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LOGICAL APPROXIMATION

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Abstract

The principle approach to the construction of approximating formulas is discussed. We suggest the generalized definition of normal forms in predicate BL-logic and prove the conditional equivalence between a formula and each of its normal form. Some mutual relations between normal forms will be also established.

Key words Disjunctive and conjunctive normal forms, BL-logic, fuzzy logic, fuzzy relation, approximation

1 Introduction

By the term "logical approximation" we understand a transformation of a formula to its simplified form such that their equivalence can be formally proved in some special logical theory. This kind of equivalence will be referred to as the conditional equivalence in this paper.

By a simplified form we mean one of normal logical forms, disjunctive or conjunctive in their generalized meaning introduced here and in [8]. Let us notice that the transformation of any logical formula to its normal forms is important not only for a logical theory where it allows to operate with unified and simplified representations. It is important also to applications which are based on different methods of approximation. Logical approach being very general, shows the principal way of constructing various approximating formulas which can be further refined by specialized theories. In fact, analyzing the way of constructing DNF, we see that other well known approximating formulas, e.g. interpolating polynomials, splines, Fourier series have the same origin.

Recall that in classical propositional logic, each formula can be equivalently transformed to one of its normal forms (disjunctive or conjunctive ones). On the semantic level, the normal forms represent the same (boolean) function as the original formula. In this paper, we consider a generalization of classical logic, known as BL-logic, which is one of fuzzy logics. We suggest the definition of normal forms in predicate BL-logic and show that the conditional equivalence between a formula and each of its normal forms takes place there. On the semantic level, it means that normal forms represent special fuzzy relations which can be regarded as approximating functions of the function represented by the original formula. The condition of the above mentioned conditional equivalence represents a precision of approximation.

Based on this and the fact that most of the known approximating schemes are special cases of normal forms, we can say that fuzzy logic can serve as a logical foundation of the approximation theory.

BL-logic has been introduced by P. Hájek ([4]). We chose this logic (among other fuzzy logics) because of its generality. In fact, each logic, based on a continuous t-norm, is a special case of BL-logic. The most known examples are Gödel logic, product logic and Łukasiewicz logic. Thus, being once proved in BL-logic, the result will be valid in any of its special cases. The other reason of our choice is the richness of the structure of truth values which allows us to express the error of an approximation.

The notion of a normal form in BL-logic is till now not well established. An implicit definition of a disjunctive normal form can be found in [3] for Łukasiewicz logic. However, the exact expression has not been presented there. The explicit definition of both kinds of normal forms in Łukasiewicz logic have been given in [8, 10] where they come out of the classical normal forms. It has been shown that for each propositional formula there exists a formula in a normal form (both disjunctive as well as conjunctive) such that the functions represented by them approximate each other with a prescribed accuracy. The proof of this fact has been given on the semantic level leaving untouched the question of the syntactical equivalence.

In this paper we extend the definition of normal forms to the predicate BL-logic and prove formally the conditional equivalence between an extensional formula and its normal form.

2 The basic fuzzy predicate logic

The basic fuzzy predicate logic (also known as BL-predicate logic) has been introduced in [4]. Since this book is regarded as a fundamental one, we will use the notation accepted there.

We will deal with some fixed language J of fuzzy predicate logic, which includes (besides others) the set of connectives $\{\neg, \&, \rightarrow, \vee, \wedge, \equiv\}$, truth constants $\bar{0}, \bar{1}$, and does not include functional symbols. By formula, we mean a formula of (fuzzy) predicate logic in the language J, including truth constants as atomic formulas.

A structure $\mathcal{M} = \langle M, (r_P)_{P\in J}, (m_c)_{c\in J}\rangle$ for the language J consists of a non-empty domain M, fuzzy relations $r_P : M^n \longrightarrow [0,1]$ assigned to each n-ary predicate symbol P, and designated elements $m_c \in M$ assigned to each object constant $\mathbf{c}$.

The interpretation of the logical connectives $\wedge, \vee, \&, \rightarrow$ is determined by the respective operations of the BL-algebra $\mathcal{L}$ on $[0,1]$

$$\mathcal{L} = \langle [0,1], \wedge, \vee, *, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$$

where $*$ stands for some continuous t-norm. The other two connectives $\neg, \equiv$ are interpreted by the respective derived operations $\neg, \leftrightarrow$:

$$\neg a = a \rightarrow \mathbf{0},$$
$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

Note, that the notation of the connectives differs from the notation of the respective operations.

