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Abstract

The paper is a contribution to the theory of fuzzy logic in narrow sense with evaluated syntax (FLn). We show that the concepts of fuzzy equality and the provability degree enable to generalize the concept of fuzzy approximation. In the second part of the paper we return to the Mamdani-Assilian formula, which is formed on the basis of the so called totally bounded fuzzy equality and using which we can approximate any function with the prescribed accuracy.

1 Introduction

An important concept specific for fuzzy logic is that of *fuzzy equality.* It is a mathematical characterization of similarity among objects. It means that we understand two objects to be approximately equal if they are similar. Apparently, the more are objects similar, the higher is the degree of their equality. The degree 1 means that the objects are indistinguishable.

From the formal point of view, fuzzy equality is a generalization of classical equality because it is subject to, formally the same axioms as in classical predicate logic with equality.

Many authors have already been dealing with fuzzy equality, for example U. Höhle [4], P. Hájek [2], F. Klawonn [5], R. Bělohlávek [1]. Their results demonstrate that this concept lays in the fundaments of the fuzzy set theory. Besides others, the authors mostly agree that the fuzzy sets should be extensional with respect to some fuzzy equality. Namely, that the following should hold for every fuzzy set $A \underset{\sim}{\subset} U$

$$A(x) \otimes [x \doteq y] \leq A(y)$$

where $\otimes$ is some t-norm and $[x \doteq y]$ denotes the truth value of $x \doteq y$. This inequality is a generalization of the obvious property of sets: if $x \in A$ and $x = y$ then $y \in A$. Let us remark that the role of imprecise equality goes even deeper. In [8], a lot of arguments have been given that meanings of words of natural language are determined by an imprecise equality.

Another interesting concept intensively studied in fuzzy logic is that of fuzzy approximation, i.e. study of the ways how a function can be imprecisely described and what are conditions for effective extraction of a function approximating it with the prescribed accuracy. Significant contribution to this field has been obtained by I. Perfilieva (see, e.g. [9, 11, 12]). It turns out that her results, which concern special models, can be well characterized on syntactical level using the machinery of fuzzy logic with evaluated syntax. Namely, that the concept of the provability degree corresponds to the concepts used in the theory of approximation.

The fuzzy approximation theory, however, has also turned out to be closely related to the concept of fuzzy equality. The reason is that fuzzy equality leads to pseudometrics and thus, in connection with the concept of the provability degree it makes possible to formulate results on the syntactical level, which therefore hold for all models and consequently, have more general validity. We may conclude that the fuzzy equality and the provability degree open an interesting area for study of the relation of topology to the formal fuzzy logic with evaluated syntax.

In this paper we study the correspondence of fuzzy equality with the fuzzy approximation theory. In Section 3 we give an overview of some of the properties of the fuzzy equality and introduce the concept of uniform formula, which leads to uniform continuity of fuzzy relations in the models. We also introduce the concept of totally bounded fuzzy equality, which corresponds with compactness of models.

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In Section 4, we introduce the concept of c-approximation of formulas and prove two results about approximation of formulas by disjunctive forms. Section 5 deals with one possibility, how a fuzzy function can be defined using the fuzzy equality, namely a way which leads to the approximation of a function using the Mamdani-Assilian type of a formula. Section 6 is a brief discussion of the question, how an approximating function can be obtained via defuzzification. The answer is ambiguous in the sense that any function passing through the support of the fuzzy relation determined by a Mamdani-Assilian formula can serve for their purpose.

2 Preliminaries

In this paper, we deal with fuzzy logic in narrow sense with evaluated syntax presented extensively in [9]. The set of truth values is supposed to form a Łukasiewicz MV-algebra.

Our notation is due to [9], which is based on the notation proposed by J. R. Shoenfield in [13]. Thus, if $\mathcal{V}$ is a structure for the language J then $\mathcal{V}(t) = v \in V$ is an element being interpretation of the term t. However, sometimes it is useful to use the other kind of notation: given a free variable x then v/x is an element assigned to x in the structure $\mathcal{V}$.

The set of all well formed formulas for the language J is denoted by F_J and the set of all closed terms by M_J. If T is a fuzzy theory then its language is denoted by $J(T)$.

Let $A(x_1, \ldots, x_n)$ be a formula and $t_1, \ldots, t_n$ be terms substitutable into A for the variables $x_1, \ldots, x_n$, respectively. Then $A_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$ is an instance of A resulting from it when replacing all the free occurrences of the variables $x_1, \ldots, x_n$ by the respective terms $t_1, \ldots, t_n$. If the variables $x_1, \ldots, x_n$ are clear from the context then we will simply write $A[t_1, \ldots, t_n]$.

