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Fuzzy closure operators

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1 Introduction

Closure operators (and closure systems) play a significant role in both pure and applied mathematics. Basically, two significant types are of importance: compact (called also finitary or algebraic) closure operators and systems (e.g. the class of all subuniverses of an algebraic structure, the consequence operator in logic etc.) and topological closure operators (e.g. the closure operator in a topological space). In the framework of fuzzy set theory, several particular examples of closure operators and systems have been considered (e.g. so-called fuzzy subalgebras, fuzzy congruences, fuzzy topology etc.). Recently, fuzzy closure operators and fuzzy closure systems themselves (i.e. operators which map fuzzy sets to fuzzy sets and the corresponding systems of closed fuzzy sets) have been studied by Gerla et al., see e.g. [6, 7, 12, 13]. As a matter of fact, a fuzzy set A is usually defined as a mapping from a universe set X into the real interval $[0, 1]$ in the above mentioned works. Therefore, the structure of truth values of the "logic behind" is fixed to $[0, 1]$ equipped with minimum being the operation corresponding to logical conjunction.

As it appeared recently in the investigations of fuzzy logic in narrow sense [15, 16] (i.e. fuzzy logic as a many-valued logical calculus), there are several logical calculi formalizing the intuitive idea of "fuzzy reasoning" which are complete with respect to the semantics over special structures of truth values. Among these structures, the most general one is that of a residuated lattice (it is worth noticing that residuated lattices (introduced originally in [19] as an abstraction in the study of ideal systems of rings) have been proposed as a suitable structure of truth values by Goguen in [14]). From this point of view, the need for a general notion of a "fuzzy closure" concept becomes apparent.

The aim of this paper is to outline a general theory of fuzzy closure operators and fuzzy closure systems. In the next section we introduce the necessary concepts. In Section 3, fuzzy closure operators and systems are defined and investigated. The notions defined here generalize the notions introduced and studied earlier in two directions: First, as indicated above, arbitrary (in fact, complete) residuated lattice is used for the structure of truth values (as a matter of fact, $[0, 1]$ equipped by minimum is itself a complete residuated lattice, cf. Section 2). Second, we generalize the usual monotonicity condition so that it reads "if A is *almost* a subset of B then the closure of A is *almost* a subset of the closure of B". Section 4 describes some relations naturally induced by fuzzy closure operators. Among others, a way to factorize a fuzzy closure system by a similarity relation in a consistent way is shown. Section 5 deals with representation of fuzzy closure operators by classical closure operators. Section 6 contains some examples. Section 7 is devoted to description of fuzzy closure operators in terms of fuzzy consequence relations.

2 Preliminaries

A fuzzy set in a universe set X is any mapping from X into L, L being an appropriate set of truth values. L has to be equipped with some structure. A general one is that of a complete residuated lattice.

Definition 1 *A* complete residuated lattice *is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *such that*

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete lattice with the least element* 0 *and the greatest element* 1*,*

(2) $\langle L, \otimes, 1\rangle$ *is a commutative monoid, i.e.* $\otimes$ *is commutative, associative, and* $x \otimes 1 = x$ *holds holds for each* $x \in L$*, and*

(3) $\otimes, \rightarrow$ *form an adjoint pair, i.e.*
$$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \tag{1}$$
holds for all $x, y, z \in L$*.*

In each residuated lattice it holds that $x \leq y$ implies $x \otimes z \leq y \otimes z$ (isotonicity), and $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ (isotonicity in the second argument) and $x \rightarrow z \geq y \rightarrow z$ (antitonicity in the first argument). $\otimes$ and $\rightarrow$ are called *multiplication* and *residuum*, respectively.

The most studied and applied set of truth values is the real interval $[0, 1]$ with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel one ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). For the role of these "building stones" in fuzzy logic see [15]. Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$. It may be easily verified that the only t-norm on $\{0, 1\}$ is the classical conjunction operation $\wedge$, i.e. $a \wedge b = 1$ iff $a = 1$ and $b = 1$, which implies that the only residuum operation is the classical implication operation $\rightarrow$, i.e. $a \rightarrow b = 0$ iff $a = 1$ and $b = 0$. Note that each of the preceding residuated lattices is complete.

Multiplication $\otimes$ and residuum $\rightarrow$ are intended for modeling of the conjunction and implication, respectively. Supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of general and existential quantifier, respectively.

The following identities of complete residuated lattices will be used (see e.g. [14]):

$$a = 1 \rightarrow a \tag{2}$$
$$a \leq (a \rightarrow b) \rightarrow b \tag{3}$$
$$a \otimes (a \rightarrow b) \leq b \tag{4}$$
$$(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) \tag{5}$$
$$a \otimes \bigwedge_{j \in J} b_j \leq \bigwedge_{j \in J} (a \otimes b_j) \tag{6}$$
$$\Big(\bigvee_{j \in J} a_j\Big) \rightarrow b = \bigwedge_{j \in J} (a_j \rightarrow b) \tag{7}$$
$$a \rightarrow \bigwedge_{j \in J} b_j = \bigwedge_{j \in J} (a \rightarrow b_j) \tag{8}$$

A nonempty subset $K \subseteq L$ is called an $\leq$*-filter* if for every $a, b \in L$ such that $a \leq b$ it holds that $b \in K$ whenever $a \in K$. An $\leq$-filter K is called a *filter* if $a, b \in K$ implies $a \otimes b \in K$. Unless otherwise stated, in what follows we denote by $\mathbf{L}$ a complete residuated lattice and by K an $\leq$-filter in $\mathbf{L}$ (both $\mathbf{L}$ and K possibly with indices).

An $\mathbf{L}$*-set* (fuzzy set) [23, 14] A in a universe set X is any map $A : X \rightarrow L$. By L^X we denote the set of all $\mathbf{L}$-sets in X. The concept of $\mathbf{L}$-relation is defined obviously. Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. Given $A, B \in L^X$, the *subsethood degree* [14] $\mathrm{S}(A, B)$ of A in B is defined by $\mathrm{S}(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$. We write $A \subseteq B$ if $\mathrm{S}(A, B) = 1$. Analogously, the *equality degree* $E(A, B)$ of A and B is defined by $E(A, B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$. It is immediate that $E(A, B) = \mathrm{S}(A, B) \wedge \mathrm{S}(B, A)$.

3 $\mathbf{L}_K$-closure operators and $\mathbf{L}_K$-closure systems

Recall that an closure operator on a set X is a mapping $\mathrm{C} : 2^X \rightarrow 2^X$ satisfying the following conditions: $A \subseteq \mathrm{C}(A)$, if $A_1 \subseteq A_2$ then $\mathrm{C}(A_1) \subseteq \mathrm{C}(A_2)$, and $\mathrm{C}(A) = \mathrm{C}(\mathrm{C}(A))$, for any $A, A_1, A_2 \in 2^X$. More generally, if $\subseteq$ denotes a partial order, we get the notion of closure operator in an ordered set [8].

Definition 2 *An* $\mathbf{L}_K$-closure operator *(*fuzzy closure operator*) on the set* X *is a mapping* $\mathrm{C} : L^X \to L^X$ *satisfying*

$$
\begin{aligned}
A &\subseteq \mathrm{C}(A) && (9)\\
\mathrm{S}(A_1, A_2) &\leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2)) \quad \textit{whenever } \mathrm{S}(A_1, A_2) \in K && (10)\\
\mathrm{C}(A) &= \mathrm{C}(\mathrm{C}(A)) && (11)
\end{aligned}
$$

for every $A, A_1, A_2 \in L^X$.

If $K = L$, we omit the subscript K and call C an $\mathbf{L}$-closure operator. The set K plays the role of the set of designated truth values. Condition (10) says that the closure preserves also partial subsethood whenever the subsethood degree is designated. Since K is an $\leq$-filter in $\mathbf{L}$, the designated truth values represent, in a sense, sufficiently high truth values. In this view. (10) reads "if A_1 is almost included in A_2 then $\mathrm{C}(A_1)$ is almost included in $\mathrm{C}(A_2)$". It is easily seen that each $\mathbf{L}_K$-closure operator on X is an closure operator on the complete lattice $\langle L^X, \subseteq \rangle$ [8].

Remark Note that for $L = \{0, 1\}$, $\mathbf{L}_K$-closure operators are precisely the classical closure operators. Clearly, if $K_1 \subseteq K_2$ then each $\mathbf{L}_{K_2}$-closure operator is an $\mathbf{L}_{K_1}$-closure operator. As we will see, the converse is not true. Note also that for $L = [0, 1]$, $\mathbf{L}_{\{1\}}$-closure operators are precisely fuzzy closure operators studied by Gerla [7, 12, 13].

Remark We show that for residuated lattices $\mathbf{L}$ with $L = [0, 1]$ of Łukasiewicz, Gödel, and product logic [15], the set K is relevant: Take $X = \{x_1, x_2\}$, and define C by $\mathrm{C}(A)(x_1) = 0, \mathrm{C}(A)(x_2) = 0.5$ for $A(x_1) = 0, A(x_2) \leq 0.5$, and $\mathrm{C}(A)(x_1) = \mathrm{C}(A)(x_2) = 1$ otherwise. An easy inspection shows that C is an $\mathbf{L}_{\{1\}}$-closure operator. However, for A_1, A_2 given by $A_1(x_1) = A_2(x_1) = 0$, $A_1(x_2) = 1$, $A_2(x_2) = 0.5$ it holds $\mathrm{S}(A_1, A_2) = 0.5 > 0 = \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2))$ (for all of the three algebras). Thus, C is not an $\mathbf{L}_{[0.5,1]}$-closure operator.

Theorem 3 $\mathrm{C} : L^X \to L^X$ *is an* $\mathbf{L}_K$*-closure operator on* X *iff it satisfies (9) and the following condition*

$$\mathrm{S}(A_1, \mathrm{C}(A_2)) \leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2)) \quad \textit{whenever } \mathrm{S}(A_1, \mathrm{C}(A_2)) \in K. \tag{12}$$

Proof. Suppose (9)–(11) hold. If $\mathrm{S}(A_1, \mathrm{C}(A_2)) \in K$ then by (10) and (11) we have $\mathrm{S}(A_1, \mathrm{C}(A_2)) \leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(\mathrm{C}(A_2))) = \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2))$, i.e. (12) holds. Conversely, let (9) and (12) hold. Suppose $\mathrm{S}(A_1, A_2) \in K$. Since, by (9), $A_2 \subseteq \mathrm{C}(A_2)$, we have $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_1, \mathrm{C}(A_2)) \in K$. Furthermore, (12) implies $\mathrm{S}(A_1, \mathrm{C}(A_2)) \leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2))$, hence $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2))$, proving (10). By (12), $1 = \mathrm{S}(\mathrm{C}(A), \mathrm{C}(A)) \leq \mathrm{S}(\mathrm{C}(\mathrm{C}(A)), \mathrm{C}(A))$, we have $\mathrm{C}(\mathrm{C}(A)) \subseteq \mathrm{C}(A)$. Since the converse inclusion holds by (9), we conclude (11). □

Definition 4 *A system* $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ *is called* closed under S_K-intersections *iff for each* $A \in L^X$ *it holds that* $\bigcap_{i \in I, \mathrm{S}(A, A_i) \in K} \mathrm{S}(A, A_i) \to A_i) \in \mathcal{S}$ *where* $(\bigcap_{i \in I, \mathrm{S}(A, A_i) \in K} \mathrm{S}(A, A_i) \to A_i)(x) = \bigwedge_{i \in I, \mathrm{S}(A, A_i) \in K} (\mathrm{S}(A, A_i) \to A_i(x))$ *for each* $x \in X$. *A system closed under* S_K*-intersections will be called an* $\mathbf{L}_K$-closure system.

For $K = L$ the subscript will again be omitted.

Remark (1) We have $\bigcap_{i \in I, \mathrm{S}(A, A_i) \in \{1\}} \mathrm{S}(A, A_i) \to A_i) = \bigcap_{i \in I, A \subseteq A_i} A_i$. Therefore, $\mathcal{S}$ is a $\mathbf{2}$-closure system iff for each $A \subseteq X$ it holds $\bigcap_{A \subseteq A_i} A_i \in \mathcal{S}$. It can be easily seen that the last condition is equivalent to being closed under arbitrary intersection for $\mathcal{S}$. Hence, $\mathbf{2}$-closure systems coincide with closure systems, i.e. systems of sets closed under intersections [8].

(2) In general, being closed under arbitrary intersections is a weaker condition then being closed under S_K-intersections. Indeed, let $\mathcal{S}$ be closed under S_K-intersections. To show that $\mathcal{S}$ is closed under arbitrary intersections, it suffices to show that $\bigwedge_{j\in J} A_j(x) = \bigwedge_{i\in I, \mathrm{S}(\bigcap_{j\in J} A_j, A_i)\in K}(\mathrm{S}(\bigcap_{j\in J} A_j, A_i) \to A_i(x))$ holds for any $J \subseteq I$. The inequality $\geq$ is clearly valid since for each $j' \in J$ we have $\mathrm{S}(\bigcap_{j\in J} A_j, A_{j'}) \to A_{j'}(x) = 1 \to A_{j'}(x) = A_{j'}(x)$. The converse inequality holds iff

$$\bigwedge_{j\in J} A_j(x) \leq \mathrm{S}(\bigcap_{j\in J} A_j, A_i) \to A_i(x)$$

for each $i \in I$ such that $\mathrm{S}(\bigcap_{j\in J} A_j, A_i)$ which is equivalent to

$$\bigwedge_{j\in J} A_j(x) \otimes \mathrm{S}(\bigcap_{j\in J} A_j, A_i) \leq A_i(x),$$

i.e.

$$\bigwedge_{j\in J} A_j(x) \otimes \bigwedge_{y\in X}(\bigwedge_{j\in J} A_j(y)) \to A_i(y) \leq A_i(x)$$

which holds because

$$\bigwedge_{j\in J} A_j(x) \otimes \bigwedge_{y\in X}(\bigwedge_{j\in J} A_j(y)) \to A_i(y) \leq \bigwedge_{j\in J} A_j(x) \otimes (\bigwedge_{j\in J} A_j(x)) \to A_i(x) \leq A_i(x).$$

On the other hand, put $X = \{x\}$, take the Łukasiewicz structure with $\mathbf{L} = \{0, \frac{1}{2}, 1\}$, $K = L$, $\mathcal{S} = \{\{{}^{0}/x\}, \{{}^{1}/x\}\}$, and $A = \{{}^{\frac{1}{2}}/x\}$. Then $\mathcal{S}$ is clearly closed under arbitrary intersections but not under S_K-intersections since $\bigcap_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i = A \notin \mathcal{S}$.

Closedness under S_K-intersections is, however, equivalent to closedness under intersections of "K-shifted" $\mathbf{L}$-sets. Let for $a \in L$, $A \in L^X$, denote by $a \to A$ the $\mathbf{L}$-set defined by $(a \to A)(x) = a \to A(x)$.

