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NORMAL FORMS FOR FUZZY LOGIC FUNCTIONS AND THEIR APPROXIMATION ABILITY[†]

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Abstract

Aiming at developing theoretical backgrounds of fuzzy logic, we consider an MV-algebra of fuzzy logic functions. Two kinds of formulas regarded as disjunctive and conjunctive normal forms are introduced as a formal translation of IF-THEN rules. Their ability for approximate representation of uniformly continuous functions investigated. The problem of transformation of any formula of propositional fuzzy logic into either of the normal forms is discussed.

Keywords: MV-algebra, fuzzy logic (FL-)function, disjunctive and conjunctive normal forms, IF-THEN rules, universal approximation

1 Introduction

The problem of explicit representation of objects by expressions in some formal language is one of the oldest in mathematics. Representations by polynomials, Taylor or Fourier series, or by boolean normal forms are among good examples of this. In general, the problem consists in finding representation of objects from some class using primitive objects from the same class and operations taken from a finite set. For example, a class of polynomials can be represented using power functions, constants as primitives, and operations of addition and multiplication.

We would like to stress that precise and approximate representation of objects should be distinguished. Classical mathematics is mainly interested in the former, while applied mathematics is interested in both. Essential for the latter is to find finite representation. To the great surprise of mathematicians, fuzzy logic suggested its contribution to this field of research. It came with a striking idea of approximate representation of objects by means of qualitative characterisation of their essential parameters. For example, the characterization "green" approximately describes a class of green objects in some universe (books, pencils, eyes, etc.). Within a logical language, this characterization is formalized using the, so called, fuzzy predicates. With a reference to the representation problem, qualitative characterizations of a domain or/and a range of variables together with logical connectives joining them give a satisfactory approximate representation of a continuous functions.

Strictly speaking, the mentioned approximate representation of functions has been obtained using linguistic prototypes of logical formulas — the so called IF-THEN rules. The machinery of approximation including description by IF-THEN rules, their logical formalization and the related computation is used to be called a *universal approximation* (see, e.g. B. Kosko [**?**]).

Once discovered, the universal approximation property for different choices of logical operations (see [**?**, **?**, **?**]) has been proved by several authors. Systematization, and also extension of the class of logical formulas, which can be used as formalizations of IF-THEN rules and thus, as approximation forms, has been suggested by the author in [**?**, **?**]. Moreover, two different forms of logical formulas which came as a direct formal translation of IF-THEN rules have been introduced there for approximate representation of functions. Thorough analysys of them led to the conclusion (see [**?**]) that they can be regarded as generalizations of boolean normal forms in fuzzy predicate logic.

As mentioned, different proofs of the universal approximation property differ one from another by a different choice of logical operations. In this paper we consider a fixed algebra of operations, namely an MV-algebra. We pursued two purposes: first, to preserve the previous history of constructing normal

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forms in algebraical systems connected with logic, and second, to use already known algebraic identities. The latter gave us a nice opportunity to introduce two equivalent conjunctive normal forms.

The paper has the following structure. In Section 2.1, an MV-algebra of fuzzy logic (FL-)functions is considered. Two special formulas over MV-algebra of FL-functions, which we call (in analogy with the boolean case) *perfect disjunctive* and *conjunctive normal forms* are introduced in Section 2.2. The approximation of uniformly continuous FL-functions by functions represented by any of the generalized normal forms is proved. Other, non-perfect normal forms are introduced in Section 2.3. It has been shown that in some cases they give less complex representations of the same FL-function than the perfect ones. Finally, perfect normal forms that can be represented by propositional formulas are introduced in Section 2.4 for Łukasiewicz propositional calculus. The problem of transformation of any formula of propositional fuzzy logic into one of the normal forms is discussed.

2 Normal forms for fuzzy logic functions

The role of disjunctive and conjunctive normal forms in classical logic and in boolean algebra of functions is well known. Both forms are used for the standard representation of elements. We would like to stress here, that though being the same, they are constructed in completely different ways. In logic, normal forms are obtained as a result of equivalent transformations of formulas. In the algebra of boolean functions, the parameters constituting normal forms are computed using tabular representation of a function. Based on the fact that any logical (propositional) formula represents a boolean function and vice versa, certain confusion in the choice of the way of construction of normal forms may arise. Usually, this happens when a function represented by a formula is given. To avoid such misunderstanding, we proclaim here that our interest is in the *computation aspect* of the problem. Partial solution of the problem of equivalent transformation of a logical formula into one of the normal forms is briefly discussed in Section 2.4 (it can also be found in [**?**, **?**]).

In this section, we will consider a normal representation of the so called fuzzy logic functions, i.e. functions defined and taking values from a set of truth values of the underlying logic. By the normal representation of a fuzzy logic function we will mean a generalization of the boolean one in the algebraic (non-logical!) sense. Thus, normal forms will be introduced here using algebraic formulas with the syntactical structures analogous to the corresponding logical ones. The distinction is confined to the use of primitive logic functions instead of propositional variables used in the classical case. Of course, the generalized normal forms reduce to the classical ones in the case of two-element set of truth values.

It is worth noticing that the expressions for normal forms suggested here came from a formal translation of IF–THEN rules to the logical language. The first translation leading to a disjunctive normal form reflects the situation when both premise (IF part) and conclusion (THEN part) of each rule are supposed to be valid. In this case, the connective joining different rules is given by disjunction. The second translation, which we call a conjunctive normal form, join premise and conlusion of each rule using implication, while different rules are joined by conjunction. This translation as we will see later, is equivalent to the conjunction of elementary disjunctions in an MV-algebra of FL-functions. Thus, we will present two equivalent expressions for the conjunctive normal form.

Two questions will be considered below: *first*, which fuzzy logic functions can be represented by normal forms and *second*, how the normal forms can be constructed from the given fuzzy logic function. Solution of the first question in the special case when the set of truth values is $[0,1]$ and the choice of logical operation can be found in [**?**]. The authors applied the Kolmogorov's theorem concerning representation of continuous functions and thus, their result cannot be used in practice. This is because some parts of the proof of Kolmogorov's theorem can hardly be realized.

On the other hand, the normal forms suggested in this paper aim at approximate representation of functions and can be constructively obtained. Moreover, in the choice of logical operations we will respect already known (see e.g. [**?**, **?**]) algebras of logical operations, such as residuated lattices, or MV-algebras. This gives us the advantage to apply identities valid in those algebras and thus, to obtain some equivalent representations of formulas. In particular, we will present two equivalent formulas for the conjunctive normal form. The first one precisely generalizes the corresponding boolean form while the second one can be regarded as logical formalization of the set of IF-THEN rules.

