which together with (9) implies

$$g^{-1}(\min(g(0), g(E(x_1,y_1)) + g(E(x_2,y_2)) + g(f(x_1,x_2)))) \leq f(y_1,y_2)$$

and therefore,

$$\min(g(0), g(E(x_1,y_1)) + g(E(x_2,y_2)) + g(f(x_1,x_2))) \geq g(f(y_1,y_2)).$$

The last inequality due to the fact that $g(0) \geq g(f(y_1,y_2))$ is equivalent to

$$g(E(x_1,y_1)) + g(E(x_2,y_2)) + g(f(x_1,x_2)) \geq g(f(y_1,y_2))$$

and thus,

$$g(E(x_1,y_1)) + g(E(x_2,y_2)) \geq g(f(y_1,y_2)) - g(f(x_1,x_2)).$$

Interchanging x_1 with y_1 and x_2 with y_2 and using the symmetry of E we obtain

$$g(E(x_1,y_1)) + g(E(x_2,y_2)) \geq g(f(x_1,x_2)) - g(f(y_1,y_2))$$

and therefore,

$$g(E(x_1,y_1)) + g(E(x_2,y_2)) \geq |g(f(x_1,x_2)) - g(f(y_1,y_2))|.$$

Using the fact that $E_d(x,y) = g^{(-1)}(d(x,y))$, we proceed with

$$|g(f(x_1,x_2)) - g(f(y_1,y_2))| \leq \min g(0), d(x_1,y_1) + \min g(0), d(x_2,y_2)$$

which taking into account that $|g(f(x_1,x_2)) - g(f(y_1,y_2))| \leq g(0)$, gives the required inequality (7)

$$|g(f(x_1,x_2)) - g(f(y_1,y_2))| \leq \min g(0), d(x_1,y_1) + d(x_2,y_2).$$

□

Particularly, if the underlying BL-algebra is the Łukasiewicz algebra $\mathcal{L}_{\text{Ł}}$ then extensional functions are Lipschitz continuous in classical sense ([4]).

Corollary 1
Suppose that the above given conditions are fulfilled, and $*$ is the Łukasiewicz t-norm with generator $g(x) = 1 - x$. Then any extensional function $f(x_1,\ldots,x_n)$ fulfils the inequality

$$|f(x_1,\ldots,x_n) - f(y_1,\ldots,y_n)| \leq \min(1, d(x_1,y_1) + \cdots d(x_n,y_n)).$$

It is worth noticing that a class of extensional functions w.r.t. one specific similarity do not form a subalgebra of $P_{\mathcal{L}}$. This can be illustrated by the following example: the function $f(x,y) = E^2(x,y)$ (this is a short for $E(x,y) * E(x,y)$) is not extensional w.r.t. similarity E. Thus, in order to obtain a BL-subalgebra of extensional functions we shall extend the set of similarities.

Let $E^k(x,y)$ abbreviate $E(x,y) * \cdots * E(x,y)$ (k times). It is easy to see that for all $k \geq 1$, $E^k(x,y)$ defines a similarity E^k on X as well. Based on the proposition proved in [5], we can establish the following two lemmas.

Lemma 1
Let $E(x,y)$ define a similarity on X. A class of extensional functions w.r.t. E^k for some $k \leq 1$, is a BL-subalgebra of $P_{\mathcal{L}}$.

On the other hand, a class of extensional functions w.r.t. one specific similarity is a subalgebra of a weaker algebra than $P_{\mathcal{L}}$.

Lemma 2
Let $E(x,y)$ define a similarity on X, $n \geq 1$ a natural number. A class of extensional w.r.t. E functions depending on n variables, forms a sublattice of $\langle P_L, \vee, \wedge, \mathbb{U}, \mathbb{O}\rangle$.