Let $A(x_1, \ldots, x_n) \in F_J$ be a formula and $\mathcal{V}$ a structure for the language J. Then the satisfaction fuzzy relation of A is

$$A_{\mathcal{V}} = \left\{ {}^{a}/\langle v_1, \ldots v_n\rangle \,\middle|\, a = \mathcal{V}(A(v_1/x_1, \ldots, v_n/x_n)), v_1, \ldots, v_n \in V \right\} \underset{\sim}{\subseteq} V^n. \tag{1}$$

As a special case, let $A(x) := x \doteq t$ where t is a closed term, which is assigned an element $v_0 = \mathcal{V}(t)$. Then its satisfaction fuzzy set in $\mathcal{V}$ is

$$A_{\mathcal{V}} = \left\{ {}^{a}/v \,\middle|\, a = \mathcal{V}(v/x \doteq v_0/t), v \in V \right\}.$$

A formula A is *crisp* in a fuzzy theory T if

$$T \vdash A \lor \neg A. \tag{2}$$

Then in every model $\mathcal{V} \models T$ either $\mathcal{V}(A) = 1$ or $\mathcal{V}(A) = 0$. However, note that both $T \vdash_0 A$ as well as $T \vdash_0 \neg A$ is possible since there may exist models $\mathcal{V}, \mathcal{V}'$ such that both $\mathcal{V}(A) = 1$ and $\mathcal{V}(\neg A) = 0$, as well as $\mathcal{V}'(A) = 0$ and $\mathcal{V}'(A) = 1$.

The following property of biresiduation

$$(a \leftrightarrow c) \wedge (b \leftrightarrow d) \leq ((a \vee b) \leftrightarrow (c \vee d)), \qquad a, b, c, d \in L \tag{3}$$

will be needed in the sequel.

Recall that a *pseudometrics* on a set V is a function $\rho : V^2 \longrightarrow \mathbb{R}$ with the following properties:

(i) $\rho(v, v) = 0$,

(ii) $\rho(u, v) = \rho(v, u)$,

(iii) $\rho(u, w) \leq \rho(u, v) + \rho(v, w)$

for all $u, v, w \in V$.

The function $\rho(a, b) = \neg(a \leftrightarrow b)$ is a pseudometrics on the set of all truth values L. In the case of Łukasiewicz algebra we obtain

$$\rho(a, b) = \neg(a \leftrightarrow b) = 1 \wedge |a - b|. \tag{4}$$

Definition 1
Let $P_1, \dots, P_k$ be unary predicate symbols and $E_{i_1 \dots i_n}$, $1 \le i_1, \dots, i_n \le k$, be some closed formulas. The following formulas of fuzzy predicate logic are called the disjunctive normal form

$$\mathrm{DNF}(x_1, \dots, x_n) = \bigvee_{i_1=1}^{k} \cdots \bigvee_{i_n=1}^{k} (P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n) \mathbin{\&} E_{i_1 \dots i_n}) \tag{5}$$

and the conjunctive normal form

$$\mathrm{CNF}(x_1, \dots, x_n) = \bigwedge_{i_1=1}^{k} \cdots \bigwedge_{i_n=1}^{k} (\neg P_{i_1}(x_1) \nabla \cdots \nabla \neg P_{i_n}(x_n) \nabla E_{i_1 \dots i_n}). \tag{6}$$

We will also use the following extended inequality relation

$$a <^* b \quad \text{iff} \quad a < b < 1 \text{ or } a = b = 1.$$

3 Fuzzy equality and some of its properties

In this section, we will extend fuzzy logic by the fuzzy equality predicate $\doteq$. In general, it is supposed to fulfill the following logical axioms:

(E1) $1/(x \doteq x)$

(E2) There are $m_1, \dots, m_n \ge 1$ such that

$$1/((x_1 \doteq y_1)^{m_1} \Rightarrow (\dots \Rightarrow ((x_n \doteq y_n)^{m_n} \Rightarrow$$
$$\Rightarrow (f(x_1, \dots, x_n) \doteq f(y_1, \dots, y_n)) \dots)$$

for every n-ary functional symbol f.

(E3) There are $m_1, \dots, m_n \ge 1$ such that

$$1/((x_1 \doteq y_1)^{m_1} \Rightarrow (\dots \Rightarrow ((x_n \doteq y_n)^{m_n} \Rightarrow$$
$$\Rightarrow (P(x_1, \dots, x_n) \Rightarrow P(y_1, \dots, y_n)) \dots)$$

for every n-ary predicate symbol P.

This definition is very general. It is reasonable to strengthen the condition for the fuzzy equality when considering the exponents $m_1, \dots, m_n$ to be equal to 1 only. On the other hand, we may confine the axioms to hold for some selected functional or predicate symbols only. If not stated otherwise, we consider axioms (E2) and (E3) to hold for all functional and predicated symbols of the language J and with the exponents $m_1, \dots, m_n$ to be equal to 1.

A special case of fuzzy equality is the crisp one $=$ which also fulfills the crispness property (2). However, when considering more than one fuzzy equality, say $\doteq_1, \doteq_2$, then we must be careful when applying axiom (E3). If we suppose that for both equalities, the predicate P can also be any of $\doteq_1$ or $\doteq_2$ then we obtain

$$T \vdash (x \doteq_1 x) \Rightarrow ((x \doteq_1 y) \Rightarrow ((x \doteq_2 x) \Rightarrow (x \doteq_2 y))), \tag{7}$$

as well as

$$T \vdash (x \doteq_2 x) \Rightarrow ((x \doteq_2 y) \Rightarrow ((x \doteq_1 x) \Rightarrow (x \doteq_1 y))), \tag{8}$$

which implies

$$T \vdash (x \doteq_1 y) \Leftrightarrow (x =_2 y)$$

and therefore, in every model $\mathcal{V} \models T$ both fuzzy equalities coincide. Thus, if we accept more than one fuzzy equality in a single fuzzy theory than the relation among them will be specified by special axioms and axiom (E3) will not be accepted for P being a fuzzy equality other than the fuzzy equality occurring on the left hand side of (E3). However, axiom (E3) makes us possible to prove the following lemma (see [9]).

