UNIVERSITY OF OSTRAVA

Institute for Research and Applications of Fuzzy Modeling

A remark on the ideal extension property

Radim Bělohlávek

Research report No. 22

April 2, 1999

Submitted/to appear:

Supported by:

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
Bráfova 7, 701 03 Ostrava 1, Czech Republic

tel.: +420-69-6160 233 fax: +420-69-6120 478
e-mail: radim.belohlavek@osu.cz

A remark on the ideal extension property

R. Bělohlávek

Abstract[1] In [1], the authors proved that for algebras of subtractive varieties, the principal ideal extension property implies the ideal extension property. We give another proof which avoids the use of the Axiom of Choice.

In a series o papers [3, 1, 2] the authors study subtractive varieties, mainly from the point of view of ideal theory. Let $\mathcal{V}$ denote a subtractive variety, i.e. there is a binary term s such that $s(x,x)=0$ and $s(x,0)=x$ hold in $\mathcal{V}$ where 0 is an equationally defined constatnt of $\mathcal{V}$. A subset I of A where A is a support of some $\mathbf{A}\in\mathcal{V}$ is called an ideal if for every ideal term $p(x_1,\dots,x_m,y_1,\dots,y_n)$ of $\mathcal{V}$ and for every $a_1,\dots,a_m\in A$, $i_1,\dots,i_n\in I$ it holds that $p(a_1,\dots,a_m,i_1,\dots,i_n)\in I$. A term p is called an ideal term of $\mathcal{V}$ if $p(x_1,\dots,x_m,0,\dots,0)=0$ holds in $\mathcal{V}$. The set of all ideals of $\mathbf{A}$ is denoted by $\mathrm{I}(\mathbf{A})$. For $X\subseteq A$, $(X)^{\mathbf{A}}$ denotes the least ideal of $\mathrm{I}(\mathbf{A})$ containing X. An ideal I is called principal if $I=(i)^{\mathbf{A}}$ for some $i\in A$. An algebra $\mathbf{A}\in\mathcal{V}$ satisfies the ideal extension property (IEP) if for each subalgebra $\mathbf{B}$ of $\mathbf{A}$ and each $I\in\mathrm{I}(\mathbf{B})$ there is $J\in\mathrm{I}(\mathbf{A})$ such that $J\cap B=I$. $\mathbf{A}$ satisfies the principal IEP if for each subalgebra $\mathbf{B}$ of $\mathbf{A}$ and any $b\in B$, $(b)^{\mathbf{A}}\cap B=(b)^{\mathbf{B}}$. For $\theta\subseteq A\times A$ and $X\subseteq A$ put $(X)_\theta=\{a\in A\,|\,\langle x,a\rangle\in\theta \text{ for some } x\in X\}$. The following proposition has been proved in [1] using Zorn Lemma.

PROPOSITION. *Let $\mathbf{A}\in\mathcal{V}$, where $\mathcal{V}$ is subtractive. If $\mathbf{A}$ has the principal IEP, then $\mathbf{A}$ has the IEP.*

We give a simple proof of Proposition without the use of Zorn Lemma.

Proof. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, $I\in\mathrm{I}(\mathbf{B})$. Obviously, we have to show $(I)^{\mathbf{A}}\cap B\subseteq I$. $a\in(I)^{\mathbf{A}}$ holds iff $a\in(i_1)^{\mathbf{A}}\vee\cdots\vee(i_n)^{\mathbf{A}}$ (since $()^{\mathbf{A}}$ is an algebraic closure system [3, p. 205]) which is by [1, 1.14] equivalent to $s_n(a,c_n,\dots,c_2)\in(i_1)^{\mathbf{A}}$ for some $c_k\in(i_k)^{\mathbf{A}}$, $k=2,\dots,n$ (where s_n is defined inductively by $s_1(x)=x$ and $s_{n+1}(x,y_1,\dots,y_n)=s(s_n(x,y_2,\dots,y_n),y_1))$. We prove by induction over n the following claim: for each $C\in\mathrm{Sub}\,\mathbf{A}$ such that $B\subseteq C$, and $a\in C$, $i_1,\dots,i_n\in I\in\mathrm{I}(\mathbf{C})$, if there are $c_k\in(i_k)^{\mathbf{A}}$, $k=2,\dots,n$, such that $s_n(a,c_2,\dots,c_n)\in(i_1)^{\mathbf{A}}$, then $a\in I$. For $C=B$ the claim obivously yields the required inclusion $(I)^{\mathbf{A}}\cap B\subseteq I$. The claim follows directly by the principal IEP for $n=1$. Suppose the claim holds for $n-1$. Take $D=(C)_{\theta_{(i_n)^{\mathbf{A}}}}\in\mathrm{Sub}\,\mathbf{A}$ where $\theta_{(i_n)^{\mathbf{A}}}$ is some congruence [3, Propositon 1.4] with the class $(i_1)^{\mathbf{A}}$. Since, by the principal IEP, $(i_n)^{\mathbf{A}}\cap C=(i_n)^{\mathbf{C}}\subseteq I$, it follows from [1, Lemma] that $J=(I)_{\theta_{(i_n)^{\mathbf{A}}}}$ is an ideal in D such that $J\cap C=I$. Clearly, $s(a,c_n)\in D$, $i_1,\dots,i_{n-1}\in J\in\mathrm{I}(\mathbf{D})$, $c_2,\dots,c_{n-1}$, satisfy the assumptions for $n-1$, hence $s(a,c_n)\in J$. Morevover, by $a\in D$, $c_n\in J\in\mathrm{I}(\mathbf{D})$, we get $a\in J$. Hence $a\in J\cap C=I$, completing the proof. □

References

[1] Agliano, P. and Ursini, A., *On subtractive varieties II: General properties,* Algebra Universalis *36*(1996), 222-259.

[2] Agliano, P. and Ursini, A., *On subtractive varieties II: From ideals to congruences,* Algebra Universalis *37*(1997), 296-333.

[3] Ursini, A., *On subtractive varieties, I,* Algebra Universalis *31*(1994), 204-222.

[1] *1991 Mathematics Subject Classification:* 08A30, 08B99
Keywords: subtractive variety, ideal extension property