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1 Introduction

In this paper we deal with syntactic aspects of two kinds of fuzzy logic, namely fuzzy logic in narrow (FLn) and that in broader sense (FLb). Fuzzy logic in narrow sense is now quite well established though the work is far from being finished. The goal of this logic is to develop means for modeling of the vagueness phenomenon. One of the partial goals is to formulate analogues of most of the classical logic theorems. This makes us possible to clarify the relation of fuzzy logic to classical logic and also, to gain a more profound understanding to both logics. Further work should concentrate on the extension of the results to comprehend better the vagueness phenomenon. One of the interesting problems are open fuzzy theories in FLn. We consider this topic important as it has direct impact to questions of provability and algorithmization and thus, also to applications. Unfortunately, as shown in [6], proving in fuzzy logic is highly computationally ineffective. Hence, we have to seek some sophisticated methods which, of course, may help us in solution of only specific problems (which, however, may be just those important for applications).

Further interesting consequences may be expected in FLb, which is an extension of FLn. The goal is to develop a logic of the commonsense human reasoning whose main characteristic feature is the use of natural language. FLb includes the concept of *computing with words*, which has been recently introduced by L. A. Zadeh.

In this paper, we define the concept of formal theory in both fuzzy logics, demonstrate some of their basic properties and mutual connection of FLb and FLn. We will focus especially to syntactic aspects and specific questions of provability. However, we assume that the reader is, at least partly, acquainted with some of the cited works [7, 11, 16, 19, 23] where precise definitions of some concepts and proofs of some theorems, which are only recalled in this paper, can be found.

2 Formal theories in fuzzy logic in narrow sense

2.1 Truth values and consequence operation

Recall that the set of truth values is considered to be the residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}\rangle, \tag{1}$$

where L is either a *finite chain*, or $L = [0, 1]$ and $\otimes, \rightarrow$ is the adjoint couple of product and residuation.

As analyzed in detail in [8], we may distinguish three main streams of FLn, namely that of Łukasiewicz style (FLn(Ł)), Gödel style (FLn(G)) and product style (FLn(P)). All three logics assume $L = [0, 1]$ and differ in the definition of the couple of operations $\otimes$ and $\rightarrow$. In this paper, we will work in FLn(Ł) (fuzzy

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logic in narrow sense of Łukasiewicz style) in which $\otimes$ is the Łukasiewicz product and $\rightarrow$ the Łukasiewicz implication. The reasons for this are widely discussed in the literature and we will mention some of them also in this paper.

Note that $\otimes$ in (1) is a particular case of the concept of t-norm (cf. e.g. [5, 9, 25]) and $\rightarrow$ is the corresponding residuation. A general feature of FLn can be characterized by the possibility to introduce more kinds of connectives than classical logic. The choice is practically unlimited but t-norms (and t-conorms) seem to be most important and interesting. However, we have to cope with the fact that we obtain various logical systems (determined especially by the implication operation) which may not always behave well with respect to our idea. When confining to continuous t-norms, we come to the three above mentioned fuzzy logics in narrow sense. However, only Łukasiewicz implication is continuous. Therefore, FLn(Ł) possesses most distinguished properties.

The general requirement in any logic is that the connectives should preserve equivalence which in FLn is naturally interpreted by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \tag{2}$$

$a, b \in L$. Furthermore, we put $a^n = \underbrace{a \otimes \cdots \otimes a}_{n}$. The we say that the operation $c : L^n \longrightarrow L$ is *logically fitting* on L if there are natural numbers $k_1 > 0, \ldots, k_n > 0$ such that

$$(a_1 \leftrightarrow b_1)^{k_1} \otimes \cdots \otimes (a_n \leftrightarrow b_n)^{k_n} \leq c(a_1, \ldots, a_n) \leftrightarrow c(b_1, \ldots, b_n) \tag{3}$$

holds for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in L$.

Using this concept, it is possible to develop fuzzy logic as an open system in which four operations are basic (given by the structure of the residuated lattice (1) and to extend it, if necessary, by some additional operations. Hence, the structure of truth values may be assumed to form an *enriched residuated lattice*

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \{c_j \mid j \in Jop\}, \mathbf{0}, \mathbf{1} \rangle, \tag{4}$$

where $\{c_j \mid j \in Jop\}$ is a set of logically fitting operations (Jop is some index set). Let us stress that introducing these operations in FLn(Ł) does not harm the whole logical system (cf. [23]). Note also that in FLn(Ł), logically fitting operations are exactly those being Lipschitz continuous (see [10]). Additional connectives are especially important in FLb where we need them to accomplish interpretation of various natural language connectives, modifiers and, possibly, other linguistic phenomena.

We will consider formal language J consisting of variables, constants, predicates, connectives and quantifiers, as defined, for example in [11, 19]. A specific feature is introducing symbols $\mathbf{a}$ for all truth values $a \in L$. However, as demonstrated in [6, 8, 15], this is only a useful technical means.

By F_J we denote the set of all the well-formed formulas (defined in a common way) and by M_J sets of all terms in the language J. The basic connectives are $\wedge, \vee$, **&** and $\Rightarrow$ interpreted by the operations $\wedge, \vee, \otimes$ and $\rightarrow$, respectively.

The operation of *sum* is defined by $a \oplus b = \neg(\neg a \otimes \neg b)$, $a, b \in L$. It can be extended to multiple na.

Syntax of fuzzy logic is evaluated by *syntactic truth values*. This makes us possible to deal with truth values in the syntax. Furthermore, the main task of fuzzy logic in narrow sense has been declared to provide tool for grasping the vagueness phenomenon. Necessary condition for that is equal importance of all the truth values. Evaluated syntax seems to be a suitable means for this purpose.

The *evaluated formula* is a couple $[A;\ a]$ where $A \in F_J$ and $a \in L$. Evaluated formulas are manipulated using the *evaluated n-ary inference rules* r which are couples

$$r = \langle r^{syn}, r^{sem} \rangle \tag{5}$$

where r^{syn} is *syntactic part* of the rule r which is a partial n-ary operation on F_J and r^{sem} is its *semantic part* which is an n-ary operation on L preserving arbitrary non-empty joins in each argument (semicontinuity). We will work with *sound* inference rules, i.e. those preserving truth evaluations (for precise definition see [11, 23]).

Let us remark that evaluated formula $[A;\, a]$ can also be seen as a fuzzy singleton $\{\, {}^{a}/A\}$. Hence, every *set of evaluated formulas* is at the same time a *fuzzy set of formulas*. This ambiguity will often be used in the sequel.