Lemma 1
Let T be a fuzzy predicate calculus with fuzzy equality. Then the following properties of the fuzzy equality are provable in T:

(a) *Symmetry*
$$T \vdash (x \doteq y) \Rightarrow (y \doteq x),$$

(b) *transitivity*
$$T \vdash ((x \doteq y) \mathbin{\&} (y \doteq z)) \Rightarrow (x \doteq z).$$

The interpretation of $\doteq$ in a structure $\mathcal{V}$ is a function $\mathcal{V}(\doteq) : V^2 \longrightarrow L$, which will analogously be denoted by $\doteq$. Let t, t' be terms such that $\mathcal{V}(t) = v$ and $\mathcal{V}(t') = v'$. Then we will often write $[v \doteq v']_{\mathcal{V}} = a$ instead of $\mathcal{V}(v/t \doteq v'/t') = a$.

As a special case, we assume that if the language of the fuzzy theory T contains both fuzzy as well as crisp equality then
$$T \vdash (\forall x)(\forall y)((x = y) \Rightarrow (x \doteq y)) \tag{9}$$
(this is provable from (7).

The following has been proved in [7].

Lemma 2
For every formula A and terms $t_1, \dots, t_n$, $s_1, \dots, s_n$ substitutable to A there are $m_1, \dots, m_n \geq 1$ such that

$$T \vdash ((t_1 \doteq s_1)^{m_1} \mathbin{\&} \cdots \mathbin{\&} (t_n \doteq s_n)^{m_n}) \Rightarrow$$
$$\Rightarrow (A_{x_1,\dots,x_n}[t_1, \dots, t_n] \Leftrightarrow A_{x_1,\dots,x_n}[s_1, \dots, s_n]). \tag{10}$$

Fuzzy equality is closely related to metrical and topological properties of models.

Lemma 3
Let T be a fuzzy theory with a fuzzy equality $\doteq$ and $\mathcal{V} \models T$ be its model.

(a) *Let us put $\rho(u, v) = \mathcal{V}(\neg(u/x \doteq v/y))$ for all $u, v \in V$. Then ρ is a pseudometrics on V.*

(b) *If $J(T)$ contains both fuzzy as well as the crisp equality and $T \vdash x \doteq y$ implies $T \vdash x = y$ then ρ is a metrics.*

PROOF: (a) Obviously, ρ is a function $\rho : V^2 \longrightarrow [0, 1]$. Then axiom (E1) gives the property $\rho(u, u) = 0$ and Lemma 1 gives symmetry and the triangular inequality of ρ by the properties of Łukasiewicz MV-algebra.

(b) Then in every model $\mathcal{V} \models T'$ and for every $u, v \in V$ it holds that $\mathcal{V}(u/x \doteq v/y) = 1$ implies $\mathcal{V}(u/x = v/y) = 1$. By the property (9), this holds iff $u = v$ and thus, we obtain $\rho(u, v) = 0$ iff $u = v$ where $\rho(u, v) = \neg\mathcal{V}(u/x \doteq v/y)$. □

The pseudometrics ρ defined in a model $\mathcal{V}$ via fuzzy equality will be denoted by $\rho_{\doteq}$. Consequently, all models of a fuzzy theory with fuzzy equality form a topological space with topology generated by $\rho_{\doteq}$.

Given an $1 > \varepsilon > 0$, let T be a fuzzy theory, in which
$$T \vdash_d A(x)$$
for some $d > 1 - \varepsilon$. Then the support of $A_{\mathcal{V}}$ in any model $\mathcal{V} \models T$ is a set
$$\mathrm{Supp}(A_{\mathcal{V}}) = \{v \in V \mid \rho_{\doteq}(v, v_0) < \varepsilon\}, \tag{11}$$
i.e. it is an open ball with the center v_0 determined by term t with respect to the pseudometrics $\rho_{\doteq}$.

The following definition characterizes uniform behaviour of formulas with respect to the fuzzy equality. Note that his is already contained in axioms (E1)–(E3) and Lemma 2. Our definition, however, is more general and is close to the concept of uniform continuity of functions in models.

Definition 2
Let T be a consistent fuzzy theory with fuzzy equality $\doteq$ and $A(x_1, \ldots, x_n) \in F_{J(T)}$. We say that A is uniform with respect to $\doteq$ if to every $0 < c < 1$ there is $1 > a > 0$ such that, if

$$T \vdash_{b_i} t_i \doteq s_i, \tag{12}$$

holds for all terms t_i, s_i and $b_i > a$, $i = 1, \ldots, n$, then

$$T \vdash_d A[t_1, \ldots, t_n] \Leftrightarrow A[s_1, \ldots, s_n] \tag{13}$$

for some $d > c$.

It follows from this definition that if A is uniform with respect to $\doteq$ then in every model $\mathcal{V} \models T$ and all elements $u_1, \ldots, u_n$, $v_1, \ldots, v_n$

$$\mathcal{V}(u_1/t_1 \doteq v_1/s_1) > a, \ldots, \mathcal{V}(u_n/t_n \doteq v_n/s_n) > a \qquad \text{implies}$$
$$A_{\mathcal{V}}(u_1, \ldots, u_n) \leftrightarrow A_{\mathcal{V}}(v_1, \ldots, v_n) > c. \tag{14}$$

The following property is a consequence of Lemma 2.

