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Research report No. 2

July 22, 1997

Submitted/to appear:

Int. J. of Approximate Reasoning 1997

Supported by:

Grant A1086501 of the GA AV ČR; (partially) COST Action 15

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ABSTRACT

The paper deals with fuzzy logic in narrow sense of Łukasiewicz style, i.e. a special kind of many-valued logic which is aimed at modeling of vagueness phenomenon. This logic is a generalization of Łukasiewicz logic with many interesting properties (for example, generalization of the deduction and completeness theorems hold in it).

Our aim is to prepare the background for the resolution in fuzzy logic. We prove an analogue of the classical Herbrand theorem as well as some other related theorems which are interesting also for the theory of approximate reasoning. The possible applications and consequences for the latter are also discussed.

1. Introduction

This paper is a contribution to the program which is development of fuzzy logic in narrow sense of Łukasiewicz style to obtain as many results generalizing the classical logic as possible. Various results have already been obtained demonstrating many striking parallels with the latter. Let us stress that this enables us to understand the logical calculus in greater depth, putting a different light also on classical logic.

Recall that the generalization of the classical completeness theorem holds in fuzzy logic which states, roughly speaking, that the provability and truth degrees of a formula in a fuzzy theory (a theory given by the fuzzy set of axioms) are equal. In this paper, we step further. Our goal is to prove also generalization of the famous Herbrand theorem. The generalization of the Hilbert–Ackermann consistency theorem, on the basis of which the Herbrand theorem follows, has been proved in [9]. The reader has certainly noticed that we follow the way of explanation provided by Shoenfield in [10]. Of course, we may think also about different way of proving these theorems. In all cases, they provide important step to the development of the resolution in fuzzy logic which has successfully been proposed for propositional logic in [4].

2. Preliminaries

We will recall only few basic notions of fuzzy logic in narrow sense. The reader may find the precise definitions and full proofs of theorems (if missing) in the cited papers.

The set of truth values forms a residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 1, 0 \rangle$$

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International Journal of Approximate Reasoning 1993 7:2–199

655 Avenue of the Americas, New York, NY 10010 0888-613X/93/$3.50

where L is either a *finite chain*, or $L = [0, 1]$, $\rightarrow$ is the Łukasiewicz implication and $\otimes$ is the Łukasiewicz product. We will use the operation of Łukasiewicz sum defined by

$$a \oplus b = \neg(\neg a \otimes \neg b) \qquad a, b \in L.$$

Furthermore, we introduce the following symbols:

$$a^n = \underbrace{a \otimes \cdots \otimes a}_{n-\text{times}}$$
$$na = \underbrace{a \oplus \cdots \oplus a}_{n-\text{times}}.$$

The *language* J of first-order fuzzy logic consists of variables, constants, n-ary functional and predicate symbols, binary predicate symbol $=$ (equality sign), symbols for truth values $\boldsymbol{a}$, binary connective $\Rightarrow$and general quantifier $\forall$.

Terms and formulas are defined as usual with the exception that all symbols for truth values are atomic formulas.

The common abbreviations of formulas $\neg A$ (negation), $A \vee B$ (disjunction), $A \wedge B$ (conjunction), $A \,\&\, B$ (Łukasiewicz conjunction), $A \Leftrightarrow B$ (equivalence), $(\exists x)A$, A^k are introduced (see [5, 6, 8]). Moreover, we will use also the abbreviation $A \nabla B$ defined by

$$A \nabla B := \neg(\neg A \,\&\, \neg B)$$

and call it Łukasiewicz disjunction.

As explained in these works, syntax of fuzzy logic is evaluated by *syntactic truth values.* An *evaluated (graded) formula* is a couple

$$[A;\, a]$$

where $A \in F_J$ and $a \in L$. The (syntactic) truth value a is the evaluation of the formula A in the syntax of fuzzy logic.