A truth value $|\varphi|_{\mathcal{M}}$ of a formula φ, given an evaluation of object variables, is computed in the same way as in the classical case except for the quantified formulas

$$|(\forall x)\varphi(x)|_{\mathcal{M}} = \inf\{|\varphi(\mathbf{c})|_{\mathcal{M}} \mid m_c \in M\},$$
$$|(\exists x)\varphi(x)|_{\mathcal{M}} = \sup\{|\varphi(\mathbf{c})|_{\mathcal{M}} \mid m_c \in M\}.$$

The predicate calculus BL$\forall$ has the following sets of axioms (see [4]).

Logical axioms on connectives:

(A1) $(\varphi \rightarrow \psi)\rightarrow((\psi\rightarrow\chi)\rightarrow(\varphi\rightarrow\chi))$

(A2) $(\varphi \,\&\, \psi)\rightarrow\varphi$

(A3) $(\varphi \,\&\, \psi)\rightarrow(\psi \,\&\, \varphi)$

(A4) $(\varphi \,\&(\varphi\rightarrow\psi))\rightarrow(\psi \,\&(\psi\rightarrow\varphi))$

(A5a) $(\varphi\rightarrow(\psi\rightarrow\chi))\rightarrow((\varphi \,\&\, \psi)\rightarrow\chi)$

(A5b) $((\varphi \,\&\, \psi)\rightarrow\chi)\rightarrow(\varphi\rightarrow(\psi\rightarrow\chi))$

(A6) $((\varphi\rightarrow\psi)\rightarrow\chi)\rightarrow(((\psi\rightarrow\varphi)\rightarrow\chi)\rightarrow\chi)$

(A7) $\bar{0}\rightarrow\varphi$

Logical axioms on quantifiers:

($\forall$1) $(\forall x)\varphi(x)\rightarrow\varphi(t)$ (t free for x in $\varphi(x)$)

($\exists$1) $\varphi(t)\rightarrow(\exists x)\varphi(x)$ (t free for x in $\varphi(x)$)

($\forall$2) $(\forall x)(\nu\rightarrow\varphi)\rightarrow(\nu\rightarrow(\forall x)\varphi)$ (x not free in ν)

(∃2) $(\forall x)(\varphi \rightarrow \nu) \rightarrow ((\exists x)\varphi \rightarrow \nu)$ (x not free in ν)

(∀3) $(\forall x)(\nu \vee \varphi) \rightarrow (\nu \vee (\forall x)\varphi)$ (x not free in ν)

and the deduction rules

- Modus ponens (from φ, $\varphi \rightarrow \psi$ infer ψ).
- Generalization (from φ infer $(\forall x)\varphi$).

The notions of *proof* and of *provable formula* in BL∀ are defined as in classical logic.

3 Normal Forms in BL-logic

In this section, we will introduce the disjunctive and conjunctive normal forms as special formulas of predicate BL-logic. It is worth to be noticed that the expressions for normal forms came from a formal translation of sets of fuzzy IF–THEN rules, which are widely used in applications of fuzzy logic for linguistic description of different kinds of dependencies. Thus, the approximation property of normal forms w.r.t. other formulas is expected.

We will extend the language J by a finite number of truth constants $\bar{d}$, for some $d \in [0,1]$.

Definition 1
Let $P_1, \ldots, P_k$ be unary predicate symbols and $E_{i_1 \ldots i_n}$, $1 \leq i_j \leq k$, $1 \leq j \leq n$, $n \geq 1$, be either truth constants or closed instances of some formula. The following formulas of fuzzy predicate logic are called the *disjunctive normal form*

$$\mathrm{DNF}(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_n = 1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n) \,\&\, E_{i_1 \ldots i_n}) \tag{1}$$

and the *conjunctive normal forms*

$$\mathrm{CNF}_I(x_1, \ldots, x_n) = \neg \bigvee_{i_1, \ldots, i_n = 1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n) \,\&(\neg E_{i_1 \ldots i_n})) \tag{2}$$

and

$$\mathrm{CNF}_{II}(x_1, \ldots, x_n) = \bigwedge_{i_1, \ldots, i_n = 1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n) \rightarrow E_{i_1 \ldots i_n}). \tag{3}$$