A question raises where the syntactic truth values come from; how they should be interpreted? As pointed out by P. Hájek (cf. [7]), it is natural to understand evaluated formulas as the formulas $\mathbf{a}{\Rightarrow}A$ ($\mathbf{a}$ is a symbol for truth value $a \in L$) which, when being true in the degree 1, means that the truth of A is greater than or equal to a. This understanding has several consequences.

First, we may interpret the evaluated formulas as shorts for the latter ones. Second, the evaluated rules of modus ponens

$$r_{MP} : \frac{[A;\, a]\,, [A{\Rightarrow}B;\, b]}{[B;\, a \otimes b]}$$

and generalization

$$r_G : \frac{[A;\, a]}{[(\forall x)A;\, a]}$$

may be embedded in non-evaluated syntax simply as special cases of classical ones. For example, r_{MP} can be obtained in FLn(Ł) (with non-evaluated syntax and truth values $\mathbf{a}$, $a \in L$, in the language) using the proof, in which transitivity and importation tautologies, and rule of modus ponens are used:

$$\begin{aligned} \mathbf{a}{\Rightarrow}A, \mathbf{b}{\Rightarrow}(A{\Rightarrow}B), \\ (\mathbf{b}{\Rightarrow}(A{\Rightarrow}B)){\Rightarrow}(A{\Rightarrow}(\mathbf{b}{\Rightarrow}B)), A{\Rightarrow}(\mathbf{b}{\Rightarrow}B), \\ (\mathbf{a}{\Rightarrow}A){\Rightarrow}((A{\Rightarrow}(\mathbf{b}{\Rightarrow}B)){\Rightarrow}(\mathbf{a}{\Rightarrow}(\mathbf{b}{\Rightarrow}B)), (\mathbf{a}{\Rightarrow}(\mathbf{b}{\Rightarrow}B), \\ (\mathbf{a}{\Rightarrow}(\mathbf{b}{\Rightarrow}B)){\Rightarrow}(\overline{a \otimes b}{\Rightarrow}B), \overline{a \otimes b}{\Rightarrow}B. \end{aligned}$$

Most important is the fact that somewhat technical and not quite natural *b-lifting rules*

$$r_{Rb} : \frac{[A;\, a]}{[\mathbf{b}{\Rightarrow}A;\, b \rightarrow a]}$$

may be omitted as they can easily be replaced by simple proofs which use modus ponens and the transitivity tautology $(\mathbf{a}{\Rightarrow}A){\Rightarrow}((\mathbf{b}{\Rightarrow}\mathbf{a}){\Rightarrow}(\mathbf{b}{\Rightarrow}A))$.

Let us stress, however, that it is not reasonable to avoid evaluated formulas completely. Consistent replacement of evaluated formulas by $\mathbf{a}{\Rightarrow}A$ would lead to a very cumbersome notation. Furthermore, the primary goal which is graded model of vagueness (fuzzy approach) would disappear.

Other reason consists in the evaluated inference rules. The semantical operation r^{syn} in (5) is required to be only upper semicontinuous and is realized on the syntactic truth evaluations. This opens the way for extension of fuzzy logic by various non-standard inference rules. However, omitting the concept of evaluated formulas would result in the restriction only to the operations defined a priori in the structure (1) (or (4)). Consequently, the semantical operation r^{syn} would have to be inherently realized as the interpretation of some logical connective.

As an example, let us consider the rule introduced already in [23]:

$$r : \frac{[A \vee \mathbf{b};\, a]}{[A;\, b \downarrow a]}$$

where $\downarrow$ is the operation defined by

$$b \downarrow a = \begin{cases} 0 & \text{if } b \geq a \\ a & \text{otherwise} \end{cases}$$

$a, b \in [0,1]$. The operation $\downarrow$ is discontinuous and thus, not logically fitting in the sense of (3). Hence, it cannot be used as a logical connective in FLn(Ł). To conclude, we may introduce various, in general n-ary, inference rules with discontinuous semantical operation. We expect here the potential, for example, for modeling of the abduction in FLn.

A *fuzzy theory* T is determined by a triple

$$T = \langle A_L, A_S, R \rangle \tag{6}$$

where A_L, A_S are sets of evaluated logical and special axioms, respectively (or, equivalently, fuzzy sets $A_L, A_S \subsetsim F_J$) and R is a set of inference rules containing at least the rules r_{MP} and r_G. In general, there may be fuzzy theories with different sets of inference rules.

The concept of provability is crucial in any logic. Let $C : F_J \longrightarrow L$ be a function such that $C(r(A_1, \ldots, A_n)) \in G$ whenever $C(A_i) \in G$, $i = 1, \ldots, n$ for every inference rule r. Let us call such a function *rules preserving*. The following definition can be introduced in classical logic.

Definition 1 *A formula A belongs to the set $\bar{C}^{syn}X$ of syntactic consequences of the set of formulas X iff $C(A) \in G$ holds for every rules preserving function such that $C(B) \in G$ holds for every $B \in X$.*

Strong definition of syntactic consequence accepted in classical as well as in many-valued logic requires existence of a proof w of A from X.

In classical logic, Definition 1 is *equivalent* with the strong definition of syntactic consequence. The main outcome of Definition 1 is the possibility to generalize syntactic consequences to the case of evaluated syntax.

Definition 2 *Let R be a set of sound evaluated inference rules. Then the fuzzy set of* syntactic consequences *of the fuzzy set $X \subsetsim F_J$ is given by*

$$(C^{syn}X)A = \bigwedge\{C(A) \mid C \subsetsim F_J,\ C \text{ preserves rules } r \in R \text{ and } A_L, X \subseteq C\}. \tag{7}$$

($\subseteq$ is classical inclusion of fuzzy sets).

An evaluated proof w is a sequence of evaluated formulas $[A_1; a_1], \ldots, [A_n; a_n]$ such that every evaluated formula $[A_i; a_i]$ is either and axiom or it is derived from previous evaluated formulas using some evaluated inference rule. The evaluation a_n of the last formula in w is called the value of the proof w and denoted by $\mathrm{Val}(w)$.

If $(C^{syn}A_S)A = a$ then we write $T \vdash_a A$ (a formula A is *provable* in the degree a in the fuzzy theory T). The following theorem, whose proof is based on (7) (see [23]), holds true.

Theorem 1

$$(C^{syn}A_S)A = a = \bigvee\{\mathrm{Val}(w) \mid w \text{ is an evaluated proof of } A \text{ in } T\}. \tag{8}$$

Note that (8) is equivalent with

$$a = \bigvee\{b \mid \mathbf{b} \Rightarrow A \text{ is provable in } T\} \tag{9}$$

in a non-evaluated syntax (with truth values). Hence, Theorem 1 generalizes the *existence* of a proof to *supremum* of the values of all the possible evaluated proofs. However, this is not equivalent with the strong definition of syntactic consequence in many-valued logic.