A *theory* T in the language J of first–order fuzzy logic in narrow sense (called a *fuzzy theory*) is a triple

$$T = \langle A_L, A_S, \{r_{MP}, r_G\}\rangle$$

where $A_L, A_S \underset{\sim}{\subset} F_J$ are *fuzzy sets* of logical and special axioms respectively and r_{MP}, r_G are many-valued inference rules of modus ponens and generalization. By $J(T)$ we denote the language of fuzzy theory. A *fuzzy predicate calculus* is the fuzzy theory with $A_S = \emptyset$. By F_J, we denote the set of all well formed formulas for the given language J of first order fuzzy logic.

If $(C_T^{sem})A = a$ then the formula A is *true* in degree a in theory T and we write

$$T \models_a A.$$

If $(C_T^{syn})A = a$ then A is a *theorem* in degree a of theory T and we write

$$T \vdash_a A.$$

We write $T \vdash A$, $T \models A$ instead of $T \vdash_1 A$, $T \models_1 A$, respectively and say that A is a theorem of (true in) theory T.

If w is a proof in theory T then we write $\mathrm{Val}_T(w)$ for its value. If A is a formula and w its proof then we will write w_A to stress this fact.

If T is a fuzzy theory and $\Gamma \underset{\sim}{\subset} F_{J(T)}$ a fuzzy set of formulas then $T' = T \cup \Gamma$ is a fuzzy theory whose special axioms are extended by formulas from Γ (as a union of fuzzy sets).

We say that a fuzzy theory T is *contradictory* (in the strong sense) iff there is a formula A and proofs w_A and $w_{\neg A}$ such that

$$\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) > 0. \tag{1}$$

The fuzzy theory T is *consistent* in the opposite case.

The proof of the following theorem can be found in [6].

Theorem 1. *A fuzzy theory T is contradictory iff $T \vdash A$ holds for every formula $A \in F_{J(T)}$.*

In [2] we have studied a weaker consistency condition (1) with $\wedge$ instead of $\otimes$. Surprizingly, even this weaker condition leads in many cases to strong result of Theorem 1.

In [9] we have introduced the equality predicate fulfilling the following (common) logical axioms:

(E1) $x = x$

(E2) $(x_1 = y_1) \Rightarrow \ldots \Rightarrow (x_n = y_n) \Rightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$

(E3) $(x_1 = y_1) \Rightarrow \ldots \Rightarrow (x_n = y_n) \Rightarrow (p(x_1, \ldots, x_n) = p(y_1, \ldots, y_n))$

for every n-ary functional symbol f and predicate symbol p. However, the axioms (E2) and (E3) seem to be too strong for various purposes. Therefore, we will replace them by the following weaker ones:

(E2') There are $m_1, \ldots, m_n$ such that

$$(x_1 = y_1)^{m_1} \Rightarrow \ldots \Rightarrow (x_n = y_n)^{m_n} \Rightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$$

(E3') There are $m_1, \ldots, m_n$ such that

$$(x_1 = y_1)^{m_1} \Rightarrow \ldots \Rightarrow (x_n = y_n)^{m_n} \Rightarrow (p(x_1, \ldots, x_n) = p(y_1, \ldots, y_n)$$

Special kind of fuzzy equality is the sharp one defined by

$$\mathcal{D}(t = s) = \begin{cases} 1 & \text{if } \mathcal{D}(t) = \mathcal{D}(s) \\ 0 & \text{otherwise} \end{cases}$$

in every model $\mathcal{D}$.

3. Some properties of fuzzy theories

The following two theorems have been proved in [9].

Theorem 1 (equality) *Let $T \vdash_{a_i} t_i = s_i$, $i = 1, \ldots, n$. Then there are $m_1, \ldots, m_n$ such that*

$$T \vdash_b A \Leftrightarrow A' \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$

where A' is a formula which is a result of replacing of terms t_i by terms s_i in A, respectively.