4 Normal Forms for the Extensional Functions

Let us fix some similarity E on a set X and consider the sublattice of the lattice $\langle P_L, \vee, \wedge, \mathbb{U}, \mathbb{O}\rangle$ consisting of extensional w.r.t. E functions depending on n variables, $n \geq 1$. We will denote this sublattice by $P^n_{\mathcal{L}}/_E$.

Lemma 3
Let $f(x_1,\ldots,x_n) \in P^n_{\mathcal{L}}/_E$, $c_1,\ldots,c_n$ arbitrary elements of X and

$$f(c_1,\ldots,c_n) = d$$

where $n \geq 1$. Then functions represented by

$$E(x_1,c_1) * \cdots * E(x_n,c_n) * d \tag{10}$$

and

$$E(x_1,c_1) * \cdots * E(x_n,c_n) \rightarrow d, \tag{11}$$

are elements of $\in P^n_{\mathcal{L}}/_E$ and moreover, for all $x_1,\ldots,x_n \in X$

$$E(x_1,c_1) * \cdots * E(x_n,c_n) * d \leq f(x_1,\ldots,x_n), \tag{12}$$

$$f(x_1,\ldots,x_n) \leq E(x_1,c_1) * \cdots * E(x_n,c_n) \rightarrow d. \tag{13}$$

PROOF: It is sufficient to consider the case $n = 2$. The first statement (10) easily follows from the transitivity of E and monotonicity of $*$.

After applying the transitivity of E and antitonicity of $\to$ w.r.t. the first argument, the proof of statement (11) continues as follows:

$$E(x_1, c_1) * E(x_2, c_2) \to d \leq$$
$$E(x_1, y_1) * E(x_2, y_2) \to (E(y_1, c_1) * E(y_2, c_2) \to d)$$

and, using the adjointness of implication (see Def.1, (iii)),

$$E(x_1, y_1) * E(x_2, y_2) * (E(x_1, c_1) * E(x_2, c_2) \to d) \leq$$
$$(E(y_1, c_1) * E(y_2, c_2) \to d). \quad (14)$$

The last two statements (12) and (13) are direct consequences of the extensionality of $f(x_1, \ldots, x_n)$. □

The functions represented by the formula $(E(x_1, c_1) * \cdots * E(x_n, c_n) * d)$ or $(E(x_1, c_1) * \cdots * E(x_n, c_n) \to d)$ will be respectively called *lower* and *upper constituents* of the function $f(x_1, \ldots, x_n)$ if $f(c_1, \ldots, c_n) = d$. Using them, we can introduce the disjunctive normal form of f as a supremum of all its lower constituents and the second conjunctive normal form of f as an infimum of all its upper constituents. The first conjunctive normal form of f will be introduced as the formula dual to DNF.

Definition 4
Let $f(x_1, \ldots, x_n) \in P^n_{\mathcal{L}}/_E$ be an extensional function. The following formulas over $P_{\mathcal{L}}$ are called the *disjunctive normal form* of f

$$f_{\mathrm{DNF}}(x_1, \ldots, x_n) = \bigvee_{c_1, \ldots, c_n \in X} (E(x_1, c_1) * \cdots * E(x_n, c_n) * f(c_1, \ldots, c_n)) \quad (15)$$

and the *conjunctive normal forms* of f

$$f_{\mathrm{CNF}_I}(x_1, \ldots, x_n) = \neg \bigvee_{c_1, \ldots, c_n \in X} (E(x_1, c_1) * \cdots * E(x_n, c_n) * (\neg f(c_1, \ldots, c_n))) \quad (16)$$

and

$$f_{\mathrm{CNF}_{II}}(x_1, \ldots, x_n) = \bigwedge_{c_1, \ldots, c_n \in X} (E(x_1, c_1) * \cdots * E(x_n, c_n) \to f(c_1, \ldots, c_n)). \quad (17)$$

