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Fuzzy Logic Deduction with Crisp Observations*

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Abstract

This paper presents an approach to approximate reasoning over the set of IF-THEN rules called fuzzy logic deduction. It understands IF-THEN rules as linguistically expressed logical implications and interprets then inside formal logical theory. Methodology and some properties are presented.

Keywords: Logical deduction, intension, linguistic description, linguistic hedges.

1 Introduction

This paper is a study of one approach to approximate reasoning over the set of IF-THEN rules called *fuzzy logic deduction*. First we present it in a general form and then concentrate on a situation when an input (in the following, we call it *observation*) is a crisp number. e.g. a result of some measurement.

We can distinguish two approaches in the general theory of fuzzy IF-THEN rules. The first one (see [13, 14]) uses linguistic form of the IF-THEN rules as a motivation for finding special formulas called *disjunctive* (DNF) and *conjunctive* (CNF) *normal forms*. The goal is to approximate a given function in some model with prescribed accuracy.

The second possibility is to take the set of IF-THEN rules (in the following, we will call it the *linguistic description*) as the set of genuine linguistic expressions, find their logical interpretation and work with this interpretation inside some formal logical theory. The theory is based on *fuzzy logic in narrow sense with evaluated syntax.*

This paper focuses on the second approach. We will show the possible interpretation of linguistic description and the way how logical deduction based on it can be performed.

2 Preliminaries

2.1 Logical preliminaries

The formal system we are working in is *fuzzy logic in narrow sense with evaluated syntax* (FLn) (see [12]). It is based on Łukasiewicz MV-algebra of truth values

$$\mathcal{L} = \langle L, \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1}\rangle.$$

where the set of truth values L is the interval $[0, 1]$ of real numbers.

Let $A(x_1, \ldots, x_n)$ be a formula and $t_1, \ldots, t_n$ be terms substitutable into A for the variables $x_1, \ldots, x_n$, respectively. By $A_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$, we denote an instance of A resulting from it when replacing all the free occurrences of the variables $x_1, \ldots, x_n$ by the respective terms $t_1, \ldots, t_n$.

A fuzzy theory T is a fuzzy set of formulas $T \underset{\sim}{\subseteq} F_J$ given by the triple $T = \langle \mathrm{LAx}, \mathrm{SAx}, R\rangle$ where $\mathrm{LAx} \underset{\sim}{\subseteq} F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \underset{\sim}{\subseteq} F_J$ is a fuzzy set of special axioms and R is a set of

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inference rules which includes the rules modus ponens (r_{MP}), generalization (r_G) and logical constant introduction (r_{LC}). If T is a fuzzy theory then its language is denoted by $J(T)$.

We will usually define a fuzzy theory only by the *fuzzy set of its special axioms*, i.e. we write

$$T = \{{}^{a}/A \mid \ldots\} \tag{1}$$

understanding that $a > 0$ in (1) and A is a special axiom of T.

The semantics is defined by generalization of the classical semantics of predicate logic. The *structure* for the language J is

$$\mathcal{V} = \langle V, f_{\mathcal{V}}, \ldots, P_{\mathcal{V}}, \ldots, u_{\mathcal{V}}, \ldots \rangle \tag{2}$$

where $f_{\mathcal{V}} : V^n \to V$ are n-ary functions on V assigned to the functional symbols $f \in J$, $P_{\mathcal{V}} \underset{\sim}{\subset} V^n$ are n-ary fuzzy relations on V assigned to the predicate symbols $P \in J$ and $u_{\mathcal{V}} \in V$ are designated elements assigned to the object constants $\mathbf{u} \in J$. If the concrete symbols $f_{\mathcal{V}}, P_{\mathcal{V}}, u_{\mathcal{V}}, \ldots$ are unimportant for the explanation then we will simplify (2) only to $\mathcal{V} = \langle V, \ldots \rangle$.

We say that the structure $\mathcal{V}$ is a *model* of the fuzzy theory T and write $\mathcal{V} \models T$ if $\mathrm{SAx}(A) \leq \mathcal{V}(A)$ holds for every formula $A \in F_{J(T)}$.

The concept of the provability degree as a generalization of the classical provability $T \vdash_a A$ and truth degree $T \models_a A$ can be introduced (for the precise definitions and a lot of properties of them — see [12]). Let us stress that the provability degree coincides with the truth due to the completeness theorem.

Theorem 1 (Completeness)

$$T \vdash_a A \iff T \models_a A$$

holds for every formula $A \in F_J$ and every consistent fuzzy theory T.

Other concepts and results from fuzzy logic in narrow sense are also presented in [10] of the this volume.

2.2 Linguistic preliminaries

A general surface structure of fuzzy IF-THEN rule is

$$\textsf{IF } \langle\text{noun}\rangle_1 \textsf{ is } \mathcal{A} \textsf{ THEN } \langle\text{noun}\rangle_2 \textsf{ is } \mathcal{B}. \tag{3}$$

It is a conditional statement characterizing relation between linguistic expressions of the form

$$\langle\text{noun}\rangle \textsf{ is } \mathcal{A}. \tag{4}$$

We will call expressions (4) the *linguistic predications.*

Examples of such predications are "temperature is quite high", "angle of the wheel is negative very big", "(breaking) force is more or less small", etc. The structure of (4) is quite general since it is possible to transform into it a great deal of more complicated expressions. For example, the former predications can be obtained from "turn the wheel very much to the left", or "break but not too much". Note that the latter possibility — transformation of commands into predications — has opened the door to the applications of fuzzy logic in control.

