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Representation of concept lattices by bidirectional associative memories

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A concept interpretation of patterns for bidirectional associative memory (BAM) is provided. Representation of hierarchical structures of concepts (concept lattices) by bidirectional associative memories (BAM) is presented. The constructive representation theorem provides a learning rule for a training set which allows a concept interpretation. Examples demonstrating the theorems are presented.

1 Introduction and preliminaries

When modeling intelligent systems, two levels have to be taken into account: the *microlevel* and the *macrolevel*. This is because there are two corresponding levels evidenced by biological systems which are held for intelligent: the level of brain and the level of mental phenomena. The macrolevel (mental level) is supposed to be implemented in the microlevel (brain level). The inspiration by the microlevel gave a rise to the paradigm of artificial neural networks—systems which are based on similar principles as their biological counterparts. On the other hand, there is a lot of models inspired by the macrolevel. A challenging goal is the development of architectures which exhibit both of the levels. On the macrolevel, a clear interpretation of the system is possible, while on the microlevel, an analysis up to an appropriate degree of exactness can be performed.

This paper deals with a (macrolevel) interpretation of the bidirectional associative memory (Kosko (1987), Kosko (1988)). The interpretation is in terms of concepts: BAM patterns are interpreted to represent concepts in the sense of Wille (1982), Ganter and Wille (1996). Section 2 and Section 3 survey bidirectional associative memories and fundamentals of concept lattices, respectively. In Section 4, a concept interpretation of BAM patterns is proposed and discussed, learning rule and representation theorem are presented. Section 5 contains illustrative examples.

2 Bidirectional associative memories

Associative memories represent a class of neural networks which aim at modeling of the association phenomenon (Arbib (1995), Bělohlávek (1998)). Based on the early models of Amari (1972) and Hopfield (1984), Kosko (1987), Kosko (1988) proposed a bidirectional associative neural network called BAM (bidirectional associative memory). BAM consists of two layers of neurons. The first and the second layers contain k and l neurons, respectively, states (signals) of which are denoted by x_i ($i = 1, \ldots, k$) and y_j ($j = 1, \ldots, l$). The states x_i and y_j take the values $-1, 1$ by bipolar and $0, 1$ by binary encoding (to which we restrict ourselves). Each (i-th) neuron of the first layer is connected to each (j-th) neuron of the second layer by a connection with the real weight w_{ij}. A real threshold θ_i^x (θ_j^y) is assigned to the

i-th neuron of the first layer (j-th neuron of the second layer). The dynamics is bidirectional: given a pair $\langle X, Y\rangle = \langle\langle x_1, \ldots, x_k\rangle, \langle y_1, \ldots, y_l\rangle\rangle \in \{0,1\}^k \times \{0,1\}^l$ of patterns of signals the signal is fed to the second layer to obtain a new pair $\langle X, Y'\rangle$, then again to the first layer to obtain $\langle X', Y'\rangle$ etc. The dynamics is given by the formulas

$$y'_j = \begin{cases} 1 & \text{for } \sum_{i=1}^k w_{ij} x_i > \theta_j^y \\ y_j & \text{for } \sum_{i=1}^k w_{ij} x_i = \theta_j^y \\ 0 & \text{for } \sum_{i=1}^k w_{ij} x_i < \theta_j^y \end{cases} , \quad x'_i = \begin{cases} 1 & \text{for } \sum_{j=1}^l w_{ij} y'_j > \theta_i^x \\ x_i & \text{for } \sum_{j=1}^l w_{ij} y'_j = \theta_i^x \\ 0 & \text{for } \sum_{j=1}^l w_{ij} y'_j < \theta_i^x \end{cases} . \tag{1}$$

The pair of patterns $\langle X, Y\rangle$ is called a stable point if the states of neurons, which are set to $\langle X, Y\rangle$, do not change under the above defined dynamics. The set of all stable points of a BAM will be denoted by $Stab(W, \Theta)$. Using appropriate energy function, Kosko proved[1] that such a network is stable for any weights w_{ij} and any thresholds θ_i^x, θ_j^y. Stability means that given any initial pattern $\langle X, Y\rangle$ of signals, the net eventually stops after a finite number of steps (feeding signal from a layer to a layer).