Theorem 5 *$\mathcal{S}$ is an $\mathbf{L}_K$-closure system iff for any $a_i \in L$, $i \in I$, it holds $\bigcap_{a_i\in K}(a_i \to A_i) \in \mathcal{S}$.*

Proof. If $\bigcap_{a_i\in K}(a_i \to A_i) \in \mathcal{S}$ for every $a_i \in L$ then taking $a_i = \mathrm{S}(A, A_i)$ if $\mathrm{S}(A, A_i) \in K$ and $a_i = 0$ otherwise for $A \in L^X$, it is easily seen that $\bigwedge_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i(x) = \bigwedge_{a_i\in K}(a_i \to A_i(x))$, i.e. $\mathcal{S}$ is an $\mathbf{L}_K$-closure system.

Conversely, let $\mathcal{S}$ be an $\mathbf{L}_K$-closure system. Let $a_i \in L$ and put $A = \bigcap_{a_i\in K}(a_i \to A_i)$. We have to show $A \in \mathcal{S}$. Clearly, it is enough to show that $\bigcap_{i\in I, \mathrm{S}(A,A_i)\in K}(\mathrm{S}(A, A_i) \to A_i) = A$. The fact $\bigcap_{i\in I, \mathrm{S}(A,A_i)\in K}(\mathrm{S}(A, A_i) \to A_i) \supseteq A$ is shown in Lemma 8. For the converse inclusion, observe first that if $a_j \in K$ then $\mathrm{S}(A, A_j) \in K$. Indeed, by the filter property it suffices to show that $a_j \leq \mathrm{S}(A, A_j)$. This holds iff for each $x \in X$ it holds $a_j \leq (\bigwedge_{a_i\in K}(a_i \to A_i(x)) \to A_j(x))$, i.e. $a_j \otimes (\bigwedge_{a_i\in K}(a_i \to A_i(x)) \leq A_j(x))$ which holds since $a_j \otimes (a_j \to A_j(x)) \leq A_j(x)$. Now, the converse inclusion holds iff for each $x \in X$ we have

$$\bigwedge_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i(x) \leq \bigwedge_{a_i\in K} a_i \to A_i(x)$$

which holds iff for each $a_j \in K$ it holds

$$a_j \otimes \bigwedge_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i(x) \leq A_j(x)$$

which holds since by the above observation $\mathrm{S}(A, A_j) \in K$, and therefore $a_j \otimes \bigwedge_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i(x) \leq a_j \otimes (\mathrm{S}(A, A_j) \to A_j(x) \leq A_j(x)$. The theorem is proved. □

Corollary 6 *A system $\mathcal{S}$ which is closed under arbitrary intersections is an $\mathbf{L}_K$-closure system iff for each $a \in K$ and $A \in \mathcal{S}$ it holds $a \to A \in \mathcal{S}$.*

The following theorem shows another way to obtain the closure in an $\mathbf{L}_K$-closure system.

Theorem 7 *Let $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ be an $\mathbf{L}_K$-closure system. Then for each $A \in L^X$ it holds*

$$\bigcap_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i = \bigcap_{i\in I, A\subseteq A_i} A_i.$$

Proof. Clearly, $\bigcap_{i\in I,\mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i \subseteq \bigcap_{i\in I,\mathrm{S}(A,A_i)=1} \mathrm{S}(A, A_i) \to A_i = \bigcap_{i\in I,A\subseteq A_i} A_i$. On the other hand, it is easy to check that $A \subseteq \bigcap_{i\in I,\mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i \in \mathcal{S}$ which immediately gives $\bigcap_{i\in I,A\subseteq A_i} A_i \subseteq \bigcap_{i\in I,\mathrm{S}(A,A_i)\in K} \mathrm{S}(A, A_i) \to A_i$. □

Lemma 8 *Let $\mathcal{S} = \{A_i \mid i \in I\}$ be an $\mathbf{L}_K$-closure system, K be a filter in $\mathbf{L}$. Then $\mathrm{C}_\mathcal{S} : L^X \to L^X$ defined by $\mathrm{C}_\mathcal{S}(A)(x) = \bigwedge_{i\in I,\mathrm{S}(A,A_i)\in K}(\mathrm{S}(A, A_i) \to A_i(x))$ is an $\mathbf{L}_K$-closure operator. Moreover, for $A \in L^X$ it holds $A \in \mathcal{S}$ iff $A = \mathrm{C}_\mathcal{S}(A)$.*

Proof. We show (9)–(11).

(9): We have to show $A(x) \leq \mathrm{C}_\mathcal{S}(A)(x)$ for each $x \in X$ which holds iff for each $i \in I$ such that $\mathrm{S}(A, A_i) \in K$ it holds $A(x) \leq \mathrm{S}(A, A_i) \to A_i(x)$. This is, by adjointness, equivalent to $A(x) \otimes \mathrm{S}(A, A_i) \leq A_i(x)$, i.e. $A(x) \otimes \bigwedge_{y\in X}(A(y) \to A_i(y)) \leq A_i(x)$ which holds because of $A(x) \otimes \bigwedge_{y\in X}(A(y) \to A_i(y)) \leq A(x) \otimes (A(x) \to A_i(x)) \leq A_i(x)$.

(10): Suppose $\mathrm{S}(A_1, A_2) \in K$. We have to show $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(\mathrm{C}_\mathcal{S}(A_1), \mathrm{C}_\mathcal{S}(A_2))$ which is equivalent to the fact that for each $x \in X$ it holds $\mathrm{S}(A_1, A_2) \leq \mathrm{C}_\mathcal{S}(A_1)(x) \to \mathrm{C}_\mathcal{S}(A_2)(x)$, i.e. by adjointness,

$$\mathrm{C}_\mathcal{S}(A_1)(x) \otimes \mathrm{S}(A_1, A_2) \leq \mathrm{C}_\mathcal{S}(A_2)(x) = \bigwedge_{i\in I,\mathrm{S}(A_2,A_i)\in K} \mathrm{S}(A_2, A_i) \to A_i(x)$$

which is true iff for each $j \in I$ with $\mathrm{S}(A_2, A_j) \in K$ it holds

$$\mathrm{C}_\mathcal{S}(A_1)(x) \otimes \mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A_j) \leq A_j(x)$$

which is true. Indeed, since $\mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A_j) \leq \mathrm{S}(A_1, A_j)$, $\mathrm{S}(A_1, A_2) \in K$ and $\mathrm{S}(A_2, A_j) \in K$, the filter property of K yields $\mathrm{S}(A_1, A_j) \in K$, and we have

$$\begin{aligned}
&\mathrm{C}_\mathcal{S}(A_1)(x) \otimes \mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A_j) = \\
&= \mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A_j) \otimes \bigwedge_{i\in I,\mathrm{S}(A_1,A_i)\in K} (\mathrm{S}(A_1, A_i) \to A_i(x)) \leq \\
&\leq \mathrm{S}(A_1, A_j) \otimes (\mathrm{S}(A_1, A_j) \to A_j(x)) \leq A_j(x).
\end{aligned}$$

(11): Clearly, we only have to show $\mathrm{C}_\mathcal{S}(\mathrm{C}_\mathcal{S}(A)) \subseteq \mathrm{C}_\mathcal{S}(A)$. Since $\mathrm{C}_\mathcal{S}(A) \in \mathcal{S}$, there is some $j \in I$ such that $A_j = \mathrm{C}_\mathcal{S}(A)$. We therefore have

$$\begin{aligned}
\mathrm{C}_\mathcal{S}(\mathrm{C}_\mathcal{S}(A))(x) &= \bigwedge_{j\in I,\mathrm{S}(\mathrm{C}_\mathcal{S}(A),A_j)\in K} \mathrm{S}(\mathrm{C}_\mathcal{S}(A), A_j) \to A_j(x) \leq \\
&\leq \mathrm{S}(\mathrm{C}_\mathcal{S}(A), \mathrm{C}_\mathcal{S}(A)) \to (\mathrm{C}_\mathcal{S}(A))(x) = (\mathrm{C}_\mathcal{S}(A))(x).
\end{aligned}$$

We now show that $A \in \mathcal{S}$ iff $A = \mathrm{C}_\mathcal{S}(A)$. Indeed, if $A = A_j \in \mathcal{S}$ then $A_j \subseteq \mathrm{C}_\mathcal{S}(A_j)$ as proved above. Conversely, $\mathrm{C}_\mathcal{S}(A_j)(x) = \bigwedge_{i\in I,\mathrm{S}(A_j,A_i)\in K}(\mathrm{S}(A_j, A_i) \to A_i(x)) \leq \mathrm{S}(A_j, A_j) \to A_j(x) \leq A_j(x)$, i.e. $\mathrm{C}_\mathcal{S}(A_j) \subseteq A_j$. If $A = \mathrm{C}_\mathcal{S}(A)$ then $A \in \mathcal{S}$ by the definition of the $\mathbf{L}_K$-closure system, completing the proof. □

Lemma 9 *Let $\mathrm{C} : L^X \to L^X$ be an $\mathbf{L}_K$-closure operator. Then $\mathcal{S}_C = \{A \in L^X \mid A = \mathrm{C}(A)\}$ is an $\mathbf{L}_K$-closure system.*

Proof. Let I be such that $\mathcal{S}_{\mathrm{C}} = \{A_i \mid i \in I\}$. We have to show that for each $A \in L^X$ it holds $\bigwedge_{i\in I,\mathrm{S}(A,A_i)\in K}(\mathrm{S}(A,A_i) \to A_i) \in \mathcal{S}_{\mathrm{C}}$. To this end it clearly suffices to show

$$\bigwedge_{i\in I,\mathrm{S}(A,A_i)\in K}(\mathrm{S}(A,A_i) \to A_i) = \mathrm{C}(A). \tag{13}$$

On the one hand, since $\mathrm{S}(A,\mathrm{C}(A)) = 1 \in K$, we have $\bigwedge_{i\in I,\mathrm{S}(A,A_i)\in K}(\mathrm{S}(A,A_i) \to A_i(x)) \leq \mathrm{S}(A,\mathrm{C}(A)) \to \mathrm{C}(A)(x)) = \mathrm{C}(A)(x)$. On the other hand, $\mathrm{C}(A)(x) \leq \bigwedge_{i\in I,\mathrm{S}(A,A_i)}(\mathrm{S}(A,A_i) \to A_i(x))$ iff for each $i \in I$ such that $\mathrm{S}(A,A_i) \in K$ it holds $\mathrm{C}(A)(x) \otimes \mathrm{S}(A,A_i) \leq A_i(x)$. This is, indeed, true since $\mathrm{C}(A)(x) \otimes \mathrm{S}(A,A_i) \leq \mathrm{C}(A)(x) \otimes \mathrm{S}(\mathrm{C}(A),\mathrm{C}(A_i)) \leq \mathrm{C}(A)(x) \otimes (\mathrm{C}(A)(x) \to \mathrm{C}(A_i)(x)) \leq \mathrm{C}(A_i)(x) = A_i(x)$. To sum up, (13) is proved. □

Theorem 10 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*,* $\mathcal{S}$ *be an* $\mathbf{L}_K$*-closure system on* X*,* K *be a filter in* $\mathbf{L}$*. Then* $\mathcal{S}_{\mathrm{C}}$ *is an* $\mathbf{L}_K$*-closure system on* X*,* $\mathrm{C}_{\mathcal{S}}$ *is an* $\mathbf{L}_K$*-closure operator on* X *and it holds* $\mathrm{C} = \mathrm{C}_{\mathcal{S}_{\mathrm{C}}}$ *and* $\mathcal{S} = \mathcal{S}_{\mathrm{C}_{\mathcal{S}}}$*, i.e. the mappings* $\mathrm{C} \mapsto \mathcal{S}_{\mathrm{C}}$ *and* $\mathcal{S} \mapsto \mathrm{C}_{\mathcal{S}}$ *are mutually inverse.*

Proof. By Lemma 8 and Lemma 9 it remains to prove $\mathrm{C} = \mathrm{C}_{\mathcal{S}_{\mathrm{C}}}$, i.e. that for any $A \in L^X$, $x \in X$, it holds

$$\mathrm{C}(A)(x) = \bigwedge_{A'\in L^X,A'=\mathrm{C}(A'),\mathrm{S}(A,A')\in K}(\mathrm{S}(A,A') \to A'(x)).$$

The inequality "$\leq$" holds iff for each $A' \in L^X$ such that $A' = \mathrm{C}(A')$ and $\mathrm{S}(A,A') \in K$ we have $\mathrm{C}(A)(x) \otimes \mathrm{S}(A,A') \leq A'(x)$ which holds since $\mathrm{C}(A)(x) \otimes \mathrm{S}(A,A') \leq \mathrm{C}(A)(x) \otimes \mathrm{S}(\mathrm{C}(A),\mathrm{C}(A')) \leq \mathrm{C}(A')(x) = A'(x)$. Conversely, putting $A' = \mathrm{C}(A)$ we get $\mathrm{S}(A,\mathrm{C}(A)) \to \mathrm{C}(A)(x) = 1 \to \mathrm{C}(A)(x) = \mathrm{C}(A)(x)$, i.e. "$\geq$" also holds. □

$\mathbf{L}_K$-closure systems as systems of "almost closed" L-sets A natural idea is to consider the property "to be closed (w.r.t. a given fuzzy closure operator C)" a graded property. An **L**-set A can be considered to be "almost closed w.r.t. C" iff "A almost equals $\mathrm{C}(A)$". This poses a question of whether fuzzy closure systems can be defined as systems of "almost closed" fuzzy sets.

Definition 11 *An* **L***-system* $\mathbf{S} \in L^{L^X}$ *is called an* $\mathbf{L}_K$-closure **L**-system in X *if for every* $A, B \in L^X$ *we have*

$$\mathbf{S}(\bigcap_{A_i\in L^X,\mathrm{S}(A,A_i)\in K}(\mathbf{S}(A_i) \otimes \mathrm{S}(A,A_i)) \to A_i) = 1, \tag{14}$$

$$\mathbf{S}(A) \otimes \mathrm{S}(A,B) \otimes \mathrm{S}(B,A) \leq \mathbf{S}(B) \tag{15}$$

$$\textit{whenever } \mathrm{S}(B,A) \in K. \tag{16}$$

Remark (1) Note that the **L**-set $\bigcap_{A_i\in L^X,\mathrm{S}(A,A_i)\in K}(\mathbf{S}(A_i) \otimes \mathrm{S}(A,A_i)) \to A_i$ in X is defined by $(\bigcap_{A_i\in L^X,\mathrm{S}(A,A_i)\in K}(\mathbf{S}(A_i) \otimes \mathrm{S}(A,A_i)) \to A_i)(x) = \bigwedge_{A_i\in L^X,\mathrm{S}(A,A_i)\in K}(\mathbf{S}(A_i) \otimes \mathrm{S}(A,A_i)) \to A_i(x)$. (2) An $\mathbf{L}_K$-closure **L**-system is therefore an **L**-set of **L**-sets in X. We interpret $\mathbf{S}(A)$ as the degree to which $A \in L^X$ is closed. Condition (15) is naturally interpreted as the requirement that an **L**-set that is both a subset and a superset of to an "almost closed" **L**-set is itself "almost closed".