For the purpose of modeling using fuzzy logic, we usually are not interested in the objects denoted by nouns occurring in the linguistic predications. In the practice, they are replaced by numbers. Therefore, we replace $\langle$noun$\rangle$ in (3) by some variable $X, Y, \ldots$, etc. Consequently, the general surface structure of fuzzy IF-THEN rule considered further is

$$\textsf{IF } X \textsf{ is } \mathcal{A} \textsf{ THEN } Y \textsf{ is } \mathcal{B}. \tag{5}$$

A special case of the expressions occurring in the linguistic predications, which deserve our attention, are *evaluating linguistic expressions* (cf. [12, 9]). These are special natural language expressions, which characterize sizes, distances, etc. In general, they characterize a position on an ordered scale. Among them, we distinguish *atomic evaluating expressions* which include any of the adjectives "small", "medium", or "big"(and possibly other cases of the same kind, such as "cold", "hot", etc.), or fuzzy quantity "approximately z". The latter is a linguistic expression characterizing some quantity z from an ordered set. Examples of fuzzy quantities are *thirty two, the value z*, etc. Simple evaluating expressions are *very small, more or less medium, roughly big, about twenty five, approximately z*, etc.

Atomic evaluating expressions usually form pairs of antonyms, i.e. the pairs

$$\langle \text{nominal adjective} \rangle — \langle \text{antonym} \rangle.$$

Of course, there are a lot of pairs of antonyms, for example "young — old", "ugly — nice", "stupid — clever", etc. When completed by the middle term, such as "medium", "average", etc., they form the so called *basic linguistic trichotomy.* Let us stress, that the basic linguistic trichotomy "small, medium, big" should be taken as canonical, which represents a lot of other corresponding trichotomies, such as "short, average, long", "deep, medium deep, shallow", etc.

The *simple evaluating expressions* are expressions of the form

$$\langle \text{linguistic hedge} \rangle \langle \text{atomic evaluating expression} \rangle.$$

The linguistic hedges are special adjectives modifying the meaning of the adjectives before which they stand. In general, we speak about the linguistic hedges with *narrowing effect* (very, highly, etc.) and those with *widening effect* (more or less, roughly, etc.).

If $\mathcal{A}$ in (4) is an evaluating expression then (4) is called the *evaluating linguistic predication.* If $\mathcal{A}$ is simple then (4) is called the *simple evaluating predication.* Examples of simple evaluating predications are, e.g. "temperature is very high" (here "high" is taken instead of "big"), "pressure is roughly small", "income is roughly three million", etc.

Various linguistic expressions can be connected by the connectives "AND" and "OR", thus forming *compound linguistic expressions*

$$\mathcal{C} := \mathcal{A} \ \langle \text{connective} \rangle \ \mathcal{B}. \tag{6}$$

If $\mathcal{A}$, $\mathcal{B}$ are evaluating linguistic expressions (or predications) then (6) is correspondingly called *compound evaluating linguistic expression (or predication).*

All the above discussed expressions of natural language (evaluating expressions, predications, etc.) form a part $\mathcal{S}$ of natural language, which is formalized using the means of fuzzy logic. The starting point are translation rules using which, the items of the surface syntax level are assigned objects from the lower levels. Besides others, we must introduce the concepts of intension, extension and possible world.

Let us stress on this place that we do not pretend that the formalism used in the subsequent sections enables us to model natural language semantics as a whole. We better say that our formalism is powerful enough to model that part of it, which covers at least the evaluating expressions, evaluating predications and simple conditional sentences formed from them. Hence, when speaking about a linguistic expression, we will have on mind usually some of the expressions from the above considered set $\mathcal{S}$ in the sequel.

2.3 The meaning of linguistic expressions

A linguistic expression may in general be understood as a name of some property. In the linguistic theory, we speak about its *intension* (instead of property named by it). Furthermore, we have to consider a *possible world* (cf. [8, 15]), which can be informally understood as "a particular state of affairs". For us, the possible world is a set of objects, which may carry the properties in concern. Hence, the intension of the linguistic expression determines in each possible world its *extension*, i.e. a grouping of objects having the given property. Since there can exists infinite number of possible worlds, one intension may lead to a class of extensions.

We will formalize these concepts using the means of predicate FLn with evaluated syntax. The level of formal syntax is identified with the syntax of FLn and the semantic level is identified with the semantics of FLn. In the sequel, we suppose some fixed predicate language J. By F_J we denote the set of well-formed formulas and by M the set of all the closed terms of J (we will further suppose that M contains at least two elements).

To define the mathematical model of the *intension* of a linguistic expression $\mathcal{A} \in \mathcal{S}$, we start by assigning some formula $A(x) \in F_J$ to $\mathcal{A}$. However, this is not sufficient since this does not grasp inherent vagueness of the property represented by $\mathcal{A}$. This can be accomplished in FLn using the concept of *evaluated formula.* Namely, if $A(x)$ is a formula with one free variable then the evaluated formula $a/A_x[t]$ means that some object represented by the term t has the property A in the degree at least $a \in L$. This renders the hint for formalization of intensions of linguistic expressions given in Definition 1 below.