Theorem 2 (equivalence) *Let A be a formula and $B_1, \ldots, B_n$ some of its subformulas. Let $T \vdash_{a_i} B_i \Leftrightarrow B_i'$, $i = 1, \ldots, n$. Then there are $m_1, \ldots, m_n$ such that*

$$T \vdash_b A \Leftrightarrow A' \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$

where A' is a formula which is a result of replacing of the formulas $B_1, \ldots, B_n$ in A by $B_1', \ldots, B_n'$.

The quantifiers in fuzzy logic of Łukasiewicz style preserve all the properties of those in classical logic, namely

$$\vdash (\forall x)(A \Rightarrow B) \Leftrightarrow (A \Rightarrow (\forall x)B) \tag{2}$$

$$\vdash (\forall x)(A \Rightarrow B) \Leftrightarrow ((\exists x)A \Rightarrow B) \tag{3}$$

$$\vdash (\exists x)(A \Rightarrow B) \Leftrightarrow (A \Rightarrow (\exists x)B) \tag{4}$$

$$\vdash (\exists x)(A \Rightarrow B) \Leftrightarrow ((\forall x)A \Rightarrow B) \tag{5}$$

where x is not free in A in (2) and (4) and in B in (3) and (5). The formula (2) is a logical axiom, the others are provable in the degree 1. Therefore, we can introduce *prenex form* of a formula which has the same meaning as in classical logic with the exception that there are more kinds of connectives (which, however, are derivable from $\Rightarrow$).

Recall that variant of A' is a formula in which $(\forall x)B$ is replaced by the formula $(\forall y)B_x[y]$ where y is not free in B.

Lemma 1. *Let A be a formula and A' its variant. Then*

$$\vdash A \Leftrightarrow A'.$$

Proof This easily follows from the substitution axiom, axiom (2) and equivalence theorem. ■

Let $Q(x)$ denote a quantifier and $Q'(x)$ its opposite. The *prenex operations* are defined as follows:

(a) Replace A by its variant.

(b) Replace $\neg Q(x)A$ by $Q'(x)\neg A$.

(c) Replace $Q(x)A \vee B$ by $Q(x)(A \vee B)$ provided that x is not free in B.

(d) Replace $Q(x)A \bigtriangledown B$ by $Q(x)(A \bigtriangledown B)$ provided that x is not free in B.

(e) Replace $Q(x)A \wedge B$ by $Q(x)(A \wedge B)$ provided that x is not free in B.

(f) Replace $Q(x)A \,\&\, B$ by $Q(x)(A \,\&\, B)$ provided that x is not free in B.

(g) Replace $Q'(x)A \Rightarrow B$ by $Q(x)(A \Rightarrow B)$ provided that x is not free in B.

(h) Replace $A \Rightarrow Q(x)B$ by $Q(x)(A \Rightarrow B)$ provided that x is not free in A.

The following theorem on prenex form can be proved using the equivalence theorem.

Theorem 3. *Let A be a formula and A' a formula obtained by some prenex operation. Then*

$$\vdash A \Leftrightarrow A'.$$

The following deduction theorem proved in [6] is the main tool for the proof of the following theorem on reduction for the consistency.

Theorem 4 (deduction) *Let A be a closed formula and $T' = T \cup \{\mathbf{1}/A\}$. Then to every B there is n such that*

$$T \vdash_a A^n \Rightarrow B \quad \textit{iff} \quad T' \vdash_a B.$$

Theorem 5 (reduction for the consistency) *A theory $T' = T \cup \Gamma$ is contradictory iff there are $m_1, \dots, m_n$ and $A_1, \dots, A_n \in \mathrm{Supp}(\Gamma)$ such that*

$$T \vdash_c \neg A_1^{m_1} \bigtriangledown \cdots \bigtriangledown \neg A_n^{m_n}$$

where $a_i = \Gamma(A_i)$, $i = 1, \dots, n$ and $c > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$ or $c = 1$ if the right-hand side is equal to 1.