We are going to investigate the relationship between $\mathbf{L}_K$-closure **L**-systems, $\mathbf{L}_K$-closure operators, and $\mathbf{L}_K$-closure systems. To this end we define the following mappings.

For an $\mathbf{L}_K$-closure operator C in X and an $\mathbf{L}_K$-closure system $\mathcal{S}$ in X we define **L**-sets $\mathbf{S}_{\mathrm{C}}$ and $\mathbf{S}_{\mathcal{S}}$ in L^X by

$$\mathbf{S}_{\mathrm{C}}(A) = E(A,\mathrm{C}(A)) \tag{17}$$

and

$$\mathbf{S}_{\mathcal{S}}(A) = E(A,\mathrm{C}_{\mathcal{S}}(A)), \tag{18}$$

respectively. Clearly, we have $\mathbf{S}_{\mathrm{C}}(A) = \mathrm{S}(\mathrm{C}(A), A)$ and $\mathbf{S}_{\mathcal{S}}(A) = \mathrm{S}(\mathrm{C}_{\mathcal{S}}(A), A)$.

For an $\mathbf{L}_K$-closure $\mathbf{L}$-system $\mathbf{S}$ in X we define a mapping $\mathrm{C}_{\mathbf{S}} : L^X \to L^X$ and a set $\mathcal{S}_{\mathbf{S}} \subseteq L^X$ by

$$(\mathrm{C}_{\mathbf{S}}(A))(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i) \in K} (\mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i)) \to A_i(x) \tag{19}$$

and

$$\mathcal{S}_{\mathbf{S}} = \{A \in L^X \mid \mathbf{S}(A) = 1\}, \tag{20}$$

respectively.

Lemma 12 *For an $\mathbf{L}_K$-closure operator* C *in X we have $\mathrm{C}_{\mathbf{S}_{\mathrm{C}}} = \mathrm{C}_{\mathcal{S}_{\mathrm{C}}}$.*

Proof. Take any $A \in L^X$, $x \in X$. We have to show $(\mathrm{C}_{\mathbf{S}_{\mathrm{C}}}(A))(x) = (\mathrm{C}_{\mathcal{S}_{\mathrm{C}}}(A))(x)$.

"$\leq$": $(\mathrm{C}_{\mathbf{S}_{\mathrm{C}}}(A))(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)\in K} \mathbf{S}_{\mathrm{C}}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x) \leq \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)\in K, \mathbf{S}_{\mathrm{C}}(A_i)=1} \mathbf{S}_{\mathrm{C}}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)\in K, A_i=\mathrm{C}(A_i)} \mathrm{S}(A, A_i) \to A_i(x) = (\mathrm{C}_{\mathcal{S}_{\mathrm{C}}}(A))(x)$.

"$\geq$": By definitions, the inequality holds iff $\mathrm{C}_{\mathcal{S}_{\mathrm{C}}}(A)(x) \leq \mathbf{S}_{\mathrm{C}}(A_j) \otimes \mathrm{S}(A, A_j) \to A_j(x)$ for any j such that $\mathrm{S}(A, A_j) \in K$. Since K is an $\leq$-filter in $\mathbf{L}$, $\mathrm{S}(A, A_j) \in K$ and $\mathrm{S}(A, A_j) \leq \mathrm{S}(A, \mathrm{C}(A_j))$ imply $\mathrm{S}(A, \mathrm{C}(A_j)) \in K$. Therefore, $\mathrm{C}_{\mathcal{S}_{\mathrm{C}}}(A)(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)\in K, A_i=\mathrm{C}(A_i)} \mathrm{S}(A, A_i) \to A_i(x) \leq \mathrm{S}(A, \mathrm{C}(A_j)) \to \mathrm{C}(A_j)(x)$. It follows that it is sufficient to show $\mathrm{S}(A, \mathrm{C}(A_j)) \to \mathrm{C}(A_j)(x) \leq \mathbf{S}_{\mathrm{C}}(A_j) \otimes \mathrm{S}(A, A_j) \to A_j(x)$. The last inequality holds iff $\mathbf{S}_{\mathrm{C}}(A_j) \otimes \mathrm{S}(A, A_j) \otimes (\mathrm{S}(A, \mathrm{C}(A_j)) \to \mathrm{C}(A_j)(x)) \leq A_j(x)$ which is true. Indeed, $\mathbf{S}_{\mathrm{C}}(A_j) \otimes \mathrm{S}(A, A_j) \otimes (\mathrm{S}(A, \mathrm{C}(A_j)) \to \mathrm{C}(A_j)(x)) = \mathrm{S}(\mathrm{C}(A_j), A_j) \otimes \mathrm{S}(A, A_j) \otimes (\mathrm{S}(A, \mathrm{C}(A_j)) \to \mathrm{C}(A_j)(x)) \leq \mathrm{S}(\mathrm{C}(A_j), A_j) \otimes \mathrm{S}(A, \mathrm{C}(A_j)) \otimes (\mathrm{S}(A, \mathrm{C}(A_j)) \to \mathrm{C}(A_j)(x)) \leq \mathrm{S}(\mathrm{C}(A_j), A_j) \otimes \mathrm{C}(A_j)(x) \leq A_j(x)$. □

Lemma 13 *For any $\mathbf{L}_K$-closure operator* C *in X, $\mathbf{S}_{\mathrm{C}}$ is an $\mathbf{L}_K$-closure $\mathbf{L}$-system in X.*

Proof. We verify (14) and (15).

(14): We have to show that for any $A \in L^X$ we have $\mathbf{S}_{\mathrm{C}}(\bigcap_{A_i \in L^X, \mathrm{S}(A,A_i)\in K} \mathbf{S}_{\mathrm{C}}(A_i) \to A_i(x)) = 1$, i.e. $\mathbf{S}_{\mathrm{C}}(\mathrm{C}_{\mathbf{S}_{\mathrm{C}}}(A)) = 1$, i.e. $\mathrm{C}(\mathrm{C}_{\mathbf{S}_{\mathrm{C}}}(A)) = \mathrm{C}_{\mathbf{S}_{\mathrm{C}}}(A)$. The last equality, however, follows from idempotency of C by observing that $\mathrm{C}_{\mathbf{S}_{\mathrm{C}}} = \mathrm{C}$ (Lemma 12).

(15): We have to show $\mathbf{S}(A) \otimes \mathrm{S}(A, B) \otimes \mathrm{S}(B, A) \leq \mathbf{S}(B)$, i.e. $\mathrm{S}(\mathrm{C}(A), A) \otimes \mathrm{S}(A, B) \otimes \mathrm{S}(B, A) \leq \mathrm{S}(\mathrm{C}(B), B)$ which holds iff for each $x \in X$ we have $\mathrm{C}(B)(x) \otimes \mathrm{S}(B, A) \otimes \mathrm{S}(\mathrm{C}(A), A) \otimes \mathrm{S}(A, B) \leq B(x)$. The last inequality is true: $\mathrm{C}(B)(x) \otimes \mathrm{S}(B, A) \otimes \mathrm{S}(\mathrm{C}(A), A) \otimes \mathrm{S}(A, B) \leq \mathrm{C}(B)(x) \otimes \mathrm{S}(\mathrm{C}(B), \mathrm{C}(A)) \otimes \mathrm{S}(\mathrm{C}(A), A) \otimes \mathrm{S}(A, B) \leq \mathrm{C}(B)(x) \otimes \mathrm{S}(\mathrm{C}(B), B) \leq B(x)$. □

Lemma 14 *Let K be a filter in $\mathbf{L}$. For any $\mathbf{L}_K$-closure $\mathbf{L}$-system $\mathbf{S}$ in X, $\mathrm{C}_{\mathbf{S}}$ is an $\mathbf{L}_K$-closure operator in X.*

Proof. We verify (9)–(11).

(9): $A \subseteq \mathrm{C}_{\mathbf{S}}(A)$ holds iff $A(x) \otimes \mathrm{S}(A, A_i) \otimes \mathbf{S}(A_i) \leq A_i(x)$ for any i such that $\mathrm{S}(A, A_i) \in K$ which is evidently true since $A(x) \otimes \mathrm{S}(A, A_i) \otimes \mathbf{S}(A_i) \leq A_i(x) \otimes \mathbf{S}(A_i) \leq A_i(x)$.

(10): Let $\mathrm{S}(A, B) \in K$. $\mathrm{S}(A, B) \leq \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), \mathrm{C}_{\mathbf{S}}(B))$ is true iff for each $x \in X$ we have $\mathrm{S}(A, B) \otimes \mathrm{C}_{\mathbf{S}}(A)(x) \leq \mathrm{C}_{\mathbf{S}}(B)(x)$ iff for any $A_j \in L^X$ such that $\mathrm{S}(B, A_j) \in K$ we have $\mathbf{S}(A_j) \otimes \mathrm{S}(B, A_j) \otimes \mathrm{S}(A, B) \otimes (\bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)\in K} \mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x)) \leq A_j(x)$. The last inequality is true: since $\mathrm{S}(A, B) \otimes \mathrm{S}(B, A_j) \leq \mathrm{S}(A, A_j)$, $\mathrm{S}(A, B) \in K$ and $\mathrm{S}(B, A_j) \in K$ yield $\mathrm{S}(A, A_j) \in K$. Therefore, $\mathbf{S}(A_j) \otimes \mathrm{S}(B, A_j) \otimes \mathrm{S}(A, B) \otimes (\bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)\in K} \mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x)) \leq \mathbf{S}(A_j) \otimes \mathrm{S}(A, A_j) \otimes (\mathbf{S}(A_j) \otimes \mathrm{S}(A, A_j) \to A_j(x)) \leq A_j(x)$.

(11): $\mathrm{C}_{\mathbf{S}}(\mathrm{C}_{\mathbf{S}}(A))(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A),A_i)\in K} \mathbf{S}(A_i) \otimes \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A_i) \to A_i(x) \leq \mathbf{S}(\mathrm{C}_{\mathbf{S}}(A)) \otimes \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), \mathrm{C}_{\mathbf{S}}(A)) \to \mathrm{C}_{\mathbf{S}}(A)(x) = 1 \to \mathrm{C}_{\mathbf{S}}(A)(x) = \mathrm{C}_{\mathbf{S}}(A)(x)$. □

The relationship between $\mathbf{L}_K$-closure operators, $\mathbf{L}_K$-closure systems, and $\mathbf{L}_K$-closure $\mathbf{L}$-systems is the subject of the following theorems.

Theorem 15 *Let* C *be an* $\mathbf{L}_K$*-closure operator in* X*,* $\mathbf{S}$ *be an* $\mathbf{L}_K$*-closure* $\mathbf{L}$*-system,* K *be a filter in* $\mathbf{L}$*. Then* $\mathbf{S}_{\mathrm{C}}$ *is an* $\mathbf{L}_K$*-closure* $\mathbf{L}$*-system in* X*,* $\mathrm{C}_{\mathbf{S}}$ *is an* $\mathbf{L}_K$*-closure operator in* X*, and* $\mathrm{C} = \mathrm{C}_{\mathbf{S}_{\mathrm{C}}}$ *and* $\mathbf{S} = \mathbf{S}_{\mathrm{C}_{\mathbf{S}}}$*, i.e. the mappings* $\mathrm{C} \mapsto \mathbf{S}_{\mathrm{C}}$ *and* $\mathbf{S} \mapsto \mathrm{C}_{\mathbf{S}}$ *are mutually inverse.*

Proof. By Lemma 13 and Lemma 14, it remains to verify $\mathrm{C} = \mathrm{C}_{\mathbf{S}_{\mathrm{C}}}$ and $\mathbf{S} = \mathbf{S}_{\mathrm{C}_{\mathbf{S}}}$. By Lemma 12 and by Theorem 10, $\mathrm{C}_{\mathbf{S}_{\mathrm{C}}} = \mathrm{C}_{\mathcal{S}_{\mathrm{C}}} = \mathrm{C}$. Since $\mathbf{S}_{\mathrm{C}_{\mathbf{S}}}(A) = \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A)$, it remains to prove $\mathbf{S}(A) = \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A)$: On the one hand, $\mathbf{S}(A) \leq \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A)$ iff for each $x \in X$ we have $\mathbf{S}(A) \otimes \mathrm{C}_{\mathbf{S}}(A)(x) \leq A(x)$, i.e. $\mathbf{S}(A) \otimes \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i) \in K} (\mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x)) \leq A(x)$ which is true since $\mathbf{S}(A) \otimes \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i) \in K} (\mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x)) \leq \mathbf{S}(A) \otimes (\mathbf{S}(A) \otimes \mathrm{S}(A, A) \to A(x)) \leq A(x)$. On the other hand, $\mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A) = \mathbf{S}(\mathrm{C}_{\mathbf{S}}(A)) \otimes \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A) \otimes \mathrm{S}(A, \mathrm{C}_{\mathbf{S}}(A)) \leq \mathbf{S}(A)$, by (15). □

Theorem 16 *Let* $\mathcal{S}$ *be an* $\mathbf{L}_K$*-closure system in* X*,* $\mathbf{S}$ *be an* $\mathbf{L}_K$*-closure* $\mathbf{L}$*-system,* K *be a filter in* $\mathbf{L}$*. Then* $\mathbf{S}_{\mathcal{S}}$ *is an* $\mathbf{L}_K$*-closure* $\mathbf{L}$*-system in* X*,* $\mathcal{S}_{\mathbf{S}}$ *is an* $\mathbf{L}_K$*-closure system in* X*, and* $\mathcal{S} = \mathcal{S}_{\mathbf{S}_{\mathcal{S}}}$ *and* $\mathbf{S} = \mathbf{S}_{\mathcal{S}_{\mathbf{S}}}$*, i.e. the mappings* $\mathcal{S} \mapsto \mathbf{S}_{\mathcal{S}}$ *and* $\mathbf{S} \mapsto \mathcal{S}_{\mathbf{S}}$ *are mutually inverse.*