The *extension* is characterized on the semantic level, which is identified with the semantics of FLn. Hence, the concept of *possible world* is understood as a special structure $\mathcal{V}$ for J

$$\mathcal{V} = \langle V, P_{\mathcal{V}}, \ldots, \rangle.$$

We will usually suppose that all the fuzzy relations assigned to predicate symbols of J in the possible world $\mathcal{V}$ have continuous membership function. Moreover, some further assumptions on $\mathcal{V}$ can be made, for example the unimodality of some membership functions, specific topological structure defined on the support V, etc. As a special case, when dealing with evaluating linguistic expressions, V is assumed to be a *linearly ordered interval* $V = [{}^{l}v, {}^{r}v]$.

Definition 1 *Let $\mathcal{A} \in \mathcal{S}$ be a natural language expression and let it be assigned a formula $A(x)$.*

(i) The intension of $\mathcal{A}$ is a set of evaluated formulas (also called multiformula)

$$\mathrm{Int}(\mathcal{A}) = \mathbf{A}_{\langle x \rangle} = \left\{ {}^{a_t}/A_x[t] \middle| \, t \in M, a_t \in L \right\}. \tag{7}$$

(ii) The extension of $\mathcal{A}$ in the possible world $\mathcal{V}$ is the satisfaction fuzzy set

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) = \left\{ {}^{\mathcal{V}(A(v/x))}/v \middle| \, v \in V \right\} \tag{8}$$

(iii) The meaning of $\mathcal{A}$ is the couple

$$\mathrm{Mean}(\mathcal{A}) = \langle \mathrm{Int}(\mathcal{A}), \mathbf{Ext}(\mathcal{A}) \rangle$$

where $\mathbf{Ext}(\mathcal{A}) = \{\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) \mid \mathcal{V}$ is a possible world$\}$ is a class of all its extensions.

3 Theories for modeling of the meaning of linguistic expressions

In this section we state several natural requirements which a logical theory aimed at characterization of the meanings of linguistic expressions should fulfill.

We denote the theory mentioned by T^{EX} and its first-order language by $J(T^{EX})$. The (finite) set of simple linguistic evaluating expressions is denoted by $\mathcal{S}$. The corresponding set of unary predicate symbols in language $J(T^{EX})$ is denoted by $\mathcal{G}$. We denote by $m : \mathcal{S} \longrightarrow \mathcal{G}$ a bijection between $\mathcal{S}$ and $\mathcal{G}$. The set of all the closed terms of the sort $i \in \vartheta$ is denoted by M. In Definition 1, the intension of linguistic expression has been generally defined as a set of (instances of) evaluated formulas. In the following, we use the provability degrees in some formal theory T as the evaluations of instances of formulas.

The intension of simple evaluating expression $\mathcal{A} \in \mathcal{S}$ in a theory T is then

$$\mathrm{Int}(\mathcal{A}) = \left\{ {}^{\tilde{\alpha}_G(t)}/G_x[t] \mid G = m(\mathcal{A}), t \in M, T \vdash_{\tilde{\alpha}_G(t)} G_x[t] \right\},$$

where $\tilde{\alpha}_G : M \longrightarrow L$ is a function called *intensional mapping*. Note that $\tilde{\alpha}_G(t)$ depends also on the theory T we are working in. Because $\mathcal{G}$ is a set of unary predicate symbols, every non-empty subset of $\mathcal{G}$ is a set of independent evaluated formulas in the sense of [12], Definition 6.16.

Definition 2 *The theory of fuzzy predicate logic T^{EX} with the set of unary predicate symbols $\mathcal{G}$ is called a* theory of evaluating expressions *if it fulfills the following conditions:*

1. *T^{EX} is consistent.*
2. *The language $J\left(T^{EX}\right)$ should contain a set of constants large enough for the representation of real numbers, e.g. let the set of closed terms M be $M = \{t_z \mid z \in [0,1]\}$.*
3. *For every formula $G(x) \in \mathcal{G}$, there should exist closed terms $t_1, t_2 \in M$ such that $T^{EX} \vdash G_x[t_1]$ and $T^{EX} \vdash \neg G_x[t_2]$.*
4. *Let $\gamma_G : [0,1] \longrightarrow [0,1]$ be functions adjoined to functions $\tilde{\alpha}_G$ by putting*

   $$\gamma_G(z) = v \iff \tilde{\alpha}_G(t_z) = v.$$

   The functions γ_G should be continuous with respect to standard topology on $\mathbb{R}$, and unimodal, i.e. they have only one maximum (point or interval) on $[0,1]$,
5. *For every $t \in M$ there is at least one $G \in \mathcal{G}$ such that $T^{EX} \vdash_c G_x[t]$ with $c > 0$.*

Item 3 of the previous definition articulates the requirement that the properties expressed by simple evaluating expressions should be completely valid for some objects and completely invalid for the other ones. Item 4 expresses the fact that vagueness cannot change abruptly (it is continuous) and that for evaluating expressions like *very small*, *more or less medium* etc. there is always only one subinterval $[t_l, t_r] \subset M$ (every real number from $[0, 1]$ has some property in a non-zero degree, i.e. that the set $\mathcal{S}$ of simple evaluating expressions is rich enough).