Lemma 4
Let T be a fuzzy theory with fuzzy equality $\doteq$. Then every open formula $A(x_1, \ldots, x_n) \in F_{J(T)}$ is uniform with respect to $\doteq$.

PROOF: If (12) holds for some $b_1, \ldots, b_n$ then it follows from (10) that (13) holds for some $d \geq b^{m_1} \otimes \cdots \otimes b^{m_n}$.

Let c, $0 < c < 1$, be given and take a such that

$$a^{m_1 + \cdots + m_n} > c,$$

where m_i are the exponents from (10). If $b_i > a$ for all $i = 1, \ldots, n$ then we get

$$d \geq b^{m_1} \otimes \cdots \otimes b^{m_n} \geq a^{m_1 + \cdots + m_n} > c,$$

which follows that A is uniform. □

The following lemma demonstrates that fuzzy theories with fuzzy equality force uniform continuity of fuzzy relations and functions in their models.

Lemma 5
Let T be a consistent fuzzy theory with fuzzy equality $\doteq$ and $\mathcal{V} \models T$ be its model.

(a) Let $A(x_1, \ldots, x_n)$ be uniform with respect to $\doteq$. Then the satisfaction fuzzy relation $A_{\mathcal{V}}$ is uniformly continuous with respect to $\rho_{\doteq}$ and the ordinary metrics.

(b) All functions $f_{\mathcal{V}}$ assigned to the functional symbols of $J(T)$ are uniformly continuous with respect to $\rho_{\doteq}$.

PROOF: a) Given a $0 < \varepsilon < 1$ and put $c = 1 - \varepsilon$. By the uniformity of A there is $1 > a > 0$ such that (14) holds. By Lemma 3 and (4) this is equivalent to the following: if $\rho(u_i, v_i) < 1 - a$, $i = 1, \ldots, n$ then

$$|A_{\mathcal{V}}(u_1, \ldots, u_n) - A_{\mathcal{V}}(v_1, \ldots, v_n)| < c = 1 - \varepsilon$$

which is the classical definition of uniform continuity of $A_{\mathcal{V}}$ with respect to ρ and the ordinary metrics.

b) Let us take c such that $1 > c > 1 - \varepsilon$ and find a such that $a^n > c$. Assume now that u_1, v_i, $i = 1, \ldots, n$ are elements such that $\mathcal{V}(u_i/x_i \doteq v_i/y_i) \geq a$. Then for any $\delta > 1 - a$ we have $\rho_{\doteq}(u_i, v_i) < \delta$. We will now show that

$$\rho_{\doteq}(f_{\mathcal{V}}(u_1, \ldots, u_n), f_{\mathcal{V}}(v_1, \ldots, v_n)) < \varepsilon. \tag{15}$$

It follows from axiom (E2) that in the model $\mathcal{V}$,

$$\mathcal{V}(u_1/x_1 \doteq v_1/y_1) \otimes \cdots \otimes \mathcal{V}(u_n/x_n \doteq v_n/y_n) \leq$$
$$\leq \mathcal{V}(f(u_1/x_1, \ldots, u_n/x_n) \doteq f(v_1/y_1, \ldots, v_n/y_n)).$$

But this implies (15) by the definition of $\rho_{\doteq}$ and the choice of ε and δ. □

The following definition characterizes a kind of compactness of the fuzzy equality.

Definition 3
Let T be a consistent fuzzy theory with fuzzy equality $\doteq$. We say that $\doteq$ is totally bounded in T if to every a there is $b >^* a$ and closed terms $t_1, \ldots, t_m$ such that

$$T \vdash_b (\forall x)(x \doteq t_1 \vee \cdots \vee x \doteq t_m). \tag{16}$$

Lemma 6
Let T be a fuzzy theory with totally bounded fuzzy equality and given a. Then to every term s and every model $\mathcal{V} \models T$ there is a term t_j such that

$$\mathcal{V}(s \doteq t_j) > a. \tag{17}$$

PROOF: By the substitution axiom, $T \vdash_{b'} (s \doteq t_1 \vee \cdots \vee s \doteq t_m)$ for some $b' > a$. Then in every model $\mathcal{V} \models T$, we have

$$\mathcal{V}(s \doteq t_1 \vee \cdots \vee s \doteq t_m) > a$$

and by the properties of supremum in the linearly ordered set there is a term t_j such that (17) holds. □

4 Approximation in fuzzy logic

As mentioned in the introduction, fuzzy equality and the provability degree make possible to generalize the concept of approximation of fuzzy relations. In this section, we will prove that any open formula of a fuzzy theory with the totally bounded fuzzy equality can be approximated by a disjunctive normal form. It should be noted that we have used the concepts and methods developed by I. Perfilieva in [9].

Definition 4
Let $c \in L$. We say that formulas $R(x_1, \ldots, x_n)$ and $S(x_1, \ldots, x_n)$ c-approximate each other in a fuzzy theory T if

$$T \vdash_c (\forall x_1) \cdots (\forall x_n)(R(x_1, \ldots, x_n) \Leftrightarrow S(x_1, \ldots, x_n)). \tag{18}$$

Definition 4 is equivalent with the requirement that in every model $\mathcal{V} \models T$ the following holds true:

$$|R_{\mathcal{V}}(v_1, \ldots, v_n) - S_{\mathcal{V}}(v_1, \ldots, v_n)| \leq 1 - c \tag{19}$$

for all $v_1, \ldots, v_n \in V$ where $R_{\mathcal{V}}$ and $S_{\mathcal{V}}$ are satisfaction fuzzy relations of the formulas R and S, respectively.