Proof. By definition, $\mathbf{S}_{\mathcal{S}} = \mathbf{S}_{\mathrm{C}_{\mathcal{S}}}$, therefore, by Lemma 13, $\mathbf{S}_{\mathcal{S}}$ is an $\mathbf{L}_K$-closure $\mathbf{L}$-system. To see that $\mathcal{S}_{\mathbf{S}}$ is an $\mathbf{L}_K$-closure system it is, by Theorem 10, sufficient to show $\mathcal{S}_{\mathbf{S}} = \mathcal{S}_{\mathrm{C}_{\mathbf{S}}}$, i.e. $\{A \in L^X \mid \mathbf{S}(A) = 1\} = \{A \in L^X \mid A = \mathrm{C}_{\mathbf{S}}(A)\}$: On the one hand, $\mathbf{S}(A) = 1$ implies $\mathrm{C}_{\mathbf{S}}(A)(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i)} \mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i) \to A_i(x) \leq \mathbf{S}(A) \otimes \mathrm{S}(A, A) \to A(x) = A(x)$, i.e. $A = \mathrm{C}_{\mathbf{S}}(A)$. On the other hand, $A = \mathrm{C}_{\mathbf{S}}(A)$ implies (using (14)) $1 = \mathbf{S}(\mathrm{C}_{\mathbf{S}}(A)) = \mathbf{S}(A)$. Therefore, $\mathcal{S}_{\mathbf{S}} = \mathcal{S}_{\mathrm{C}_{\mathbf{S}}}$. We show $\mathcal{S} = \mathcal{S}_{\mathbf{S}_{\mathcal{S}}}$: We have $A \in \mathcal{S}$ iff $A = \mathrm{C}_{\mathcal{S}}(A)$ iff $\mathbf{S}_{\mathrm{C}_{\mathcal{S}}}(A) = 1$ iff $\mathbf{S}_{\mathcal{S}}(A) = 1$ iff $A \in \mathcal{S}_{\mathbf{S}_{\mathcal{S}}}$. It remains to show $\mathbf{S}(A) = \mathbf{S}_{\mathcal{S}_{\mathbf{S}}}(A)$: We have $\mathrm{C}_{\mathbf{S}} = \mathrm{C}_{\mathcal{S}_{\mathrm{C}_{\mathbf{S}}}}$ and (by the above observation) $\mathcal{S}_{\mathrm{C}_{\mathbf{S}}} = \mathcal{S}_{\mathbf{S}}$. Therefore, $\mathrm{C}_{\mathbf{S}} = \mathrm{C}_{\mathcal{S}_{\mathbf{S}}}$. Using $\mathbf{S}(A) = \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A)$ (see the end of the proof of Theorem 15), we conclude $\mathbf{S}(A) = \mathrm{S}(\mathrm{C}_{\mathbf{S}}(A), A) = \mathrm{S}(\mathrm{C}_{\mathcal{S}_{\mathbf{S}}}(A), A) = \mathbf{S}_{\mathcal{S}_{\mathbf{S}}}(A)$ completing the proof. □

$\mathbf{L}_K$-Galois connections

Definition 17 *An* $\mathbf{L}_K$-Galois connection *(*fuzzy Galois connection*) between the sets* X *and* Y *is a pair* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *of mappings* ${}^{\uparrow} : L^X \to L^Y$*,* ${}^{\downarrow} : L^Y \to L^X$*, satisfying*

$$\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_2^{\uparrow}, A_1^{\uparrow}) \quad \textit{whenever } \mathrm{S}(A_1, A_2) \in K \tag{21}$$

$$\mathrm{S}(B_1, B_2) \leq \mathrm{S}(B_2^{\downarrow}, B_1^{\downarrow}) \quad \textit{whenever } \mathrm{S}(B_1, B_2) \in K \tag{22}$$

$$A \subseteq (A^{\uparrow})^{\downarrow} \tag{23}$$

$$B \subseteq (B^{\downarrow})^{\uparrow}\ . \tag{24}$$

for every $A, A_1, A_2 \in L^X$*,* $B, B_1, B_2 \in L^Y$*.*

If $K = L$ then we again omit the subscript K. Note also that an $\mathbf{L}_K$-Galois connection between X and Y forms a Galois connection between the complete lattices $\langle L^X, \subseteq \rangle$ and $\langle L^Y, \subseteq \rangle$ [8, 17].

Remark Note that Galois connections between sets [8, 17] are just $\mathbf{L}$-Galois connections for $\mathbf{L} = \mathbf{2}$.

For the following simple characterization see [2].

Theorem 18 *A pair* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *forms an* $\mathbf{L}_K$*-Galois connection between* X *and* Y *iff* $\mathrm{S}(A, B^{\downarrow}) \in K$ *or* $\mathrm{S}(B, A^{\uparrow}) \in K$ *implies*

$$\mathrm{S}(A, B^{\downarrow}) = \mathrm{S}(B, A^{\uparrow}) \tag{25}$$

for all $A \in L^X$*,* $B \in L^Y$*.*

Call two systems $\mathcal{S}_1 \subseteq L^X$ and $\mathcal{S}_2 \subseteq L^Y$ of $\mathbf{L}$-sets in X and $\mathbf{L}$-sets in Y, respectively, S_K*-dually isomorphic* iff there is a bijective mapping $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ such that for $A_1, A_2 \in \mathcal{S}_1$, $B_1 = \varphi(A_1), B_2 = \varphi(A_2) \in \mathcal{S}_2$ it holds that $\mathrm{S}(A_1, A_2) = \mathrm{S}(B_2, B_1)$ whenever $\mathrm{S}(A_1, A_2) \in K$ or $\mathrm{S}(B_2, B_1) \in K$.

Theorem 19 *Let* $\langle ^{\uparrow}, ^{\downarrow} \rangle$ *be an* $\mathbf{L}_K$*-Galois connection between* X *and* Y*, let* C^X *and* C^Y *be* $\mathbf{L}_K$*-closure operators on* X *and* Y*, respectively, such that* $\mathcal{S}_{\mathrm{C}^X}$ *and* $\mathcal{S}_{\mathrm{C}^Y}$ *are* S_K*-dually isomorphic with* φ *being the isomorphism. Put* $\mathrm{C}^X_{\langle \uparrow, \downarrow \rangle} = {}^{\uparrow\downarrow}$ *and* $\mathrm{C}^Y_{\langle \uparrow, \downarrow \rangle} = {}^{\downarrow\uparrow}$ *(the composite mappings), and let* $(A)^{\uparrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}} = \varphi(\mathrm{C}(A))$ *and* $(B)^{\downarrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}} = \varphi^{-1}(\mathrm{C}(B))$ *for* $A \in L^X$*,* $B \in L^Y$*. Then the following is true.*

(1) $\mathrm{C}^X_{\langle \uparrow, \downarrow \rangle}$ *and* $\mathrm{C}^Y_{\langle \uparrow, \downarrow \rangle}$ *are* $\mathbf{L}_K$*-closure operators on* X *and* Y*, respectively, and* $\mathcal{S}_{\mathrm{C}^X_{\langle \uparrow, \downarrow \rangle}}$ *and* $\mathcal{S}_{\mathrm{C}^Y_{\langle \uparrow, \downarrow \rangle}}$ *are* S_K*-dually isomorphic.*

(2) $\langle ^{\uparrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}}, ^{\downarrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}} \rangle$ *is an* $\mathbf{L}_K$*-Galois connection.*

(3) The correspondences defined by $\langle ^{\uparrow}, ^{\downarrow} \rangle \mapsto \langle \mathrm{C}^X_{\langle \uparrow, \downarrow \rangle}, \mathrm{C}^Y_{\langle \uparrow, \downarrow \rangle} \rangle$ *and* $\langle \mathrm{C}^X, \mathrm{C}^Y \rangle \mapsto \langle ^{\uparrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}}, ^{\downarrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}} \rangle$ *are mutually inverse mappings.*

Proof. (1): (23) implies (9). If $\mathrm{S}(A_1, A_2) \in K$ then, by (21), by the $\leq$-filter property of K, and by (22), we have $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_2^{\uparrow}, A_1^{\uparrow}) \leq \mathrm{S}(A_1^{\uparrow\downarrow}, A_2^{\uparrow\downarrow})$, i.e. (10) holds. The fact that $\langle ^{\uparrow}, ^{\downarrow} \rangle$ is a Galois connection between the lattices L^X and L^Y immediately gives (11) [8], i.e. $\mathrm{C}^X_{\langle \uparrow, \downarrow \rangle}$ is an $\mathbf{L}_K$-closure operator on X. The proof for $\mathrm{C}^Y_{\langle \downarrow, \uparrow \rangle}$ is completely analogous. The rest easily follows by Theorem 18 observing that $A = A^{\uparrow\downarrow}$ and $B = B^{\downarrow\uparrow}$ for $A \in \mathcal{S}_{\mathrm{C}^X_{\langle \uparrow, \downarrow \rangle}}$ and $B \in \mathcal{S}_{\mathrm{C}^Y_{\langle \uparrow, \downarrow \rangle}}$.

(2): For simplicity, we write only $^{\uparrow}$ instead of $^{\uparrow_{\langle \mathrm{C}^X, \mathrm{C}^Y \rangle}}$, the same for $^{\downarrow}$. First, since $\mathrm{C}^X(A) = \varphi^{-1}(\varphi(\mathrm{C}^X(A))) = \varphi^{-1}(\mathrm{C}^Y(\varphi(\mathrm{C}^X(A)))) = A^{\uparrow\downarrow}$, we have $\mathrm{C}^X = \mathrm{C}^X_{\langle \uparrow, \downarrow \rangle}$ whence (21) follows. If $\mathrm{S}(A_1, A_2) \in K$ then by (10) and the assumption $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(\mathrm{C}^X(A_1), \mathrm{C}^X(A_2)) = \mathrm{S}(\varphi(\mathrm{C}^X(A_2)), \varphi(\mathrm{C}^X(A_1))) = \mathrm{S}(A_2^{\uparrow}, A_1^{\uparrow})$, i.e. (22) holds. The rest of the statement can be proved analogously.

Part (3) now follows easily from the proof of (2) and the fact that $A^{\uparrow} = A^{\uparrow\downarrow\uparrow}$ and $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$ for $A \in L^X$, $B \in L^Y$. □

For $K = L$ the foregoing theorem can be strengthened.

Theorem 20 *Let* C *be an* $\mathbf{L}$*-closure operator, and* $Y = \{\mathrm{C}(A) \mid A \in L^X\}$*. Then the pair of mappings* $^{\uparrow_\mathrm{C}} : L^X \to L^Y$*,* $^{\downarrow_\mathrm{C}} : L^Y \to L^X$ *defined for* $A \in L^X$*,* $B \in L^Y$ *and* $x \in X$*,* $A' \in Y$ *by*

$$\begin{aligned} A^{\uparrow_\mathrm{C}}(A') &= \mathrm{S}(A, A') \\ B^{\downarrow_\mathrm{C}}(x) &= \bigwedge_{A \in Y, B(A) \in K} B(A) \to (A)(x) \end{aligned}$$

forms an $\mathbf{L}$*-Galois connection such that* $C = {}^{\uparrow_\mathrm{C}\downarrow_\mathrm{C}}$*.*

Proof. For brevity we write $^{\uparrow}$ and $^{\downarrow}$ instead of $^{\uparrow_\mathrm{C}}$ and $^{\downarrow_\mathrm{C}}$, respectively. We first verify (21)–(24).

(21): $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_1^{\uparrow}, A_2^{\uparrow})$ holds iff for each $A' \in Y$ it holds $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_2, A') \to \mathrm{S}(A_1, A')$ which holds iff $\mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A') \leq \mathrm{S}(A_1, A')$ iff for each $x \in X$ it holds $A_1(x) \otimes \mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A') \leq A'(x)$ which is true since $A_1(x) \otimes \mathrm{S}(A_1, A_2) \otimes \mathrm{S}(A_2, A') \leq A_1(x) \otimes (A_1(x) \to A_2(x)) \otimes (A_2(x) \to A'(x)) \leq A'(x)$.

(22): Let $B_1, B_2 \in L^Y$. We have to prove $\mathrm{S}(B_1, B_2) \leq \mathrm{S}(B_2^{\downarrow}, B_1^{\downarrow})$ which holds iff for each $x \in X$ it holds $\mathrm{S}(B_1, B_2) \otimes B_2^{\downarrow}(x) \leq B_1^{\downarrow}(x) = \bigwedge_{A \in Y}(B_1(A) \to A(x))$ iff for each $A \in Y$ it holds $B_1(A) \otimes \mathrm{S}(B_1, B_2) \otimes B_2^{\downarrow}(x) \leq A(x)$ which is valid since $B_1(A) \otimes \mathrm{S}(B_1, B_2) \otimes B_2^{\downarrow}(x) \leq B_1(A) \otimes (B_1(A) \to B_2(A)) \otimes (B_2(A) \to A(x)) \leq A(x)$.

To show (23), it suffices to show that $\mathrm{C} = {}^{\uparrow\downarrow}$. Let thus $A \in L^X$, $x \in X$. We show $\mathrm{C}(A)(x) = A^{\uparrow\downarrow}(x)$ by proving both of the inequalities. "$\leq$": $\mathrm{C}(A)(x) \leq A^{\uparrow\downarrow}(x) = \bigwedge_{A' \in Y}(A^{\uparrow}(A') \to A'(x))$ holds iff for each $A' \in Y$ it holds $A^{\uparrow}(A') \otimes \mathrm{C}(A)(x) \leq A'(x)$ which holds iff $A^{\uparrow}(A') \leq \mathrm{C}(A)(x) \to A'(x)$, i.e. $\mathrm{S}(A, A') \leq \mathrm{C}(A)(x) \to A'(x)$ which holds since $\mathrm{S}(A, A') \leq \mathrm{S}(\mathrm{C}(A), \mathrm{C}(A')) \leq \mathrm{C}(A)(x) \to \mathrm{C}(A')(x) = \mathrm{C}(A)(x) \to A'(x)$ as $\mathrm{C}(A') = A'$. "$\geq$": $A^{\uparrow\downarrow}(x) \leq \mathrm{C}(A)(x)$, i.e. $\bigwedge_{A' \in Y}(\mathrm{S}(A, A') \to A'(x)) \leq \mathrm{C}(A)(x)$ holds since for $A' = \mathrm{C}(A)$ we have $\mathrm{S}(A, \mathrm{C}(A)) \to \mathrm{C}(A)(x) = 1 \to \mathrm{C}(A)(x) = \mathrm{C}(A)(x)$.

Ad (24): We have $B(A) \leq B^{\downarrow\uparrow}(A)$ iff $B(A) \leq \mathrm{S}(B^{\downarrow}, A) = \bigwedge_{x \in X}(B^{\downarrow}(x) \to A(x))$ which holds iff for each $x \in X$ we have $B(A) \otimes B^{\downarrow}(x) \leq A(x)$, i.e. $B(A) \otimes \bigwedge_{A' \in Y}(B(A') \to A'(x)) \leq A(x)$ which holds (putting $A' = A$ the inequality is evident). The proof is complete. □

4 Induced relations: quasiorder and similarity

An $\mathbf{L}$-relation R on a set X is called

$$\begin{array}{rl} \text{reflexive if} & R(x,x)=1 \\ \text{symmetric if} & R(x,y)=R(y,x) \\ K\text{-transitive if} & R(x,y)\otimes R(y,z)\leq R(x,z) \\ & \text{whenever } R(x,y)\in K \text{ and } R(y,z)\in K. \end{array}$$

An $\mathbf{L}_K$*-quasiorder* on X is an $\mathbf{L}$-relation on X that is reflexive and K-transitive. An $\mathbf{L}_K$*-similarity* (or $\mathbf{L}_K$-equivalence) on X is an $\mathbf{L}$-relation on X that is reflexive, symmetric, and K-transitive.

Remark (1) Clearly, putting $\mathbf{L}=\mathbf{2}$ we get the usual (bivalent) notion of a quasiorder and an equivalence relation (no matter what K). Thus, the notions of $\mathbf{L}_K$-quasiorder and $\mathbf{L}_K$-similarity are generalizations of of the bivalent notions.