Corollary 1. *A theory $T' = T \cup \{\neg a/\neg A\}$ is contradictory iff $T \vdash_b mA$ for some m and $b > ma$ or $b = 1$ if $ma = 1$.*

We say that a formula A is a *fuzzy quasitautology* in the degree a if

$$\models_a B_1 \,\&\, \cdots \,\&\, B_k \Rightarrow A$$

where B_i are closed instances of the equality axioms. Formally, we will write

$$\models_a^Q A.$$

The following is a generalization of the Hilbert–Ackermann consistency theorem.

Theorem 6 (consistency) *Open theory T is contradictory iff there are $p_1, \dots, p_n$ and special axioms $A_1, \dots, A_n$ of the theory T such that*

$$\models_b^Q \neg \bar{A}_1^{p_1} \bigtriangledown \cdots \bigtriangledown \neg \bar{A}_n^{p_n}$$

where $\bar{A}_i$ are instances of the special axioms and $b > \neg(a_1^{p_1} \otimes \cdots \otimes a_n^{p_n})$ where $a_i = A_S(A_i)$, $i = 1, \dots, n$.

4. Herbrand theorem in fuzzy logic

THEOREM 1. *Let T be a fuzzy predicate calculus with equality, A be a closed existential formula*

$$A := (\exists x)B$$

and $a \geq A_S(A)$. Then

$$T \vdash_b mA \quad \textit{iff} \quad \models^Q_d p_1 B_1 \triangledown \cdots \triangledown p_n B_n$$

is a fuzzy quasitautology for some m and $p_1, \dots, p_n$ where $b > ma$ (possibly b=1), $B_1, \dots, B_n$ are instances of the formula B and $d > (p_1 + \cdots + p_n)a$ (or $d = 1$).

Proof Note that

$$\vdash \neg(\exists x)B \Leftrightarrow (\forall x)\neg B.$$

By Corollary 1,

$$T \vdash_b mA \quad \text{iff} \quad T' = T \cup \{\, {}^{\neg a}/(\forall x)\neg B\}$$

is contradictory for some m and $b > ma$ (we have also used the equivalence theorem). By the closure theorem, this holds iff $T' = T \cup \{\, {}^{\neg a}/\neg B\}$ is contradictory. By Theorem 6 there is a quasitautology of instances of $\neg B$

$$\models^Q_d \neg(\neg B_1)^{p_1} \triangledown \cdots \triangledown \neg(\neg B_n)^{p_n}$$

where $d > \neg((\neg a)^p_1 \otimes \cdots \otimes (\neg a)^{p_n})$ (or $d = 1$) if the right hand side is equal to 1). But this is equivalent with

$$\models^Q_d p_1 B_1 \triangledown \cdots \triangledown p_n B_n$$

and $d > \neg\neg(p_1 a \oplus \cdots \oplus p_n a) = (p_1 + \cdots + p_n)a$.

■

Let us now introduce the special equality axiom

$$(\forall x)(B \Leftrightarrow C) \Rightarrow \boldsymbol{r} = \boldsymbol{s}$$

where $\boldsymbol{r}$ and $\boldsymbol{s}$ are special constants for $(\forall x)B$ $(\forall x)C$, respectively. Analogous axiom is considered in classical logic in the proof of the Herbrand theorem. We will denote by T_H the Henkin extension of the theory T and

$$T'_H = T_H \cup \{\, {}^{1}/(\forall x)(B \Leftrightarrow C) \Rightarrow \boldsymbol{r} = \boldsymbol{s} \mid B, C \in F_J\}$$

further extension of T_H by special equality axioms.