To summarize, FLn(Ł) can be interpreted in many-valued logic if we introduce names for the truth values in its language and translate the evaluated formulas as mentioned above[†)]. However, to obtain all the results mentioned below, we still need to generalize the concept of provability (cf. (7), (8)).

Classical logic can be obtained within FLn as its special case because formulas and inference rules in classical logic may always be evaluated by the truth value 1. However, in many-valued logic, the strong

[†)]Note that this may concern only the rational truth values; we can even designate some formulas to serve us as the above names though the language needs not explicitly contain them.

definition of the consequence operation is used. The bridge between classical logic and FLn is also other reason for keeping the concept of evaluated formulas; it can be more clearly seen that the latter is direct generalization of the former.

At the end of this section, we will repeat some definitions from FLn(Ł), keeping on mind the discussion above. The *fuzzy theory* T is given by the triple (6) where $A_L, A_R \underset{\sim}{\subseteq} F_J$ are *fuzzy sets* of logical and special axioms (sets of evaluated formulas). At the same time we may see fuzzy theory T as a fuzzy set of formulas

$$T = (C^{syn}(A_S \cup A_L)) \underset{\sim}{\subseteq} F_J.$$

The *evaluated proof* w of a formula A in the theory T is a sequence of evaluated formulas, the value of the last one is the *value* $\mathrm{Val}_T(w)$ of the proof w. The provability degree (=evaluation) of an evaluated formula A is supremum of the values of all its evaluated proofs.

In the sequel, when defining a fuzzy theory, we will usually write only fuzzy set of its special axioms, i.e.

$$T = \{\, {}^{a_i}/A_i \mid i \in I\}$$

where I is some index set.

By $T \models_a A$ we mean that a formula $A \in F_J$ is *true* in the degree a in the fuzzy theory T, i.e.

$$a = \bigwedge \{\mathcal{D}(A) \mid \mathcal{D} \models T\}$$

where $\mathcal{D} \models T$ means that $\mathcal{D}$ is a model of T (for the precise definition see [11]).

2.2 Few theorems of FLn(Ł)

Fundamental theorems characterizing the provability are the *validity* and *closure* ones. The first one says that the provability degree of a formula cannot exceed its truth. By the second one, we may confine ourselves to closed formulas, analogously as in the classical logic.

If T is a theory and $E \underset{\sim}{\subseteq} F_J$ a fuzzy set of formulas then $T' = T \cup E$ is an extension of the theory T, i.e. its fuzzy set A'_S of special axioms is $A'_S = A_S \cup E$.

Proofs of the following theorems can be found in the cited literature ([6, 11, 18, 23].

Theorem 2 (deduction) *Let A be a closed formula and $T' = T \cup \{\, {}^{1}/A\}$. Then to every B there is n such that*

$$T \vdash_a A^n \Rightarrow B \quad \textit{iff} \quad T' \vdash_a B.$$

A fuzzy theory is *contradictory* if there is a formula $A \in F_{J(T)}$ and proofs w_A and $w_{\neg A}$ such that $\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) > 0$ (other equivalent characterizations of inconsistency can also be introduced).

Theorem 3 *A theory T is contradictory iff $T \vdash A$ holds for every formula $A \in F_{J(T)}$.*

Henkin extension T_H of a theory T by a fuzzy set of Henkin axioms

$$T_H = T \cup \{\, {}^{1}/(A_x[\mathbf{r}] \Rightarrow (\forall x)A(x)) \mid \mathbf{r} \text{ is special for } (\forall x)A\}$$

is proved to be conservative.

Theorem 4 *A fuzzy theory $T \cup \{\, {}^{a}/A\}$ is contradictory iff to every $b \in L$ and every formula B there is m such that $T \vdash_b A^m \Rightarrow B$.*

Theorem 5 (completeness)

(a) A theory T is consistent iff it has a model.

(b) $T \vdash_a A$ *iff* $T \models_a A$.

The equality predicate fulfilling the following (common) axioms can be introduced: There are natural numbers $m_1 > 0, \dots, m_n > 0$ such that

(E1)	$x = x$
(E2)	$(x_1 = y_1)^{m_1} \Rightarrow \dots \Rightarrow (x_n = y_n)^{m_n} \Rightarrow (f(x_1, \dots, x_n) = f(y_1, \dots, y_n))$
(E3)	$(x_1 = y_1)^{m_1} \Rightarrow \dots \Rightarrow (x_n = y_n)^{m_n} \Rightarrow (p(x_1, \dots, x_n) \Leftrightarrow p(y_1, \dots, y_n))$

for every n-ary functional symbol f and predicate symbol p.

Theorem 6 (equality) *Let* $T \vdash_{a_i} t_i = s_i$, $i = 1, \dots, n$. *Then there are natural numbers* $m_1 > 0, \dots, m_n > 0$ *such that*

$$T \vdash_b A \Leftrightarrow A', \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$

where A' is a formula which is a result of replacing of the terms t_i by the term s_i in A, respectively.

Theorem 7 (equivalence) *Let A be a formula and $B_1, \dots, B_n$ some of its subformulas. Let* $T \vdash_{a_i} B_i \Leftrightarrow B_i'$, $i = 1, \dots, n$. *Then there are natural numbers* $m_1 > 0, \dots, m_n > 0$ *such that*

$$T \vdash_b A \Leftrightarrow A', \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$

where A' is a formula which is a result of replacing of the formulas $B_1, \dots, B_n$ in A by $B_1', \dots, B_n'$.

Let $\Gamma \subsetsim F_J$ be a fuzzy set of formulas. By $\mathrm{Supp}(\Gamma)$ we denote its support, i.e. $\mathrm{Supp}(\Gamma) = \{A \mid \Gamma(A) > 0\}$. The $\triangledown$ denotes Łukasiewicz disjunction given by $A \triangledown B := \neg(\neg A \,\&\, \neg B)$.

Theorem 8 (reduction for the consistency) *A theory* $T' = T \cup \Gamma$ *is contradictory iff there are natural numbers* $m_1 > 0, \dots, m_n > 0$ *and* $A_1, \dots, A_n \in \mathrm{Supp}(\Gamma)$ *such that*

$$T \vdash_c \neg A_1^{m_1} \triangledown \cdots \triangledown \neg A_n^{m_n}$$

where $a_i = \Gamma(A_i)$, $i = 1, \dots, n$ *and* $c > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$ *or* $c = 1$ *if the right-hand side is equal to 1.*

Corollary 1 *A theory* $T' = T \cup \{ {}^{\neg a}/\neg A\}$ *is contradictory iff* $T \vdash_b mA$ *for some natural number* $m > 0$ *and* $b > ma$ *or* $b = 1$ *if* $ma = 1$.