Remark 1

1. Actually, normal forms are situated on the right sides of expressions (15)–(17). On their left sides we have functions represented by the normal forms.
2. The first conjunctive normal form does not have the form of conjunction of disjunctions. We have introduced this expression as the dual to the disjunctive normal form (simply, we negated the DNF of the negated function). Although de Morgan laws are valid in BL-algebra, and thus, we can rewrite CNF_I as conjunction of negated terms, we prefer to work with the form given by (16), because it simplifies the proofs.
3. The expression $f(c_1, \ldots, c_n)$ in normal forms represents the constant from L. Namely, this constant is a value of f at the fixed point $(c_1, \ldots, c_n)$.
4. Two conjunctive normal forms are generally not equivalent except for the case of Łukasiewicz algebra.

At first, we will investigate whether the functions represented by normal forms, belong to the same sublattice $P_{\mathcal{L}}/_E$?

Lemma 4
Let $f(x_1, \ldots, x_n)$ be an extensional function from $P_{\mathcal{L}}/_E$. Then functions represented by normal forms of f, are extensional w.r.t. similarity E and thus, belong to $P_{\mathcal{L}}/_E$.

PROOF: To simplify the notation, we will consider the case $n = 2$.

(i) The proof of extensionality of DNF. By Lemma 3, lower constituents of the function f are extensional w.r.t. E, i.e.

$$E(x_1, y_1) * E(x_2, y_2) * (E(x_1, c_1) * E(x_2, c_2) * d) \leq E(y_1, c_1) * E(y_2, c_2) * d.$$

where $d = f(c_1, \ldots, c_n)$. Then

$$E(x_1, y_1) * E(x_2, y_2) * (E(x_1, c_1) * E(x_2, c_2) * d) \leq \bigvee_{c_1, c_2 \in X} (E(y_1, c_1) * E(y_2, c_2) * d),$$

and thus,

$$E(x_1, y_1) * E(x_2, y_2) * \bigvee_{c_1, c_2 \in X} (E(x_1, c_1) * E(x_2, c_2) * d) \leq \bigvee_{c_1, c_2 \in X} (E(y_1, c_1) * E(y_2, c_2) * d). \quad (18)$$

Note that due to the inequality (12) proved in Lemma 3, the suprema given above exist.

(ii) We begin the proof of extensionality of CNF_I by replacement of the occurrence of d in (18) by $\neg d$.

$$E(x_1, y_1) * E(x_2, y_2) * \bigvee_{c_1, c_2 \in X} (E(x_1, c_1) * E(x_2, c_2) * \neg d) \leq \bigvee_{c_1, c_2 \in X} (E(y_1, c_1) * E(y_2, c_2) * \neg d). \quad (19)$$

Applying the adjointness property of implication to (19) and the law of contraposition (valid in BL-algebra), we will obtain the extensionality of CNF_I.

$$E(x_1, y_1) * E(x_2, y_2) \leq \bigvee_{c_1, c_2 \in X} (E(x_1, c_1) * E(x_2, c_2) * \neg d) \to \bigvee_{c_1, c_2 \in X} (E(y_1, c_1) * E(y_2, c_2) * \neg d),$$

$$E(x_1, y_1) * E(x_2, y_2) \leq \neg \bigvee_{c_1, c_2 \in X} (E(y_1, c_1) * E(y_2, c_2) * \neg d) \to \neg \bigvee_{c_1, c_2 \in X} (E(x_1, c_1) * E(x_2, c_2) * \neg d).$$

(iii) We begin the proof of extensionality of CNF_{II} by recalling the extensionality of upper constituents of f, see (14).

$$E(x_1, y_1) * E(x_2, y_2) * (E(x_1, c_1) * E(x_2, c_2) \to d) \leq (E(y_1, c_1) * E(y_2, c_2) \to d).$$

Then the required property will be obtained in two steps

$$\bigwedge_{c_1, c_2} (E(x_1, y_1) * E(x_2, y_2) * (E(x_1, c_1) * E(x_2, c_2) \to d)) \leq$$

$$\bigwedge_{c_1,c_2} (E(y_1,c_1) * E(y_2,c_2) \to d),$$

$$E(x_1,y_1) * E(x_2,y_2) * \bigwedge_{c_1,c_2} (E(x_1,c_1) * E(x_2,c_2) \to d) \leq$$

$$\bigwedge_{c_1,c_2} \to (E(y_1,c_1) * E(y_2,c_2) \to d).$$

Note that due to the inequality (13) proved in Lemma 3 the infima given above exist.