2.1 MV-algebras

Since the most important structure considered in this paper is that of MV-algebra, we will briefly overview some of the main properties of it. The notion of MV-algebra ('MV' means "many-valued") was introduced by C. C. Chang in [?] for an algebraic system, which generalizes Boolean algebra from one side and corresponds to the $\aleph_0$-valued propositional calculus from the other side. It turned out that MV-algebras stand for the algebraic structures of truth values for various non-classical logical calculi including fuzzy logics.

The following definition of an MV-algebra given by C. C. Chang, is transparent though not the most economic one.

Definition 1
An MV-algebra is an algebra

$$\mathcal{L} = \langle L, \oplus, \otimes, \neg, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

where basic operations of $\oplus, \otimes$ *stand for addition and multiplication respectively and* $\neg$ *stands for logical negation. The following identities characterise these operations:*

$$a \oplus b = b \oplus a, \qquad a \otimes b = b \otimes a, \tag{2}$$
$$a \oplus (b \oplus c) = (a \oplus b) \oplus c, \qquad a \otimes (b \otimes c) = (a \otimes b) \otimes c, \tag{3}$$
$$a \oplus \mathbf{0} = a, \qquad a \otimes \mathbf{1} = a, \tag{4}$$
$$a \oplus \mathbf{1} = \mathbf{1}, \qquad a \otimes \mathbf{0} = \mathbf{0}, \tag{5}$$
$$a \oplus \neg a = \mathbf{1}, \qquad a \otimes \neg a = \mathbf{0}, \tag{6}$$
$$\neg(a \oplus b) = \neg a \otimes \neg b, \qquad \neg(a \otimes b) = \neg a \oplus \neg b, \tag{7}$$
$$a = \neg\neg a, \qquad \neg\mathbf{0} = \mathbf{1}, \tag{8}$$
$$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a. \tag{9}$$

Some of these identities have obvious logical meaning. Note that the left identity of (6) is the law of the excluded middle, identities (7) are the de Morgan law for the operations $\otimes$ and $\oplus$ and the left identity of (8) is the law of double negation.

The additinal operations of $\vee$ (sup), $\wedge$ (inf) and $\to$ (implication) can be introduced using formulas.

$$a \vee b = \neg(\neg a \oplus b) \oplus b = (a \otimes \neg b) \oplus b, \tag{10}$$
$$a \wedge b = \neg(\neg a \vee \neg b) = (a \oplus \neg b) \otimes b, \tag{11}$$
$$a \to b = \neg a \oplus b. \tag{12}$$

It can be shown that every MV-algebra is a distributive lattice w.r.t. $\vee$ and $\wedge$. Therefore, concerning the similarity between MV- and boolean algebras, we can say that the full list of identities characterizing boolean disjunction and conjunction is shared by couples $(\oplus, \vee)$ and $(\otimes, \wedge)$, so that each couple may be regarded as a generalization of the corresponding boolean operations.

The following are the examples of MV-algebras.

Example 1 (Łukasiewicz algebra)
Lukasiewicz algebra $\mathcal{L}_L$ *can be formally written as*

$$\mathcal{L}_L = \langle [0, 1], \oplus, \otimes, \neg, 0, 1 \rangle \tag{13}$$

where basic operations are defined by

$$a \oplus b = \min(1, a + b) \qquad a \otimes b = \max(0, a + b - 1), \tag{14}$$
$$\neg a = 1 - a, \tag{15}$$

and so are the additional operations

$$a \vee b = \max(a, b) \qquad a \wedge b = \min(a, b), \tag{16}$$
$$a \to b = \min(1, 1 - a + b). \tag{17}$$

□

Example 2 (MV-algebra of fuzzy logic (FL-)functions)
Let L be an MV-algebra and P_L be the set of functions which are defined and take values in L. Then the following algebra

$$\mathcal{P}_L = \langle P_L, \oplus, \otimes, \neg, \mathbf{1}_L, \mathbf{0}_L \rangle$$

where each operation over functions is defined pointwise accordingly to (14), (15) and constants are the respective constant functions, is an induced MV-algebra. The additional operations of $\vee$, $\wedge$ and $\rightarrow$ are defined in accordance with (16), (17). □

Given L, the elements of P_L will be called *fuzzy logic functions* or simply, *FL-functions*. The algebra $\mathcal{P}_L$ will be called *MV-algebra of FL-functions*. Furthermore, in order to simplify the notation, we will not distinguish between constant functions and constants.

2.2 Perfect normal forms in MV-algebra of FL-functions

Let us fix some MV-algebra $\mathcal{L}$ with the support L and consider the set P_L of FL-functions. Let F be a fixed maximal filter of $\mathcal{L}$ and $\cong_F$ be the corresponding congruence relation on $\mathcal{L}$. Moreover, let $\mathcal{L}|F$ be the corresponding factor algebra with elements denoted by $[v]$, so that $[v] = \{u \in L \mid u \cong_F v\}$ and thus, $[v]$ is a class of elements from L which are equivalent to v w.r.t. $\cong_F$. According to [**?**], $\mathcal{L}|F$ is locally finite and linearly ordered. The local finiteness means that for each element $v \in L$, $v \notin [\mathbf{0}]$, there exists an integer p_v so that $[p_v v] = [\mathbf{1}]$. (Here and in what follows we use the abbreviation kv where k is any positive integer, for the expression $(v \oplus \cdots \oplus v)$, k-times.) If p_v is the least integer with that property then p_v is the *order* of v, $p_v = \mathrm{ord}(v)$. The linear ordering gives $[\mathbf{0}] < [v] < [2v] \ldots < [p_v v]$.

For each element $v \in L$, $v \notin [\mathbf{0}]$, and each integer $0 \le k \le p_v$ where $p_v = \mathrm{ord}(v)$, the following FL-functions $I_k^v(x)$, which will be called *interval functions*, can be introduced:

$$I_k^v(x) \in [\mathbf{1}], \qquad \text{if } [kv] \le [x] < [(k+1)v], \tag{18}$$
$$I_k^v(x) = \mathbf{0}, \qquad \text{otherwise} \tag{19}$$

where $0 \le k \le p_v - 1$ and moreover,

$$I_{p_v}^v(x) \in [\mathbf{1}], \qquad \text{if } x \in [\mathbf{1}], \tag{20}$$
$$I_k^v(x) = \mathbf{0}, \qquad \text{otherwise.} \tag{21}$$

Relations (18) and (20) mean that an arbitrary (but one!) element from $[\mathbf{1}]$ corresponds to the value of x. Clearly, for all $x \in L$

$$\bigvee_{k=0}^{p_v} I_k^v(x) \in [\mathbf{1}].$$

Each function I_k^v, $0 \le k \le p_v - 1$, can be regarded as a generalized characteristic function of the subset $\{x \mid [kv] \le [x] < [(k+1)v]\}$.