Why are these formulas called the normal forms? First of all, our disjunctive normal form is represented by the disjunction of elementary conjunctions and thus, resembles the analogous boolean formula. Moreover, each elementary conjunction $P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n) \,\&\, E_{i_1 \ldots i_n}$ consists of the characterization of an area $P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n)$ combined with the description of a selected truth value $E_{i_1 \ldots i_n}$ of some formula when its arguments lie inside the respective area. Second, the expression of the first conjunctive normal form is obtained by the dual transformation of the disjunctive normal form (simply, we negated the DNF of the negated function). Of course, CNF_I can be equivalently transformed to the conjunction of elementary components (since de Morgan laws are provable in BL-logic). However, it is better for our proofs to work with the form given by (2). The second expression for the conjunctive normal form came from the formal translation of the fuzzy IF–THEN rules mentioned above. In BL-logic, two conjunctive normal forms are, in general, not equivalent, whereas they are equivalent in Łukasiewicz logic — a special case of BL. The relationship between both conjunctive normal forms will be investigated below.

The similarity between classical and fuzzy normal forms can be seen better in two particular cases when we take $E_{i_1 \ldots i_n}$ either as $\bar{1}$ for all $1 \leq i_j \leq k$, $1 \leq j \leq n$, $n \geq 1$, or as $\bar{0}$. In the first case, DNF can be equivalently transformed to

$$\mathrm{DNF}(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_n = 1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n)), \tag{4}$$

while in the second case, two conjunctive normal forms are equivalent and can be equivalently transformed to

$$\mathrm{CNF}_I(x_1,\ldots,x_n) = \neg \bigvee_{i_1,\ldots,i_n=1}^{k} (P_{i_1}(x_1)\,\&\cdots\&\,P_{i_n}(x_n)). \tag{5}$$

or to

$$\mathrm{CNF}_{II}(x_1,\ldots,x_n) = \bigwedge_{i_1,\ldots,i_n=1}^{k} \neg(P_{i_1}(x_1)\,\&\cdots\&\,P_{i_n}(x_n)), \tag{6}$$

The normal forms are distinguished among the other formulas because they can be used as *universal approximators*. By this, we mean that any continuous fuzzy relation on a compact domain can be approximated by the fuzzy relation represented in one of these normal forms provided with certain interpretation of all included symbols (for different proofs regarding DNF see [1, 2, 5, 6] and regarding CNF see [9]).

4 Predicate BL-logic with Similarity

Aiming at proving the approximation property of normal forms, we have to restrict formulas which can be approximately represented. Having on mind the result concerning a universal approximation cited above, we will introduce an analogy to the continuity property. This will be called by "extensionality w.r.t similarity" which for some special choices of similarity is equivalent to Lipschitz continuity.

Let us further extend the language J by the binary predicate symbol of similarity $\approx$. The corresponding atomic formula will be written as $x \approx y$ and has the meaning "x is similar to y". The following axioms characterize similarity:

(S1) $(\forall x)(x \approx x)$,

(S2) $(\forall x, y)(x \approx y \rightarrow y \approx x)$,

(S3) $(\forall x, y, z)(x \approx y\,\&\,y \approx z \rightarrow x \approx z)$.

Let $(x \approx^k y)$ be the abbreviation of $(x \approx y)\,\&\cdots\&(x \approx y)$, k times, $k \geq 1$. It is easy to see that in the theory of similarity over BL$\forall$ with special axioms (S1)-(S3), the same axioms with $\approx$ replaced by $(x \approx^k y)$ can be proved.

The following axiom characterizes the property of *extensionality* for a predicate P of arity n w.r.t. $\approx$:

$$x_1 \approx y_1\,\&\cdots\&\,x_n \approx y_n \rightarrow P(x_1,\ldots,x_n) \equiv P(y_1,\ldots,y_n).$$

Let us show that the extensionality is somehow analogous to the continuity. Let $\langle M, r\rangle$ be a model of the *similarity* axioms (over a BL-algebra). Then

$$r(a,a) = 1$$
$$r(a,b) = r(b,a)$$
$$r(a,b) * r(b,c) \leq r(a,c)$$

holds for all $a, b, c \in M$.