□

Second, we will investigate the relation between the original function f and the functions represented by normal forms of f.

Theorem 2
Let $f(x_1,\ldots,x_n)$ be an extensional function from $P_{\mathcal{L}}/_E$. Then

$$f(x_1,\ldots,x_n) = f_{\mathrm{DNF}}(x_1,\ldots,x_n), \tag{20}$$
$$f(x_1,\ldots,x_n) = f_{\mathrm{CNF}_{II}}(x_1,\ldots,x_n), \tag{21}$$
$$f(x_1,\ldots,x_n) \leq f_{\mathrm{CNF}_{I}}(x_1,\ldots,x_n). \tag{22}$$

PROOF: Again, we will consider the case $n = 2$. By inequalities (12),(13) proved in Lemma 3, easily follows

$$f_{\mathrm{DNF}}(x_1,x_2) \leq f(x_1,x_2),$$
$$f(x_1,x_n) \leq f_{\mathrm{CNF}_{II}}(x_1,x_n).$$

Then, based on tautologies

$$f(x_1,x_2) = E(x_1,x_1) * E(x_2,x_2) * f(x_1,x_2),$$
$$f(x_1,x_2) = E(x_1,x_1) * E(x_2,x_2) \to f(x_1,x_2),$$

we deduce

$$f(x_1,x_2) \leq \bigvee_{c_1,c_2\in X} (E(x_1,c_1) * E(x_2,c_2) * f(c_1,c_2)),$$
$$f(x_1,x_2) \geq \bigwedge_{c_1,c_2\in X} (E(x_1,c_1) * E(x_2,c_2) \to f(c_1,c_2))$$

which prove equalities (20) and (21).

The inequality (22) easily follows from (20) applied to the negated function $\neg f(x_1,x_2)$, then

$$\neg\neg f(x_1,x_2) = \bigvee_{c_1,c_2\in X} (E(x_1,c_1) * E(x_2,c_2) * \neg f(c_1,c_2))$$

and then $f(x_1,x_2) \leq \neg\neg f(x_1,x_2)$. □

Let us stress that Theorem 2 demonstrates very convincing results, asserting that even in the fuzzy case, normal forms can be equal to the original function. Moreover, the construction of normal forms gives us the idea of how they can be simplified without significant loss of their ability to represent (at least approximately) the original function.

5 Discrete Normal Forms and Universal Approximation

The normal forms introduced above can hardly be used in practice because they are based on the full knowledge of the represented function in all its points. Having on mind practical applications, we have to simplify normal forms. We will do this by removing some of its elementary terms. Of course, after such removing we cannot expect that thus obtained simplification will represent the original function precisely. But we expect an approximate representation which in many cases is sufficient. Aiming at this, we will introduce discrete normal forms which are based on partial knowledge of the represented function in some nodes.

Definition 5
Let $f(x_1,\ldots,x_n)$ be an extensional function from $P_{\mathcal{L}}/_E$ defined on X, $c_1,\ldots,c_k$ some chosen elements (*nodes*) from X. Let elements $d_{i_1\ldots i_n}\in L$ be chosen so that

$$d_{i_1\ldots i_n}\leftrightarrow f(c_{i_1},\ldots,c_{i_n})$$

for each collection of indices $(i_1\ldots i_n)$ where $1\le i_1,\ldots,i_n\le k$. Then the following formulas over $P_{\mathcal{L}}$ are called the *discrete disjunctive normal form* of f