Definition 2
Let $f(x_1, \ldots, x_n)$ be an FL-function and v be an element from L such that $v \notin [\mathbf{0}]$ and $p_v = \mathrm{ord}(v)$. The formula over $\mathcal{P}_L$

$$\mathrm{PDNF}(x_1, \ldots, x_n) = \bigvee_{k_1=0}^{p_v} \cdots \bigvee_{k_n=0}^{p_v} (I_{k_1}^v(x_1) \otimes \cdots \otimes I_{k_n}^v(x_n) \otimes f(k_1 v, \ldots, k_n v)) \tag{22}$$

is called the perfect MV-disjunctive normal form *for $f(x_1, \ldots, x_n)$ and*

$$\mathrm{PCNF}(x_1, \ldots, x_n) = \bigwedge_{k_1=0}^{p_v} \cdots \bigwedge_{k_n=0}^{p_v} (\neg I_{k_1}^v(x_1) \oplus \cdots \oplus \neg I_{k_n}^v(x_n) \oplus f(k_1 v, \ldots, k_n v)) \tag{23}$$

is called the perfect MV-conjunctive normal form *for* $f(x_1, \dots, x_n)$.

We will call the disjunctive component $I^v_{k_1}(x_1) \otimes \cdots \otimes I^v_{k_n}(x_n) \otimes f(k_1 v, \dots, k_n v)$ *the* elementary conjunction *and the conjunctive component* $\neg I^v_{k_1}(x_1) \oplus \cdots \oplus \neg I^v_{k_n}(x_n) \oplus f(k_1 v, \dots, k_n v)$ *the* elementary disjunction.

Note, that the expression $f(k_1 v, \dots, k_n v)$ occurring in PDNF and PCNF denote the corresponding constant. It is not difficult to see that for $L = \{0, 1\}$, expressions (22) and (23) transform into boolean perfect normal forms. Moreover, for $L = \{0, 1, \dots, k-1\}$ the expression (22) transforms into the so called Rosser-Turquette formula, which is a generalization of the boolean perfect disjunctive normal form to the algebra of k-valued logical functions, $k > 2$. To see it, let us note that the difference between (22), (23) and the corresponding boolean normal forms consists in the replacement of characteristic functions of single points by characteristic functions I^v_k of intervals $[kv] \leq [x] < [(k+1)v]$ (see Appendix for the precise expressions).

Let us apply the equality $\neg a \oplus b = a \to b$ (see (12)) to the expression (23) of PCNF and thus derive the following equivalent form

$$\mathrm{PCNF}(x_1, \dots, x_n) = \bigwedge_{k_1=0}^{p_v} \cdots \bigwedge_{k_n=0}^{p_v} ((I^v_{k_1}(x_1) \otimes \cdots \otimes I^v_{k_n}(x_n)) \to f(k_1 v, \dots, k_n v)). \tag{24}$$

Remark 1

In order to respect the logical origin of the introduced normal forms, we will distinguish formulas and functions represented by them. The above introduced expressions $\mathrm{PDNF}(x_1, \dots, x_n)$ *and* $\mathrm{PCNF}(x_1, \dots, x_n)$ *are used to denote the corresponding formulas over* $\mathcal{P}_L$. *When dealing with FL-functions represented by* PDNF *and* PCNF, *we will use the notation* $f_{\mathrm{PDNF}}(x_1, \dots, x_n)$ *and* $f_{\mathrm{PCNF}}(x_1, \dots, x_n)$, *respectively where* f *refers to the original FL-function. The dependence on the parameter* v *will be indicated separately.*

Three facts have to be noticed when comparing the normal representations of functions in boolean and MV-algebras.

- Contrary to the boolean case, the FL-functions $f_{\mathrm{PDNF}}(x_1, \dots, x_n)$ and $f_{\mathrm{PCNF}}(x_1, \dots, x_n)$ are not generally equal to their origin $f(x_1, \dots, x_n)$.
- The expressions for $f_{\mathrm{PDNF}}(x_1, \dots, x_n)$ and $f_{\mathrm{PCNF}}(x_1, \dots, x_n)$ include symbols I^v_k for elementary functions depending on one variable, while boolean normal forms include only the variables. This is because the MV-algebra of FL-functions (not logical formulas) has been chosen. In both cases normal representatons show the way of decomposition of a complex function into elementary parts.
- The expressions for the perfect normal forms in an MV-algebra of FL-functions contain the parameter v, which, as will be shown below, indicates a proximity between the original FL-function and that represented by a certain normal form.

Example 3

Let the underlying MV-algebra $\mathcal{L}$ *be the Lukasiewicz algebra* $\mathcal{L}_L$. *The induced MV-algebra of functions* $\mathcal{P}_{[0,1]}$ *is then based on the set* $P_{[0,1]} = \{f(x_1, \dots, x_n) \mid f : [0,1]^n \longrightarrow [0,1], \quad n \geq 0\}$. *In this example, we will show how the perfect normal forms* PDNF *and* PCNF *look and moreover, how the functions represented by them can be characterized. For simplicity, let us consider the case when the functions depend on one variable only.*

First, we will take into account the peculiarity of $\mathcal{L}_L$, *namely that the support* $[0,1]$ *is a linearly ordered set and that its only ultrafilter coincides with* $\{1\}$. *The consequence is that for each element* $a \in [0,1]$, $[a] = \{a\}$ *holds true and thus, the parameter* v *can be chosen as any one which differs from* 0. *Let* $v = \frac{1}{m}$ *where* m *is a positive integer such that* $m = \mathrm{ord}(v)$. *In this case, the interval functions can be expressed as follows:*

$$I^m_k(x) = \begin{cases} 1, & \text{if } \frac{k}{m} \leq x < \frac{(k+1)}{m}, \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

where $0 \leq k \leq m-1$, and

$$I_m^m(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

If the normal forms include one variable then their representation is a simple reduction of (22)–(23) such that

$$\mathrm{PDNF}(x) = \bigvee_{k=0}^{m} \left(I_k^m(x) \otimes f\left(\frac{k}{m}\right)\right), \tag{27}$$

$$\mathrm{PCNF}(x) = \bigwedge_{k=0}^{m} \left(\neg I_k^m(x) \oplus f\left(\frac{k}{m}\right)\right). \tag{28}$$