The aim of learning in the context of associative memories is to set the parameters of the net so that a prescribed training set of patterns is in some relation to the set of all stable points. Usually, all training patterns have to become stable points. Kosko proposes a kind of Hebbian learning, by which the weights w_{ij} are determined from the training set $T = \{\langle X^p, Y^p\rangle \mid p \in P\}$ by

$$w_{ij} = \sum_{p \in P} \mathrm{bip}(x_i^p) \cdot \mathrm{bip}(y_j^p) \tag{2}$$

where bip maps 1 to 1 and 0 to -1, i.e. it changes the binary encoding to bipolar one. Thresholds are set to 0. As may be easily checked, a one-element training set will be learned completely by this rule. Another rule has been proposed by Wang *et al.* (1990) where also a further theoretical analysis of BAM dynamics is provided.

3 Concepts and concept lattices

The notion of concept is central in human thinking. Also, the notion of concept appears in the context of neural networks. Networks are seen as if extracting characteristic features of input data. These characteristic features are considered to represent concepts. Other attributes than the agregation function (selection of characteristic features) are usually ignored.

In the programmatic paper Wille (1982) started the theory of concept lattices which serves as a foundation of formal concept analysis (Ganter and Wille (1996)). It is based on the traditional understanding of concepts of the Port-Royal school by which a concept is determined by its *extent* and its *intent*. The extent of a concept (e.g. DOG) is the collection of all objects which are covered by the concept (the collection of all dogs) while the intent is the collection of all attributes (e.g. "to bark", "to be a mammal", etc.) covered by the concept. The starting point of the formalization is that of context, i.e. a triple $\langle G, M, I\rangle$, where $I \subseteq G \times M$. Elements of G are interpreted as *objects*, elements of M as *attributes*, the fact $\langle g, m\rangle \in I$ is interpreted as "the object g has the attribute m". According to the philosophical tradition, a (formal) *concept* in a given context is any pair $\langle A, B\rangle$ of extent $A \subseteq G$ and intent $B \subseteq M$ such that $A = B^{\downarrow} :=_{\mathrm{def}} \{g \in G \mid \text{ for all } m \in B \text{ it holds } \langle g, m\rangle \in I\}$ and $B = A^{\uparrow} :=_{\mathrm{def}} \{m \in M \mid \text{ for all } g \in A \text{ it holds } \langle g, m\rangle \in I\}$. In other words, $\langle A, B\rangle$ is a concept if $A = B^{\downarrow}$ and $B = A^{\uparrow}$, i.e. A is the set of all objects which

[1] In fact, in the original paper, only the dynamics without the thresholds is discussed.

have all the attributes of B and, conversely, B is the set of all attributes which are shared by all the objects of A.

The crucial relation between concepts is that of a hierarchical ordering. The hierarchy of concepts plays a crucial role in conceptual reasoning. Denote $\mathcal{B}(G,M,I)$ the set of all concepts in the context $\langle G,M,I\rangle$, i.e.

$$\mathcal{B}(G,M,I)=\{\langle A,B\rangle \mid A=B^{\downarrow}, B=A^{\uparrow}\}$$

and for $\langle A_1,B_1\rangle,\langle A_2,B_2\rangle\in\mathcal{B}(G,M,I)$ put $\langle A_1,B_1\rangle\leq\langle A_2,B_2\rangle$ iff $A_1\subseteq A_2$ (which is equivalent to $B_1\supseteq B_2$). The relation $\leq$ naturally models the relation "to be a subconcept" (as an example, consider the concepts DOG and MAMMAL). The fundamental structure of the set of all concepts given by a context is given by the following proposition which is a part of the so called Main Theorem of Conceptual Data Analysis.