A fuzzy relation $s : M^n \longrightarrow [0,1]$ is *extensional* (w.r.t.a similarity r) if for each $u_1,\ldots,u_n$ and $v_1,\ldots,v_n \in M$

$$r(u_1,v_1) * \cdots * r(u_n,v_n) * s(u_1,\ldots,u_n) \leq s(v_1,\ldots,v_n).$$

It has been proved in [4] that a structure $\langle M, r, s\rangle$ is a model of the similarity axioms and extensionality iff s is extensional w.r.t. r. Moreover, due to [7], in Łukasiewicz logic a fuzzy relation s on D is extensional w.r.t. r iff it is Lipschitz continuous w.r.t. the pseudo-metric d, where $d(x,y) = 1 - r(x,y)$.

Till now, we spoke about extensionality for predicates. Having on mind that a formula with n free variables defines an n-ary predicate we can formally consider the extensionality axiom for formulas too. It has been proved (P. Hájek, [4]) that if a theory T over BL$\forall$ contains the similarity axioms for $\approx$ and extensionality axioms for all its predicates then the extensionality for any formula of T can be proved w.r.t. $(x \approx^k y)$ for some $k \geq 1$. Further we will use the term *extensional formula* when the extensionality axiom for this formula can be proved in the respective theory.

5 Normal Forms Based on Similarity and What Can Be Approximated by Them

Here we will specify normal forms which will be used for approximation of extensional formulas. Of course, they should be connected with the approximated formula. How? Two parts will be distinguished in the constructions of normal forms. One part characterizes the universe and is independent on any represented formula, while the other part actually characterizes it.

Let T be a consistent theory over BL$\forall$ with the similarity predicate $\approx$ and the respective axioms. We suppose that the language $J(T)$ of the theory T is extended by a finite number of truth constants as new atomic formulas, and stipulate that an evaluation of constants is the same in each interpretation.

Let $\mathbf{c}_1, \dots, \mathbf{c}_k$, $k \geq 1$, be object constants and $\overline{d}_{i_1 \dots i_n}$, $1 \leq i_j \leq k$, $1 \leq j \leq n$, be truth constants in the language $J(T)$. In expressions (1)–(3) of normal forms we will specify:

$$P_i(x) \quad \text{is} \quad x \approx \mathbf{c}_i \quad \text{and} \quad E_{i_1 \dots i_n} \quad \text{is} \quad \overline{d}_{i_1 \dots i_n}.$$

The result will be demonstrated on the expression for DNF

$$\mathrm{DNF}(x_1, \dots, x_n) = \bigvee_{i_1, \dots, i_n = 1}^{k} ((x_1 \approx \mathbf{c}_{i_1}) \,\&\, \cdots \&(x_n \approx \mathbf{c}_{i_n}) \,\&\, \overline{d}_{i_1 \dots i_n}). \tag{7}$$

It can be easily verified that with such specification, the predicates $P_i(x)$ are extensional w.r.t $\approx$. Moreover, below theorem shows that the extensionality for normal forms w.r.t. $\approx$ can be established too.

Theorem 1
Let T be a theory over BL$\forall$ containing the similarity axioms for $\approx$ and extended by $\mathbf{c}_1, \dots, \mathbf{c}_k$ as object constants and $\overline{d}_{i_1 \dots i_n}$ as truth constants. Moreover, let $P_i(x)$ and $E_{i_1 \dots i_n}$ be specified by $x \approx \mathbf{c}_i$ and $\overline{d}_{i_1 \dots i_n}$ respectively. Then T proves the extensionality for the normal forms $\mathrm{DNF}(x_1, \dots, x_n)$, $\mathrm{CNF}_I(x_1, \dots, x_n)$ and $\mathrm{CNF}_{II}(x_1, \dots, x_n)$ (with the given specification of their subformulas) w.r.t. $\approx$.

PROOF: For the simplicity we will restrict our consideration to the case of one free variable ($n = 1$).

(1) The proof of the extensionality for DNF(x).