(2) If K is interpreted as the set of sufficiently high (designated) truth values, then K-transitivity means: "if the facts that $\langle x,y\rangle$ belongs to R and $\langle y,z\rangle$ belongs to R are sufficiently high then $\langle x,z\rangle$ also belongs to R (and is sufficiently high in case K is a filter)". For $K=L$ (in which case we omit the subscript K) we get the usual notions of $\mathbf{L}$-quasiorder and $\mathbf{L}$-similarity (called usually fuzzy quasiorder and fuzzy similarity).

Induced L-relations on X

Theorem 21 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*. Then the relation* Q_{C} *in* X *defined by*

$$Q_{\mathrm{C}}(x,y)=\mathrm{C}(\{\,{}^{1}\!/x\})(y)$$

is an $\mathbf{L}_K$*-quasiorder.*

Proof. Since $\{\,{}^{1}\!/x\}\subseteq\mathrm{C}(\{\,{}^{1}\!/x\})$, reflexivity follows by $Q(x,x)=\mathrm{C}(\{\,{}^{1}\!/x\})(x)=1$. K-transitivity: Let $Q_{\mathrm{C}}(x,y)\in K$, $Q_{\mathrm{C}}(y,z)\in K$. We have to show $Q_{\mathrm{C}}(x,y)\otimes Q_{\mathrm{C}}(y,z)\leq Q_{\mathrm{C}}(x,z)$. By adjointness and by definition of Q_{C} we thus have to show that $\mathrm{C}(\{\,{}^{1}\!/x\})(y)\leq\mathrm{C}(\{\,{}^{1}\!/y\})(z)\to\mathrm{C}(\{\,{}^{1}\!/x\})(z)$ whenever $\mathrm{C}(\{\,{}^{1}\!/x\})(y)$, $\mathrm{C}(\{\,{}^{1}\!/y\})(z)\in K$. We have $\mathrm{S}(\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/\},\mathrm{C}(\{\,{}^{1}\!/x\}))=1$, therefore, by (10) and using $\mathrm{C}(\mathrm{C}(\{\,{}^{1}\!/x\}))=\mathrm{C}(\{\,{}^{1}\!/x\})$, also $\mathrm{S}(\mathrm{C}(\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/\}),\mathrm{C}(\{\,{}^{1}\!/x\}))=1$. Furthermore, by $\mathrm{C}(\{\,{}^{1}\!/x\})(y)\in K$ and (10), $\mathrm{C}(\{\,{}^{1}\!/x\})(y)=\mathrm{S}(\{\,{}^{1}\!/y\},\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/y\})\leq\mathrm{S}(\mathrm{C}(\{\,{}^{1}\!/y\}),\mathrm{C}(\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/y\}))$. Therefore, $\mathrm{C}(\{\,{}^{1}\!/x\})(y)\leq\mathrm{S}(\mathrm{C}(\{\,{}^{1}\!/y\}),\mathrm{C}(\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/y\}))=\mathrm{S}(\mathrm{C}(\{\,{}^{1}\!/y\}),\mathrm{C}(\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/y\}))\otimes\mathrm{S}(\mathrm{C}(\{\,{}^{\mathrm{C}(\{{}^{1}\!/x\})(y)}\!/\}),\mathrm{C}(\{\,{}^{1}\!/x\}))\leq\mathrm{S}(\mathrm{C}(\{\,{}^{1}\!/y\}),\mathrm{C}(\{\,{}^{1}\!/x\}))\leq\mathrm{C}(\{\,{}^{1}\!/y\})(z)\to\mathrm{C}(\{\,{}^{1}\!/x\})(z)$. The proof is complete. □

Remark Note that Q_{C} actually satisfies a stronger condition than K-transitivity. Namely, as it follows from the proof of Theorem 21, $Q_{\mathrm{C}}(x,y)\otimes Q_{\mathrm{C}}(y,z)\leq Q_{\mathrm{C}}(x,z)$ whenever $Q_{\mathrm{C}}(x,y)\in K$. This property is typical for Pavelka style fuzzy logic (see [18] and also [15]): Take X to be the set of all formulas; put $K=\{1\}$; let C be the operator of syntactic consequence, i.e. for an $\mathbf{L}$-set A of formulas and a formula $x\in X$, let $\mathrm{C}(A)(x)$ be the degree of provability of x from A. One easily verifies that C satisfies the above condition stronger than K-transitivity. On the other hand, C does not satisfy the in a sense symmetric condition, i.e. it is not true that if $Q_{\mathrm{C}}(y,z)\in K$ then $Q_{\mathrm{C}}(x,y)\otimes Q_{\mathrm{C}}(y,z)\leq Q_{\mathrm{C}}(x,z)$.

$Q_{\mathrm{C}}(x,y)$ is naturally interpreted as the truth degree to which y belongs to the closure of a singleton containing x. One might wonder what is the relationship between Q_{C} and $Q_{\mathcal{S}_{\mathrm{C}}}$ defined by

$$Q_{\mathcal{S}_{\mathrm{C}}}(x,y)=\bigwedge_{A\in\mathcal{S}_{\mathrm{C}},A(x)\in K}A(x)\to A(y),$$

i.e. the truth degree to which it holds that whenever it is sufficiently true that x belongs to some closure then y belongs to that closure as well.

Theorem 22 *For any* $\mathbf{L}_K$*-closure operator* C *we have* $Q_{\mathrm{C}} = Q_{\mathcal{S}_{\mathrm{C}}}$*. Therefore,* $Q_{\mathcal{S}_{\mathrm{C}}}$ *is an* $\mathbf{L}_K$*-quasiorder.*

Proof. On the one hand, $\mathrm{C}(\{{}^1/x\})(x) = 1 \in K$ yields $Q_{\mathcal{S}_{\mathrm{C}}}(x,y) = \bigwedge_{A\in\mathcal{S}_{\mathrm{C}},A(x)\in K} A(x) \to A(y) \leq \mathrm{C}(\{{}^1/x\})(x) \to \mathrm{C}(\{{}^1/x\})(y) = 1 \to \mathrm{C}(\{{}^1/x\})(y) = \mathrm{C}(\{{}^1/x\})(y) = Q_{\mathrm{C}}(x,y)$. On the other hand, $Q_{\mathrm{C}}(x,y) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y)$ is true iff for each $A \in \mathcal{S}_{\mathrm{C}}$ such that $A(x) \in K$ we have $\mathrm{C}(\{{}^1/x\})(y) \leq A(x) \to A(y)$. Applying adjointness twice, the last inequality is equivalent to $A(x) \leq \mathrm{C}(\{{}^1/x\})(y) \to A(y)$ which is true. Indeed, since $A(x) \in K$, (10) gives $A(x) = 1 \to A(x) = \{{}^1/x\}(x) \to A(x) = \mathrm{S}(\{{}^1/x\}, A) \leq \mathrm{S}(\mathrm{C}(\{{}^1/x\}), \mathrm{C}(A)) = \mathrm{S}(\mathrm{C}(\{{}^1/x\}), A) \leq \mathrm{C}(\{{}^1/x\}(y) \to A(y)$. □

If K is, moreover, a filter in $\mathbf{L}$, the fact that Q_{C} is an $\mathbf{L}_K$-quasiorder follows from Theorem 22 and the following statement.

Lemma 23 *Let* K *be an filter in* $\mathbf{L}$*,* $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ *be a system of* $\mathbf{L}$*-sets. Then the* $\mathbf{L}$*-relation* $Q_{\mathcal{S}}$ *on* X *given by*

$$Q_{\mathcal{S}}(x,y) = \bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y)$$

is an $\mathbf{L}_K$*-quasiorder.*

Proof. Reflexivity follows from the fact that $A_i(x) \to A_i(x) = 1$ and from $\bigwedge \emptyset = 1$. K-transitivity: By definition, we have to show that if $\bigwedge_{i\in I,A_i(x)\in K} A_i(x) \to A_i(y) \in K$ and $\bigwedge_{i\in I,A_i(x)\in K} A_i(y) \to A_i(z) \in K$ then $(\bigwedge_{i\in I,A_i(x)\in K} A_i(x) \to A_i(y)) \otimes (\bigwedge_{i\in I,A_i(x)\in K} A_i(y) \to A_i(z)) \leq (\bigwedge_{i\in I,A_i(x)\in K} A_i(x) \to A_i(z))$, i.e. to show that for each $i \in I$ such that $A_i(x) \in K$ we have $A_i(x) \otimes (\bigwedge_{i\in I,A_i(x)\in K} A_i(x) \to A_i(y)) \otimes (\bigwedge_{i\in I,A_i(x)\in K} A_i(y) \to A_i(z)) \leq A_i(z)$. This inequality is true. Indeed, $\bigwedge_{i\in I,A_i(x)\to A_i(y)} \in K$ and $\bigwedge_{i\in I,A_i(x)\to A_i(y)} \leq A_i(x) \leq A_i(x) \to A_i(y)$ gives $A_i(x) \to A_i(y) \in K$. As also $A_i(x) \in K$, we have $A_i(x)\otimes(A_i(x) \to A_i(y)) \in K$. From $A_i(x)\otimes(A_i(x) \to A_i(y)) \leq A_i(y)$ we thus have $A_i(y) \in K$. Therefore, we conclude $A_i(x) \otimes (\bigwedge_{i\in I,A_i(x)\in K} A_i(x) \to A_i(y)) \otimes (\bigwedge_{i\in I,A_i(x)\in K} A_i(y) \to A_i(z)) \leq A_i(x) \otimes (A_i(x) \to A_i(y)) \otimes (A_i(y) \to A_i(z)) \leq A_i(z)$, completing the proof. □

Indeed, putting $\mathcal{S} = \mathcal{S}_{\mathrm{C}}$, Lemma 23 yields that $Q_{\mathcal{S}_{\mathrm{C}}}$ is an $\mathbf{L}_K$-quasiorder. Theorem 22 then completes the argument.

Remark (1) A closer look at the proof of Lemma 23 shows that like Q_{C}, $Q_{\mathcal{S}}$ it satisfies a stronger form of K-transitivity: $Q_{\mathcal{S}}(x,y) \otimes Q_{\mathcal{S}}(y,z) \leq Q_{\mathcal{S}}(x,z)$ whenever $Q_{\mathcal{S}}(x,y) \in K$.

(2) The assumption of Lemma 23 that K be closed w.r.t. $\otimes$ is essential. As a counterexample, consider $I = \{i,j\}$, $X = \{x,y,z\}$, $L = [0,1]$ equipped with Łukasiewicz structure, $A_i(x) = 0.9$, $A_i(y) = 0.8$, $A_i(z) = 0.7$, $A_j(x) = A_j(y) = A_j(z) = 1$. Taking $K = [0.9, 1]$ which is an $\leq$-filter not closed w.r.t. $\otimes$, we have $Q_{\mathcal{S}}(x,y) \otimes Q_{\mathcal{S}}(y,z) = 0.9 \otimes 1 = 0.9 \not\leq 0.8 = Q_{\mathcal{S}}(x,z)$.

It can be easily seen that every $\mathbf{L}_K$-quasiorder on X induces an $\mathbf{L}_K$-similarity E_Q on X by putting $E_Q(x,y) = Q(x,y) \wedge Q(y,x)$. Therefore, for any $\mathbf{L}_K$-closure operator C on X, the $\mathbf{L}$-relation E_{C} on X defined by

$$E_{\mathrm{C}}(x,y) = Q_{\mathrm{C}}(x,y) \wedge Q_{\mathrm{C}}(y,x)$$

is an $\mathbf{L}_K$-similarity on X. We say that an $\mathbf{L}_K$-similarity E on X is compatible with $A \in L^X$ if $A(x) \otimes E(x,y) \leq A(y)$ holds for any $x,y \in X$ such that $A(x) \in K$. The condition of compatibility translates verbally to "if it is sufficiently true that x belongs to A and if x and y are similar then y belongs to A as well".

Theorem 24 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*. Then* E_{C} *is the largest* $\mathbf{L}_K$*-similarity on* X *that is compatible w.r.t. every* C*-closed* $\mathbf{L}$*-set (i.e. w.r.t. every* $A \in \mathcal{S}_{\mathrm{C}}$*).*

Proof. The fact that E_{C} is an $\mathbf{L}_K$-similarity on X was established in the above paragraph. Let $x, y \in X$, $A \in \mathcal{S}_{\mathrm{C}}$, $A(x) \in K$. By Theorem 22, $E_{\mathrm{C}}(x,y) = Q_{\mathcal{S}_{\mathrm{C}}}(x,y) \wedge Q_{\mathcal{S}_{\mathrm{C}}}(y,x) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y) = \bigwedge_{A\in\mathcal{S}_{\mathrm{C}},A(x)\in K} A(x) \to A(y) \leq A(x) \to A(y)$, i.e. $A(x) \otimes E_{\mathrm{C}}(x,y) \leq A(y)$ by adjointness. We proved that E_{C} is compatible w.r.t. any $A \in \mathcal{S}_{\mathrm{C}}$.

Let E be an $\mathbf{L}_K$-similarity that is compatible w.r.t. any $A \in \mathcal{S}_{\mathrm{C}}$. Take $x, y \in X$ and an $A \in \mathcal{S}_{\mathrm{C}}$ such that $A(x) \in K$. Compatibility of E yields $A(x) \otimes E(x,y) \leq A(y)$, i.e. $E(x,y) \leq A(x) \to A(y)$. Since x, y, and A were chosen arbitrarily, we conclude $E(x,y) \leq \bigwedge_{A\in\mathcal{S}_{\mathrm{C}},A(x)\in K} A(x) \to A(y) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y)$. Due to the symmetry of E we finally have $E(x,y) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y) \wedge Q_{\mathcal{S}_{\mathrm{C}}}(y,x) = E_{\mathrm{C}}(x,y)$ proving that E_{C} is the largest $\mathbf{L}_K$-similarity compatible with all $A \in \mathcal{S}_{\mathrm{C}}$. □

Factorization of $\mathbf{L}_K$-closure systems by similarity For an $\mathbf{L}_K$-closure operator on X, $\mathcal{S}_{\mathrm{C}}$ is a complete lattice w.r.t. $\subseteq$ (follows from Remark (2) after Definition 4). For various reasons (e.g. for computational ones), it might not be desirable to distinguish the particular $\mathbf{L}$-sets in X. Rather, it can be advantageous to treat $\mathbf{L}$-sets which are similar in terms of membership degrees of elements of X as if they were the same, i.e. one might desire to perform a kind of abstraction by factorization w.r.t. to a suitable similarity defined on $\mathbf{L}$-sets. A suitable $\mathbf{L}$-similarity relation is described in the following assertion (see e.g. [4]).

Lemma 25 *The $\mathbf{L}$-relation E on X defined by*

$$E(A,B) = \bigwedge_{x\in X} A(x) \leftrightarrow B(x)$$

is an $\mathbf{L}$-similarity on X.

For $A, B \in L^X$, $E(A,B)$ is the truth degree to which it is true that for any $x \in X$, x belongs to A iff x belongs to B. The first observation states that sufficiently high similarity between $\mathbf{L}$-sets is preserved by $\mathbf{L}_K$-closure operators.