Both these forms represent the same function

$$f_{\mathrm{PDNF}}(x) = f_{\mathrm{PCNF}}(x) = \begin{cases} f(\frac{k}{m}), & \text{if } \frac{k}{m} \leq x < \frac{k+1}{m}, \quad 0 \leq k \leq m-1, \\ f(1), & \text{if } x = 1. \end{cases} \tag{29}$$

□

Remark 2
*As can be seen, a quotient MV-algebra $\mathcal{L}|F$ can be homomorphically embedded to Lukasiewicz algebra (Example 1) by the natural homomorphism with kernel $\cong_F$ (this fact can be deduced already from C. C. Chang's results — cf. [**?**, **?**]). It follows that the normal forms from Example 3 can be considered as typical representatives of the general ones.*

In the case of a general MV-algebra, the relation between perfect PDNF and PCNF is established in the following lemma.

Lemma 1
Let the perfect normal forms $\mathrm{PDNF}(x_1,\dots,x_n)$ *and* $\mathrm{PCNF}(x_1,\dots,x_n)$, $n \geq 1$, *be constructed for the same FL-function* $f(x_1,\dots,x_n)$ *and use the same parameter* v *of the order* p_v. *Then for any values* $x_1,\dots,x_n \in L$ $f_{\mathrm{PDNF}}(x_1,\dots,x_n) \cong_F f_{\mathrm{PCNF}}(x_1,\dots,x_n)$.

Proof: It is sufficient to restrict ourselves to the case when $n = 1$. Suppose that $\mathrm{PDNF}(x)$ and $\mathrm{PCNF}(x)$ are constructed for some common FL-function $f(x)$ and are based on $v \in L$ so that $v \notin [\mathbf{0}]$ and $p_v = \mathrm{ord}(v)$. Choose an arbitrary $x \in L$ and verify that $f_{\mathrm{PDNF}}(x) \cong_F f_{\mathrm{PCNF}}(x)$. If $x \notin [\mathbf{1}]$ then find a unique integer k, $0 \leq k \leq p_v - 1$, such that $[kv] \leq [x] < [(k+1)v]$, otherwise let $k = p_v$. For the chosen k, the corresponding value $I_k^v(x)$ belongs to $\mathbf{1}$ whereas values of other interval functions are equal to $\mathbf{0}$. This implies that $I_k^v(x) \otimes f(kv) \in [f(kv)]$ and $\neg I_k^v(x) \oplus f(kv) \in [f(kv)]$ while the other elementary conjunctions are equal to $\mathbf{0}$ and elementary disjunctions are equal to $\mathbf{1}$. Hence, $f_{\mathrm{PDNF}}(x) \in [f(kv)]$ and $f_{\mathrm{PCNF}}(x) \in [f(kv)]$, which imply $f_{\mathrm{PDNF}}(x) \cong_F f_{\mathrm{PCNF}}(x)$. □

It has already been mentioned that in general, the perfect normal form (disjunctive or conjunctive) for the function $f(x_1,\dots,x_n)$ does not represent the same function as f. We can only prove that there is an approximation effect. To be correct, we will use the notion which plays the role of a distance in MV-algebra. According to the definition of C. C. Chang in [**?**], the FL-function

$$d(x,y) = (\neg x \otimes y) \oplus (\neg y \otimes x) \tag{30}$$

can be taken as a distance. The following properties justify well this choice (see [**?**]).

Lemma 2
The following holds true for every $x, y, z \in L$.

(a) $d(x,x) = \mathbf{0}$,

(b) if $d(x,y) = \mathbf{0}$, *then* $x = y$,

(c) $d(x, y) = d(y, x)$,

(d) $d(x, z) \leq d(x, y) \oplus d(y, z)$.

Definition 3
An FL-function $f(x_1, \ldots, x_n)$ is uniformly continuous on L if for any $u \in L$, $[\mathbf{0}] < [u] < [\mathbf{1}]$, there exists $v \in L$, $[v] > [\mathbf{0}]$, such that $[d(x_1, y_1)] < [v], \ldots, [d(x_n, y_n)] < [v]$ implies $[d(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))] < [u]$.

Note, that in the case of $L = [0, 1]$, the distance $d(x, y) = |x - y|$ and thus, the notion of uniform continuity coincides with the classical one for real valued real functions.

Definition 4
Let $u \in L$ and $[u] < [\mathbf{1}]$. We say that an FL-function $g(x_1, \ldots, x_n)$ u-approximates the FL-function $f(x_1, \ldots, x_n)$, or in other words $g(x_1, \ldots, x_n)$ approximates $f(x_1, \ldots, x_n)$ with the accuracy u, if the inequality

$$[d(g(x_1, \ldots, x_n), f(x_1, \ldots, x_n))] \leq [u]$$

holds for all $(x_1, \ldots, x_n) \in L^n$.

Theorem 1
Let an FL-function $f(x_1, \ldots, x_n)$ be uniformly continuous on L. Then for any $u \in L$, $[\mathbf{0}] < [u] < [\mathbf{1}]$, there exists $v \in L$, $[v] > [\mathbf{0}]$, such that the function $f_{\mathrm{PDNF}}(x_1, \ldots, x_n)$ based on the parameter v u-approximates $f(x_1, \ldots, x_n)$.

PROOF: For simplicity, let us consider the case of one variable. Fix an element $u \in L$, $[\mathbf{0}] < [u] < [\mathbf{1}]$, and, taking into account the uniform continuity of $f(x)$, find an element $v \in L$, $[v] > [\mathbf{0}]$, such that $[d(x, y)] < [v]$ implies $[d(f(x), f(y))] < [u]$. On the basis of v let us construct the perfect normal form

$$\mathrm{PDNF}(x) = \bigvee_{k=0}^{p_v} (I_k^v(x) \otimes f(kv))$$

where $p_v = \mathrm{ord}(v)$. We will show that $f_{\mathrm{PDNF}}(x)$ is the function we are looking for.