$T \vdash x \approx y \,\&\, x \approx \mathbf{c}_i \rightarrow y \approx \mathbf{c}_i$ by axiom (S3)
$T \vdash x \approx y \rightarrow (x \approx \mathbf{c}_i \rightarrow y \approx \mathbf{c}_i)$ by axiom (A5b)
$T \vdash x \approx y \rightarrow ((x \approx \mathbf{c}_i) \,\&\, \bar{d}_i \rightarrow (y \approx \mathbf{c}_i) \,\&\, \bar{d}_i)$ by properties of &

$$T \vdash x \approx y \rightarrow ((x \approx \mathbf{c}_i) \,\&\, \bar{d}_i \rightarrow \bigvee_{j=1}^{k} (y \approx \mathbf{c}_j) \,\&\, \bar{d}_j) \quad \text{by properties of } \vee$$

$$T \vdash x \approx y \rightarrow \bigwedge_{i=1}^{k} \left((x \approx \mathbf{c}_i) \,\&\, \bar{d}_i \rightarrow \bigvee_{j=1}^{k} ((y \approx \mathbf{c}_j) \,\&\, \bar{d}_j) \right) \quad \text{by properties of } \wedge$$

$$T \vdash x \approx y \rightarrow \left(\bigvee_{i=1}^{k} ((x \approx \mathbf{c}_i) \,\&\, \bar{d}_i) \rightarrow \bigvee_{j=1}^{k} ((y \approx \mathbf{c}_j) \,\&\, \bar{d}_j) \right) \quad \text{by the fact that}$$

the formula $\bigwedge_{i=1}^{k} (a_i \rightarrow b) \equiv (\bigvee_{i=1}^{k} a_i \rightarrow b)$ is provable in BL-logic.

Thus, we have proved that

$$T \vdash x \approx y \rightarrow (\mathrm{DNF}(x) \rightarrow \mathrm{DNF}(y)).$$

The opposite implication can be obtained if we replace x by y.

(2) The proof of the extensionality for $\mathrm{CNF}_I(x)$ is analogous to the preceding one and is based on the following theorem of propositional BL-logic (see [4]):

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

(3) The proof of the extensionality for $\mathrm{CNF}_{II}(x)$.

$$T \vdash x \approx y \,\&\, y \approx \mathbf{c}_i \to x \approx \mathbf{c}_i \quad \text{by axiom (S3)}$$
$$T \vdash ((x \approx \mathbf{c}_i) \to \bar{d}_i) \to ((x \approx y) \,\&\, (y \approx \mathbf{c}_i) \to \bar{d}_i) \quad \text{by axiom (A1)}$$
$$T \vdash ((x \approx \mathbf{c}_i) \to \bar{d}_i) \to ((x \approx y) \to ((y \approx \mathbf{c}_i) \to \bar{d}_i))) \quad \text{by axiom (A5b)}$$
$$T \vdash (x \approx y) \,\&\, ((x \approx \mathbf{c}_i) \to \bar{d}_i) \to ((y \approx \mathbf{c}_i) \to \bar{d}_i) \quad \text{by axiom (A5a)}$$
$$T \vdash \bigwedge_{i=1}^{k} ((x \approx y) \,\&\, ((x \approx \mathbf{c}_i) \to \bar{d}_i)) \to \bigwedge_{i=1}^{k} ((y \approx \mathbf{c}_i) \to \bar{d}_i) \quad \text{by properties of } \wedge, \&$$
$$T \vdash (x \approx y) \,\&\, \bigwedge_{i=1}^{k} ((x \approx \mathbf{c}_i) \to \bar{d}_i) \to \bigwedge_{i=1}^{k} ((y \approx \mathbf{c}_i) \to \bar{d}_i) \quad \text{by properties of } \wedge, \&$$

Thus, we have proved that

$$T \vdash (x \approx y) \,\&\, \mathrm{CNF}_{II}(x) \to \mathrm{CNF}_{II}(y) \quad \text{or}$$
$$T \vdash x \approx y \to (\mathrm{CNF}_{II}(x) \to \mathrm{CNF}_{II}(y)).$$

The opposite implication can be obtained if we replace x by y.