Theorem 26 *For an $\mathbf{L}_K$-closure operator C on X and $A, B \in L^X$ we have $E(A,B) \leq E(\mathrm{C}(A),\mathrm{C}(B))$ whenever $E(A,B) \in K$.*

Proof. It is easy to see that $E(A,B) = \mathrm{S}(A,B) \wedge \mathrm{S}(B,A)$. Therefore, $E(A,B) \in K$ yields $\mathrm{S}(A,B) \in K$ and $\mathrm{S}(B,A) \in K$. Applying (10) we get $\mathrm{S}(A,B) \leq \mathrm{S}(\mathrm{C}(A),\mathrm{C}(B))$ and $\mathrm{S}(B,A) \leq \mathrm{S}(\mathrm{C}(B),\mathrm{C}(A))$, and thus $E(A,B) = \mathrm{S}(A,B) \wedge \mathrm{S}(B,A) \leq \mathrm{S}(\mathrm{C}(A),\mathrm{C}(B)) \wedge \mathrm{S}(\mathrm{C}(B),\mathrm{C}(A)) = E(\mathrm{C}(A),\mathrm{C}(B))$. □

Intuitively, the complete lattice $\mathcal{S}_{\mathrm{C}}$ can be simplified by putting similar closed $\mathbf{L}$-sets together, i.e. putting together $\mathbf{L}$-sets A and B for which $E(A,B)$ is high. Putting the $\mathbf{L}$-sets together should be compatible w.r.t. to the complete lattice structure on $\mathcal{S}_{\mathrm{C}}$. Recall that for any $a \in L$, an a-cut of E is a (bivalent) relation ${}^aE$ on X defined by $\langle x,y\rangle \in {}^aE$ iff $a \leq E(x,y)$. It is easy to see that ${}^aE$ is always a tolerance on X (i.e. a reflexive and symmetric relation on X). If $\otimes = \wedge$, ${}^aE$ is, moreover, transitive, i.e. an equivalence relation. In general, ${}^aE$ is not transitive. Factorization of a structure by a compatible tolerance relation is, in general, not possible (one needs transitivity so that operations on the factor set can be defined). Surprisingly, Czédli [10] showed a way to factorize lattices by compatible tolerances (for factorization of complete lattices by tolerances see [21]). We now recall the necessary concepts: Let T be a tolerance relation on a support V of a complete lattice $\mathbf{V} = \langle V, \leq\rangle$. T is called compatible if it is preserved under arbitrary infima and suprema, i.e. if $\langle u_i, v_i\rangle \in T$ $(i \in I)$ implies $\langle \bigwedge_{i\in I} u_i, \bigwedge_{i\in I} v_i\rangle \in T$ and $\langle \bigvee_{i\in I} u_i, \bigvee_{i\in I} v_i\rangle \in T$. For $v \in V$, denote $v_T = \bigwedge_{\langle v,v'\rangle\in T} v'$ and $v^T = \bigvee_{\langle v,v'\rangle\in T} v'$, and call each set of the form $[v]_T = [v_T, v^T] = \{v' \in V \mid v_T \leq v' \leq v^T\}$ a block of T. Denote $V/T = \{[v]_T \mid v \in V\}$ the set of all blocks of T and call it the factor set of V by T. Introduce a relation $\leq_T$ defined on V/T by $[v]_T \leq [v']_T$ iff $\bigwedge[v]_T \leq \bigwedge[v']_T$ (iff $\bigvee[v]_T \leq \bigvee[v']_T$). The following assertion follows from [21].

Proposition 27 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*,* T *be a compatible tolerance relation on* $\langle \mathcal{S}_{\mathrm{C}}, \subseteq \rangle$*. (1)* $\mathcal{S}_{\mathrm{C}}/T$ *is the set of all maximal blocks of* T*, i.e.* $\mathcal{S}_{\mathrm{C}}/T = \{B \subseteq \mathcal{S}_{\mathrm{C}} \mid B \times B \subseteq T \ \& \ ((\forall B' \subset B) B' \times B' \not\subseteq T)\}$*. (2)* $\langle \mathcal{S}_{\mathrm{C}}/T, \subseteq_T \rangle$ *is a complete lattice (factor lattice) where infima and suprema are given by*

$$\bigwedge_{i\in I}[A_i]_T = [(\bigcap_{i\in I} A_i)^T]_T \quad \textit{and} \quad \bigvee_{i\in I}[A_i]_T = [\bigvee_{i\in I} A_i]_T.$$

One may easily verify that if T is, moreover, transitive (i.e. a complete congruence on $\mathbf{V}$), then $\langle \mathcal{S}_{\mathrm{C}}, \subseteq_T \rangle$ is the well-known factor lattice.

To show that the a-cuts ${}^aE$ can be used to factorize $\mathcal{S}_{\mathrm{C}}$ by the above described procedure, we need to verify that ${}^aE$ is compatible w.r.t. $\subseteq$.

Lemma 28 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*. For any* $a \in K$*,* ${}^aE$ *is a compatible tolerance relation on the complete lattice* $\langle \mathcal{S}_{\mathrm{C}}, \subseteq \rangle$*.*

Proof. We show that ${}^aE$ is compatible both with infima and suprema, i.e. we show that $\langle A_i, B_i \rangle \in {}^aE$ ($i \in I$) implies both $\langle \bigwedge_{i\in I} A_i, \bigwedge_{i\in I} B_i \rangle \in {}^aE$ and $\langle \bigvee_{i\in I} A_i, \bigvee_{i\in I} B_i \rangle \in {}^aE$.

Infima: Suppose $\langle A_i, B_i \rangle \in {}^aE$, i.e. $a \leq \bigwedge_{x\in X}(A_i(x) \leftrightarrow B_i(x))$ ($i \in I$). We have to show $a \leq \bigwedge_{x\in X}(\bigwedge_{i\in I} A_i(x) \leftrightarrow \bigwedge_{i\in I} B_i(x))$, i.e. to show that for each $i \in I$ we have $a \leq (\bigwedge_{i\in I} A_i(x) \leftrightarrow \bigwedge_{i\in I} B_i(x))$. The last inequality is true iff both $a \leq (\bigwedge_{i\in I} A_i(x) \to \bigwedge_{i\in I} B_i(x))$ and $a \leq (\bigwedge_{i\in I} B_i(x) \to \bigwedge_{i\in I} A_i(x))$ are valid. Due to symmetry we verify only $a \leq (\bigwedge_{i\in I} A_i(x) \to \bigwedge_{i\in I} B_i(x))$ which is equivalent to $a \otimes \bigwedge_{i\in I} A_i(x) \leq \bigwedge_{i\in I} B_i(x)$. This inequality is true. Indeed, by assumption, $\langle A_i, B_i \rangle \in {}^aE$, i.e. $a \leq \bigwedge_{x\in X}(A_i(x) \leftrightarrow B_i(x))$, from which it follows $a \otimes A_i(x) \leq B_i(x)$ for any $i \in I$, $x \in X$. We therefore have

$$\begin{aligned} & a \otimes \bigwedge_{i\in I} A_i(x) \leq \\ \leq \quad & \bigwedge_{i\in I}(a \otimes A_i(x)) \leq \bigwedge_{i\in I} B_i(x). \end{aligned}$$

Suprema: We have to show $a \leq E(\bigvee_{i\in I} A_i, \bigvee_{i\in I} B_i)$, i.e. $a \leq E(\mathrm{C}(\bigcup_{i\in I} A_i), \mathrm{C}(\bigcup_{i\in I} B_i))$. First, observe that $(*)$ $a \leq E(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$: Indeed, the inequality is true iff both $a \leq \mathrm{S}(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ and $a \leq \mathrm{S}(\bigcup_{i\in I} B_i, \bigcup_{i\in I} A_i)$ hold. Because of symmetry, we verify only the former one: by definition, $a \leq \mathrm{S}(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ is true iff $a \leq (\bigcup_{i\in I} A_i)(x) \to (\bigcup_{i\in I} B_i)(x)$, i.e. iff $a \otimes (\bigcup_{i\in I} A_i)(x) \leq (\bigcup_{i\in I} B_i)(x)$ holds for each $x \in X$. By assumption, $a \otimes (\bigcup_{i\in I} A_i)(x) = \bigvee_{i\in I}(a \otimes A_i(x)) \leq \bigvee_{i\in I} B_i(x) = (\bigcup_{i\in I} B_i)(x)$ establishing $(*)$.

Now, $a \leq E(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ implies both $a \leq \mathrm{S}(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ and $a \leq \mathrm{S}(\bigcup_{i\in I} B_i, \bigcup_{i\in I} A_i)$. Since $a \in K$, (10) implies $a \leq \mathrm{S}(\mathrm{C}(\bigcup_{i\in I} A_i), \mathrm{C}(\bigcup_{i\in I} B_i))$ and $a \leq \mathrm{S}(\mathrm{C}(\bigcup_{i\in I} B_i), \mathrm{C}(\bigcup_{i\in I} A_i))$, i.e. $a \leq E(\mathrm{C}(\bigcup_{i\in I} A_i), \mathrm{C}(\bigcup_{i\in I} B_i)) = E(\bigvee_{i\in I} A_i, \bigvee_{i\in I} B_i)$ completing the proof. □

Remark (1) Note that the tolerance relation ${}^aE$ used to factorize $\mathcal{S}_{\mathrm{C}}$ need not to be supplied from the outside. It is determined by selecting an appropriate $a \in K$.

(2) The role of $a \in K$ is to control the granularity of the factorization: since $a \leq b$ implies ${}^bE \subseteq {}^aE$, the rule is "the bigger a, the finer the factorization". Clearly, for the extreme cases of a, i.e. $a = 0$ (note that ${}^0E = \mathcal{S}_{\mathrm{C}} \times \mathcal{S}_{\mathrm{C}}$ which is always a compatible relation on $\mathcal{S}_{\mathrm{C}}$) and $a = 1$ we obtain $\mathcal{S}_{\mathrm{C}}/{}^0E$ which is a one-element lattice and $\mathcal{S}_{\mathrm{C}}/{}^1E$ which is a lattice isomorphic to $\mathcal{S}_{\mathrm{C}}$.

(3) Note also that the fact $a \in K$ not needed in the proof of compatibility with infima in $\mathcal{S}_{\mathrm{C}}$.

5 Representation by 2-closure operators

We show that there is a natural one-to-one correspondence between $\mathbf{L}_{\{1\}}$-closure operators on X and special closure operators on $X \times L$. Call a subset $A \subseteq X \times L$ *(*$\mathbf{L}$*-set)-representative* if (1) for each $x \in X$

it holds $\langle x,a\rangle \in A$ and $b \leq a$ implies $\langle x,b\rangle \in A$, and (2) for each $x \in X$ the set $\{a \in L \mid \langle x,a\rangle \in A\}$ has the greatest element.

For any **L**-set $A \in L^X$ put

$$\lfloor A \rfloor = \{\langle x,a\rangle \in X \times L \mid a \leq A(x)\}. \tag{26}$$

For any set $A \subseteq X \times L$ put

$$\lceil A \rceil = \{\langle x,a\rangle \in X \times L \mid a = \bigvee_{\langle x,b\rangle \in A} b\}. \tag{27}$$

We have immediately the following result.

Lemma 29 *Let $A \in L^X$ be an **L**-set, $A' \subseteq X \times L$ be a representative set. Then (1) $\lfloor A \rfloor \subseteq X \times L$ is an representative set, (2) $\lceil A' \rceil$ is an **L**-set such that (3) $A = \lceil \lfloor A \rfloor \rceil$, $A' = \lfloor \lceil A' \rceil \rfloor$.*

Definition 30 *A **2**-closure operator* D *on $X \times L$ is called* commutative w.r.t. $\lfloor\lceil\ \rceil\rfloor$ *if*

$$\lfloor \lceil \mathrm{D}(A) \rceil \rfloor = \mathrm{D}(A) = \mathrm{D}(\lfloor \lceil A \rceil \rfloor) \tag{28}$$

holds for each $A \in X \times L$.

Remark It is easy to verify that (28) holds iff both $\lfloor \lceil \mathrm{D}(A) \rceil \rfloor \subseteq \mathrm{D}(A)$ and $\mathrm{D}(\lfloor \lceil A \rceil \rfloor) \subseteq \mathrm{D}(A)$ hold.

For an operator $\mathrm{D} : X \times L \to X \times L$ define an operator $\mathrm{C_D} : L^X \to L^X$ by

$$\mathrm{C_D}(A) = \lceil \mathrm{D}(\lfloor \mathrm{A} \rfloor) \rceil$$

for $A \in L^X$. For an operator $\mathrm{C} : L^X \to L^X$ define an operator $\mathrm{D_C} : X \times L \to X \times L$ by

$$\mathrm{D_C}(A) = \lfloor \mathrm{C}(\lceil \mathrm{A} \rceil) \rfloor$$

for $A \in X \times L$.

Theorem 31 *Let* C *be an $\mathbf{L}_{\{1\}}$-closure operator on X and* D *be a **2**-closure operator on $X \times L$ which is commutative w.r.t. $\lfloor\lceil\ \rceil\rfloor$. Then (1) $\mathrm{D_C}$ is a **2**-closure operator on X which is commutative w.r.t. $\lfloor\lceil\ \rceil\rfloor$, (2) $\mathrm{C_D}$ is an $\mathbf{L}_{\{1\}}$-closure operator on X, and (3) $\mathrm{C} = \mathrm{C_{D_C}}$ and $\mathrm{D} = \mathrm{D_{C_D}}$.*

Proof. (1) Let $A, A_1, A_2 \subseteq X \times L$. We have $A \subseteq \lfloor \lceil A \rceil \rfloor \subseteq \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D_C}(A)$, proving extensionality of $\mathrm{D_C}$. If $A_1 \subseteq A_2$ then clearly $\mathrm{S}(\lceil A_1 \rceil, \lceil A_2 \rceil) = 1$, hence $\mathrm{S}(\mathrm{C}(\lceil A_1 \rceil), \mathrm{C}(\lceil A_2 \rceil)) = 1$, so $\mathrm{D_C}(A_1) = \lfloor \mathrm{C}(\lceil A_1 \rceil) \rfloor \subseteq \lfloor \mathrm{C}(\lceil A_2 \rceil) \rfloor = \mathrm{D_C}(A_2)$, proving monotonicity of $\mathrm{D_C}$. $\mathrm{D_C}(\mathrm{D_C}(A)) = \lfloor \mathrm{C}(\lceil \lfloor \mathrm{C}(\lceil A \rceil) \rfloor \rceil) \rfloor = \lfloor \mathrm{C}(\mathrm{C}(\lceil A \rceil)) \rfloor = \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D_C}(A)$, proving idempotency. Finally, $\lfloor \lceil \mathrm{D_C}(A) \rceil \rfloor = \lfloor \lceil \lfloor \mathrm{C}(\lceil A \rceil) \rfloor \rceil \rfloor = \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D_C}(A)$, and $\mathrm{D_C}(\lfloor \lceil A \rceil \rfloor) = \lfloor \mathrm{C}(\lceil \lfloor \lceil A \rceil \rfloor \rceil) \rfloor = \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D_C}(A)$, verifying commutativity of $\mathrm{D_C}$.