Proposition (Wille (1982)**)** *Let $\langle G,M,I\rangle$ be a context. Then the set $\mathcal{B}(G,M,I)$ is under the relation $\leq$ introduced above a complete lattice where*

$$\bigwedge\{\langle A_j,B_j\rangle;\, j\in J\}=\langle\bigcap\{A_j;\, j\in J\},(\bigcap\{A_j;\, j\in J\})^{\uparrow}\rangle\ ,$$
$$\bigvee\{\langle A_j,B_j\rangle;\, j\in J\}=\langle(\bigcap\{B_j;\, j\in J\})^{\downarrow},\bigcap\{B_j;\, j\in J\}\rangle\ .$$

The lattice $\mathcal{B}(G,M,I)$ is called a *concept lattice* given by the context $\langle G,M,I\rangle$. The complete lattice Birkhoff (1967) structure is a very natural one for conceptual structures. Informally, it states that for each set of concepts there is their direct generalization (supremum) as well as their direct specialization (infimum). Not that the visualizable hierarchical structure of the revealed concepts is the basic tool of conceptual data analysis. Further results can be found in Ganter and Wille (1996).

4 Representation and learning of concept lattices

Our aim now is to provide a conceptual interpretation of BAM and to show that BAM's can represent lattices of concepts. To this end we accept the following convention. For a set $Z=\{z_1,\ldots,z_n\}$ and a subset $A\subseteq Z$ we denote by $s_Z(A)=\langle a_1,\ldots,a_n\rangle\in\{0,1\}^n$ the characteristic vector of A, i.e. $a_i=1$ if $z_i\in A$ and $a_i=0$ if $z_i\notin A$.

Let us have a context $\langle G,M,I\rangle$ with both G and M finite, $G=\{g_1,\ldots,g_k\}$, $M=\{m_1,\ldots,m_l\}$. Using the grandmother-cell idea (Arbib (1995)) we can consider a BAM with k neurons in the first layer and l neurons in the second layer with the interpretation: Both of the layers represent subsets of G and M, respectively. The set $A\subseteq G$ is represented by the vector $s_G(A)\in\{0,1\}^k$ of the states of the first layer and similarly $B\subseteq M$ is represented by $s_M(B)\in\{0,1\}^l$. The i-th neuron of the first layer therefore represents the object g_i and the j-th neuron of the second layer represents the property m_j. The pairs of subsets of G and M are thus in a one-to-one correspondence with the pairs of states of the first and the second layer.

Example 1 In general, the concept interpretation of the patterns of states is not possible. Namely, we easily find a BAM stable points of which cannot be interpreted as concepts. Take e.g. $k=1$, $m=2$, $w_{11}=1$, $w_{12}=-2$, all thresholds set to 0. Then $\mathit{Stab}(W,\Theta)=\{\langle 0,\langle 0,0\rangle\rangle,\langle 0,\langle 0,1\rangle\rangle,\langle 0,\langle 1,1\rangle\rangle,\langle 1,\langle 1,0\rangle\rangle\}$. For $\langle s_G(A_1),s_M(B_1)\rangle=\langle 0,\langle 0,0\rangle\rangle$, $\langle s_A(A_2),s_M(B_2)\rangle=\langle 1,\langle 1,1\rangle\rangle$ we have $A_1\subset A_2$ but $B_1\not\supseteq B_2$ which contradicts the rule "the more common objects the less common properties".

On the other hand, taking e.g. $k = l = 2$, $w_{11} = w_{12} = 1$, $w_{12} = w_{21} = -3$, all thresholds set to $-\frac{1}{2}$. Then $Stab(W, \Theta) = \{\langle\langle 0,0\rangle, \langle 1,1\rangle\rangle, \langle\langle 0,1\rangle, \langle 0,1\rangle\rangle, \langle\langle 1,0\rangle, \langle 1,0\rangle\rangle, \langle\langle 1,1\rangle, \langle 0,0\rangle\rangle\}$. As can be easily verified, $Stab(W, \Theta)$ corresponds to $\mathcal{B}(G, M, I)$ for $I = \{\langle g_1, m_1\rangle, \langle g_2, m_2\rangle\}$ by $Stab(W, \Theta) = \{\langle s_G(A), s_M(B)\rangle \mid \langle A, B\rangle \in \mathcal{B}(G, M, I)\}$.