□

Lemma 1
Let T fulfill the assumptions of Theorem 1 and $\langle M, r, c_1, \ldots, c_k \rangle$ be a model of T where r models $\approx$ and $c_1, \ldots, c_k$ are designated elements from M which correspond to the constants $\mathbf{c}_1, \ldots, \mathbf{c}_k$. Then the fuzzy relations modeling normal forms can be expressed by

$$R_{\mathrm{DNF}}(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_n = 1}^{k} (r(c_{i_1}, x_1) * \cdots * r(c_{i_n}, x_n) * d_{i_1 \ldots i_n})$$

$$R_{\mathrm{CNF}_I}(x_1, \ldots, x_n) = \neg \bigvee_{i_1, \ldots, i_n = 1}^{k} (r(c_{i_1}, x_1) * \cdots * r(c_{i_n}, x_n) * (\neg d_{i_1 \ldots i_n}))$$

and

$$R_{\mathrm{CNF}_{II}}(x_1, \ldots, x_n) = \bigwedge_{i_1, \ldots, i_n = 1}^{k} (r(c_{i_1}, x_1) * \cdots * r(c_{i_n}, x_n) \to d_{i_1 \ldots i_n}).$$

PROOF: Obvious. □

The following theorem gives a partial answer to the problem of approximation ability of normal forms. In fact, it shows that DNF and CNF can be respectively considered as lower and upper approximate representations of a given formula.

Theorem 2
Let T be a theory over BL∀ containing the similarity axioms for $\approx$ and extended by $\mathbf{c}_1, \ldots, \mathbf{c}_k$ as object constants and $\overline{d}_{i_1 \ldots i_n}$ as truth constants. Moreover, let T contains the extensionality axiom for $\varphi(x_1, \ldots, x_n)$ w.r.t. $\approx$ and

$$T \vdash \varphi(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_n}) \equiv \overline{d}_{i_1 \ldots i_n}. \tag{8}$$

Then

$$T \vdash \mathrm{DNF}(x_1, \ldots, x_n) \to \varphi(x_1, \ldots, x_n), \tag{9}$$

$$T \vdash \varphi(x_1, \ldots, x_n) \to \mathrm{CNF}_{II}(x_1, \ldots, x_n) \tag{10}$$

and moreover,

$$T \vdash \varphi(x_1, \ldots, x_n) \to \mathrm{CNF}_{I}(x_1, \ldots, x_n). \tag{11}$$

PROOF: The proofs of (9) and (10) are analogous to the ones given in [4]. Therefore, we will only demonstrate the proof of (9) and give the proof of (11). As before, we consider the case of one free variable ($n = 1$).

(1) Proof of (9).

$$T \vdash x \approx \mathbf{c}_i \to (\bar{d}_i \to \varphi(x)) \quad \text{from extensionality for } \varphi(x) \text{ and by (8)}$$
$$T \vdash (x \approx \mathbf{c}_i) \,\&\, \bar{d}_i \to \varphi(x) \quad \text{by axiom (A5a)}$$
$$T \vdash \bigvee_{i=1}^{k} ((x \approx \mathbf{c}_i) \,\&\, \bar{d}_i) \to \varphi(x) \quad \text{by properties of } \vee$$

(2) Proof of (11).

First of all, let us note that from the fact

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

it follows that $\neg\varphi(x)$ is also extensional and

$$T \vdash \neg\varphi(\mathbf{c}_i) \equiv \neg\overline{d}_i$$

holds true. Then

$$T \vdash \bigvee_{i=1}^{k} ((x \approx \mathbf{c}_i) \,\&\, \neg\bar{d}_i) \to \neg\varphi(x) \quad \text{by (9)}$$
$$T \vdash \neg\neg\varphi(x) \to \neg \bigvee_{i=1}^{k} ((x \approx \mathbf{c}_i) \,\&\, \neg\bar{d}_i) \quad \text{by properties of } \neg \text{, and finally}$$
$$T \vdash \varphi(x) \to \neg \bigvee_{i=1}^{k} ((x \approx \mathbf{c}_i) \,\&\, \neg\bar{d}_i) \quad \text{also by properties of } \neg.$$