$$f_{\mathrm{DNF}}(x_1,\ldots,x_n)=\bigvee_{i_1,\ldots,i_n=1}^{k}(E(x_1,c_{i_1})*\cdots*E(x_n,c_{i_n})*d_{i_1\ldots i_n}) \tag{23}$$

and the *discrete conjunctive normal forms* of f

$$f_{\mathrm{CNF}_I}(x_1,\ldots,x_n)=\neg\bigvee_{i_1,\ldots,i_n=1}^{k}(E(x_1,c_{i_1})*\cdots*E(x_n,c_{i_n})*(\neg d_{i_1\ldots i_n})) \tag{24}$$

and

$$f_{\mathrm{CNF}_{II}}(x_1,\ldots,x_n)=\bigwedge_{i_1,\ldots,i_n=1}^{k}(E(x_1,c_{i_1})*\cdots*E(x_n,c_{i_n})\to d_{i_1\ldots i_n}). \tag{25}$$

Further on, we will show that the discrete normal forms do approximate the original function if the nodes are chosen properly. It is interesting that, independently on the choice of nodes, DNF and both CNF's can be considered as lower and upper approximate representations of the given function, respectively. This fact is proved in the theorem given below.

Theorem 3
Let $f(x_1,\ldots,x_n)$ be an extensional function from $P_{\mathcal{L}}/_E$ defined on X, $c_1,\ldots,c_k$ be nodes from X. Let elements $d_{i_1\ldots i_n}\in L$ be chosen so that

$$d_{i_1\ldots i_n}\leftrightarrow f(c_{i_1},\ldots,c_{i_n})$$

for each collection of indices $(i_1\ldots i_n)$ where $1\le i_1,\ldots,i_n\le k$. Then discrete normal forms given by (23)–(25) fulfil the following inequalities:

$$f_{\mathrm{DNF}}(x_1,\ldots,x_n)\le f(x_1,\ldots,x_n), \tag{26}$$
$$f(x_1,\ldots,x_n)\le f_{\mathrm{CNF}_I}(x_1,\ldots,x_n), \tag{27}$$
$$f(x_1,\ldots,x_n)\le f_{\mathrm{CNF}_{II}}(x_1,\ldots,x_n). \tag{28}$$

PROOF: All three inequalities are proved in the same way as their infinite analogies are proven in Theorem 2. □

The following theorem exposes the approximation property of discrete normal forms. By this we mean, that the equivalence between an extensional function and the functions represented by each of its discrete normal forms can be estimated from below. Thus, in the language of BL-algebra $P_{\mathcal{L}}$, the approximation means the *conditional equivalence*.

Theorem 4
Let $f(x_1,\dots,x_n)$ be an extensional function from $P_{\mathcal{L}}/_E$ defined on X, $c_1,\dots,c_k$ be nodes from X. Let elements $d_{i_1\dots i_n}\in L$ be chosen so that

$$d_{i_1\dots i_n}\leftrightarrow f(c_{i_1},\dots,c_{i_n})$$

for each collection of indices $(i_1\dots i_n)$ where $1\le i_1,\dots,i_n\le k$. Then functions represented by the discrete normal forms given by (23) and (25) are conditionally equivalent to the original function f which means

$$f_{\mathrm{DNF}}(x_1,\dots,x_n)\leftrightarrow f(x_1,\dots,x_n)\ge E^2(x_1,c_{i_1})*\dots*E^2(x_n,c_{i_n}) \tag{29}$$

$$f_{\mathrm{CNF}_{II}}(x_1,\dots,x_n)\leftrightarrow f(x_1,\dots,x_n)\ge E^2(x_1,c_{i_1})*\dots*E^2(x_n,c_{i_n}) \tag{30}$$

PROOF: Again, the case $n=2$ will be considered.