Given a Henkin fuzzy theory T, the fuzzy set of formulas $\Delta(T)$ contains instances of special, equality, Henkin and substitution axioms in the membership degree 1.

A formula A is a *fuzzy quasitautology* in the degree a if $\models_a B_1 \,\&\, \cdots \,\&\, B_k \Rightarrow A$ where B_i are closed instances of the equality axioms.

Theorem 9 (consistency) *Open theory T is contradictory iff there are natural numbers* $p_1 > 0, \dots, p_n > 0$ *and special axioms* $A_1, \dots, A_n$ *of the theory T such that*

$$\neg \bar{A}_1^{p_1} \triangledown \cdots \triangledown \neg \bar{A}_n^{p_n}$$

is a fuzzy quasitautology in the degree b where $\bar{A}_i$ are instances of the special axioms and $b > \neg(a_1^{p_1} \otimes \cdots \otimes a_n^{p_n})$ *where* $a_i = A_S(A_i)$ *(or $b = 1$ if the right hand side is equal to 1).*

This theorem is a basis for a fuzzy analogy of the Herbrand theorem proved in [20].

3 Formal theories in fuzzy logic in broader sense

Unlike FLn, which is graded generalization of the classical logic, FLb can be motivated by modeling of the human deduction in which crucial role is played by the natural language. We develop it as a certain extension of FLn.

In our considerations, we confine to a suitable part of natural language, i.e. we will work with selected linguistic expressions (*syntagms*) from some set $\mathcal{S}$. These are translated into formulas of fuzzy logic in narrow sense and finally lead to construction of a special fuzzy theory T of FLn (cf. [19]).

In formal part of FLb, we deal with many-sorted language J of FLn (cf. [3]). We consider a finite number of sorts ι, $\iota = 1, \dots, p$ of variables and the corresponding constants as well as functions. We thus obtain sets of terms $M_{\iota J}$ of the sort ι in the language J. However, in FLb we will need only that part of $M_{\iota J}$ which does not contain variables. Therefore, M_ι will denote *sets of terms of the sort ι without variables* (the subscript J will often be omitted) in the sequel. As in the previous section, F_J is a set of well formed formulas in J.

Natural language expressions are, in general, names of some properties φ of objects. The property $\varphi(x)$ of objects x is assigned a formula $A(x) \in F_J$. Hence, each syntagm $\mathcal{A} \in \mathcal{S}$ is assigned a couple

$$\langle A(x), \mathbf{A} \rangle \tag{10}$$

where $\mathbf{A} = \{[A_x[t]; a_t] \mid t \in M_\iota\}$ is a set of evaluated formulas being closed instances of $A(x)$, i.e. $A_x[t]$ is obtained from $A(x)$ when replacing the variable x (of the sort ι) by the term $t \in M_\iota$. We will call $\mathbf{A}$ a *multiformula*. Recall from the previous section that a multiformula $\mathbf{A}$ can at the same time be seen as a fuzzy set of closed instances $A_x[t]$ of the formula $A(x)$.

This model is motivated by the use of natural language, and also by the potential of fuzzy logic in narrow sense, which enables us to work with fuzzy sets of formulas *included in the syntax*. This makes us possible to introduce and formalize (at least partly) the distinction between the concepts of *intension* and *extension* which are very important in the study of natural language semantics. In this paper, the intension is the multiformula $\mathbf{A}$ together with the formula $A(x)$ (see (10)). To simplify the formalism, we will omit $A(x)$ from most of our deliberation because it is inherently present in all elements of $\mathbf{A}$.

In intensional logic, intensions are (losely speaking) functions assigning truth values to objects in each possible world and time moment. Possible worlds as well as time are not explicitly included in our formalism. However, they are hidden behind the assignment of truth values to the instances $A_x[t]$ in the multiformula $\mathbf{A}$. Hence, the truth values are assigned using certain function

$$\Omega \times \tau \longrightarrow \{\mathbf{A} : A|M_\iota \longrightarrow L \,\big|\, \mathcal{A} \in \mathcal{S}, \mathcal{A} \mapsto A\}$$

where Ω is the set of possible worlds, τ time and $A|M_\iota$ a set of all instances of the form $A_x[t]$, $t \in M_\iota$. In this paper, however, the multiformulas $\mathbf{A}$ are considered to be given a priori.

It follows from the previous discussion that the basic scheme in the fuzzy logic in broader scheme is

$$\text{syntagm } \mathcal{A} \mapsto \text{formula } A, \text{intension } \mathbf{A} \mapsto \text{extension } \mathcal{D}(\mathbf{A})$$

where $\mathcal{D}$ is an interpretation (model) of the formal language J in concern. The extension is thus the fuzzy set

$$\mathcal{D}(\mathbf{A}) = \left\{ \left. \mathcal{D}(A_x[t])/\mathcal{D}(t) \right| t \in M_\iota \right\}.$$

It is clear that one intension $\mathbf{A}$ may lead to (infinitely) many extensions $\mathcal{D}(\mathbf{A})$. Obviously, $\mathcal{D}(t)$ is a concrete object and $\mathcal{D}(A_x[t])$ is a truth degree in which the object $\mathcal{D}(t)$ has the property $A(x)$. In the case of $A(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are variables of various sorts, we obtain a fuzzy relation

$$\left\{ \left. \mathcal{D}(A_{x_1,\dots,x_n}[t_1, \dots, t_n])/\langle \mathcal{D}(t_1), \dots, \mathcal{D}(t_n) \rangle \right| t_1 \in M_1, \dots, t_n \in M_n \right\}.$$

Example.
Let $\mathcal{A}$:= *Young* and $M = \{t_0, \dots, t_{100}\}$ be a set of terms representing years. We may define a multiformula **Young** by

$$\mathbf{Young} = \{[Young(t_0);\ 1], \dots, [Young(t_{20});\ 1], \dots, [Young(t_{30});\ 0.6], \dots, [Young(t_{45});\ 0.2], \dots, [Young(t_{60});\ 0]\}.$$