The provability degree c thus takes a natural interpretation as the degree of precision of the approximation. Obviously, the higher c, the more precise approximation. In the interpretation, we usually set c indirectly as the *accuracy level* ε for which $\varepsilon = \neg c = 1 - c$ holds.

The following is obvious.

Lemma 7
Let $\mathcal{L}_L$ be Łukasiewicz algebra and f be 1 *or* 0*. Then*

$$(a \leftrightarrow f) \rightarrow (b \leftrightarrow f) = 1 \tag{20}$$

iff one of the following holds:

(a) $a < b$ *and* $f = 1$,

(b) $a > b$ *and* $f = 0$,

(c) $a = b$ *and either* $f = 1$ *or* $f = 0$.

The following lemma by can be proved using the properties of residuated lattices and the completeness theorem.

Lemma 8
Let $T \vdash_a A \Leftrightarrow C$, $T \vdash_b B \Leftrightarrow D$. *Then* $T \vdash_c (A \vee C) \Leftrightarrow (B \vee D)$ *where* $c \geq a \wedge b$.

Theorem 1
Let T *be a consistent fuzzy theory with a totally bounded fuzzy equality* $\doteq$ *and let* $R(x_1, \ldots, x_n) \in F_{J(T)}$ *be a formula. Then to every* $c < 1$ *there is a conservative extension* T' *of* T *and a disjunctive normal form* $R_{DNF} \in F_{J(T')}$ *such that* R_{DNF} *c-approximates* R.

PROOF: Given $c < 1$, find $a > 0$ for which R is uniform with respect to $\doteq$ (by Lemma 4). Since $\doteq$ is totally bounded, there are closed terms $t_1, \ldots, t_m$ such that

$$T \vdash_d (\forall x)(x \doteq t_1 \vee \cdots \vee x \doteq t_m) \tag{21}$$

where $d > a$. Then in every model $\mathcal{V} \models T$ and for every $v \in V$ there is t_i such that

$$\mathcal{V}(v/x \doteq t_i) > a.$$

Now, let us extend $J(T)$ by unary predicate symbols $P_1, \ldots, P_m$ and put

$$T' = T \cup \{1/(\forall x)(P_i(x) \vee \neg P_i(x)) \mid i = 1, \ldots, m\} \cup$$
$$\cup \{1/((\mathbf{a} \Leftrightarrow P_i[s]) \Rightarrow ((s \doteq t_i) \Leftrightarrow P_i[s])) \mid i = 1, \ldots, m, s \in M_{J(T)}\}$$

By Lemma 7, in every model $\mathcal{V}' \models T'$ either $\mathcal{V}'(P_i[s]) = 1$ or $\mathcal{V}'(P_i[s]) = 0$ holds for all terms $s \in M_{J(T)}$.

A model of T' can be constructed from any model $\mathcal{V} \models T$ by assigning fuzzy sets $P_{\mathcal{V},i} \underset{\sim}{\subset} V$ to the predicate symbols P_i, $i = 1, \ldots, n$, for which $P_{\mathcal{V},i}(v)$, $v \in V$ is equal either to 0 or to 1 and which have the properties described below.

We will now define the following normal form:

$$R_{DNF}(x_1, \ldots, x_n) := \bigvee_{1 \leq i_1, \ldots, i_n \leq m} P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P(x_n) \mathbin{\&} R[t_{i_1}, \ldots, t_{i_n}]. \tag{22}$$

Let $\mathcal{V}' \models T'$ be a model of T' and s_k, $k = 1, \ldots, n$ be terms, which are assigned elements $\mathcal{V}'(s_k) = v_k \in V$. By Lemma 6, to every such v_k there is term t_{j_k} such that

$$\mathcal{V}'(s_k \doteq t_{j_k}) > a \tag{23}$$

and thus, $\mathcal{V}'(P_{j_k}[s_k]) = 1$. For all other terms t_j, $j = 1, \ldots, m$, $j \neq j_k$, we mostly have $\mathcal{V}'(P_j[s_k]) = 0$. (In case that $\mathcal{V}'$ is constructed from $\mathcal{V} \models T$, we set $P_{\mathcal{V},j}(v_k)$ equal to 1 or 0 in accordance with the above description.)

It follows from the described procedure that there is a set of subscripts $J \subset \{1, \ldots, m\}^n$ such that

$$\mathcal{V}'(s_1 \doteq t_{j_1}) > a, \ldots, \mathcal{V}'(s_n \doteq t_{j_n}) > a$$

for all $\langle j_1, \ldots, j_n \rangle \in J$ and thus

$$\mathcal{V}'(R[t_{j_1}, \ldots, t_{j_n}]) \leftrightarrow \mathcal{V}'(R[s_1, \ldots, s_n]) > c.$$

Furthermore, because $\mathcal{V}'(P_{j_1}[s_1]) = 1, \ldots, \mathcal{V}'(P_{j_n}[s_n]) = 1$ for $\langle j_1, \ldots, j_n \rangle \in J$, we get

$$\mathcal{V}'(R_{DNF}[s_1, \ldots, s_n]) = \mathcal{V}'\left(\bigvee_{\langle j_1, \ldots, j_n \rangle \in J} R[t_{j_1}, \ldots, t_{j_n}] \right). \tag{24}$$