(i) Proof of (29).

$$E^2(x_1,c_{i_1})*E^2(x_2,c_{i_2})*f(x_1,x_2)\le E(x_1,c_{i_1})*E(x_2,c_{i_2})*f(c_1,c_2)\le$$
$$\le\bigvee_{i_1,i_2=1}^{k}E(x_1,c_{i_1})*E(x_2,c_{i_2})*f(c_{i_1},c_{i_2}).$$

By the adjointness,

$$E^2(x_1,c_{i_1})*E^2(x_2,c_{i_2})\le f(x_1,x_2)\to\bigvee_{i_1,i_2=1}^{k}(E(x_1,c_{i_1})*E(x_2,c_{i_2})*f(c_{i_1},c_{i_2}))$$

which together with (26) gives us (29).

(ii) Proof of (30).

$$E(x_1,c_{i_1})*E(x_2,c_{i_2})*\min(E(x_1,c_{i_1})*E(x_2,c_{i_2}),f(c_{i_1},c_{i_2}))\le$$
$$E(x_1,c_{i_1})*E(x_2,c_{i_2})*f(c_{i_1},c_{i_2})\le f(x_1,x_2)$$

By the property (iv) of BL-algebra and then by the adjointness,

$$E^2(x_1,c_{i_1})*E^2(x_2,c_{i_2})*(E(x_1,c_{i_1})*E(x_2,c_{i_2})\to f(c_{i_1},c_{i_2})\le f(x_1,x_2),$$
$$E^2(x_1,c_{i_1})*E^2(x_2,c_{i_2})\le(E(x_1,c_{i_1})*E(x_2,c_{i_2})\to f(c_{i_1},c_{i_2}))\to f(x_1,x_2).$$

Finally, by the antitonicity of implication we obtain

$$E^2(x_1,c_{i_1})*E^2(x_2,c_{i_2})\le\bigwedge_{i_1,i_2=1}^{k}(E(x_1,c_{i_1})*E(x_2,c_{i_2})\to f(c_{i_1},c_{i_2}))\to f(x_1,x_2)$$

which together with (28) gives us (30). □

Note, that the restriction $E^2(x_1,c_{i_1})*\dots*E^2(x_n,c_{i_n})$ of equivalences (29) and (30) describes the degree of a similarity between two points $(x_1,\dots,x_n)$ and $(c_{i_1},\dots,c_{i_n})$. The direct estimation of the quality of approximation of the original function by functions represented by the discrete normal forms can be given below.

Corollary 2
Let the conditions of the above theorem be fulfilled. Moreover, let $L=[0,1]$ and the multiplication $*$ be a continuous Archimedean t-norm with continuous generator $g:[0,1]\longrightarrow[0,\infty]$. Let $d:X\longrightarrow[0,\infty]$ be a pseudo-metric on X and the similarity E on X be given by $E(x,y)=t^{(-1)}(d(x,y))$. Then we can

estimate the quality of approximation of the original function f by functions represented by the discrete normal forms given by (23) or (25)

$$|g(f(x_1,\dots,x_n)) - g(f_{\mathrm{DNF}}(x_1,\dots,x_n))| \le \min(t(0), \sum_{j=1}^{n} 2d(x_j, c_{i_j})), \tag{31}$$

$$|g(f(x_1,\dots,x_n)) - g(f_{\mathrm{CNF}_{II}}(x_1,\dots,x_n))| \le \min(t(0), \sum_{j=1}^{n} 2d(x_j, c_{i_j})). \tag{32}$$

PROOF: The proof of this Corollary can be obtained in the same way as the proof of Theorem 1 if we notice that the restricted equivalence (29) implies

$$E^2(x_1, c_{i_1}) * \dots * E^2(x_n, c_{i_n}) \le f_{\mathrm{DNF}}(x_1,\dots,x_n) \to f(x_1,\dots,x_n)$$

or, equivalently,

$$E^2(x_1, c_{i_1}) * \dots * E^2(x_n, c_{i_n}) * f_{\mathrm{DNF}}(x_1,\dots,x_n) \le f(x_1,\dots,x_n).$$