The intensional mappings of the atomic formulas $G(x) \in \mathcal{G}$ can be computed inside some other theory, e.g. the *theory of evaluating syntagms* T^{EV} described in [9]. There are formulas which correspond to simple evaluating expressions composed from predicate symbol for horizon L, a set of unary connectives $\lhd_\alpha$ representing linguistic hedges, and functional symbol η for order-reversing automorphism. It means that these formulas are not independent, in general.

Definition 3 Canonical model $\mathcal{V}$ *of the theory of evaluating expressions* T^{EX} *is a model* $\mathcal{V} = \langle [0, 1], \ldots, \rangle$ *for which the following property should hold for all* $G \in \mathcal{G}$ *and* $t \in M$*:*

$$\mathcal{V}(G_x[t]) = \tilde{\alpha}_G(t). \tag{9}$$

The question of existence of such a canonical model is addressed by the following proposition.

Proposition 1 *Let the theory* T *fulfill the conditions of the Definition 2. Suppose that the fuzzy set of special axioms of the theory* T *has the following property: if a predicate symbol* $G \in \mathcal{G}$ *is contained in a special axiom* $^a\!/A$*, then* A *has the form*

$$\{\tilde{\alpha}_G(t)/G_x[t]\}. \tag{10}$$

Then there exists a canonical model $\mathcal{V} \models T$ *in the sense of the Definition 3.*

PROOF: The theory T is consistent, so it has a model $\mathcal{U}$. It follows from Lemma 6.1 in [12], that a fuzzy theory $T^\star$ with the set of special axioms $\mathrm{SAx} = \left\{\tilde{\alpha}_G(t)/G_x[t] \mid G \in \mathcal{G}\right\}$ has a model $\mathcal{U}^\star \models T^\star$ for which property (9) holds. We can construct a structure $\tilde{\mathcal{U}}$ such that $\tilde{U} = U^\star$,

$$\tilde{\mathcal{U}}(G_{i,x}[t]) = \mathcal{U}^\star(G_{i,x}[t]) = \tilde{\alpha}_{G_i}(t),$$

which is possible because $G(x)$ are atomic formulas. Other special axioms $^a\!/A$ of the theory T have not G_i as their subformulas, and we put interpretation of special symbols of $J(T)$ different from G_i in $\tilde{\mathcal{U}}$ equal to their interpretation in $\mathcal{U}$, from which it follows that $\tilde{\mathcal{U}}(A) = \mathcal{U}(A)$, $A \in \mathrm{Supp}(\mathrm{SAx}_{\mathcal{U}})$, and therefore $\tilde{\mathcal{U}} \models T$. □

This proposition says that canonical model exists if atomic predicate symbols $G \in \mathcal{G}$ are not subformulas of special axioms other than of the form (10). Existence of canonical models is important, because it guarantees the existence of possible world where membership functions of fuzzy relations — interpretations of predicate symbols $G \in \mathcal{G}$ — have the properties defined in Definition 2.

4 Linguistic descriptions

This section presents the treatment of linguistic descriptions on linguistic, syntactic and semantic levels. On the linguistic level, linguistic description is a set of linguistic expressions IF $\mathcal{A}$ THEN $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are evaluating linguistic predications.

Definition 4 Linguistic description *in FLb is a finite set* $\mathcal{LD}^I = \{\mathcal{R}_1^I, \mathcal{R}_2^I, \ldots, \mathcal{R}_r^I\}$ *of the conditional clauses*

$$\mathcal{R}_i^I := \text{IF } \mathcal{A}_i \text{ THEN } \mathcal{B}_i, \quad i = 1, 2, \ldots, r \tag{11}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ *are evaluating predications. If all evaluating predications* $\mathcal{A}_i$, $\mathcal{B}_i$ *are simple then also the linguistic description* $\mathcal{LD}^I$ *is called simple.*

The intensions of individual conditional clauses $\mathcal{R}_i^I$ are determined by the theories of evaluating expressions T_1^{EX} and T_2^{EX} for the antecedent and consequent parts of IF-THEN rules, respectively. The theories T_1^{EX} and T_2^{EX} could be identical, but sometimes it is advantageous to use different theories for antecedents and for consequents. The language in which the intension of IF-THEN rule is written down is two-sorted first-order language

$$J(T_I) = \langle \mathcal{G}_1, \mathcal{G}_2, \{t_{1,z} \mid z \in [0,1]\}, \{t_{2,z} \mid z \in [0,1]\}, \ldots \rangle$$

where $\mathcal{G}_i$, $i = 1,2$ are the sets of atomic predicate symbols of type i, discussed in the Section 3. The sets of all closed terms of the sort i are denoted by M_i. The intension of one IF-THEN rule is

$$\begin{aligned} \mathbf{R}_{i,\langle x,y\rangle} &= \mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle} = \\ &= \left\{ \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s) / A_{i,x}[t] \Rightarrow B_{i,y}[s] \mid t \in M_1, s \in M_2, A_i, B_i \in \mathcal{G} \right\}, \end{aligned} \tag{12}$$

where $\tilde{\alpha}_{A_i \Rightarrow B_i} = \tilde{\alpha}_{A_i}(t) \rightarrow \tilde{\alpha}_{B_i}(s)$ ($\rightarrow$ denotes Łukasiewicz implication). These intensions $\mathbf{R}_{i,\langle x,y\rangle}$ are the (only) special axioms of fuzzy theory T_I of language $J(T_I)$,

$$T_I = \{ \mathbf{R}_{i,\langle x,y\rangle} \mid i = 1,2,\ldots,r \} \tag{13}$$

We call this theory T_I *the theory of linguistic description* $\mathcal{LD}^I$.