This is the logical representation of the intension of the word "Young". The extensions can be, for example, the following:

$$\mathcal{D}(\mathbf{Young}) = \{\ ^1/1, \dots,\ ^1/20, \dots,\ ^{0.6}/30, \dots,\ ^{0.2}/45, \dots,\ ^0/60\} \tag{11}$$

where $\mathcal{D}(t_0) = 1, \dots$, $\mathcal{D}(t_{20}) = 20, \dots$, $\mathcal{D}(t_{30}) = 30, \dots$, $\mathcal{D}(t_{45}) = 45, \dots$, $\mathcal{D}(t_{60}) = 60$ are interpretations of the terms from M when representing age of people. When representing age of dogs, we may obtain the following extension of *Young*:

$$\mathcal{D}(\mathbf{Young}) = \{\ ^1/0.1, \dots,\ ^1/4, \dots,\ ^{0.7}/6, \dots,\ ^{0.3}/8, \dots,\ ^0/14\} \tag{12}$$

where $\mathcal{D}(t_0) = 0.1, \dots$, $\mathcal{D}(t_{20}) = 4, \dots$, $\mathcal{D}(t_{30}) = 6, \dots$, $\mathcal{D}(t_{45}) = 8, \dots$, $\mathcal{D}(t_{60}) = 14$. Note that the truth degrees in (12) are grater than the corresponding ones in (11) to illustrate that only the inequality $\mathcal{D}(A) \geq a$ should be fulfilled where a is the truth evaluation of the formula A in the evaluated formula $[A;\ a]$. Note also that fuzzy sets of the form (11) and (12) are introduced in various examples in the literature on fuzzy set theory[†)]. From our point of view, some concrete, usually not explicitly characterized, extension is considered there.

Let us remark that a slightly simplified interpretation of fuzzy logic in broader sense which concerns only the logical aspect without linguistics has been proposed in [3]. However, we are convinced that linguistics should not be excluded from fuzzy logic and fuzzy techniques in general (let us remind the concept of soft computing where the main stress is given to "computing with words").

The *proof* in FLb is a sequence of linguistic statements (syntagms from $\mathcal{S}$) together with their intensions

$$\mathcal{B}_1[\mathbf{B}_1], \dots, \mathcal{B}_n[\mathbf{B}_n] \tag{13}$$

each of which is a linguistically formulated axiom (logical or special), or it is derived using some inference rule.

A formal theory of FLb is given by the set of linguistically expressed special axioms together with their intensions

$$\mathcal{T} = \{\mathcal{A}_0[\mathbf{A}_0], \dots, \mathcal{A}_m[\mathbf{A}_m]\}, \tag{14}$$

where $\mathcal{A}_i \in \mathcal{S}$, $i = 1, \dots, m$. The reasoning uses proofs of the form (13). As these deal with multiformulas, i.e. sets of evaluated formulas, we face a multiple inference in the adjoint fuzzy theory T determined by the intensions $\mathbf{A}_i$, $i = 1, \dots, m$ from (14), i.e.

$$T = \mathbf{A}_0 \cup \dots \cup \mathbf{A}_m. \tag{15}$$

In the sequel, we will denote the fuzzy theory in FLb by the script letter $\mathcal{T}$ and the adjoint theory of FLn by the italic letter T.

To simplify the formalism, we may omit the linguistic statements from (13) and write proof using only the intensions

$$\mathbf{B}_1, \dots, \mathbf{B}_n. \tag{16}$$

This has also other side: To consider the corresponding syntagms at each step of (13) would mean that the reasoning proceeds in words all the time. In practice, it would force us to find a suitable syntagm $\mathcal{B}_i$ to each multiformula $\mathbf{B}_i$, $i = 1, \dots, m$, which is a task of linguistic approximation to be solved at each reasoning step. But this is unrealistic. Hence, realistic view is to begin with natural language,

[†)]See, e.g. [26] and a lot of other papers and books.

translate its syntagms into multiformulas, then make proofs of the form (16) and translate only the final multiformula $\mathbf{B}_n$ into the syntagm $\mathcal{B}_n$. More precisely, we form a fuzzy theory $\mathcal{T}$ in (14) using natural language and then, within the adjoint fuzzy theory T in (15), we realise the reasoning whose result being a multiformula $\mathbf{B}$ may be formulated using a corresponding syntagm $\mathcal{B} \in \mathcal{S}$.

Intensions of the syntagms should be constructed from the other (simpler) ones. We face here the problem of truth functionality which is subject to a long and still unfinished discussion between logicians and linguists. Truth functionality cannot, in general, be accepted in the model of semantics of natural language. However, for some parts of it, the truth functionality holds, or at least may be bypassed by accepting various kinds of connectives in the local cases (recall our discussion about additional operations in Section 2).

Let $\mathcal{S}_A, \mathcal{S}_B$ be two disjoint sets of syntagms in the form

$$[\langle\text{linguistic modifier}\rangle]\langle\text{adjective}\rangle\langle\text{noun}\rangle \tag{17}$$

where each $\mathcal{A} \in \mathcal{S}_A$, $\mathcal{B} \in \mathcal{S}_B$ is assigned a formula A and B, respectively and the intensions being the respective multiformulas $\mathbf{A}$ and $\mathbf{B}$ (more exactly, $\mathcal{A}, \mathcal{B}$ are interpreted by the couples (10)). The following construction is important.

The *linguistic description* in FLb is a set of linguistic conditional statements of the form

$$\mathsf{IF}\ \mathcal{A}_j\ \mathsf{THEN}\ \mathcal{B}_j\ [\mathbf{A}_j \Rightarrow \mathbf{B}_j], \qquad j = 1, \ldots m \tag{18}$$

where $\mathcal{A}_j \in \mathcal{S}_A$, $\mathcal{B}_j \in \mathcal{S}_B$, $j = 1, \ldots, m$. These statements can be joined by the connective AND interpreted using conjunction. The intension of the whole linguistic description (18) is thus a multiformula

$$\left\{ \left[\bigwedge_{j=1}^{m} (A_{jx}[t] \Rightarrow B_{jy}[s]);\ c_{ts} = \bigwedge_{j=1}^{m} (a_{jt} \rightarrow b_{js}) \right] \middle|\ t \in M_1, s \in M_2 \right\}$$

where x, y are variables of the sorts 1 and 2, respectively.

Recall that a formula A' is a *variant* of A if it is the result of replacing of all subformulas of A of the form $(\forall y)B$ by formulas $(\forall x)B_y[x]$ where x is substitutible into A.

Lemma 1 *Let T be a fuzzy theory and A' be a variant of A. Then*

$$T \vdash_a A \quad \textit{iff} \quad T \vdash_a A'.$$

This lemma justifies the following concepts.

We say that two formulas A and B are *independent* if no variant or instance of one is a subformula of the other one.

Let F_0 be a set of evaluated formulas such that, if $[A;\ a], [B;\ b] \in F_0$ then A, B are independent and to each A there is at most one a such that $[A;\ a] \in F_0$. We will call F_0 a set of independent evaluated formulas.