The crucial question therefore is: Is there for each concept lattice $\mathcal{B}(G, M, I)$ a BAM such that the set of all concepts of $\mathcal{B}(G, M, I)$ is (modulo the above correspondence) precisely the set of all stable points of this BAM? The positive answer is given by the following theorem.

Theorem 1 *Let $\mathcal{B}(G, M, I)$ be a concept lattice given by the context $\langle G, M, I\rangle$ with G and M finite. Then there is a BAM given by the weights W and thresholds Θ such that $Stab(W, \Theta) = \{\langle s_G(A), s_M(B)\rangle \mid \langle A, B\rangle \in \mathcal{B}(G, M, I)\}$.*

Proof Let $G = \{g_1, \ldots, g_k\}$, $M = \{m_1, \ldots, m_l\}$. Define a BAM by the matrix W given by

$$w_{ij} = \begin{cases} 1 & \text{if } \langle g_i, m_j\rangle \in I \\ -q & \text{if } \langle g_i, m_j\rangle \notin I \end{cases} \tag{3}$$

for $i = 1, \ldots, k$, $j = 1, \ldots, l$ where $q = \max\{k, l\} + 1$. All the thresholds are set to $-\frac{1}{2}$.

Let $\langle A, B\rangle \in \mathcal{B}(G, M, I)$. We show that $\langle s_G(A), s_M(B)\rangle$ is a stable point of the BAM. Initialize the network with $\langle s_G(A), s_M(B)\rangle$, i.e. $x(0) = \langle x_1(0), \ldots, x_k(0)\rangle = s_G(A)$, $y(0) = \langle y_1(0), \ldots, y_l(0)\rangle = s_M(B)$. We show $\langle x(1), y(1)\rangle = \langle x(0), y(0)\rangle$. Clearly, $x(1) = x(0)$ (if the signal is fed forward the first layer does not change its state). Consider now any y_j. We distinguish two cases. First, let $y_j(0) = 1$. Since $\langle A, B\rangle$ is a concept, we have $\langle g_i, m_j\rangle \in I$ (i.e. $w_{ij} = 1$) for each i such that $g_i \in A$ (i.e. $x_i(0) = 1$). We therefore have

$$\sum_{i=1}^{k} w_{ij} x_i(0) = |A| \geq -\frac{1}{2} = \theta_j^y \ .$$

By the activation dynamics of BAM we have $y_j(1) = 1$. For $y_j(0) = 1$ the state therefore does not change. Second, let $y_j(0) = 0$. Since $\langle A, B\rangle$ is a concept, there is some i such that $x_i(0) = 1$ (i.e. $g_i \in A$) but $\langle g_i, m_j\rangle \notin I$ (i.e. $w_{ij} = -q$). Denote by K the set of all such i. Denote furthermore by K^* the set of i such that $x_i(0) = 1$ (i.e. $g_i \in A$) and $\langle g_i, m_j\rangle \in I$ (i.e. $w_{ij} = 1$). We have

$$\sum_{i=1}^{k} w_{ij} x_i(0) = \sum_{i \in K} w_{ij} x_i(0) + \sum_{i \in K^*} w_{ij} x_i(0) = -|K|q + |K^*| < -\frac{1}{2}$$

since $q > k \geq |K^*|$, $|K| \geq 1$, and $-|K|q + |K^*|$ is an integer. By the BAM dynamics we have $y_j(1) = 0$, i.e. the state does not change. We have proved $y(1) = y(0)$. We should now show that $x(2) = x(1)$ by the backward phase. The proof is completely symmetric and we omit it. We have thus proved that $\langle s_G(A), s_M(B)\rangle$ is stable.