On the *semantic level*, given a possible world

$$\mathcal{V} = \langle \langle V_1, V_2 \rangle, \{\mathrm{A}_i \mid i \in \{1,\ldots,r\}\}, \{\mathrm{B}_i \mid i \in \{1,\ldots,r\}\}, \ldots \rangle,$$

where A_i, B_i are fuzzy sets – interpretations of atomic predicate symbols $A_i \in \mathcal{G}_1$, $B_i \in \mathcal{G}_2$, each fuzzy IF-THEN rule $\mathcal{R}_i \in \mathcal{LD}^I$ is assigned an *extension* in $\mathcal{V}$:

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{R}) = \left\{ \mathcal{V}(A_i(u/x) \Rightarrow B_i(v/y)) / \langle u,v \rangle \,\middle|\, \langle u,v \rangle \in V_1 \times V_2 \right\}. \tag{14}$$

Note that extension of the fuzzy rule $\mathcal{R}_i$ is a *fuzzy relation* $\mathrm{Ext}_{\mathcal{V}}(\mathcal{R}_i) = R \underset{\sim}{\subset} V_1 \times V_2$.

Proposition 2 *Let $\mathcal{LD}^I$ be linguistic description. Then the theory T_I is consistent and there exists the possible world $\mathcal{W} = \langle \langle W_1, W_2 \rangle, \{\mathrm{A}_i \mid i \in \{1,\ldots,r\}\}, \{\mathrm{B}_i \mid i \in \{1,\ldots,r\}\}, \ldots \rangle$, such that*

$$\mathcal{W}(A_{i,x}[t] \Rightarrow B_{i,y}[s]) = \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s).$$

PROOF: Let us construct the model $\mathcal{W}$: define $W_j = [0,1]$, $j = 1,2$ and $\mathrm{A}_i(x) = \tilde{\alpha}_{A_i}(t_x)$ for $x = \mathcal{W}(t_x)$, $t_x \in M_1$ and $x \in [0,1]$. Analogously, $\mathrm{B}_i(y) = \tilde{\alpha}_{B_i}(s_y)$ for $y = \mathcal{W}(s_y)$, $s_y \in M_2$ and $y \in [0,1]$. Then

$$\begin{aligned} \mathcal{W}(A_{i,x}[t] \Rightarrow B_{i,y}[s]) &= \mathcal{W}(A_{i,x}[t]) \rightarrow \mathcal{W}(B_{i,x}[s]) = \mathrm{A}_i(\mathcal{W}(t)) \rightarrow \mathrm{B}_i(\mathcal{W}(s)) = \\ &= \tilde{\alpha}_{A_i}(t) \rightarrow \tilde{\alpha}_{B_i}(s) = \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s). \end{aligned}$$

It follows that $\mathcal{W} \models T_I$ and, therefore, T_I is a consistent theory. □

The possible worlds isomorphic to the model $\mathcal{W}$ from the proof of Proposition 2 will be called *canonical models* (or possible worlds) of T_I.

5 Fuzzy logic deduction – basic schema

The basic schema of fuzzy logic deduction is the following: We have fuzzy theory T_I composed from implications and fuzzy theory T' which represents an observation. From these theories we form a theory $T_D = T_I \cup T'$. The theory T_I expresses the relationship between antecedent and succedent variables. The problem (addressed in the next section) is how an observation u' measured in some possible world $\mathcal{V}$ can be transformed to its logical counterpart T'. The general form of T' is

$$T' = \{\mathbf{A}'_i \mid i \in J\}$$

where $J \subseteq \{1,2,\ldots,r\}$ and

$$\mathbf{A}'_i = \left\{ \tilde{\alpha}_{A_i}(t) / A_{i,x}[t] \mid A_i = m(\mathcal{A}_i) \right\}$$

If the theory $T_D = T_I \cup T'$ is consistent then we can derive the *conclusion*

$$\mathbf{B}' = \{\mathbf{B}'_i \mid i \in J\},$$

where

$$\mathbf{B}'_i = \left\{ \tilde{\alpha}_{B_i}(s)/B_{i,y}[s] \mid s \in M_2 \right\},$$

and $\tilde{\alpha}_{B_i}(s) = c$ iff $T_D \vdash_c B_{i,y}[s]$. Generally, the theory T' can contain several multiformulas $\mathbf{A}'_i$, and then also the conclusion $\mathbf{B}'$ is composed of several parts.