□

In general, two conjunctive normal forms can hardly be compared. But if we assume some additional requirements on object constants $\mathbf{c}_1, \ldots, \mathbf{c}_k$, such that

$$T \vdash (\mathbf{c}_i \approx \mathbf{c}_j) \equiv \overline{0} \quad \text{if} \quad i \neq j, \tag{12}$$

then we can easily deduce that

$$T \vdash \mathrm{DNF}(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_n}) \equiv \overline{d}_{i_1 \ldots i_n}, \tag{13}$$

$$T \vdash \mathrm{CNF}_{II}(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_n}) \equiv \overline{d}_{i_1 \ldots i_n}. \tag{14}$$

From (14) and Theorem 2 the relation between two conjunctive normal forms immediately follows:

$$T \vdash \mathrm{CNF}_{II}(x_1, \ldots, x_n) \to \mathrm{CNF}_{I}(x_1, \ldots, x_n). \tag{15}$$

The following theorem exposes the desired approximation property of normal forms. By this, we mean that under special conditions, any extensional formula can be equivalently represented by each of its normal forms. Formally, we will prove that the formula which describes the above mentioned special conditions implies the other formula which describes the equivalence between a given extensional formula and each of its normal forms. Thus, on the syntactical level, the approximation means the conditional equivalence.

Let us now discuss a semantical aspect of the conditional equivalence. In essence, normal forms represent an aggregated description of an available local information about formula. This information is given by each elementary conjunction (or implication in case of CNF_{II}) and consists of the characterization of a domain combined with the description of a selected truth value of a formula when its arguments lie inside the respective domain. The theorem below states that if the whole universe can be covered by a finite number of local domains (this is the meaning of our special condition, in other words, compactness) then an original extensional formula can be equivalently represented by each of its normal forms. In other words, the truth value of the equivalence is at least as high as is the truth value of the special condition. Moreover, if we refine the covering (and increase the truth value of the special condition), then the respective equivalence will be more precise (its truth value will increase, as well).

Theorem 3
Let T be a theory over $BL\forall$ containing the similarity axioms for $\approx$ and extended by $\mathbf{c}_1, \dots, \mathbf{c}_N$ as object constants and $\overline{d}_{i_1\dots i_n}$ as truth constants. Moreover, let T contains the extensionality axiom for $\varphi(x_1, \dots, x_n)$ w.r.t. $\approx$ and

$$T \vdash \varphi(\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_n}) \equiv \overline{d}_{i_1\dots i_n}.$$

In the expressions of normal forms, let $P_i(x)$ and $E_{i_1\dots i_n}$ stand for $x \approx \mathbf{c}_i$ and $\overline{d}_{i_1\dots i_n}$ respectively. Then

$$T \vdash (\forall x_1, \dots, x_n) \left(\bigvee_{i_1\dots i_n} (P^2_{i_1}(x_1) \,\&\, \cdots \,\&\, P^2_{i_n}(x_n)) \rightarrow (\mathrm{DNF}(x_1, \dots, x_n) \equiv \varphi(x_1, \dots, x_n)) \right) \quad (16)$$

and

$$T \vdash (\forall x_1, \dots, x_n) \left(\bigvee_{i_1\dots i_n} (P^2_{i_1}(x_1) \,\&\, \cdots \,\&\, P^2_{i_n}(x_n)) \rightarrow (\mathrm{CNF}_{II}(x_1, \dots, x_n) \equiv \varphi(x_1, \dots, x_n)) \right). \quad (17)$$

PROOF: To simplify the notation, we will consider the case of one free variable ($n = 1$). The idea of the proof has been borrowed from [4].

$$T \vdash ((x \approx \mathbf{c}_i) \rightarrow \bar{d}_i) \rightarrow ((x \approx^2 \mathbf{c}_i) \rightarrow (x \approx \mathbf{c}_i) \,\&\, \bar{d}_i)$$
$$T \vdash (x \approx^2 \mathbf{c}_i) \rightarrow (((x \approx \mathbf{c}_i) \rightarrow \bar{d}_i) \rightarrow (x \approx \mathbf{c}_i) \,\&\, \bar{d}_i)$$
$$T \vdash (x \approx^2 \mathbf{c}_i) \rightarrow \left(\bigwedge_{i=1}^{k} ((x \approx \mathbf{c}_i) \rightarrow \bar{d}_i) \rightarrow \bigvee_{i=1}^{k} ((x \approx \mathbf{c}_i) \,\&\, \bar{d}_i) \right).$$
$$T \vdash \bigvee_{i=1}^{k} (x \approx^2 \mathbf{c}_i) \rightarrow (\mathrm{CNF}_{II}(x) \rightarrow \mathrm{DNF}(x)).$$