Then by (3) we have

$$c < \bigwedge_{\langle j_1,\dots,j_n\rangle\in J} \mathcal{V}'(R[t_{j_1},\dots,t_{j_n}]) \leftrightarrow \mathcal{V}'(R[s_1,\dots,s_n]) \le$$

$$\le \mathcal{V}'\left(\bigvee_{\langle j_1,\dots,j_n\rangle\in J} R[t_{j_1},\dots,t_{j_n}]\right) \leftrightarrow \mathcal{V}'(R[s_1,\dots,s_n]) =$$

$$= \mathcal{V}'(R_{DNF}[s_1,\dots,s_n] \Leftrightarrow R[s_1,\dots,s_n]).$$

Consequently,

$$T \vdash_d R_{DNF}[s_1,\dots,s_n] \Leftrightarrow R[s_1,\dots,s_n]$$

for some $d > c$.

The conservativeness of the extension T' follows from the construction of (some) models $\mathcal{V}' \models T$ from the models $\mathcal{V} \models T$ since $\mathcal{V}'(A) = \mathcal{V}(A)$ for all $A \in F_{J(T)}$. □

Theorem 2
Let T be a fuzzy theory with totally bounded fuzzy equality $\doteq$ and $A(x)$ be a formula. Then to every $0 < c < 1$ there are terms $t_1,\dots,t_m$ such that

(a)

$$T \vdash_d (\exists x)A(x) \Leftrightarrow \bigvee_{j=1}^{m} A_x[t_j], \qquad d > c, \tag{25}$$

(b)

$$T \vdash_d (\forall x)A(x) \Leftrightarrow \bigwedge_{j=1}^{m} A_x[t_j], \qquad d > c. \tag{26}$$

PROOF: a) Let T_H be a conservative Henkin extension of T and $\mathbf{r}$ a special constant for $(\exists x)A(x)$. Let some c, $0 < c < 1$ be given. Because $A(x)$ is uniform with respect to $\doteq$, there is $1 > a > 0$ and terms $t_1,\dots,t_m$ such that, first,

$$T_H \vdash_b (\forall x)(x \doteq t_1 \vee \cdots \vee x \doteq t_m), \qquad b >^* a \tag{27}$$

and second, if $T_H \vdash_{b'_j} \mathbf{r} \doteq t_j$ where $b'_j > a$ then

$$T_H \vdash_d A_x[\mathbf{r}] \Leftrightarrow A_x[t_j]$$

for some $d > c$. By (27), in every model $\mathcal{V}_H \models T_H$ there is t_j such that

$$\mathcal{V}_H(\mathbf{r} \doteq t_j) > a, \tag{28}$$

which implies that

$$\mathcal{V}_H(A_x[\mathbf{r}]) \leftrightarrow \mathcal{V}_H(A_x[t_j]) \ge d > c. \tag{29}$$

Furthermore, $\mathcal{V}_H(A_x[t_j]) \le \mathcal{V}_H(A_x[\mathbf{r}])$ (because $\mathbf{r}$ is special for $(\exists x)A$) and thus, (29) is equivalent with

$$\mathcal{V}_H(A_x[\mathbf{r}]) \to \mathcal{V}_H(A_x[t_j]) \ge d > c.$$

From the properties of $\to$ we thus obtain that

$$\mathcal{V}_H(A_x[\mathbf{r}]) \to \bigvee_{j=1}^{m} \mathcal{V}_H(A_x[t_j]), \ge d > c,$$

which after rewriting leads to

$$\mathcal{V}_H(A_x[\mathbf{r}] \Leftrightarrow \bigvee_{j=1}^{m} A_x[t_j]) \ge d > c. \tag{30}$$

Since (29) can be derived using the same reasoning for every model $\mathcal{V}_H \models T_H$, we conclude from the properties of Henkin fuzzy theories that

$$T_H \vdash_d (\exists x)A(x) \Leftrightarrow \bigvee_{j=1}^{m} A_x[t_j], \qquad d > c,$$

which due to conservativeness of T_H gives (25).

b) can be proved quite analogously. □

According to this theorem, in a fuzzy theory with totally bounded fuzzy equality, every existential formula can be approximated by a finite disjunction of closed instances of its matrix and similarly, every general formula can be approximated by a finite conjunction of closed instances of its matrix.

5 Fuzzy functions and fuzzy equality

F. Klawonn and R. Kruse in [6] have shown that the well known Mamdani-Assilian formula, which is used in fuzzy control, is closely related to the concept of fuzzy equality. We will show using formal means of FLn that this formula defines a fuzzy relation which approximates fuzzy set elements being fuzzy-equal to functional values of certain given function. Throughout this section T is a consistent fuzzy theory with a totally bounded fuzzy equality $\doteq$. The parameter $0 < c < 1$ takes the role of the accuracy level of the approximation.

Let g be a functional symbol and let us define a formula

$$A(u, x, y) := (x \doteq u) \wedge (y \doteq g(u)). \tag{31}$$

The functional symbol g represents a function to be approximated.