Theorem 2 *Let $\mathcal{LD}^I$ be a simple linguistic description and the theory $T_D = T_I \cup T'$ be constructed as above. Then it is consistent and we may derive a conclusion $\mathbf{B}' = \{\mathbf{B}'_i \mid i \in J\}$, where the intensions $\mathbf{B}'_i$ are*

$$\mathbf{B}'_i = \left\{ \tilde{\alpha}_{B_i}(s)/B_{i,y}[s] \mid s \in M_2,\ B_i = m(\mathcal{B}_i),\ i \in J \right\}, \tag{15}$$

where

$$\tilde{\alpha}_{B_i}(s) = \bigvee_{t \in M_1} (\tilde{\alpha}_{A_i}(t) \otimes \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s)),$$

and all $\tilde{\alpha}_{B_i}(s)$ in $\mathbf{B}_i$, $i \in J$ are maximal.

PROOF: Similar as the proof of Theorem 6.1 in [12], page 249. □

6 Deduction with crisp observations – implementation

Our primary interest is the case of crisp observations (in some possible world). Then the theory T' has the following form:

$$T' = \{1/A_i[t_o] \mid A_i = m(\mathcal{A}_i),\ i \in \{1, \dots, r\}\} \tag{16}$$

where $\mathcal{A}_i$ is the most suitable expression among evaluating expressions occurring in the antecedents of IF-THEN rules in the linguistic description $\mathcal{LD}^I$, which corresponds to the term t_0. Informally, this means that in every possible world the element assigned to t_0 intuitively corresponds to the meaning of the found suitable expression; for example, if t_0 is assigned the expression "big" then every interpretation v of t_0 in every possible world is intuitively indeed "big".

The algorithm which selects the formula $G(x) \in \mathcal{G}$ (and, consequently, also the evaluating expression $\mathcal{A}_i \in \mathcal{S}$, $\mathcal{A}_i = m^{-1}(G)$) given an observation u_0 in some possible world $\mathcal{W}$ is described as follows.

We will suppose some structure of the set $\mathcal{G}$ of atomic predicate symbols which model the meanings of linguistic evaluating expressions. This structure is motivated by the linguistic intuition (see Section 2.2) — that there are several atomic evaluating expressions (adjectives like *small*, *big* etc.). Their meaning can be modified by *linguistic hedges* (see e.g. [5]) – the adverbs *very*, *more or less* etc. Then the set of simple evaluating expressions is divided into several groups with the same atomic expressions, and these groups are totally ordered by an order relation characterizing a *sharpness* of the hedges.

The sharpness characterizes the degree of precision imposed by hedge on the atomic expression, e.g. *very small* is more specific than *roughly small* and, therefore, the hedge *very* is sharper than *roughly*. Therefore in the following we will denote predicate symbols from $\mathcal{G}$ by $G_{i,\alpha}$, where the first subscript $i \in P$ denotes the group of predicate symbols corresponding to the evaluating expressions with the same atomic one. The second subscript $\alpha \in Q_i$ denotes the sharpness, the smaller is this subscript, the sharper is the corresponding linguistic hedge. There is also a total order relation $\prec$ defined on the set $\{G_i \mid i \in P\}$ which determines formula G to be chosen in the situation where there are several candidates for the most suitable formula G for the given term $t \in M$.

Let us define the operation $p : M \longrightarrow L$ for the threshold $c_0 \in (0, 1]$ by

$$p_{T,c_0}(A) = \begin{cases} c, & T \vdash_c A \text{ and } c \le c_0 \\ c_0, & T \vdash_c A \text{ and } c > c_0. \end{cases}$$

Definition 5 *Let us denote the maximal value of operation p on the set $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ for the given term $t \in M$ by m_t, i.e.*

$$m_t = \max_{G \in \tilde{\mathcal{G}}} \{p_{T,c_0}(G_x[t])\}.$$

The most suitable linguistic expression operation Suit $: M \longrightarrow \tilde{\mathcal{G}}$ *for the given theory T and the threshold $c_0 \in (0, 1]$ is defined by the following formula:*

$$\mathrm{Suit}^{c_0}_{\tilde{\mathcal{G}}}(t_0) = \begin{cases} \min_{i \in P}\{G_{i,\alpha_0} \mid \alpha_0 = \min\{\alpha \mid p_{T,c_0}(G_{i,\alpha,x}[t_0]) = m_{t_0},\ G_{i,\alpha} \in \tilde{\mathcal{G}}\}\} & \textit{if} \quad m_{t_0} > 0 \\ \text{undefined} & \textit{if} \quad m_{t_0} = 0. \end{cases} \tag{17}$$

The first minimum in the formula (17) is taken with respect to the ordering $\prec$ discussed above.

The next proposition shows that if the theory T is the theory of evaluating expressions in the sense of Definition 2, then the result of Suit operation is always defined, provided that $\tilde{\mathcal{G}} = \mathcal{G}$.

Proposition 3 *Let the theory T^{EX} fulfills the conditions of Definition 2 and $c_0 \in (0,1]$. Then*

$$\mathrm{Suit}^{c_0}_{\mathcal{G}}(t) = G, \qquad G \in \mathcal{G}$$

holds for all $t \in M$.