We say that F_0 is *directed*, if:

(a) If $[(\forall x)A;\ a] \in F_0$ and $[A_x[t];\ b] \in F_0$, then $a \leq b$, where $t \in M_\iota$.

(b) If A is a logical axiom then $[A;\ a] \in F_0$ implies $a = A_L(A)$.

Note that if $A(x), B(y)$ are independent then also all their respective instances are independent.

The proof of the following lemma was inspired by the paper of E. Turunen [24].

Lemma 2 *Let F_0 be directed set of independent evaluated formulas Let $T = \left\{ {}^{a}/A \,\middle|\, [A;\, a] \in F_0 \right\}$. Then there is a model $\mathcal{D} \models T$ such that*

$$\mathcal{D}(A) = a \tag{19}$$

holds for all $[A;\, a] \in F_0$.

PROOF: We construct a Henkin extension T_H of the theory T and a Lindenbaum algebra $\mathcal{L}(T_H)$ using the equivalence

$$A \approx B, \quad \text{iff} \quad T \vdash A \Leftrightarrow B.$$

By Theorem 13 in [11], $\mathcal{L}(T_H)$ is a residuated lattice. Let $|\cdot|$ denote the elements from $\mathcal{L}(T_H)$.

Now, we construct an algebra Q generated by the set

$$Q_0 = \{|A| \mid (\exists A)(\exists a)([A;\, a] \in F_0)\} \cup \{|\mathbf{0}|\}.$$

The Q is determined by the following conditions:

(a) $Q_0 \subseteq Q$.

(b) If $|A|, |B| \in Q$ then $|A| \to |B| := |A \Rightarrow B| \in Q$.

Using the rule of modus ponens, logical axioms and formulas provable in the degree 1 (theorems) we can show that Q is a residuated lattice (analogously as in the proof of Theorem 13 in [11]).

Let us now define the function $f : Q \longrightarrow L$ as follows:

(a) $f(|A|) = a$ if $[A;\, a] \in F_0$.

(b) $f(|\mathbf{0}|) = 0$.

(c) $f(|A| \to |B|) = f(|A|) \to f(|B|)$.

Since F_0 is directed set of independent formulas, the function f exists and it is a homomorphism. Using the results of [2], the lattice of truth values L in consideration is injective and thus, f can be extended to homomorphism

$$g : \mathcal{L}(T_H) \longrightarrow L.$$

Finally, we define the truth evaluation $H : F_J \longrightarrow L$ by $H(A) = g(|A|)$. Obviously, $H(A) = a$ for every $[A;\, a] \in F_0$. We will also show that $H((\forall x)B) = \bigwedge_{t \in M_\iota} H(B_x[t])$.

As T_H is Henkin and H is a homomorphism, it follows from the logical and Henkin axioms that

$$H((\forall x)B) = H(B_x[\mathbf{r}])$$

where $\mathbf{r}$ is a special constant for $(\forall x)B$, both of the same sort ι. At the same time,

$$H((\forall x)B) \leq H(B_x[t])$$

holds for every term t of the sort ι. If $H(C) \leq H(B_x[t])$ holds for all terms t then,

$$H(C) \leq H(B_x[\mathbf{r}]) = H((\forall x)B)$$

as a special case, i.e. $H((\forall x)B)$ is infimum of all the truth evaluations $H(B_x[t])$, $t \in M_{\iota J}$. Analogously we proceed for suprema, using the negation.

Hence, using H, we can construct a canonical structure $\mathcal{D}$, which is a model of the theory T_H and has the property (19). But then $\mathcal{D} \models T$ follows from the fact that T_H is a conservative extension of T. □

This lemma plays an important role in proving some theorems about approximate reasoning. Using it and the completeness theorem we can prove the following lemma.

Lemma 3 *Let $A_j(x), B_j(y)$, $j = 1, \ldots$ be formulas, x, y variables of the sorts 1 and 2 such that for every $j \neq k$, A_j and A_k, as well as B_j and B_k are independent, respectively. Let k, $1 \leq k \leq m$ be given and*

$$T = \left\{ \left. {}^{a_{kt}}/A_{kx}[t],\ {}^{c_{ts}}/\bigwedge_{j=1}^{m}(A_{jx}[t] \Rightarrow B_{jy}[s]) \right| t \in M_1, s \in M_2 \right\}$$

be a fuzzy theory. Then

$$T \vdash_{b_{ks}} B_{ky}[s], \quad b_{ks} = \bigvee_{t \in M_1} (a_{kt} \otimes c_{ts}), \qquad s \in M_2.$$

PROOF: Put

$$F_0 = \{[A_{jx}[t];\, a_{jt}]\,, \left[B_{jy}[s];\, b_{js} = \bigvee_{t \in M_1} (a_{jt} \otimes c_{ts})\right] \mid t \in M_1, s \in M_2, j = 1, \ldots m\}.$$

It follows from the assumptions that F_0 is a set of independent evaluated formulas which, obviously, is also directed. By Lemma 2, there exists a structure $\mathcal{D}$ such that

$$\begin{aligned} \mathcal{D}(A_{jx}[t]) &= a_{jt} \\ \mathcal{D}(B_{jy}[s]) &= b_{js}. \end{aligned}$$

We will show that $\mathcal{D} \models T$.

Obviously,

$$a_{jt} \otimes c_{ts} \leq \bigvee_{t \in M_1} (a_{jt} \otimes c_{ts})$$

for all $t \in M_1$ and $s \in M_2$ and j. By the adjunction, we obtain

$$c_{ts} \leq a_{jt} \to \bigvee_{t \in M_1} (a_{jt} \otimes c_{ts}) = \mathcal{D}(A_{jx}[t]) \to \mathcal{D}(B_{jy}[s]) = \mathcal{D}(A_{jx}[t] \Rightarrow B_{jy}[s])$$

for all $j = 1, \ldots m$, and thus

$$c_{ts} \leq \bigwedge_{j=1}^{m} \mathcal{D}(A_{jx}[t] \Rightarrow B_{jy}[s]) = \mathcal{D}(\bigwedge_{j=1}^{m} (A_{jx}[t] \Rightarrow B_{jy}[s])),$$

i.e. $\mathcal{D} \models T$.