The formula $F(x, y) := y \doteq g(x)$ represents a fuzzy set of all values approximately equal to the functional values $g(x)$. By by the transitivity of $\doteq$ we can prove that $T \vdash (F(x, y) \,\&\, F(x, y')) \Rightarrow y \doteq y'$, and so the formula $F(x, y)$ determines a *fuzzy function*. Note that $F(x, y)$ is also extensional, i.e.

$$T \vdash F(x, y) \,\&\, y \doteq y' \Rightarrow F(x, y').$$

Lemma 9
To every $0 < c < 1$ there are terms $t_1, \ldots, t_m$ such that

$$T \vdash_d (\exists u)A(u, x, y) \Leftrightarrow \bigvee_{j=1}^{m} A_u[t_j](x, y), \qquad d > c. \tag{32}$$

PROOF: This is a corollary of Lemma 4 and Theorem 2. □

The formula

$$MA(x, y) := \bigvee_{j=1}^{m} A_u[t_j](x, y) := \bigvee_{j=1}^{m} (x \doteq t_j) \wedge (y \doteq g(t_j))$$

will be called the Mamdani-Assilian formula.

For the clarity of the lemma below, we will denote

$$B(x, y) := (\exists u)A(u, x, y).$$

Lemma 10
The following is provable in the fuzzy theory T

$$T \vdash (\forall x)(\exists y)B(x, y), \tag{33}$$

$$T \vdash (\forall x)(\forall y)(\forall y')((B \,\&\, B_y[y']) \Rightarrow (y \doteq y')), \tag{34}$$

$$T \vdash (\forall x)(\forall y)(\forall y')((B \,\&(y \doteq y')) \Rightarrow B_y[y']). \tag{35}$$

PROOF: This can be proved using the properties of the fuzzy equality or by verification in models (using the completeness theorem). □

Due to the results of the paper [7], Lemma 10 makes us possible to define a new functional symbol f by

$$1/(\forall x)(\forall y)(y \doteq f(x) \Leftrightarrow B(x,y)). \tag{36}$$

This functional symbol has only an auxiliary role. It enables us to define a new fuzzy theory T' as a conservative extension of T by the axiom (36).

Lemma 11

$$T' \vdash (\forall x)(g(x) \doteq f(x))$$

PROOF: This follows from

$$T' \vdash (\forall x)(\exists u)((x \doteq u) \wedge (g(x) \doteq g(u)).$$

by an easy verification in models. □

Theorem 3
To every $0 < c < 1$ there are terms $t_1, \ldots, t_m$ such that

$$T \vdash_d F(x,y) \Leftrightarrow MA(x,y), \qquad d > c, \tag{37}$$

where $F(x,y) = y \doteq g(x)$ is a formula representing the fuzzy function and $MA(x,y) = \bigvee_{j=1}^{m}((x \doteq t_j) \wedge (y \doteq g(t_j))$ is the Mamdani-Assilian formula.

PROOF: Using the transitivity of $\doteq$ from Lemma 1 Lemma 11 we prove that $T' \vdash (g(x) \doteq y) \Leftrightarrow (f(x) \doteq y)$. The result then follows from (36) using the equivalence theorem and conservativity of the extension T'. □

According to this theorem, to every c we can find $d > c$ and the Mamdani-Assilian formula, which d-approximates the fuzzy function determined by the formula $F(x,y) := y \doteq g(x)$.

Given a model $\mathcal{V} \models T$, let a function $g_{\mathcal{V}}$ be assigned to g. Then the fuzzy function $F_{\mathcal{V}}$ determined by $F(x,y)$ characterizes all elements $v \in V$, which are "close" to the functional values $g_{\mathcal{V}}(u)$ for all $u \in V$. If $\doteq$ is totally bounded then, given some precision $1 > \varepsilon > 0$, we may approximate the fuzzy function $F_{\mathcal{V}}(x,y)$ by the fuzzy relation due to Mamdani-Assilian formula with the error

$$|F_{\mathcal{V}}(u,v) - MA_{\mathcal{V}}(u,v)| < \varepsilon$$

for all $u, v \in V$, i.e. the difference between the degree of truth that v is approximately equal to $g(u)$ and its estimation using the Mamdani-Assilian formula is at most ε.

The following easily follows from the equivalence theorem.

Corollary 1
Let (37) hold and let

$$T \vdash_a MA(x,y).$$

Then

$$T \vdash_b y \doteq g(x) \text{ where } b \geq a \otimes d.$$

This corollary provides estimation of the provability degree that an element y is close to the functional value of the approximated function g if we use the Mamdani-Assilian formula instead of the fuzzy function formula F. It follows from it that in every model $\mathcal{V} \models T$, if

$$MA_{\mathcal{V}}(u,v) \geq a$$

then

$$\rho_{\doteq}(v, g_{\mathcal{V}}(u)) < \neg a \oplus \varepsilon$$

where $\varepsilon = \neg c$ (recall that the provability degree d in (37) fulfills $d > c$).

6 Approximation of function via defuzzification

A specific concept in the applications of fuzzy logic is that of the *defuzzification*. We will follow the reasoning provided by I. Perfilieva in [9]. According to her, the defuzzification is a function

$$\Theta : \mathcal{F}(X) \setminus \{\emptyset\} \longrightarrow X$$

such that $\Theta(X) = x \in \mathrm{Supp}(X)$ where $\mathcal{F}(X)$ is a set of all fuzzy sets on X. Hence, as defuzzification we can use any function assigning elements which are "sufficiently close" to the values of the approximated function. In terms of the (evaluated) syntax of fuzzy logic, we arrive at the concept of defuzzification as follows.