Then, applying the proof of Theorem 1, we will obtain (31). Analogously, the proof of (32) can be obtained. □

Remark 2
It is worth noticing that the discrete normal forms model what is known in fuzzy literature as fuzzy systems. Usually, fuzzy systems are developed with the help of IF–THEN rules which describe a behaviour of a dynamic system in a language close to the natural one. The proof that fuzzy systems are really able to do this job is given in a number of papers unified by the key words "universal approximation" (see e.g.,[1, 2, 6, 10]). The result proved in Theorem 4 also belongs to this family. However, the method of a proof suggested here, differs from the other ones by its algebraic origin. It does not use the assumption of compactness of the basic set X which, on the other hand, automatically appears in the conditions of equivalences (29) and (30). This means that values of both equivalences in a point $(x_1,\dots,x_n)$ are as high as the degree of similarity between $(x_1,\dots,x_n)$ and the nearest node $(c_{i_1},\dots,c_{i_n})$. This correspondence is also expressed in Corollary 2 where the quality of approximation by discrete normal forms is evaluated.

6 Conclusions

In this paper we have introduced the disjunctive and conjunctive normal forms as special formulas of BL-algebra of functions. On the one side, the expressions for the normal forms generalize boolean ones which are used for the representation of logical functions. On the other side, they generalize formulas known in fuzzy literature as universal approximation formulas. The latter has been used for approximate description of continuous functions defined on compact domains.

In our investigation, normal forms are defined in two variants: infinite and finite. It has been proved that infinite normal forms are universal representation formulas whereas finite normal forms are universal approximation formulas for extensional functions. The estimation of the quality of approximation has been provided.

In some sense, our result is weaker that the above cited because the set of extensional functions is smaller that the set of continuous functions on the same domain. We are going to extend our results to the case of continuous functions in the next paper.

References

[1] Bucley J.J. (1993) Sugeno type controllers are universal controllers. Fuzzy Sets and Systems, 53, 299–304

[2] Castro J.L., M. Delgado (1996) Fuzzy systems with defuzzification are universal approximators. IEEE Trans. Syst., Man, Cybern., 26(1), 149–152

[3] Cignoli R., D'Ottaviano I.M.L., D. Mundici (2000) Algebraic foundations of many-valued reasoning, Kluwer, Dordrecht

[4] Daňková M. (2000) Extensionality and Continuity of Fuzzy Relations. Jour. of Electrical Engineering, 51(12/s), 33–35

[5] Hájek P. (1998) Metamathematics of fuzzy logic, Kluwer, Dordrecht

[6] Kosko B. (1992) Fuzzy systems as universal approximators. Proc. IEEE Int. Conf. on Fuzzy Syst., San Diego, CA, 1153– 1162

[7] Klement P., Mesiar R., L. Papp (2001) Triangular norms, Kluwer, Dordrecht

[8] Kreinovich V., Nguyen H.T., D.A. Sprecher (1996) Normal forms for fuzzy logic – an application of Kolmogorov's theorem. Int. Journ. of Uncert., Fuzz. and Knowl.-Based Syst., 4, 331–349

[9] Novák V., Perfilieva I. and J. Močkoř (1999) Mathematical Principles of Fuzzy Logic, Kluwer, Boston-Dordrecht

[10] Perfilieva I. (1999) Fuzzy logic normal forms for control law representation. In: Verbruggen H., Zimmermann H.-J., Babuska R. (eds.) Fuzzy Algorithms for Control, Kluwer, Boston, 111–125

[11] Perfilieva I. (2001) Normal Forms for Fuzzy Logic Functions and Their Approximation Ability. Fuzzy Sets and Systems, to appear

[12] Perfilieva I. (2001) Logical Approximation. Soft Computing, to appear.