Consider the proofs

$$w_{ts} := [A_{kx}[t];\, a_{kt}]_{SA}\,, \left[\bigwedge_{j=1}^{m} (A_{jx}[t] \Rightarrow B_{jy}[s]);\, c_{ts}\right]_{SA}, [B_{ky}[s];\, a_{kt} \otimes c_{ts}]_{r_{MPC}}\,,$$

$t \in M_1$, $s \in M_2$ where r_{MPC} is a modified rule of modus ponens for the conjunction of implications (cf. [19, 16]). Then

$$b_{ks} \geq \bigvee_{t \in M_1} \mathrm{Val}_T(w_{ts}) = \bigvee_{t \in M_1} (a_{kt} \otimes c_{ts}).$$

But at the same time, $b_{ks} \leq \mathcal{D}(B_{ky}[s]) = \bigvee_{t \in M_1}(a_{kt} \otimes c_{ts})$, and we obtain the theorem using the completeness property. □

This lemma states that for special kinds of formulas, we may obtain the maximal provability degree only on the basis of multiformulas used in the definition of the fuzzy theory in concern. In other words, the formula used in the generalized modus ponens for FLb gives the maximal possible value if we confine only to linguistic expressions of the special kind. This is explicitly formulated in the following theorem.

Let $\mathcal{S}_A, \mathcal{S}_B$ be two disjoint sets of syntagms (17). Using the translation rules from [19], the intension of each syntagm is a set of closed evaluated instances of a formula of the form $\mathbf{c}(p(x))$ where $\mathbf{c}$ is a logically sound unary connective.

Theorem 10 *Let the theory of FLb*

$$\mathcal{T} = \left\{ \mathcal{A}_k[\mathbf{A}_k],\ \underset{j=1}{\overset{m}{\mathsf{AND}}}(\mathsf{IF}\ \mathcal{A}_j\ \mathsf{THEN}\ \mathcal{B}_j) \left[\bigwedge\nolimits_{j=1}^{m} (\mathbf{A}_j \Rightarrow \mathbf{B}_j) \right] \right\}$$

be given using the above syntagms for some k, $1 \leq k \leq m$. Let

$$\mathbf{A}_k = \{[A_{kx}[t];\ a_{kt}] \mid t \in M_1\}$$

and

$$\bigwedge\nolimits_{j=1}^{m} (\mathbf{A}_j \Rightarrow \mathbf{B}_j) = \left\{ \left[\bigwedge_{j=1}^{m} (A_{jx}[t] \Rightarrow B_{jy}[s]);\ c_{ts} \right] \middle| \ t \in M_1, s \in M_2 \right\}.$$

Then we may derive the conclusion $\mathcal{B}_k$ with the intension

$$\mathbf{B}_k = \left\{ \left[B_{ky}[s];\ b_s = \bigvee_{t \in M_1} (a_{kt} \otimes c_{ts}) \right] \middle| \ s \in M_2 \right\} \tag{20}$$

such that all b_s for $s \in M_2$ in the multiformula $\mathbf{B}_k$ are maximal.

PROOF: The linguistic description determines the fuzzy theory

$$T = \left\{ \left. {}^{a_{kt}}\!/A_{kx}[t],\ {}^{c_{ts}}\!/\bigwedge_{j=1}^{m} (A_{jx}([t] \Rightarrow B_{jy}[s]) \right| t \in M_1, s \in M_2 \right\}.$$

The theorem then follows from Lemma 3. □

In this theorem, we do not consider modification of the premise. However, this is possible when using special inference rule in fuzzy logic in narrow sense (cf. [19]).

The following lemma is proved using the same methods. It demonstrates that standard Mamdani's Max-Min rule can be obtained as a consequence of some special axioms (see [14]) in which we consider a new predicate $R(x, y)$ representing some function to be approximated. Furthermore, for the formulas in concern (i.e. those occurring in all the linguistic descriptions so far) the resulting computation formula gives the best possible result in the same sense as above.

Lemma 4 *Given a fuzzy theory*

$$\begin{aligned} T' &= \{\mathbf{A}_j \wedge \mathbf{B}_j \mid j = 1, \ldots, m\} = \\ &= \{\{[A_{jx}[t] \wedge B_{jy}[s];\ a_{jt} \wedge b_{js}] \mid t \in M_1, s \in M_2\} \mid j = 1, \ldots, m\} \end{aligned}$$

where x, y are variables of different sorts, M_1, M_2 are the corresponding sets of terms (without variables) and for every $j \neq k$ are $A_j \wedge B_j$ and $A_k \wedge B_k$ independent. Furthermore, put

$$T = T' \cup \{[(\forall x)(\forall y)((A_j(x) \wedge B_j(y)) \Rightarrow R(x, y));\ 1] \mid j = 1, \ldots, m\}. \tag{21}$$

If

$$T \vdash_{a'_t} A'_x[t] \qquad t \in M_1$$

where $A'(x)$ is either $A_j(x)$ for some $j = 1, \ldots, m$ or it is independent on all $A_1(x) \wedge B_j(y)$ then

$$T \vdash_{b'_s} B'_y[s] \qquad s \in M_2$$

where

$$b'_s = \bigvee_{t \in M_1} (a'_t \wedge \bigvee_{j=1}^{m} (a_{jt} \wedge b_{js}))$$

and $B'(y) := (\exists x)(A'(x) \wedge R(x, y))$.

PROOF: Using the instances of the substitution axiom, we obtain the provable evaluated formula $[(A_{jx}[t]\wedge B_{jy}[s])\Rightarrow R_{x,y}[t,s];\ 1]$, from which it follows that

$$T \vdash_{d_{ts}} R_{xy}[t,s] \qquad t\in M_1, s\in M_2$$

where $d_{ts}\geq \bigvee_{j=1}^{m}(a_{jt}\wedge b_{js})$. Then there is a set of proofs in T

$$\begin{aligned} w_{jt} \ :=\ & [A'_x[t];\ a'_t]\,,\ [R_{xy}[t,s];\ a_{jt}\wedge b_{js}]\,,\ \ldots,\ [A'_x[t]\wedge R_{xy}[t,s];\ a'_t\wedge a_{jt}\wedge b_{js}]\,,\ldots \\ & [(\exists x)(A'(x)\wedge R(x)_y[s]);\ a'_t\wedge(a_{jt}\wedge b_{js})]\,, \end{aligned}$$

$t\in M_1$, $j=1,\ldots,m$ where we have used the rule r_{MP}, substitution axiom and its consequences. From it follows that

$$\bigvee_{\substack{t\in M_1\\ j=1,\ldots,m}} \mathrm{Val}_T(w_{jt}) = \bigvee_{t\in M_1}\bigvee_{j=1}^{m}(a'_t\wedge(a_{jt}\wedge b_{js})) = \bigvee_{t\in M_1}(a'_t\wedge\bigvee_{j=1}^{m}(a_{jt}\wedge b_{js}))$$

which gives

$$T\vdash_{c_s} B'_y[s]$$

where

$$c_s\geq\bigvee_{t\in M_1}(a'_t\wedge\bigvee_{j=1}^{m}(a_{jt}\wedge b_{js})). \tag{22}$$