Take an arbitrary element $x \in L$ and find an integer k, $0 \leq k < p_v$, such that either $[kv] \leq [x] < [(k+1)v]$, or $k = p_v$. It is true that $[d(kv, x)] < [v]$. Now evaluate the desired distance and see that

$$[d(f_{\mathrm{PDNF}}(x), f(x))] = [d(f(kv), f(x))] < [u].$$

□

On the basis of Lemma 1, the following theorem can be established.

Theorem 2
Let an FL-function $f(x_1, \ldots, x_n)$ be uniformly continuous on L. Then for any $u \in L$, $[\mathbf{0}] < [u] < [\mathbf{1}]$, there exists $v \in L$, $[v] > [\mathbf{0}]$, such that the function $f_{\mathrm{PCNF}}(x_1, \ldots, x_n)$ based on the parameter v u-approximates $f(x_1, \ldots, x_n)$.

Example 4
Let us consider the algebra of functions $\mathcal{P}_{[0,1]}$. Take the function $f(x) = x^2$ restricted to $[0, 1]$ and find its approximation using the functions $f_{\mathrm{PDNF}}(x)$ or $f_{\mathrm{PCNF}}(x)$, which in this case are equal (cf. (29)). Moreover, the concepts of continuity, approximation and its accuracy are the classical ones.

Let us set the accuracy of approximation $\varepsilon = .2$ and see that the integer $m = 10$ fulfils the requirement $x^2 < \varepsilon$ whenever $\frac{k}{m} \leq x < \frac{k+1}{m}$. The following formula has the form of PDNF *(see (27))and represent the function $f_{\mathrm{PDNF}}(x)$ which approximates x^2 with the accuracy ε.*

$$\mathrm{PDNF}(x) = \bigvee_{k=0}^{10} \left(I_k^{10}(x) \otimes f\left(\frac{k}{10}\right)\right) =$$
$$= I_0^{10}(x) \otimes 0 \vee I_1^{10}(x) \otimes 0.01 \vee \ldots I_9^{10}(x) \otimes 0.81 \vee I_{10}^{10}(x) \otimes 1$$

and

$$f_{\mathrm{PDNF}}(x) = \begin{cases} (\frac{k}{10})^2, & \textit{if } \frac{k}{10} \le x < \frac{k+1}{10}, \quad 0 \le k \le 9, \\ 1, & \textit{if } x = 1. \end{cases}$$

□

2.3 Other normal forms in MV-algebra of FL-functions

As can be seen from Example 4, the approximation of a continuous function by a function represented in one of perfect normal forms can be simplified in the sense of reduction of the number of elementary conjunctions or disjunctions. It can be done by decreasing of the number of interval functions and "joining together" those ones which characterize intervals with a global difference between maximal and minimal values of a given function is not greater than a chosen accuracy. Doing this, we will come to the notions of (non-perfect) disjunctive ans conjunctive normal forms.

Preserving the system of notation of the previous subsection, let $\mathcal{L}$ be some MV-algebra with the support L. For each element $v \in L$, $v \notin [\mathbf{0}]$, integer $0 \le k_1 \le p_v - 1$ where $p_v = \mathrm{ord}(v)$ and integer $1 \le k_2 \le p_v - k_1 + 1$, we introduce FL-functions $I^v_{k_1,k_2}(x)$ which will be called *integrated interval functions*

$$\begin{aligned} I^v_{k_1,k_2}(x) &\in [\mathbf{1}], && \text{if } [k_1 v] \le [x] < [(k_1 + k_2 - 1)v], \text{ or} \\ & && [k_1 v] \le [x] \le [1]; \\ I^v_{k_1,k_2}(x) &= \mathbf{0}, && \text{otherwise.} \end{aligned}$$

Definition 5
Let $f(x_1, \dots, x_n)$ be an FL-function and v be an element from L such that $v \notin [\mathbf{0}]$ and $p_v = \mathrm{ord}(v)$. As above, the formula

$$I^v_{k_{11},k_{12}}(x_1) \otimes \cdots \otimes I^v_{k_{n1},k_{n2}}(x_n) \otimes f(k_{11}v, \dots k_{n1}v)$$

over $\mathcal{P}_L$ is called the elementary conjunction, *and*

$$\neg I^v_{k_{11},k_{12}}(x_1) \oplus \cdots \oplus \neg I^v_{k_{n1},k_{n2}}(x_n) \oplus f(k_{11}v, \dots k_{n1}v)$$

the elementary disjunction, *both w.r.t. $f(x_1, \dots, x_n)$.*

A disjunction of elementary conjunctions is called an MV-disjunctive normal form *and a conjunction of elementary disjunctions is called an* MV-conjunctive normal form, *both for $f(x_1, \dots, x_n)$:*

$$\begin{aligned} \mathrm{DNF}(x_1, \dots, x_n) = {} & I^v_{0,k_{11}}(x_1) \otimes \cdots \otimes I^v_{0,k_{n1}}(x_n) \otimes f(0, \dots, 0) \vee \\ & I^v_{k_{11},k_{12}}(x_1) \otimes \cdots \otimes I^v_{k_{n1},k_{n2}}(x_n) \otimes f(k_{11}v, \dots, k_{n1}v) \vee \\ & I^v_{k_{11}+k_{12},k_{13}}(x_1) \otimes \cdots \otimes I^v_{k_{n1}+k_{n2},k_{n3}}(x_n) \otimes f((k_{11} + k_{12})v, \dots, (k_{n1} + k_{n2})v) \vee \cdots \end{aligned}$$

$$\begin{aligned} \mathrm{CNF}(x_1, \dots, x_n) = {} & (\neg I^v_{0,k_{11}}(x_1) \oplus \cdots \oplus \neg I^v_{0,k_{n1}}(x_n) \oplus f(0, \dots, 0)) \wedge \\ & (\neg I^v_{k_{11},k_{12}}(x_1) \oplus \cdots \oplus \neg I^v_{k_{n1},k_{n2}}(x_n) \oplus f(k_{11}v, \dots, k_{n1}v)) \wedge \\ & (\neg I^v_{k_{11}+k_{12},k_{13}}(x_1) \oplus \cdots \oplus \neg I^v_{k_{n1}+k_{n2},k_{n3}}(x_n) \oplus f((k_{11} + k_{12})v, \dots, (k_{n1} + k_{n2})v) \wedge \cdots \end{aligned}$$

Let us remark that

- the expressions $f(0, \dots, 0)$, $f(k_{11}v, \dots, k_{n1}v), \dots$ occurring in DNF and CNF denote the corresponding constants;
- the number (always finite!) of elementary conjunctions (disjunctions) in DNF (CNF) depends on the choice of parameters v, k_{ij} which at the same time indicate a proximity between the original FL-function and that represented by a certain normal form;

- another representation of CNF based on implication can be constructed similarly to (24).