Conversely, let $\langle s_G(A), s_M(B)\rangle = \langle x, y\rangle \in Stab(W, \Theta)$. We show that $\langle A, B\rangle$ is a concept of $\mathcal{B}(G, M, I)$, i.e. $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Again, due to symmetry we show only $A^{\uparrow} = B$. We reason as follows: $m_j \in A^{\uparrow}$ iff for each i such that $g_i \in A$ (i.e. $x_i \in 1$) we have $\langle g_i, m_j\rangle \in I$ (i.e. $w_{ij} = 1$). The last assertion holds iff $\sum_{i=1}^{k} w_{ij} x_i > -\frac{1}{2}$. (Indeed, the direction "$\Rightarrow$" is clear. Conversely, let $\sum_{i=1}^{k} w_{ij} x_i > -\frac{1}{2}$. If there would be some i such that $g_i \in A$ $(x_i = 1)$ and $\langle g_i, m_j\rangle \notin I$ $(w_{ij} = -q)$ then $\sum_{i=1}^{k} w_{ij} x_i < -\frac{1}{2}$, a contradiction.) By the BAM dynamics, $\sum_{i=1}^{k} w_{ij}x_i > -\frac{1}{2}$ iff $y_j = 1$. To sum up, $m_j \in A^{\uparrow}$ iff $y_j = 1$, hence $A^{\uparrow} = B$. The theorem is proved. □

Corollary 2 *For each finite lattice $\mathcal{L} = \langle L, \leq \rangle$ there is a BAM given by the weights W and thresholds Θ such that under the relation $\leq$ defined on $Stab(W, \Theta)$ by*

$$\langle x_1, y_1 \rangle \leq \langle x_2, y_2 \rangle \quad \text{iff} \quad x_{1i} \leq x_{2i}\ (\forall i)(\text{ iff } y_{1j} \geq y_{2j}\ (\forall j))$$

$\langle Stab(W, \Theta), \leq \rangle$ and $\langle L, \leq \rangle$ are isomorphic lattices.

Proof The proof follows from Theorem 1 by the fact that each complete lattice $\mathcal{L} = \langle L, \leq \rangle$ is isomorphic to the concept lattice $\mathcal{B}(L, L, \leq)$ (cf. Wille (1982)). □

The proof of the Theorem 1 gives also the rule (3) for learning the set of all patterns (concepts) $\langle A, B \rangle$ of $\mathcal{B}(G, M, I)$ from the relation I. However, one might be concerned with a situation where the training information is not in the form of a binary relation but in the form of a training set

$$T = \{\langle A^p, B^p \rangle \mid A \subseteq G,\, B \subseteq M,\, p \in P\}$$

of patterns. In this case, the fundamental question is whether T can be interpreted as a (consistent) structure of concepts. Call T *conceptually consistent* if there is a concept lattice $\mathcal{B}(G, M, I)$ such that $T \subseteq \mathcal{B}(G, M, I)$. In the following we concentrate on the problem of finding necessary and sufficient condition for a training set T to be conceptually consistent and on the problem of learning a conceptually consistent training set.

Lemma 3 *Let $T = \mathcal{B}(G, M, I)$. Then for $I_T \subseteq G \times M$ defined by*

$$\langle g, m \rangle \in I_T \quad \text{iff} \quad \exists \langle A, B \rangle \in T : g \in A, m \in M \tag{4}$$

it holds $I = I_T$.

Proof If $\langle g, m \rangle \in I_T$ then for $\langle A, B \rangle$ of (4) we conclude from $A^{\uparrow} = B$ that $\langle g, m \rangle \in I$, i.e. $I_T \subseteq I$. Conversely, if $\langle g, m \rangle \in I$ then for $\langle A, B \rangle = \langle \{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow} \rangle \in \mathcal{B}(G, M, I)$ we have $g \in A, m \in B$, i.e. $\langle g, m \rangle \in I_T$, which proves $I \subseteq I_T$. □

Lemma 4 *For a conceptually consistent T let I_T be defined by (4). Then it holds $T \subseteq \mathcal{B}(G, M, I_T)$.*