PROOF: Condition 5 from Definition 2 says that for every $t \in M$ there is at least one $G \in \mathcal{G}$ such that $T^{EX} \vdash_c G_x[t]$ with $c > 0$, which means that $m_t > 0$. It follows that there always exists some $G_{i,\alpha}$ such that $p_{T^{EX},c_0}(G_{i,\alpha,x}[t]) = m_t$ and therefore the set

$$S^{c_0}_{\mathcal{G}}(t_0) = \{G_{i,\alpha_0} \mid \alpha_0 = \min\{\alpha \mid p_{T,c_0}(G_{i,\alpha,x}[t_0]) = m_{t_0},\ G_{i,\alpha} \in \tilde{\mathcal{G}}\}\} \tag{18}$$

is non-empty for every $t \in T$ and have the property that if $G_{i,\alpha} \in S^{c_0}_{\mathcal{G}}(t_0)$ and $G_{j,\beta} \in S^{c_0}_{\mathcal{G}}(t_0)$ then $i \neq j$. Because $\prec$ is total order, the set $S^{c_0}_{\mathcal{G}}(t_0)$ has unique minimum with respect to $\prec$, and the claim follows. □

The meaning of the threshold c_0 appearing in the definition of the Suit operation is the following: all the provability degrees with respect to the theory T, which are greater than c_0, are regarded as maximal, and therefore they are considered as possible candidates for the result of the Suit operation. This is necessary e.g. in situations when there are for some $t \in M$ two (or more) predicate symbols $G_1 = G_{i,\alpha_j}$ and $G_2 = G_{i,\alpha_k}$ from $\mathcal{G}$, $i \in P$, $\alpha_j < \alpha_k$ with high values of $\tilde{\alpha}_{G_i}(t)$ and also $\tilde{\alpha}_{G_1}(t) \leq \tilde{\alpha}_{G_2}(t)$ for all $t \in M$. This correspond to the so-called *inclusive interpretation* of linguistic hedges [5]. Then it is unsatisfactory to prefer *small* to *very small* in the situation when $\tilde{\alpha}_{G_2}(t)$ is 1 and $\tilde{\alpha}_{G_1}(t)$ is, say, 0.99, $G_1 = m(\textit{very small})$ and $G_2 = m(\textit{small})$. The threshold c_0 allows us to adjust the behavior of fuzzy logic deduction in such situations.

In the following we denote by An a set of all atomic predicate symbols $G_{i,\alpha}$ – meanings of simple evaluating expressions appearing in the antecedents of rules from the linguistic description $\mathcal{LD}^I$. Formally, if the linguistic description $\mathcal{LD}^I$ has the form (11), then

$$An = \{m(\mathcal{A}_i) \mid i = 1, \ldots, r\}. \tag{19}$$

Proposition 4 *Given linguistic description $\mathcal{LD}^I$, $\mathrm{Suit}^{c_0}_{An}(t)$ is total function iff the set An defined by (19) have the following property (P): For every $t \in M_1$ there exists $G \in An$ such that $\tilde{\alpha}_G(t) > 0$.*

PROOF: Suppose that P holds. Then the set $S^{c_0}_{An}(t)$ defined by (18) is non-empty and has unique minimum with respect to $\prec$ for every $t \in M_1$ and, therefore $\mathrm{Suit}^{c_0}_{An}(t)$ is a total function. Vice-versa, is P does not hold, then there exists some $t_0 \in M_1$ such that $\tilde{\alpha}_G(t_0) = 0$ for all $G \in An$. It follows that $m_{t_0} = 0$ and, according to (17), $\mathrm{Suit}^{c_0}_{An}(t_0)$ is not defined for t_0 and $\mathrm{Suit}^{c_0}_{An}(t)$ is not a total function. □

Now, the operation Suit is used in the algorithm of the fuzzy logic deduction. We start in some possible world $\mathcal{W} = \langle\{W_1, W_2\}, \ldots\}$, where $W_i = [{}^l w_i, {}^r w_i]$, $W_i \subset \mathbb{R}$, $i = 1, 2$. For the observation $u' \in W_1$ we find the corresponding closed term $t_0 \in M_1$ such that $\mathcal{W}(t_0) = u'$. We may always assume that there are enough terms in M_1 so that t_0 always exists. We suppose that there is some theory of evaluating expressions T^{EX} at our disposal. Last, we choose the threshold $c_0 \in (0,1]$. Then we find atomic predicate symbol $G_i \in An$, $\mathrm{Suit}^{c_0}_{An}(t_0) = G_i$, if the result of $\mathrm{Suit}^{c_0}_{An}(t_0)$ is defined. It means that the i-th IF-THEN rule IF $\mathcal{A}_i$ THEN $\mathcal{B}_i$ is selected for performing the inference. The situation in which the result of $\mathrm{Suit}^{c_0}_{An}(t_0) = G$ is undefined corresponds to situation in which we have no relevant information for the decision about the result of inference. Then also the result of fuzzy logic deduction for such a t_0 is left undefined.

Now we form the theory $T_D = T' \cup T_I$, where T' is defined by (16). Next we derive a conclusion $\mathbf{B}'$ defined by formula (15). In this special case, when only one IF-THEN rule is used, (15) has the form

$$\mathbf{B}' = \left\{\tilde{\alpha}_{B'}(s)/B_{i,y}[s] \mid s \in M_2,\ B_i = m(\mathcal{B}_i)\right\}, \tag{20}$$

where

$$\tilde{\alpha}_{B'}(s) = \tilde{\alpha}_{A_i \Rightarrow B_i}(t_0, s) = \tilde{\alpha}_{A_i}(t_0) \to \tilde{\alpha}_{B_i}(s),$$

where $A_i = m(\mathcal{A}_i)$.