Moreover, it is easy to see that analogues of Theorems 1,2 obtained when replacing PDNF and PCNF by DNF and CNF, respectively hold true. For demonstration we will choose the same function as in the Example 4 and construct its approximate representation by DNF. We will see that the number of elementary conjunction in this new representation can be reduced.

Example 5
As in Example 4, choose the function $f(x) = x^2$ restricted to $[0,1]$ and find its approximation using the function $f_{\mathrm{DNF}}(x)$. Remember that for the approximation accuracy $\varepsilon = .2$, the integer $m = 10$ fulfils the condition $x^2 < \varepsilon$ whenever $\frac{k}{m} \le x < \frac{k+1}{m}$. Put $v = \frac{1}{m}$ and find integers $k_0, k_1, \ldots$ such that $k_0 = 0$, and the inequality $x^2 < \varepsilon$ is valid whenever $\frac{k_i}{m} \le x < \frac{k_{i+1}}{m}$, $i \ge 0$. Moreover, none of $k_1, k_2, \ldots$ can be increased without harming the above inequality, at least on one interval. It easy to see that $k_1 = 4$, $k_2 = 5$, $k_3 = 6$, $k_4 = 7$, $k_5 = 8$, $k_6 = 9$, $k_7 = 10$.

The following formula has the form of DNF *and represents the function $f_{\mathrm{DNF}}(x)$, which approximates x^2 with the accuracy ε.*

$$\mathrm{DNF}(x) = I^{10}_{0,4}(x) \otimes 0 \vee I^{10}_{4,5}(x) \otimes .16 \vee I^{10}_{5,6}(x) \otimes 0.25 \vee \ldots I^{10}_{9,10}(x) \otimes 0.81$$

It is worth noting that with the same value of the parameter v, the number of elementary conjunctions in the above formula is equal to 7 *against* 11 *in* PDNF(x). *Thus, by the appropriate choice of parameters k_{ij} we can decrease the complexity of* DNF(x) *in comparison with* PDNF(x). □

2.4 Normal forms in MV-algebra of propositional formulas

The above introduced normal forms are special expressions in MV-algebra of FL-functions. We used them as uniform approximate description of continuous FL-functions. They can also be regarded as appropriate decomposition formulas in the sense that their structure can be easily looked over. However, we can also study normal forms as precise logical expressions, i.e. propositional formulas.

In this subsection, we will show how the above introduced perfect normal forms can be modified in order to obtain the desired formulas. Moreover, we will show that the new obtained normal forms preserve the ability to approximate arbitrary uniformly continuous function.

We must step aside from the general case of MV-algebras to the case of Łukasiewicz algebra (see Example 1) and the corresponding propositional calculus because the relationship between formulas and functions is known there. Namely, according to McNaughton theorem [**?**] (proved constructively by D. Mundici [**?**] and then extended by the author [**?**, **?**]) each formula of Łukasiewicz propositional calculus represents a piecewise linear function on $[0,1]$ with integer coefficients, and also conversely — each piecewise linear function on $[0,1]$ with integer coefficients can be represented by some formula of Łukasiewicz propositional calculus. Therefore, normal forms which we are looking for can be introduced as expressions in MV-algebra of piecewise linear functions with Łukasiewicz operations

$$\mathrm{PL}_{[0,1]} = \langle \mathrm{PL}_{[0,1]}, \oplus, \otimes, \neg, \mathbf{1}_{[0,1]}, \mathbf{0}_{[0,1]} \rangle$$

where $\mathrm{PL}_{[0,1]}$ denotes the set of all piecewise linear (continuous) functions with integer coefficients, whose domain and range is equal to $[0,1]$, and constants $\mathbf{1}_{[0,1]}, \mathbf{0}_{[0,1]}$ are the respective constant functions.

After the examination of perfect normal forms (22)–(24), we see that they must be modified in all parts containing symbols I^v_k, which stand for the non-continuous functions $I^v_k(x)$. For $L = [0,1]$, these functions have been denoted by $I^m_k(x)$ and defined by (25).

To be able to obtain normal forms over $\mathrm{PL}_{[0,1]}$, we need to approximate functions $I^m_k(x)$ by piecewise linear (continuous) ones. The resulting expressions will be substituted into PDNF and PCNF for the symbols $I^v_k(x)$.

Let $m > 0$ and $0 \le k \le m-1$ be integers. We will consider closed intervals $[\frac{k}{m}, \frac{k+1}{m}]$ and establish a correspondence between them and FL-functions $\hat{I}^m_k(x)$ on $[0,1]$ given by

$$\hat{I}^m_k(x) = \neg\left(-x + \frac{k}{m}\right)^* \wedge \neg\left(x - \frac{k+1}{m}\right)^*,$$

where the operations '$-$' and '$+$' are ordinary operations on reals and '$*$' is a truncation operation on the set of real numbers:

$$x^* = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{31}$$

Obviously, $\hat{I}_k^m(x) \in \mathrm{PL}_{[0,1]}$, and moreover, $\hat{I}_k^m(x) = 1$ iff $x \in [\frac{k}{m}, \frac{k+1}{m}]$, $0 \le k \le m-1$.

For each function $\hat{I}_k^m(x)$, $0 \le k \le m-1$, $m > 0$, and each integer $r \ge 1$ let us introduce the function

$$(\hat{I}_k^m(x))^{rm} = \left(\neg\left(-x + \frac{k}{m})^* \wedge \neg(x - \frac{k+1}{m}\right)^*\right)^{rm}. \tag{32}$$

Similarly, $(\hat{I}_k^m(x))^{rm} \in \mathrm{PL}_{[0,1]}$ and can be described by the below placed formula.

Lemma 3
Let function $(\hat{I}_k^m(x))^{rm}$ be given by (32) where $0 \le k \le m-1$, $m > 0$, and $r \ge 1$. Then $(\hat{I}_k^m(x))^{rm} \in \mathrm{PL}_{[0,1]}$ and its linear parts are given by the formula

$$(\hat{I}_k^m(x))^{rm} = \begin{cases} 1 & \text{iff } x \in [\frac{k}{m}, \frac{k+1}{m}], 0 \le k \le m-1, \\ (rm)x - kr + 1 & \text{iff } x \in (\frac{k}{m} - \frac{1}{rm}, \frac{k}{m}), 1 \le k \le m-1, \\ -(rm)x + (k+1)m + 1 & \text{iff } x \in (\frac{k+1}{m}, \frac{k+1}{m} + \frac{1}{rm}), 0 \le k \le m-2, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of this lemma is a technical exercise and is omitted.