(2) Let $A, A_1, A_2 \in L^X$. We have $A = \lceil \lfloor A \rfloor \rceil \subseteq \lceil \mathrm{D}(\lfloor A \rfloor) \rceil = \mathrm{C_D}(A)$, thus $\mathrm{C_D}$ is extensional. If $\mathrm{S}(A_1, A_2) = 1$ then $\lfloor A_1 \rfloor \subseteq \lfloor A_2 \rfloor$, therefore $\mathrm{D}(\lfloor A_1 \rfloor) \subseteq \mathrm{D}(\lfloor A_2 \rfloor)$ and thus $\mathrm{C_D}(A_1) = \lceil \mathrm{D}(\lfloor A_1 \rfloor) \rceil \subseteq \lceil \mathrm{D}(\lfloor A_2 \rfloor) \rceil = \mathrm{C_D}(A_2)$, monotonicity of $\mathrm{C_D}$. Using commutativity we further get $\mathrm{C_D}(\mathrm{C_D}(A)) = \lceil \mathrm{D}(\lfloor \lceil \mathrm{D}(\lfloor A \rfloor) \rceil \rfloor) \rceil = \lceil \mathrm{D}(\mathrm{D}(\lfloor A \rfloor)) \rceil = \lceil \mathrm{D}(\lfloor A \rfloor) \rceil = \mathrm{C_D}(A)$, idempotency of $\mathrm{C_D}$.

(3) For any $A \in L^X$ we have $\mathrm{C_{D_C}}(A) = \lceil \lfloor \mathrm{C}(\lceil \lfloor A \rfloor \rceil) \rfloor \rceil = \mathrm{C}(A)$. For any $A \subseteq X \times L$ we have by commutativity of D that $\mathrm{D_{C_D}}(A) = \lfloor \lceil \mathrm{D}(\lfloor \lceil A \rceil \rfloor) \rceil \rfloor = \mathrm{D}(A)$. □

Remark Note that commutativity of D is essential in the foregoing proposition (a counterexample is easy to get).

6 Some examples

In this section we introduce two further properties of $\mathbf{L}_K$-closure operators and show some examples. Call an $\mathbf{L}$-set $A \in L^X$ *finite* if $\{x \in X \mid A(x) > 0\}$ is a finite set.

Definition 32 *An $\mathbf{L}_K$ closure operator* C *on* X *is called* compact *(finitary, or algebraic) if*

$$\mathrm{C}(A) = \bigcup\{\mathrm{C}(B) \mid B \in L^X, B \subseteq A, B \ \ \textit{is finite}\ \}$$

holds for each $A \in L^X$.

Remark For $L = \{0, 1\}$ we get the compact closure operators.

Definition 33 *An $\mathbf{L}_K$-closure operator* C *on* X *is called* topological *if it satisfies*

$$\mathrm{C}(A \cup B) = \mathrm{C}(A) \cup \mathrm{C}(B)$$

for any $A, B \in L^X$.

Remark For $L = \{0, 1\}$, topological $\mathbf{L}_K$-closure operators are just closure operators of topological spaces. The corresponding system $\mathcal{S}_{\mathrm{C}}$ consists of the closed sets of the topology.

Fuzzy subalgebras Let $\mathbf{A} = \langle A, F\rangle$ be an algebra, i.e. A is a nonempty set and F is a system of operations on A. An $\mathbf{L}$-set $B \in L^A$ is called an $\mathbf{L}$-subalgebra of $\mathbf{A}$ if for each $f : A^n \to A$ of F and every $a_1, \ldots, a_n \in A$, it holds

$$B(a_1) \otimes \cdots \otimes B(a_n) \leq B(f(a_1, \ldots, a_n)).$$

Denote by $\mathbf{L}$-Sub $\mathbf{A}$ the set of all $\mathbf{L}$-subalgebras of $\mathbf{A}$.

Remark For $\otimes = \wedge$, $\mathbf{L}$-subalgebras and their systems are introduced and investigated in [6]. Note that $\mathbf{2}$-subalgebras coincide with the usual subalgebras.

Observation 34 *For any algebra $\mathbf{A} = \langle A, F\rangle$, $\mathbf{L}$-Sub $\mathbf{A}$ is an $\mathbf{L}_{\{1\}}$-closure system and the corresponding operator $\mathrm{C} = \mathrm{C}_{\mathbf{L}\text{-Sub}\,\mathbf{A}}$ is an algebraic $\mathbf{L}_{\{1\}}$-closure operator. Moreover, if $\otimes = \wedge$ (i.e. $\mathbf{L}$ is an algebra of intuitionistic logic (Heyting algebra)), C is an $\mathbf{L}$-closure operator.*

Proof. It is easy to see that $\mathbf{L}$-Sub $\mathbf{A}$ is closed under arbitrary intersections, hence $\mathbf{L}$-Sub $\mathbf{A}$ is an $\mathbf{L}_{\{1\}}$-closure system and the corresponding C is an $\mathbf{L}_{\{1\}}$-closure operator. It remains to verify the compactness of C. To this end, let $B \in L^A$ and put

$$[B](a) = \bigvee\{B(a_1)^{|x_1|_t} \otimes \cdots \otimes B(a_n)^{|x_n|_t} \mid t \in T_n,\ a_i \in A, t(a_1, \ldots, a_n) = a\}$$

where $\alpha^k = \alpha \otimes \cdots \otimes \alpha$ (k-times), T_i denotes the set of all i-ary terms of the type of $\mathbf{A}$, and $|x_i|_t$ denotes the number of occurrences of the variable x_i in t. It is a matter of routine to prove by induction over rank of the term t (defined by $\mathrm{ran}\,(x_i) = 0$ and $\mathrm{ran}\,(f(t_1, \ldots, t_n)) = 1 + \max\{\mathrm{ran}\,(t_1), \ldots, \mathrm{ran}\,(t_n)\}$) that $[B] = \mathrm{C}(B)$ which implies the compactness of C.

Let $\otimes = \wedge$. We have to prove $\mathrm{S}(B_1, B_2) \leq \mathrm{S}([B_1], [B_2])$ for every $B_1, B_2 \in L^A$ which holds iff for each $a \in A$ we have $[B_1](a) \wedge \mathrm{S}(B_1, B_2) \leq [B_2](a)$. Since $\wedge$ is idempotent we have

$$[B_1](a) \wedge \mathrm{S}(B_1, B_2) = \tag{29}$$

$$= \bigvee_{t \in T_n, t(a_1,\dots,a_n)=a} (B_1(a_1) \wedge \dots \wedge B_1(a_n)) \wedge \mathrm{S}(B_1, B_2) = \tag{30}$$

$$= \bigvee_{t \in T_n, t(a_1,\dots,a_n)=a} (B_1(a_1) \wedge \mathrm{S}(B_1, B_2) \wedge \dots \wedge B_1(a_n) \wedge \mathrm{S}(B_1, B_2)) \leq \tag{31}$$

$$\leq \bigvee_{t \in T_n, t(a_1,\dots,a_n)=a} (B_2(a_1) \wedge \dots \wedge B_2(a_n)) = [B_2](a). \tag{32}$$

□

Remark Note that in general, $\mathrm{C} = \mathrm{C}_{\mathbf{L}\text{-}\mathrm{Sub}\,\mathbf{A}}$ is not an **L**-closure system. As a counterexample consider $L = \{0, \frac{1}{2}, 1\}$ with a Łukasiewicz structure, and a four-element lattice with the support $A = \{a, b, c, d\}$ with the least element a, the greatest element d, and two mutually incomparable elements b and c, i.e. its Hasse diagram is a 45^o-rotated square. Let $B_1, B_2 \in L^A$ be given by $B_1(a) = 0$, $B_1(b) = B_1(c) = B_1(d) = 1$, $B_2(a) = 0$, $B_2(b) = B_2(c) = B_2(d) = \frac{1}{2}$. Clearly, B_2 itself is an **L**-subalgebra of **A**. On the other hand, $[B_1](a) = 1$ since $B_1(b) \otimes B_1(c) == 1 \leq [B_1](b \wedge c) = [B_1](a)$. We therefore have $\mathrm{S}(B_1, B_2) = \frac{1}{2} \not\leq 0 = [B_1](a) \to [B_2](a) = \mathrm{S}([B_1], [B_2])$, i.e. $[\ ] = \mathrm{C}$ is not an **L**-closure operator.

Fuzzy relational closures Let R be an **L**-relation on the set X, i.e. $R \in L^{X \times X}$. By a reflexive (symmetric, transitive) closure of R it is meant the least **L**-relation on X which is itself reflexive (symmetric, transitive) and contains R. The reflexive, symmetric, and transitive closure of R is denoted by R^r, R^s, and R^t, respectively. Recall that R is reflexive if $R(x,x) = 1$, symmetric if $R(x,y) = R(y,x)$, and transitive if $R(x,y) \otimes R(y,z) \leq R(x,z)$.

It is immediate that $R^r = R \cup_{x \in X} \{ {}^1/\langle x,x \rangle \}$, $R^s = R \cup R^{-1}$ where $R^{-1}(x,y) = R(y,x)$. Since $R \subseteq R^r$, $R \subseteq R^s$, $\mathrm{S}(R,S) \leq \mathrm{S}(R^r, S^r)$, $\mathrm{S}(R,S) \leq \mathrm{S}(R^s, S^s)$ (both of the inequalities are easy to verify), and $R^r = R^{rr}$, $R^s = R^{ss}$, we conclude that both ^{r} and ^{s} are **L**-closure operators on $X \times X$. Moreover, by the above description of ^{r} and ^{s} we conclude that both of them are compact as well as topologic.

To show that $R^t = \bigcup_{i=1}^{\infty} R^i = R \cup R \circ R \cup R \circ R \circ R \cup \cdots$ (where $(R \circ S)(x,y) = \bigvee_{z \in X} (R(x,z) \otimes S(z,y))$ it is enough to observe that $R \subseteq \bigcup_{i=1}^{\infty} R^i$; if $R \subseteq S$ and S is transitive then $\bigcup_{i=1}^{\infty} R^i \subseteq S$; and that $\bigcup_{i=1}^{\infty} R^i$ is transitive. Since the two former are evident, we only verify the last condition. $\bigcup_{i=1}^{\infty} R^i$ is transitive iff

$$(\bigcup_{i=1}^{\infty} R^i)(x,y) \circ (\bigcup_{j=1}^{\infty} R^j)(y,z) = (\bigvee_{i=1}^{\infty} R^i(x,y)) \circ (\bigvee_{j=1}^{\infty} R^j(y,z)) =$$

$$= \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} R^i(x,y) \circ R^j(y,z) \leq \bigvee_{k=1}^{\infty} R^k(x,z)$$

which holds iff for every i,j we have $R^i(x,y) \circ R^j(y,z) \leq \bigvee_{k=1}^{\infty} R^k(x,z)$. The last statement is true because $R^i(x,y) \circ R^j(y,z) \leq R^{i+j}(x,z)$. It is easy to see that for $K = \{1\}$, the conditions (9)–(11) are satisfied, hence ^{t} is an $\mathbf{L}_K$-closure operator on $X \times X$. In general, ^{t} is not an **L**-closure operator (consider $L = \{0, \frac{1}{2}, 1\}$ with Łukasiewicz structure, $X = \{a,b,c\}$, $R(a,b) = R(b,c) = 1$, $S(a,b) = S(b,c) = \frac{1}{2}$, and $R(x,y) = S(x,y) = 0$ otherwise). ^{t} is since since

$$R^t(x,y) = \bigvee_{i=1}^{\infty} R^i(x,y) =$$

$$= \bigvee_{i=1}^{\infty} \bigvee_{x=z_1, z_2, \dots, z_{i+1}=y} R(z_1, z_2) \otimes \cdots \otimes R(z_i, z_{i+1}) =$$

$$= \bigvee_{i=1}^{\infty} \bigvee_{x=z_1, z_2, \dots, z_{i+1}=y} \{ {}^{R(z_1,z_2)}/\langle z_1, z_2 \rangle, \dots, {}^{R(z_i,z_{i+1})}/\langle z_i, z_{i+1} \rangle \}^t(x,y).$$

As it is well-known from the classical case ($L = \{0, 1\}$), ^{t} is not topologic.

Remark An easy inspection shows that if $\mathbf{L}$ is an algebra of intuitionistic logic ($\otimes = \wedge$) then ^{t} is even an $\mathbf{L}$-closure operator on $X \times X$.

Fuzzy concept lattices By Port-Royal logic [1], a concept is determined by its extent (the collection of all objects which fall under the concept) and its intent (the collection of all attributes which fall under the concept). For instance, the extent of the concept DOG is the collection of all dogs while its intent is the collection of all attributes common to dogs (like "to be a mammal", "to bark" etc.). Port-Royal theory of concepts has been formalized and developed into a logico-algebraical theory of conceptual data analysis and knowledge representation by Wille [20, 11]. Wille's theory was extended to fuzzy setting by the present author (see e.g. [2, 3, 4]). The theory goes as follows: Let X and Y be non-empty sets interpreted as the set of objects and the set of attributes, respectively, I be an $\mathbf{L}$-relation between X and Y. The triple $\langle X, Y, I\rangle$ is called a (formal) $\mathbf{L}$-context. A (formal) $\mathbf{L}$-concept in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle \in L^X \times L^Y$ (i.e. A is an $\mathbf{L}$-set of attributes, B is an $\mathbf{L}$-set of attributes) such that B is the $\mathbf{L}$-set of all attributes common to all objects from A, and A is the $\mathbf{L}$-set of all objects sharing all the attributes from B. These verbal conditions translate formally as follows: Let $^{\uparrow_I} : L^X \to L^Y$ and $^{\downarrow_I} : L^Y \to L^X$ be defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x \in X} (A(x) \to I(x, y)) \tag{33}$$

$$B^{\downarrow_I}(x) = \bigwedge_{y \in Y} (B(y) \to I(x, y)). \tag{34}$$

Then $\langle A, B\rangle \in L^X \times L^Y$ is an $\mathbf{L}$-concept in $\langle X, Y, I\rangle$ iff $A^{\uparrow} = B$ and $B^{\downarrow} = A$ (one easily verifies that the verbal conditions are expressed exactly by the latter formulas). The set $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$ equipped with the partial order $\leq$ defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff } A_1 \subseteq A_2 \quad (\text{iff } B_2 \subseteq B_1)$$

is called the $\mathbf{L}$-concept lattice (fuzzy concept lattice) determined by $\langle X, Y, I\rangle$. Fuzzy concept lattice (which is in fact a complete lattice, see [3] for a complete characterization) is the basic derived structure which reveals the conceptual knowledge present in (the input data) $\langle X, Y, I\rangle$ (for more information on the principles of conceptual data analysis see [11]). The following theorem was proved in [2].