Then by (9) and (10) we will obtain

$$T \vdash \bigvee_{i=1}^{k} (x \approx^2 \mathbf{c}_i) \rightarrow (\mathrm{DNF}(x) \equiv \varphi(x_1, \dots, x_n))$$

and

$$T \vdash \bigvee_{i=1}^{k} (x \approx^2 \mathbf{c}_i) \rightarrow (\mathrm{CNF}_{II}(x) \equiv \varphi(x_1, \ldots, x_n)).$$

Now, the conclusion of the theorem follows from the rule of generalization. □

Based on deduction theorem and the fact that formula $q \rightarrow (q \rightarrow q^2)$ is provable in propositional BL-logic, we can put the condition of equivalences (16) and (17) among special axioms of theory T and thus obtain the following corollary.

Corollary 1
Let T, φ and normal forms fulfill the assumptions of Theorem 3. Then

$$T \cup \{(\forall x_1, \ldots, x_n)(\bigvee_{i_1 \ldots i_n} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash$$
$$(\forall x_1, \ldots, x_n)(\mathrm{DNF}(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)) \quad (18)$$

and

$$T \cup \{(\forall x_1, \ldots, x_n)(\bigvee_{i_1 \ldots i_n} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash$$
$$(\forall x_1, \ldots, x_n)(\mathrm{CNF}_{II}(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)) \quad (19)$$

The natural question arises: can we increase truth values of equivalences (16) and (17) by improving, in some sense, their special condition? Let us notice that the only possible way to do this is to choose the objects $\mathbf{c}_1, \ldots, \mathbf{c}_N$ in a more appropriate way. This is theoretically possible if we require additional properties of these objects. However, this special question should be left to the approximation theory and not to logic. Using the logical means, we can establish the following.

Corollary 2
Let T, φ and normal forms fulfill the assumptions of Theorem 3. Further on, let T be extended by $\mathbf{c}'_1, \ldots, \mathbf{c}'_N$ as additional object constants and $\overline{d}'_{i_1 \ldots i_n}$ as additional truth constants, so that

$$T \vdash \varphi(\mathbf{c}'_{i_1}, \ldots, \mathbf{c}'_{i_n}) \equiv \overline{d}'_{i_1 \ldots i_n}$$

and for each $1 \leq i \leq N$

$$T \vdash \mathbf{c}_i \approx \mathbf{c}'_i.$$

Let us consider other expressions of normal forms, DNF' and CNF', such that $P_i(x)$ and $E_{i_1 \ldots i_n}$ stand for $x \approx \mathbf{c}'_i$ and $\overline{d}'_{i_1 \ldots i_n}$, respectively. Then

$$T \vdash \mathrm{DNF}(x_1, \ldots, x_n) \equiv \mathrm{DNF}'(x_1, \ldots, x_n),$$
$$T \vdash \mathrm{CNF}(x_1, \ldots, x_n) \equiv \mathrm{CNF}'(x_1, \ldots, x_n).$$

□

6 Conclusions

In this paper we have introduced the concepts of disjunctive and conjunctive normal forms as special formulas of predicate BL-logic. The expressions for normal forms came from a formal translation of sets of IF–THEN rules which are widely used in applications of fuzzy logic for linguistic description of different kinds of dependencies.

We have restricted ourselves to the so called extensional formulas and, moreover, we have specified the basic predicates in normal forms by particular cases of similarity predicates. Given this, we have established the approximation property of normal forms and showed that on the syntactical level the approximation means the conditional equivalence. Thus, we have justified that in the algebra of logical operations on $[0,1]$ determined by a continuous t-norm, any extensional fuzzy relation on a compact domain can be approximated by the fuzzy relation represented in one of the normal forms. The precision of the approximation can be extracted from a truth value of the respective condition. The exact expression for the precision of the approximation as well as concrete examples of constructing normal forms will be presented in the next paper.

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