Corollary 1
For each function $\hat{I}_k^m(x)$, $0 \le k \le m-1$, $m > 0$, and each integer $r \ge 1$, the function $(\hat{I}_k^m(x))^{rm}$ represented by (32) coincides with $I_k^m(x)$ for all $x \in [0,1]$ except for those belonging to the set

$$E_k^m = \begin{cases} \left(\frac{k}{m} - \frac{1}{rm}, \frac{k}{m}\right) \cup \left[\frac{k+1}{m}, \frac{k+1}{m} + \frac{1}{rm}\right) & \text{if } 1 \le k \le m-2, \\ \left[\frac{k+1}{m}, \frac{k+1}{m} + \frac{1}{rm}\right) & \text{if } k = 0, \\ \left(\frac{k}{m} - \frac{1}{rm}, \frac{k}{m}\right) & \text{if } k = m-1. \end{cases}$$

By virtue of Lemma 3 and Corollary 1, the function $(\hat{I}_k^m(x))^{rm}$ approximates $I_k^m(x)$ in the sense that they differ on the set E_k^m, which consists of one or two intervals of arbitrary small length. Of course, such kind of approximation is different from the u-approximation introduced earlier, but this sufficiently corresponds to our purposes.

We are now able to give new definitions of the perfect disjunctive and conjunctive normal forms in $\mathrm{PL}_{[0,1]}$ (MV-algebra of piecewise linear functions) for a given FL-function.

Definition 6
Let $L = [0,1]$ and $f(x_1, \ldots, x_n)$ be an FL-function. Moreover, let $m > 0, r \ge 2$ be integers. The following is the perfect MV-disjunctive normal form in $\mathrm{PL}_{[0,1]}$ for $f(x_1, \ldots, x_n)$

$$\mathrm{PDNF}(x_1, \ldots, x_n) = \bigvee_{k_1=0}^{m-1} \cdots \bigvee_{k_n=0}^{m-1} \Big((\hat{I}_{k_1}^m(x_1))^{rm} \otimes \cdots \otimes (\hat{I}_{k_n}^m(x_n))^{rm} \otimes \otimes f\left(\frac{k_1}{m}, \ldots, \frac{k_n}{m}\right)\Big), \tag{33}$$

and the perfect MV-conjunctive normal form in $\mathrm{PL}_{[0,1]}$ for $f(x_1, \ldots, x_n)$

$$\mathrm{PCNF}(x_1, \ldots, x_n) = \bigwedge_{k_1=0}^{m-1} \cdots \bigwedge_{k_n=0}^{m-1} \Big(\neg(\hat{I}_{k_1}^m(x_1))^{rm} \oplus \cdots \oplus \neg(\hat{I}_{k_n}^m(x_n))^{rm} \oplus \oplus f\left(\frac{k_1}{m}, \ldots, \frac{k_n}{m}\right)\Big). \tag{34}$$

Using the identity $\neg a \oplus b = a \to b$ we can again equivalently obtain

$$\mathrm{PCNF}(x_1,\dots,x_n) = \bigwedge_{k_1=0}^{m-1} \cdots \bigwedge_{k_n=0}^{m-1} \left((\hat{I}_{k_1}^m(x_1))^{rm} \otimes \cdots \otimes (\hat{I}_{k_n}^m(x_n))^{rm} \to \right.$$
$$\left. \to f\left(\frac{k_1}{m},\dots,\frac{k_n}{m}\right)\right). \quad (35)$$

It can be seen from the definition that both the perfect disjunctive as well as the perfect conjunctive normal forms depend on the parameters m and r. The first parameter m controls proximity between $f(x_1,\dots,x_n)$ and $f_{\mathrm{PDNF}}(x_1,\dots,x_n)$ (or $f_{\mathrm{PCNF}}(x_1,\dots,x_n)$), and the second parameter r controls local fitness between them.

Below we will formulate the approximation theorem which is similar to that proved in Section 2.1. For its proof we refer to [?] where a similar, but slightly less general result has been presented.

Theorem 3
Let $L = [0,1]$ and $f(x_1,\dots,x_n)$ be an FL-function. Then for any $0 < \varepsilon < 1$ there exists an integer $m > 0$ such that the function $f_{\mathrm{PDNF}}(x_1,\dots,x_n)$ ($f_{\mathrm{PCNF}}(x_1,\dots,x_n)$) based on the parameters m and $r = 2$ approximates the function $f(x_1,\dots,x_n)$ with the accuracy ε.

The following corollary concerns the problem of transformation of any formula of propositional fuzzy logic into one of the normal forms.

Corollary 2
For any formula of the propositional fuzzy logic and any ε, $0 < \varepsilon < 1$, there exists a formula over $\mathrm{PL}_{[0,1]}$ of the form of PDNF (PCNF) such that the functions represented by these formulas ε-approximate each other.

The proof of this corollary is based on Theorem 3 and the fact that each formula of propositional fuzzy logic represents a piecewise linear function from $\mathrm{PL}_{[0,1]}$. In other words, this corollary states that each formula of propositional fuzzy logic can be transformed into the form of PDNF (PCNF) by means of the function represented by the former. However, some precision is lost during this transformation.

3 Conclusion

An MV-algebra of fuzzy logic (FL-)functions is considered. Two special formulas over MV-algebra of FL-functions which (by analogy with the boolean case) we call perfect disjunctive and conjunctive normal forms are introduced. Their origin steming from a formal translation of IF-THEN rules is pointed out. The approximation of uniformly continuous FL-functions by functions represented by any of generalized normal forms is proved. Other, non-perfect normal forms are defined as well. It has been shown that in some cases they give less complex representations for the same FL-function than the perfect ones. The concrete examples of approximate representation of a uniformly continuous FL-function by a function represented by the disjunctive normal form (perfect and non-perfect) are given. Finally, perfect normal forms that can be represented by propositional formulas are introduced for Łukasiewicz propositional calculus. The problem of transformation of any formula of propositional fuzzy logic into one of the normal forms has been discussed.