Let $\mathcal{D} \models T$ be a model. We put

- $\mathcal{D}'_H(A) = \mathcal{D}_H(A)$ for all the formulas which do not contain the equality $\boldsymbol{r} = \boldsymbol{s}$ for any special constants $\boldsymbol{r}$, $\boldsymbol{s}$. Note that $\mathcal{D}'_H(A) = \mathcal{D}(A)$ for all formulas $A \in F_{J(T)}$.
- $\mathcal{D}'_H(\boldsymbol{r} = \boldsymbol{s}) = \mathcal{D}_H(B_x[\boldsymbol{r}] \Leftrightarrow C_x[\boldsymbol{s}])$ where $\boldsymbol{r}$ is a special constant for $(\forall x)B$ and $\boldsymbol{s}$ is that for $(\forall x)C$.

LEMMA 1.

$$\mathcal{D}'_H \models T'_H.$$

Proof The lemma obviously holds for the axioms (E1) and (E2').

We now verify (E3'). Let $\vdash (\boldsymbol{r} = \boldsymbol{s})^m \Rightarrow (p_x(\boldsymbol{r}) \Rightarrow p_x(\boldsymbol{s}))$ for some m and predicate p. Then we require

$$\mathcal{D}'_H(B_x[\boldsymbol{r}] \Leftrightarrow C_x[\boldsymbol{s}])^m \leq \mathcal{D}'_H(p_x(\boldsymbol{r})) \to \mathcal{D}'_H(p_x(\boldsymbol{s}))$$

to hold. However, we always can find appropriate m.

Let us now verify the special equality axiom, i.e. that

$$\mathcal{D}'_H((\forall x)(B \Leftrightarrow C)) \leq \mathcal{D}'_H(\boldsymbol{r} = \boldsymbol{s}) = \mathcal{D}_H(B_x[\boldsymbol{r}] \Leftrightarrow C_x[\boldsymbol{s}])$$

which holds because

$$(\forall x)(B\Leftrightarrow C),\quad (\forall x)B\Leftrightarrow(\forall x)C,\quad B_x[\boldsymbol{r}]\Leftrightarrow C_x[\boldsymbol{s}$$

are provable in the degree 1 (we have to use Henkin axioms and the tautology $\vdash (\forall x)(A\Rightarrow B)\Rightarrow(\forall x)A\Rightarrow(\forall x)B$).
■

THEOREM 2. *The theory T'_H is a conservative extension of T.*

Proof As proved in [7], Henkin theory T_H is a conservative extension of T. Furthermore,

$$\mathcal{D}(A)\leq\mathcal{D}_H(A)\leq\mathcal{D}'(A)$$

holds for every $A\in F_{J(T)}$. By the assumption, to every model $\mathcal{D}\models T$ there is a model $\mathcal{D}_H\models T_H$ such that $\mathcal{D}(A)=\mathcal{D}_H(A)$ and $\mathcal{D}(A)=\mathcal{D}'_H(A)$ by the construction of $\mathcal{D}'_H$. Hence, using Lemma 1 we obtain

$$\bigwedge\{\mathcal{D}'_H(A)\mid\mathcal{D}'\models T'_H\}=\bigwedge\{\mathcal{D}(A)\mid\mathcal{D}\models T\}.$$

The theorem the follows from the completeness. ■

LEMMA 2.

$$\vdash(A_1\Rightarrow C)\Rightarrow(\cdots((A_n\Rightarrow C)\Rightarrow(A_1\bigtriangledown\cdots\bigtriangledown A_n\Rightarrow nC)\cdots).$$

Proof The formulas

$$\vdash\neg C\mathbin{\&}(A\Rightarrow C)\Rightarrow\neg A,$$
$$\vdash\neg A\mathbin{\&}(\neg A\Rightarrow B)\Rightarrow B$$
$$\vdash((A\Rightarrow C)\mathbin{\&}(B\Rightarrow C)\mathbin{\&}(\neg A\Rightarrow B)\mathbin{\&}(C\Rightarrow\mathbf{0}))\Rightarrow C$$

are provable in the degree 1 where the latter is equivalent with

$$\vdash(A\Rightarrow C)\Rightarrow((B\Rightarrow C)\Rightarrow(A\bigtriangledown B\Rightarrow 2C)).$$