Proof If T is conceptually consistent then $T \subseteq \mathcal{B}(G, M, I)$ for some $I \subseteq G \times M$. By Lemma 3, $I_T \subseteq I$. Denote the operators corresponding to I and I_T by ${}^{\uparrow}, {}^{\downarrow}$ and ${}^{\uparrow_T}, {}^{\downarrow_T}$, respectively. Take $\langle A, B \rangle \in T$. Since $I_T \subseteq I$, we have $B = A^{\uparrow} = \{m \in M \mid \forall g \in A : \langle g, m \rangle \in I\} \supseteq \{m \in M \mid \forall g \in A : \langle g, m \rangle \in I_T\} = A^{\uparrow_T}$. On the other hand, since for every $g \in A$, $m \in B$ we have $\langle g, m \rangle \in I_T$ (by (4)), it holds $A^{\uparrow_T} \supseteq B$, i.e. $A^{\uparrow_T} = B$. One would similarly obtain $B^{\downarrow_T} = A$. To sum up, $A^{\uparrow_T} = B$ and $B^{\downarrow_T} = A$, i.e. $\langle A, B \rangle \in \mathcal{B}(G, M, I_T)$. □

We have the following criterion for a training set to be conceptually consistent.

Theorem 5 *A training set $T = \{\langle A^p, B^p \rangle \mid p \in P\}$ is conceptually consistent iff for each $p \in P$ it holds*

$$\begin{aligned} A^p &= \{g \in G \mid \forall m \in B^p\, \exists p' \in P : g \in A^{p'}, m \in B^{p'}\} \\ B^p &= \{m \in M \mid \forall g \in A^p\, \exists p' \in P : g \in A^{p'}, m \in B^{p'}\}. \end{aligned}$$

	size			from sun		has moon	
	small (ss)	medium (sm)	large (sl)	near (dn)	far (df)	yes (my)	no (mn)
Mercury (Me)	×			×			×
Venus (V)	×			×			×
Earth (E)	×			×		×	
Mars (Ma)	×			×		×	
Jupiter (J)			×		×	×	
Saturn (S)			×		×	×	
Uranus (U)		×			×	×	
Neptune (N)		×			×	×	
Pluto (P)	×				×	×	

Table 1: Planets and their attributes.

Proof If T is conceptually consistent then the assertion follows from Lemma 4. Conversely, if for each $p \in P$ the above equalities hold then $T \subseteq \mathcal{B}(G, M, I_T)$. □

By the previous results, we have for a training set $T = \{\langle A^p, B^p \rangle \mid A \subseteq G, B \subseteq M, p \in P\}$ with $|G| = k$, $|M| = l$ the following learning algorithm: For $i = 1, \ldots, k$, $j = 1, \ldots, l$, set the weights by

$$w_{ij} = \begin{cases} 1 & \text{if } \exists p \in P : g \in A^p, m \in B^p \\ -(\max k, l+1) & \text{otherwise} \end{cases}$$

and the thresholds by

$$\theta_i^x = -\frac{1}{2}, \qquad \theta_j^y = -\frac{1}{2}.$$

Call T *learnable* (by our algorithm) if for the weights and the thresholds set according the algorithm it holds $s(T) \subseteq Stab(W, \Theta)$ where $s(T) = \{\langle s_G(A), s_M(B) \rangle \mid \langle A, B \rangle \in T\}$. The scopes and limits of the algorithm are desribed by the following assertion.

Corollary 6 *A training set T is learnable iff it is conceptually consistent.*

Proof If T is conceptually consistent then the assertion follows from Lemma 4. Conversely, if T is learnable then, by definition, $s(T) \subseteq Stab(W, \Theta)$. The assertion then follows from the fact that if W and Θ are learned from T by our algorithm then $Stab(W, \Theta) = s(\mathcal{B}(G, M, I_T))$ by Theorem 1. □