Proposition 35 *For a binary* $\mathbf{L}$*-relation* $I \in L^{X \times Y}$ *denote* $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ *the mappings defined for* $A \in L^X$, $B \in L^Y$ *by (33) and (34). For an* $\mathbf{L}$*-Galois connection* $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ *between* X *and* Y *denote* $I_{\langle \uparrow, \downarrow\rangle}$ *the binary* $\mathbf{L}$*-relation* $I \in L^{X \times Y}$ *defined for* $x \in X$, $y \in Y$ *by* $I(x, y) = \{ {}^1/x\}^{\uparrow}(y) (= \{ {}^1/y\}^{\downarrow}(x))$. *Then* $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ *is an* $\mathbf{L}$*-Galois connection and it holds*

$$\langle {}^{\uparrow}, {}^{\downarrow}\rangle = \langle {}^{\uparrow_{I_{\langle \uparrow, \downarrow\rangle}}}, {}^{\downarrow_{I_{\langle \uparrow, \downarrow\rangle}}}\rangle \quad \textit{and} \quad I = I_{\langle \uparrow_I, \downarrow_I\rangle}.$$

Putting in other words, $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ forms a representative form of $\mathbf{L}$-Galois connection. Theorem 19 implies that the composite mappings $^{\uparrow_I \downarrow_I} : L^X \to L^X$ and $^{\downarrow_I \uparrow_I} : L^Y \to L^X$ are $\mathbf{L}$-closure operators on X and Y, respectively, and that the sets $\{A \in L^X \mid A = A^{\uparrow_I \downarrow_I}\}$ and $\{B \in L^X \mid B = B^{\uparrow_I \downarrow_I}\}$ are S_L-dually isomorphic $\mathbf{L}$-closure systems. Moreover, Theorem 20 implies that each $\mathbf{L}$-closure operator on X ($\mathbf{L}$-closure system in X) can be viewed as being of the form $^{\uparrow_I \downarrow_I}$ (or $^{\downarrow_I \uparrow_I}$) (as the set of all extents (or intents) of an $\mathbf{L}$-concept lattice) for some $\mathbf{L}$-context $\langle X, Y, I\rangle$.

7 Fuzzy closure operators and consequence relations

The concept of a closure operator is an important one from the point of view of logic. Typically, a closure operator C in a given logical calculus arises as follows: for a given set collection A of formulas, the closure $\mathrm{C}(A)$ is defined to be the collection of all formulas provable from A. A detailed study of closure operators in the context of two-valued logic can be found in [22]. The situation is analogous in fuzzy logic. Namely, following the fundamental work of Pavelka [18], provability degree of a formula from a fuzzy set of formulas is defined in fuzzy logic. Then, a fuzzy closure operator is naturally induced by a fuzzy logical calculus as follows: for a given fuzzy set A of formulas and a given formula φ, the degree to which φ belongs to the closure $\mathrm{C}(A)$ of A is defined to be the provability degree of φ from A. For a special structure of truth values (namely, for $L = [0,1]$ equipped with min as the connective modeling conjunction, i.e. the standard Gödel algebra [15]), fuzzy closure operators and fuzzy consequence relations have been studied by Chakraborty (see e.g. [9]) and Gerla (see e.g. [12]). However, the study of general fuzzy closure operators and its relations to fuzzy logic is still an open goal (a paper on this topic is in preparation).

Our aim in this section is to present a general result on the relationship between fuzzy closure operators and fuzzy consequence relations. First, we show that each binary $\mathbf{L}$-relation between L^X (the set of all $\mathbf{L}$-sets in a given set X) and X induces in a natural way a fuzzy closure system (and the corresponding fuzzy closure operator). We say that an $\mathbf{L}$-set $A \in L^X$ is R_K-closed ($K \subseteq L$, R be an $\mathbf{L}$-relation $R \in L^{L^X \times X}$) if for any $B \in L^X$ and each $x \in X$ we have

$$\mathrm{S}(B, A) \otimes R(B, x) \leq A(x)$$

whenever $\mathrm{S}(B, A) \in K$.

Lemma 36 *For any $R \in L^{L^X \times X}$ and $K \subseteq L$, the set $\mathcal{S}_R$ of all $\mathbf{L}$-sets in X that are R_K-closed forms an $\mathbf{L}_K$-closure system.*

Proof. By definition, we have to show that $\mathcal{S}_R$ is closed w.r.t. S_K-intersections, i.e. we have to show that for any $A \in L^X$, $(\bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B)$ is R_K-closed. Take any $C \in L^X$ such that $\mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B) \in K$. We have to show $\mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B) \otimes R(C, x) \leq \bigwedge_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B(x)$ is true for any $x \in X$. The last inequality holds iff for any $B \in L^X$ such that $\mathrm{S}(A, B) \in K$ we have $\mathrm{S}(A, B) \otimes \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B) \otimes R(C, x) \leq B(x)$ which is clearly true provided both $\mathrm{S}(A, B) \otimes \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B) \leq \mathrm{S}(C, B)$ and $\mathrm{S}(C, B) \in K$ are valid. We have $\mathrm{S}(A, B) \otimes \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B) \leq \mathrm{S}(A, B) \otimes \mathrm{S}(C, \mathrm{S}(A, B) \to B) = \mathrm{S}(A, B) \otimes \bigwedge_{y \in X}(C(y) \to (\mathrm{S}(A, B) \to B(y))) = \mathrm{S}(A, B) \otimes \bigwedge_{y \in X}(\mathrm{S}(A, B) \to (C(y) \to B(y))) \leq \bigwedge_{y \in X}(\mathrm{S}(A, B) \otimes (\mathrm{S}(A, B) \to (C(y) \to B(y)))) \leq \bigwedge_{y \in X} C(y) \to B(y) = \mathrm{S}(C, B)$. Since $\mathrm{S}(A, B) \in K$ and $\mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A, B) \to B) \in K$, we conclude $\mathrm{S}(C, B) \in K$. □

In fact, each $\mathbf{L}_K$-closure system is induced in the way described in Lemma 36 by its corresponding $\mathbf{L}_K$-closure operator: Each $\mathbf{L}_K$-closure operator C on X induces an $\mathbf{L}$-relation $R_{\mathrm{C}} \in L^{L^X \times X}$ by

$$R_{\mathrm{C}}(A, x) = \mathrm{C}(A)(x).$$

Now, applying Lemma 36 to R_{C} we get the $\mathbf{L}_K$-closure system corresponding to C:

Lemma 37 *For an $\mathbf{L}_K$-closure operator* C *on X we have $\mathcal{S}_{\mathrm{C}} = \mathcal{S}_{R_{\mathrm{C}}}$.*

Proof. Let $A \in \mathcal{S}_{\mathrm{C}}$. We show that A is R_K-closed. Let $\mathrm{S}(B, A) \in K$. Then, by (10), $\mathrm{S}(B, A) \leq \mathrm{S}(\mathrm{C}(B), \mathrm{C}(A))$, i.e. for each $x \in X$ we have $\mathrm{S}(B, A) \otimes R_{\mathrm{C}}(B, x) = \mathrm{S}(B, A) \otimes \mathrm{C}(B)(x) \leq \mathrm{C}(A)(x) = A(x)$, whence A is R_K-closed. Conversely, if A is R_K-closed then for any B such that $\mathrm{S}(B, A) \in K$ we have $\mathrm{S}(B, A) \otimes R_{\mathrm{C}}(B, x) \leq A(x)$. Putting $B = A$ and considering $\mathrm{S}(A, A) = 1 \in K$, we conclude $\mathrm{S}(B, A) \otimes R_{\mathrm{C}}(B, x) = 1 \otimes R_{\mathrm{C}}(A, x) = \mathrm{C}(A)(x) \leq A(x)$, thus $A \in \mathcal{S}_{\mathrm{C}}$. □

Recall (see e.g. [12]) that a (bivalent) relation $\vdash$ between 2^X and X is called a consequence relation in (a given set) X if (i) $X \vdash \varphi$ holds for each $\varphi \in X$; (ii) $X \vdash \varphi$ and $X \subseteq Y$ imply $Y \vdash \varphi$; and (iii) $X \cup Y \vdash \varphi$ and $X \vdash \psi$ for any $\psi \in Y$ imply $X \vdash \varphi$.

Definition 38 *An $\mathbf{L}$-relation $\vdash$ between L^X and X is called an $\mathbf{L}_K$-consequence relation provided it satisfies* *(i)* $A(\varphi) \leq \vdash (A, \varphi)$

(ii) $\vdash (A, \varphi) \otimes \mathrm{S}(A, B) \leq \vdash (B, \varphi)$ *whenever* $\vdash (B, \varphi) \in K$

(iii) $(\bigwedge_{\varphi \in X} B(\psi) \to \vdash (A, \psi)) \otimes \vdash (A \cup B, \varphi) \leq \vdash (A, \varphi)$ *whenever* $(\bigwedge_{\varphi \in X} B(\psi) \to \vdash (A, \psi)) \in K$

for any $A, B \in L^X$, $\varphi \in X$.

Remark Note that condition (iii) may be equivalently replaced by (iii'): $(\bigwedge_{\varphi \in X} B(\psi) \to \vdash (A, \psi)) \otimes \vdash (B, \varphi) \leq \vdash (A, \varphi)$ whenever $(\bigwedge_{\varphi \in X} B(\psi) \to \vdash (A, \psi)) \in K$. Indeed, $\vdash (B, \varphi) \leq \vdash (A \cup B, \varphi)$ by (ii) whence (iii) implies (iii'). Conversely, suppose (iii') and take $B' = A \cup B$. By $A(\psi) \leq \vdash (A, \psi)$ we have $\bigwedge_{\psi \in X}((A \cup B)(\psi) \to \vdash (A, \psi)) = \bigwedge_{\psi \in X}(A(\psi) \to \vdash (A, \psi)) \wedge \bigwedge_{\psi \in X}(B(\psi) \to \vdash (A, \psi)) = \bigwedge_{\psi \in X}(B(\psi) \to \vdash (A, \psi))$. Therefore, $(\bigwedge_{\psi \in X}(B(\psi) \to \vdash (A, \psi))) \otimes \vdash (A \cup B, \varphi) = (\bigwedge_{\psi \in X}((A \cup B)(\psi) \to \vdash (A, \psi))) \otimes \vdash (A \cup B, \varphi) = (\bigwedge_{\psi \in X}(B'(\psi) \to \vdash (A, \psi))) \otimes \vdash (B', \varphi) \leq \vdash (A, \varphi)$ by (iii').

For a mapping $\mathrm{C} : L^X \to L^X$, define an $\mathbf{L}$-relation $\vdash_{\mathrm{C}} \in L^{L^X \times X}$ by $\vdash_{\mathrm{C}} (A, \varphi) = \mathrm{C}(A)(\varphi)$. For an $\mathbf{L}$-relation $\vdash \in L^{L^X \times X}$ define a mapping $\mathrm{C}_{\vdash} : L^X \to L^X$ by $\mathrm{C}_{\vdash}(A)(\varphi) = \vdash (A, \varphi)$.

Theorem 39 *Let* $\mathrm{C} : L^X \to L^X$ *be a mapping,* $\vdash \in L^{L^X \times X}$ *be an* $\mathbf{L}$*-relation. Then (1)* C *is an* $\mathbf{L}_K$*-closure operator iff* $\vdash_{\mathrm{C}}$ *is an* $\mathbf{L}_K$*-consequence relation; (2)* $\vdash$ *is an* $\mathbf{L}_K$*-consequence relation iff* $\mathrm{C}_{\vdash}$ *is an* $\mathbf{L}_K$*-closure operator; (3)* $\mathrm{C} = \mathrm{C}_{\vdash_{\mathrm{C}}}$ *and* $\vdash = \vdash_{\mathrm{C}_{\vdash}}$.

Proof. Clearly, (3) is true. Therefore, it is sufficient to establish the "⇒"-parts of (1) and (2).

(1): We verify that $\vdash_{\mathrm{C}}$ is an $\mathbf{L}_K$-consequence operator. (i) and (ii) are direct consequences of (9) and (10). We now verify (iii'), a condition equivalent to (iii) (see Remark following Definition 38): $(\bigwedge_{\psi \in X} B(\psi) \to \vdash_{\mathrm{C}} (A, \psi)) \otimes \vdash_{\mathrm{C}} (B, \varphi) \leq \vdash_{\mathrm{C}} (A, \varphi)$ is by definition equivalent to $(\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi)) \otimes \mathrm{C}(B)(\varphi) \leq \mathrm{C}(A)(\varphi)$ which is true iff $\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi) \leq \mathrm{C}(B)(\varphi) \to \mathrm{C}(A)(\varphi)$. The last inequality is valid since $\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi) \in K$ implies $\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi) = \mathrm{S}(B, \mathrm{C}(A)) \leq \mathrm{S}(\mathrm{C}(B), \mathrm{C}(\mathrm{C}(A))) \leq \mathrm{C}(B)(\varphi) \to \mathrm{C}(A)(\varphi)$.

(2): We check that $\mathrm{C}_{\vdash}$ is an $\mathbf{L}_K$-closure operator. (9) and (10) are direct consequences of (i) and (ii). Putting $B = \mathrm{C}_{\vdash}(A)$, (iii') yields $(\mathrm{C}_{\vdash}(\mathrm{C}_{\vdash}(A)))(\varphi) = 1 \otimes \vdash (\mathrm{C}_{\vdash}(A), \varphi) = \bigwedge_{\psi \in X}(B(\psi) \to \vdash (A, \psi)) \otimes \vdash (B, \varphi) \leq \vdash (A, \varphi) = \mathrm{C}_{\vdash}(A)(\varphi)$. □

Remark Theorem 39 thus establishes a one-to-one correspondence between fuzzy closure operators and fuzzy consequence relations. In [12], Gerla defines graded consequence relation in X as a fuzzy relation (Gerla takes $L = [0, 1]$ and $\otimes = \min$) between $\mathbf{2}^X$ (the power set of X) and X. Gerla then establishes a one-to-one correspondence between graded consequence relations and special fuzzy closure operators (Gerla deals, in our terms, with $\mathbf{L}_{\{1\}}$-closure operators), i.e. not all fuzzy closure operators. As it can be easily seen, the difficulty is in condition (iii) of the definition of $\mathbf{L}_K$-consequence relation: particularly, the condition $(\bigwedge_{\varphi \in X} B(\psi) \to \vdash (A, \psi)) \in K$ is missing in Gerla's definition. Instead, Gerla uses (iii"): $\bigwedge_{\psi \in B} \vdash (A, \psi) \otimes \vdash (A \cup B, \varphi) \leq \vdash (A, \varphi)$ where $A, B \in 2^X$ (i.e. are A, B are subsets of X). Clearly, for $A, B \in 2^X$, (iii") is equivalent to $(\bigwedge_{\varphi \in X} B(\psi) \to \vdash (A, \psi)) \otimes \vdash (A \cup B, \varphi) \leq \vdash (A, \varphi)$. Now, it is not true (as Gerla observes) that (iii") is satisfied by $\vdash_{\mathrm{C}}$ for any $\mathbf{L}_{\{1\}}$-closure operator C, i.e. (iii") is too strong and the above natural relations do not establish a one-to-one correspondence. Theorem 39 shows a way to have a one-to-one correspondence, generalizing fully the bivalent case.

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