As the formulas $A_j\wedge B_j$ and A', $j=1,\ldots,m$ are independent, there exists a model $\mathcal{D}'\models T'$ such that

$$\begin{aligned} \mathcal{D}'(A_{jx}[t]\wedge B_{jy}[s]) &= a_{jt}\wedge b_{js}, \\ \mathcal{D}'(A'_x[t]) &= a'_t, \end{aligned}$$

$j=1,\ldots m, t\in M_1, s\in M_2$. Let us construct a model $\mathcal{D}$ as follows. We put $D=D'$ and, furthermore,

$$\begin{aligned} \mathcal{D}(A_{jx}[t]\wedge B_{jy}[s]) &= \mathcal{D}'(A_{jx}[t]\wedge B_{jy}[s]) \\ \mathcal{D}(R_{xy}[t,s]) &= \bigvee_{j=1}^{m}(a_{jt}\wedge b_{js}) \end{aligned}$$

$t\in M_1$, $s\in M_2$, $j=1,\ldots m$ and $\mathcal{D}(C)=\mathcal{D}'(C)$ for every formula C containing no instance of $R(x,y)$. Then

$$\begin{aligned} &\mathcal{D}(\forall x)(\forall y)((A_j(x)\wedge B_j(y))\Rightarrow R(x,y))) = \\ &= \bigwedge_{t\in M_1,s\in M_2}(\mathcal{D}(((A_{jx}[t]\wedge B_{jy}[s])\Rightarrow R_{x,y}[t,s])) = \\ &= \bigwedge_{t\in M_1,s\in M_2}((a_{jt}\wedge b_{js})\to\bigvee_{j=1}^{m}(a_{jt}\wedge b_{js}))=1 \end{aligned}$$

and thus, $\mathcal{D}\models T$. Finally,

$$\begin{aligned} \mathcal{D}(B_y[s]) &= \mathcal{D}((\exists x)(A'(x)\wedge R(x)_y[s])) = \\ &= \bigvee_{t\in M_1}\mathcal{D}(A'_x[t]\wedge R_{xy}[t,s]) = \bigvee_{t\in M_1}(a'_t\wedge\bigvee_{j=1}^{m}(a_{jt}\wedge b_{js})), \end{aligned}$$

i.e.

$$c_s\leq\bigvee_{t\in M_1}(a'_t\wedge\bigvee_{j=1}^{m}(a_{jt}\wedge b_{js}))$$

which which together with (22) gives the required equality. □

On the basis of Lemma 4 and analogously to Theorem 10 we can formulate the following theorem.

Theorem 11 *Let the theory of FLb*

$$\mathcal{T} = \{\mathcal{A}'[\mathbf{A}'], \underset{j=1}{\overset{m}{\mathsf{OR}}}(\mathcal{A}_j \ \mathsf{AND}\ \mathcal{B}_j) \left[\bigvee_{j=1}^{m}(\mathbf{A}_j \wedge \mathbf{B}_j)\right]\}$$

be given using the same syntagms as in Theorem 10. Furthermore, let the axioms from (21) be added. Then we may derive a conclusion $\mathcal{B}'$ assigned a formula $B'(y) := (\exists x)(A'(x) \wedge R(x,y))$ and having the intension

$$\mathbf{B}' = \left\{ \left[B'_y[s];\ b'_s = \bigvee_{t \in M_1} (a'_t \wedge \bigvee_{j=1}^{m} (a_{jt} \wedge b_{js})) \right] \middle|\, s \in M_2 \right\}$$

where all b'_s for $s \in M_2$ in the multiformula $\mathbf{B}'$ are maximal.

This theorem, analogously as Theorem 10, explicitly states that for the syntagms widely used in fuzzy control, and provided that we assume (21), the Mamdani's Max-Min rule can be used to derive a conclusion which is the best possible one (in the sense of maximalization of truth values). Hence, we have basically two (from the point of view of truth values) effective inference procedures: the first one is based on sound inference rules of fuzzy logic in narrow sense and deals with logical implications, and the second one is based on additional assumptions and deals with conjunctions. Note that the latter is linguistic representation of the concept of *fuzzy graph.*

4 Conclusion

In this paper, we reviewed some formal aspects of fuzzy logics in narrow as well as in broader sense. The former can be viewed as a special many-valued logic aimed at modelling of the vagueness phenomenon. Therefore, it is slightly modified to fulfil this goal. Most important is its ability to derive conclusions concerning any truth value, i.e. all the truth values are equally important. As a consequence, we obtain evaluated syntax in which evaluated formulas $[A; a]$ are considered. Such formulas, however, can be interpreted as non-evaluated ones of the form $\mathbf{a} \Rightarrow A$. We have discussed outcomes of this approach, and also showed that this logic may be considered as a direct generalization of the classical one due to the definition of the syntactic consequence operation (Definition 1) which is equivalent with the classical definition. In many-valued logic, however, the requirement that a provable formula must have a proof with the designated truth value is quite strong and restrictive for fuzzy logic in narrow sense. Slight weakening (cf. Definition 2 and Theorem 1) makes possible to keep the syntactico-semantical completeness. Unfortunately, this holds only in the case that we use Łukasiewicz implication since it is continuous. Interesting problem might be to classify fuzzy logics in narrow sense with respect to some "degrees of completeness" which would be based on the general implication $1 \wedge (1 - a^p + b^p)^{\frac{1}{p}}$, $p \neq 0$.

The last section is devoted to fuzzy logic in broader sense which should be the logic of commonsense human deduction and thus, it is non-separably connected with linguistics. As natural language inherently encompasses vagueness, FLn becomes its frame and FLb can thus be seen as an extension of FLn. Natural language expressions are translated into multiformulas (sets of evaluated instances of formulas of FLn) which are interpreted as intensions of the former. We have proved two theorems which demonstrate that when confining ourselves to restricted kinds of syntagms, the formulas widely used for generalized modus ponens both in implicational as well as Mamdani's forms give the best possible values.

Godo and Hájek [3] derived generalized modus ponens on the basis of purely logical considerations. They present several forms of this rule. Note that it is possible to express their rules also in the aggregated form using the concept of multiformula. Theorem 11 is based on the assumption (21) stating that, roughly speaking, the linguistic description concerns some relation between input and output (predicate $R(x,y)$). Godo and Hájek use weaker assumption but they still keep a condition which inherently assumes some relation between the premise and the consequent.

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