Appendix

a) Normal forms for boolean functions

We recall the disjunctive and conjunctive normal forms for boolean functions and explain our way of generalization to FL-functions. Let $f(x_1,\dots,x_n)$, $n \geq 1$, be a boolean function, i.e. its domain and range is the set $\{0,1\}$. We denote

$$x^\sigma = \begin{cases} x' & \text{if } \sigma = 0, \\ x & \text{if } \sigma = 1. \end{cases}$$

It is easy to see that

$$x^\sigma = 1 \text{ iff } x = \sigma.$$

If $f(x_1, \dots, x_n) \not\equiv 0$ then it can be represented in the perfect disjunctive normal form

$$\begin{aligned} f(x_1, \dots, x_n) = & \bigvee_{f(\sigma_1, \dots, \sigma_n)=1} (x_1^{\sigma_1} \wedge \dots \wedge x_n^{\sigma_n}) = \\ = & \bigvee_{(\sigma_1, \dots, \sigma_n)} (x_1^{\sigma_1} \wedge \dots \wedge x_n^{\sigma_n} \wedge f(\sigma_1, \dots, \sigma_n)) \end{aligned}$$

and if $f(x_1, \dots, x_n) \not\equiv 1$, it can be represented in the perfect conjunctive normal form

$$\begin{aligned} f(x_1, \dots, x_n) = & \bigwedge_{f(\tau_1, \dots, \tau_n)=0} (\neg x_1^{\tau_1} \vee \dots \vee \neg x_n^{\tau_n}) = \\ = & \bigwedge_{(\tau_1, \dots, \tau_n)} (\neg x_1^{\tau_1} \vee \dots \vee \neg x_n^{\tau_n} \vee f(\tau_1, \dots, \tau_n)). \end{aligned}$$

It is easy to see that the second perfect conjunctive normal form based on implication is valid in boolean algebra as well.

Analyzing each elementary conjunction $x_1^{\sigma_1} \wedge \dots \wedge x_n^{\sigma_n} \wedge f(\sigma_1, \dots, \sigma_n)$ in the perfect disjunctive normal form we see that it contains two parts joined by conjunction, namely the characteristic function $x_1^{\sigma_1} \wedge \dots \wedge x_n^{\sigma_n}$ of the point $(\sigma_1, \dots \sigma_n)$ and the value of the represented function defined on it. Analogously, analyzing each elementary disjunction $\neg x_1^{\tau_1} \vee \dots \vee \neg x_n^{\tau_n} \vee f(\tau_1, \dots, \tau_n)$ in the perfect conjunctive normal form we see that it contains a negation of the characteristic function $\neg x_1^{\tau_1} \vee \dots \vee \neg x_n^{\tau_n}$ of the point $(\tau_1, \dots \tau_n)$, which is joined by disjunction with the value of the represented function.

Thus, in the generalization of the perfect disjunctive and conjunctive normal forms given above we preserved two parts in each elementary construction (conjunction or disjunction), which consists of the characterization $I_{k_1}^v(x_1) \otimes \dots \otimes I_{k_n}^v(x_n)$ (positive or negative respectively) of a certain set joined with the chosen value of a represented function $f(k_1 v, \dots, k_n v))$ on it.

b) Normal forms for k-valued logical functions

Let $L = \{0, 1, \dots, k-1\}$, $k > 2$, and $f(x_1, \dots, x_n)$, $n \geq 1$, be a k-valued logical function, i.e. its domain and range is the set L. We denote

$$I_\sigma(x) = \begin{cases} k-1 & \text{if } x = \sigma, \\ 0 & \text{oterwise.} \end{cases}$$

The following is known as Rosser-Turquette formula (see [?])

$$f(x_1, \dots, x_n) = \bigvee_{(\sigma_1, \dots, \sigma_n)} I_{\sigma_1}(x_1) \wedge \dots \wedge I_{\sigma_n}(x_n) \wedge f(\sigma_1, \dots, \sigma_n). \tag{36}$$

Taking into the consideration the explanation given above, it is easily seen that (??) is a generalization of a perfect disjunctive normal form in boolean algebra to the algebra of k-valued logical functions. Moreover, the way of generalization which we suggested for FL-functions looks similar to that of the Rosser-Turquette one.

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References

[1] J.J. Buckley, Sugeno type controllers are universal controllers, Fuzzy Sets and Syst., 53 (1993) 299–304.

[2] J.L. Castro, M. Delgado, Fuzzy systems with defuzzification are universal approximators, IEEE Trans.Syst., Man, Cybern., 26 (1996) 149–152.

[3] C.C. Chang, Algebraic analysis of many valued logics, Trans. AMS, 88 (1958) 467–490.

[4] C.C. Chang, A new proof of the completeness of the Łukasiewicz axioms, Trans. AMS, 93 (1959) 74–80.

[5] B. Kosko, Fuzzy systems as universal approximators, IEEE Int. Conf. on Fuzzy Systems. San Diego, CA (1992) 1143–1162.

[6] V. Kreinovich, H.T. Nguyen, D.A. Sprecher, Normal forms for fuzzy logic – an application of Kolmogorov's theorem, Int. Journ. of Uncert., Fuzz. and Knowl.-Based Syst. 4 (4) (1996) 331–349.

[7] R. McNaughton, A theorem about infinite-valued sentential logic, J. Symb. Log. 16 (1951) 1–13.

[8] D. Mundici, A constructive proof of McNaughton's theorem in infinite-valued logic, J. of Symbolic Logic 59 (1994) 596–602.

[9] D. Mundici, R. Cignoli, and I.M.L. D'Ottaviano, Algebraic Foundations of Many-valued Reasoning, Kluwer, Dordrecht, 2000.

[10] H.T. Nguyen, V. Kreinovich, O. Sirisaengtaksin, Fuzzy control as a universal control tool, Fuzzy Sets and Syst., 80 (1996).

[11] V. Novák, V., I. Perfilieva and J. Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer, Boston, 1999.

[12] I. Perfilieva, Fuzzy Logic Normal Forms for Control Law Representation. In: Verbruggen H., Zimmermann H.-J., Babuska R. (eds.) Fuzzy Algorithms for Control, Kluwer, Boston, (1999) 111-125.

[13] J.B. Rosser, A.R. Turquette, Many-Valued Logics, Amsterdam 1952.

[14] A.S. Tonis, I.G. Perfilieva, Functional System of Infinite-valued Propositional Calculus, Discretnij Analiz i Issled. Oper., Novosibirsk 7(2) (2000) 75–85 (in Russian).