Let h be a functional symbol of $J(T)$ and suppose that $J(T)$ contains also the crisp equality $=$ discussed in Section 3. Recall that the formula $MA(x, y)$ depends on the specification of the accuracy level $0 < c < 1$ since the latter determines the terms $t_1, \dots, t_n$ occurring in MA.

Let us define a new fuzzy theory

$$T_c = T \cup \left\{ {}^1\!/MA_{x,y}[t,s] \;\middle|\; t, s \text{ are terms and } T \vdash s = h(t) \right\}. \tag{38}$$

Theorem 4
Let $T \vdash s = h(t)$ for some terms t, s. Then to every $0 < c < 1$ there is an extension T_c of T such that

$$T_c \vdash_b h(t) \doteq g(t), \qquad b > c. \tag{39}$$

PROOF: By the equality theorem we get $T' \vdash MA_{x,y}[t,s] \Leftrightarrow MA_{x,y}[t, h(t)]$. This, (38) and Corollary 1 imply (39). □

This simple theorem provides characterization of all crisp functions represented by h, which approximate the function represented by g with the given accuracy c. The function represented by h is tied with the Mamdani-Assilian formula MA via the fuzzy set of axioms in (38), which, in fact, state that in every model $\mathcal{V}' \models T_c$, given the terms t, s there is a term t_j such that

$$\mathcal{V}'(t \doteq t_j) = 1 \quad \text{and} \quad \mathcal{V}'(s \doteq g(t_j)) = 1.$$

Let us now specify h as follows:

(i) If $T \vdash s = h(t)$ for some terms t, s then $T \vdash_a MA_{x,y}[t,s]$ where $a > 0$.

(ii) $T \vdash h(t_j) = g(t_j)$ for all $j = 1, \dots, m$.

Then in every model $\mathcal{V}' \models T_c$, the function $h_{\mathcal{V}'}$ (assigned to h in $\mathcal{V}'$) is a crisp function which is close to $g_{\mathcal{V}}$ in the sense that

$$\rho_{\doteq}(h_{\mathcal{V}'}(u), g_{\mathcal{V}}(u)) < \varepsilon = \neg c$$

holds for all $u \in V$, for which $MA_{\mathcal{V}'}(u, h_{\mathcal{V}'}(u)) > 0$. Moreover, it coincides with $g_{\mathcal{V}}$ at all the points $\mathcal{V}'(t_j)$, $j = 1, \dots, m$.

Note that the model $\mathcal{V}' \models T'$ may always be constructed from the given model $\mathcal{V} \models T$. Indeed, we put $V' = V$ and specify the function $h_{\mathcal{V}'}$ as follows:

$$h_{\mathcal{V}'}(u) = v \quad \text{implies} \quad MA_{\mathcal{V}}(u, v) > 0, \tag{40}$$

$$h_{\mathcal{V}'}(u_j) = g_{\mathcal{V}}(u_j) \quad \text{for all} \quad u_j = \mathcal{V}(t_j), \quad j = 1, \dots, m. \tag{41}$$

Finally, we set

$$MA_{\mathcal{V}'}(u, v) = 1$$

for all u, v such that $h_{\mathcal{V}'}(u) = v$ and $MA_{\mathcal{V}'}(u, v) = MA_{\mathcal{V}}(u, v)$ otherwise. From it and the completeness theorem follows that T_c is consistent relatively to T.

As a consequence, each value $h_{\mathcal{V}'}(u)$ can be considered as a result of the defuzzification $h_{\mathcal{V}'}(u) = \Theta(A_u)$ of the fuzzy set

$$A_u = \left\{ {}^{MA_{\mathcal{V}}(u,v)}\!/v \;\middle|\; v \in V \right\}.$$

In other words, any function $h_{\mathcal{V}'}$ "passing through" the support of $MA_{\mathcal{V}}$ and all the points $g_{\mathcal{V}}(u_j)$, $j = 1, \dots, m$ is a "good" function, which approximates $g_{\mathcal{V}}$ with the accuracy $\varepsilon = \neg c$ and which has been derived on the basis of the Mamdani-Assilian formula MA using the defuzzification. The question, which of many possible defuzzification functions Θ is the best one has been solved by I. Perfilieva in [10].

7 Conclusion

In this paper, we have shown that the theory of fuzzy approximation of relations and functions can be elaborated inside formal theory of fuzzy logic with evaluated syntax. It is thus demonstrated that fuzzy logic is not only a generalization of classical logic, but that it departs from the latter by solving specific problems, which have no sense (or are trivial) in classical one.

The main concept applied in our reasoning was that of fuzzy equality. It is known that this fuzzy equality leads to the pseudometrics in the models. Then all functions and open formulas of fuzzy theories with fuzzy equality are uniformly continuous with respect to it. We have also introduced the concept of totally bounded fuzzy equality, which corresponds with the totally bounded pseudometrics. This enabled us to prove theorem on the approximation of any open formula by a disjunctive normal form.

We have also studied approximation of a functions by the Mamdani-Assilian formula, which can be obtained on the basis of the totally bounded fuzzy equality. On the syntax level we have proved that in this case, every function can be approximated by such a formula and moreover, we can find a function approximating the given function with the prescribed accuracy.

Our results indicate that there is a deeper relation of fuzzy logic to topology, which seems to be worth of study. It can also be expected that similar results concerning the disjunctive normal form can be proved also for the conjunctive one.

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