Then the extension of linguistic expression $\mathcal{B}'$ with intension $\mathbf{B}'$ in possible world $\mathcal{W}$ is found,

$$\mathrm{Ext}_{\mathcal{W}}\, \mathcal{B}' = \left\{ \mathcal{W}(B'(w/x))/w \,\middle|\, w \in W_2 \right\},$$

We can assign linguistic expression $\mathcal{B}'$ to $\mathbf{B}'$ by means of some *linguistic approximation* algorithm [2]. The last step is to find $v' \in W_2$, $v' = \mathrm{DEF}(\mathbf{B}')$.

The defuzzification operation is defined on the semantic level: for the possible world $\mathcal{W}$, it is an operation $\mathrm{DEF} : L^W \longrightarrow W$ such that $\mathcal{W}(\mathrm{DEF}(A)) > 0$ ([12], p. 214, see also [7]). The problem of logically well-founded defuzzification method is one of directions of our further research. Currently we are using defuzzification which is a combination of the well known defuzzification methods LOM (Last of Maxima), MOM (Mean of Maxima) and FOM (First of Maxima). It takes the right edge of the kernel, center of gravity, or the left hedge of the kernel for the non-increasing (type "small"), with one peak (type "medium") and non-decreasing (type "big") membership function, respectively.

The generalization of the results in this section for the (in practice much more important situation) where there are several antecedent variables $X_1, \ldots, X_n$ and the IF-THEN rules have the form

$$\mathcal{R}_i^I := \text{IF } \mathcal{A}_{i,1} \text{ AND } \ldots \text{ AND } \mathcal{A}_{i,n} \text{ THEN } \mathcal{B}_i, \quad i = 1, 2, \ldots, r$$

is left to the reader.

7 The behavior of the fuzzy logic deduction

Generally speaking, the behavior of the fuzzy logic deduction depends on several factors, namely:

1. concrete setting of the functions $\tilde{\alpha}_G$,
2. the value of the threshold c_0,
3. the defuzzification method DEF.

We can study the behaviour of the fuzzy logic deduction by examining a function $LD : W_1 \longrightarrow W_2$, which assigns a conclusion $v' \in W_2$ to a given observation $u' \in W_1$. We suppose that W_1 and W_2 are real intervals $[{}^l w_i, {}^r w_i]$, $i = 1, 2$. Moreover, we also assume that the possible world $\mathcal{W} = \langle\langle W_1, W_2\rangle, \ldots\rangle$ is canonical model of the theory T_I.

It follows from Proposition 4 that if the linguistic description $\mathcal{LD}^I$ is such that it fulfills the property (P), $\mathrm{Suit}^{c_0}_{An}(t)$ is a total function, which means that for every observation is one IF-THEN rule selected and, consequently, the conclusion $\mathbf{B}'$ is defined and therefore also LD is a total function.

Because only one IF-THEN rule from the linguistic description is selected for the given observation $u' \in W_1$, the function LD is only piece-wise continuous. However, this non-continuity is not necessarily a major drawback. In decision-type problems, for example, we are not looking for continuity, but for well-defined behavior in every possible situation. Consider, e.g. the linguistic description

$$\begin{aligned} &\mathcal{R}_1 := \text{IF } X \text{ is small THEN } Y \text{ is big,} \\ &\mathcal{R}_2 := \text{IF } X \text{ is very small THEN } Y \text{ is small,} \\ &\qquad \ldots \end{aligned}$$

If we interpret X as "distance to obstacle" and Y as "angle of steering wheel", then it can be dangerous to use for small observations more than one rule, because it can result in "medium" conclusion and crash with the obstacle. This can be amended by a special defuzzification method adapted to such a situations. However, we are convinced that such a behavior conforms with the way of human reasoning and so, we prefer to find a convenient inference mechanism.

In the next paper, we will study the behavior of fuzzy logic deduction in more details, we concentrate on the modifications of it which allow the function LD to be continuous and study also approximation capabilities of it. Another field of study, which can be well formulated using our formalism, is fuzzy logic deduction with linguistically expressed observations.

8 Conclusion

We have presented in this paper methodology which treats fuzzy inference as a logical deduction in formal logical system of fuzzy logic in narrow sense. This methodology have its counterpart in algorithms implemented in software system LFLC (Linguistic Fuzzy Logic Controller) developed in Institute for Research and Applications of Fuzzy Modeling at University of Ostrava. It uses three atomic evaluating expressions *small*, *medium* and *big* and linguistic hedges (with decreasing sharpness) *extremely*, *highly*, *very*, *more or less*, *roughly*, *quite roughly*, *very roughly* interpreted inclusively with respect to sharpness, i.e. if x is *very small* then x is *small* also.

The LFLC system and fuzzy logic deduction implemented in it proved itself useful also in practical applications [11]. Fuzzy logic deduction have been also used in methods for learning linguistic descriptions from data [1, 3].

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