5 Examples

Example 2 Let a context be given by the set G of nine planets (Mercury, ..., Pluto), the set M of seven attributes ("size small", ..., "does not have a moon") and the relation I between them depicted in Tab. 1 (see Wille (1982)). The BAM learned from this context by Theorem 1 consists of nine and seven neurons in the first and the second layer, respectively. The set of all stable points (i.e. concepts) is depicted in Tab. 2. The approaximate linguistic description of the stable points is as follows: 1 – the empty concept, 2 – "small planet without moon near to sun", 3 – "small planet with moon(s) near to sun", 4 – "small planet with moon(s) far from sun", 5 – "large planet with moon(s) far from sun", 6 – "medium planet with moon(s) far from sun", 7 – "small planet near to sun", 8 – "small planet with moon(s)", 9 – "planet far from sun", 10 – "small planet", 11 – "planet with moon",

no.	extent									intent						
	Me	V	E	Ma	J	S	U	N	P	ss	sm	sl	dn	df	my	mn
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
2	1	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1
3	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0
4	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0
5	0	0	0	0	1	1	0	0	0	0	0	1	0	1	1	0
6	0	0	0	0	0	0	1	1	0	0	1	0	0	1	1	0
7	1	1	1	1	0	0	0	0	0	1	0	0	1	0	0	0
8	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	0
9	0	0	0	0	1	1	1	1	1	0	0	0	0	1	1	0
10	1	1	1	1	0	0	0	0	1	1	0	0	0	0	0	0
11	0	0	1	1	1	1	1	1	1	0	0	0	0	0	1	0
12	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

Table 2: Stable points of the “planet example”.

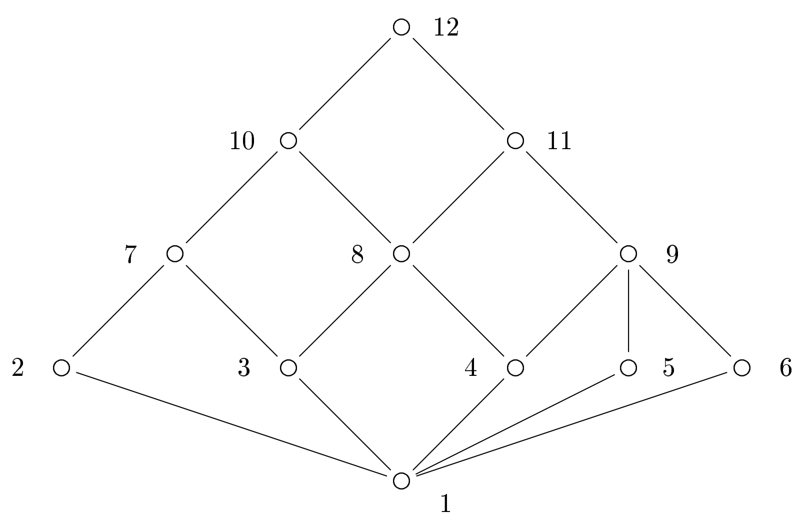

Figure 1: Hierarchical structure (lattice) of stable points of the “planet example”.

12 – "planet". The conceptual hierarchy of the stable points (concept lattice) is depicted in Fig. 1.

Example 3 Consider a BAM with 9 neurons in the first layer and 25 neurons in the second layer. We will represent each state $\langle X, Y \rangle$ of the net by a pair of bitmap pictures, a 3×3-bitmap for X and 5×5-bitmap for Y. If the neuron state is 0 (1) then the corresponding pixel is white (black). Consider the training set T consisting of six pairs $\langle A, B \rangle$ such that $\langle s_G(A), s_M(B) \rangle$ are represented by the pairs 5, 6, 9, 19, 22, and 23 of Fig. 2. By Theorem 5 it is easy to verify that T is conceptually consistent. The learning algorithm yields a BAM with 24 stable points depicted in Fig. 2. The conceptual hierarchy of the stable points is visualized in Fig. 3. The patterns of T have been learned and completed into a complete conceptual structure. Note that the concepts 6 and 19 of the training set are complementary concepts. The learned structure of concepts contains additional pairs of complementary concepts ("oposite concepts" in conceptual terms), e.g. 5 and 20 (5 is in T), 7 and 18 (none of them is in T) etc. Note that the pairs of complementary concepts, viewed from the lattice point of view, form complementary elements in the lattice of concepts.

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1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21
22	23	24

Figure 2: Stable points.

24

21 22 23 16

18 19 20 13 14 15 8

17 10 11 12 5 6 7

9 2 3 4

1

Figure 3: Hierarchical structure (lattice) of stable points.