From this, we get the proposition of lemma using induction. ■

LEMMA 3. *If $\vdash_a A\Rightarrow B$ then $\vdash_a(\exists x)A\Rightarrow(\exists x)B$.*

Proof Immediately using the substitution axiom for the existential quantifier, rule of generalization and provable formula (3). ■

The following is a generalization of the classical Herbrand theorem to fuzzy logic in narrow sense of Łukasiewicz style. Recall that the matrix of a formula $A:=(\forall x_1)\cdots(\forall x_n)B(x_1,\ldots,x_n)$ in the prenex form is the formula $B(x_1,\ldots,x_n)$. The Herbrand existential formula A_H is constructed from A by substitution of new functional symbols in the same way as in classical logic.

THEOREM 3. *Let T be a fuzzy theory and $A\in F_{J(T)}$ a closed formula in prenex form and $a=A_S(A)$. Then*

$$T\vdash_b mA\quad\text{iff}\quad\models^Q_d p_1A_H^{(1)}\bigtriangledown\cdots\bigtriangledown p_nA_H^{(n)}$$

is a fuzzy quasitautology for some m and $p_1,\ldots,p_n$ where $b>ma$ (or $b=1$), $d>(p_1+\cdots+p_n)a$ (or $d=1$) where $A_H^{(i)}$ are instances of the matrix of the formula A_H.

Proof We construct a fuzzy theory T'_H being a Henkin extension of T extended further by special equality axioms. By Theorem 2, this is conservative. Then we prove that

$$T \vdash_a A \quad \text{iff} \quad T'_H \vdash_a A_H.$$

The implication from left to right is proved in a classical way using the substitution axiom and Lemma 3.

Analogously as in classical way, we also prove the opposite implication. We need the provable formulas $\mathbf{a} = \mathbf{a}' \Rightarrow \mathbf{r}(\mathbf{a}) = \mathbf{r}(\mathbf{a}')$ and $\mathbf{a} = \mathbf{a}' \Rightarrow \mathbf{b} = \mathbf{b}' \Rightarrow \mathbf{s}(\mathbf{a}, \mathbf{b}) = \mathbf{s}(\mathbf{a}', \mathbf{b}')$, which follow immediately from the equality theorem.

Finally, let $T'_H \vdash_b mA_H$. Then using Theorem 1, there is a fuzzy quasitautology

$$\models_d^Q p_1 B_1 \nabla \cdots \nabla p_n B_n.$$

To finish, we realize exchanges of B_i in the same way as in the classical proof and use Lemma 2. ■

This theorem is not as strong as in classical logic mainly due to the multiplication constant m which is not directly known. This appears after application of the deduction Theorem 4 in the proofs of Theorem 5 and its Corollary 1 (cf. [9]); only existence of some m is assured in the former. Further elaboration of this theorem, if possible, might be useful together with its generalization using, e.g., Mac Naughton functions.

5. Conclusion

In this paper, we continue our program to develop fuzzy logic in narrow sense in a direction to cover most of the (basic) results of classical logic. We focus on fuzzy logic of Łukasiewicz style because, as is demonstrated on lot of places, it is the most developed formal theory of fuzzy logic in narrow sense which seems to be the closest nontrivial generalization of classical logic. Moreover, it also meets well the requirement to provide syntactical reasoning on truth values and thus, it fulfils the program of being formal tool for modeling of the vagueness phenomenon.

We dealt mainly with open fuzzy theories and proved fuzzy analogy of the famous Herbrand theorem. Further development should focus on consequences and generalizations of this theorem with the goal to establish well formally founded resolution in fuzzy logic. To achieve this, we may start with the results of [4].

Acknowledgment

This paper has been supported by grant A1086501 of the GA AV ČR and partially also by COST Action 15.

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