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Arithmetics of Extensional Fuzzy Numbers – Part II: Algebraic framework

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***Abstract*—In the first part of this contribution, we proposed extensional fuzzy numbers and a working arithmetic for them that may be abstracted to so-called many identities algebras (MI-algebras, for short).**

In this second part, we show that the proposed MI-algebras give a framework not only for the arithmetic of extensional fuzzy numbers, but also for other arithmetics of fuzzy numbers and even more general sets of real vectors used in mathematical morphology. This entitles us to develop a theory of MI-algebras to study general properties of structures for which the standard algebras are not appropriate. Some of the basic concepts and properties are presented here.

I. Introduction

As it has been discussed in the first part of this contribution [1], the standard arithmetics of fuzzy numbers often stemming from the Zadeh's extension principle or arithmetic of real intervals do not satisfy several common properties of real numbers. It is well know that an inverse element defined for fuzzy sets is not closely connected with the identity element as in the case of real numbers (e.g., $a + (-a) \neq 0$) and only the subdistributivity is satisfied (see [2],[3],[4]), i.e.,

$$x(y + z) \subseteq xy + xz.$$

These drawbacks and a natural view on the generalization of real numbers gave rise to the concept of extensional fuzzy number. An extensional fuzzy number x_S is the extensional hull of a singleton $\tilde{x}$ with respect to a similarity relation S (see Definition 4 in [1]). Simply speaking, each extensional fuzzy number x_S may be interpreted as a real number x over which an uncertainty (imprecision, inaccuracy, etc.) is expressed by a similarity relation S in such way that $x_S(y) = S(x, y)$ for any $y \in \mathbb{R}$. Hence, the arithmetic of extensional fuzzy numbers may be divided to the standard arithmetic of real numbers and appropriate fuzzy set operations with similarity relations. This construction significantly simplifies the counting with fuzzy numbers and, moreover, it imitates the practical calculation with the imprecisely defined numbers, where intuitively

"*about five*" plus "*about three*" gives "*about eight*".

In the case of the extensional fuzzy numbers, the adverb "about" is naturally modelled by a similarity relation. Although, the new arithmetic of extensional fuzzy numbers provides the satisfaction of the distributivity, the problem of inverse elements still remains. Observing that the equality $a + (-a) = 0$ is rather artificial for (extensional) fuzzy numbers and the element $a+(-a)$ expresses a (extensional) fuzzy number about 0 (later called pseudoidentity), we introduced in [1] several types of so-called many identities algebras (MI-algebras, for short), namely, the MI-monoid, MI-group and MI-field, without any further accompanied theory.

In this second part of our contribution, we will provide a further justification for our step to introduce the concept of MI-algebras. First, we will show that, although, the MI-algebras were initiated by our analysis of properties of the arithmetical operations with extensional fuzzy numbers, the proposed definitions may be successfully used to classify the properties of other arithmetical operations with real intervals, fuzzy numbers and even (convex) sets of real vectors used in the mathematical morphology (see [5]). Further, we will demonstrate that a reasonable theory of MI-algebras may be provided with similar results well known in the group theory. Finally, we will propose a weaker definition of MI-field introduced in [1].

II. MI-monoids

Although, it is well known that the set of (extensional) fuzzy numbers endowed by the addition or multiplication forms a monoid, we will start with the extension of this basic algebraic structure. The motivation for this step is an elimination of the drawback with the property of inverse elements mentioned above and discussed in [1]. Recall that MI stands for "Many Identities" and these "identities" are used to model a fuzziness (inaccuracy, uncertainty) in numbers.

Definition 1. A triplet $(G, \circ, E)$ is said to be an *MI-monoid*, if E is a non-empty subset of G and $\circ$ is a binary operation on G such that for all $x, y, z \in G$ and $a, b \in E$

(M1) $x \circ (y \circ z) = (x \circ y) \circ z$,
(M2) $\exists e \in E, \forall x \in G : x \circ e = e \circ x = x$,
(M3) $a \circ b \in E$,
(M4) $x \circ x = a$ then $x \in E$,

(M5) $x \circ a = a \circ x$.

Elements from E are called *pseudoidentity elements* (*pseudoidentities*, for short) and the element $e \in E$ satisfying (M2) is called *(strong) identity element*. An MI-monoid G is said to be *commutative* or *abelian*, if $x \circ y = y \circ x$ holds for any $x, y \in G$.

Standardly, we write $G = (G, \circ, E)$ and $x \circ y = xy$. Clearly, for commutative MI-monoids the axiom (M5) may be omitted. One can simply verify that each MI-monoid possesses a unique identity element (consider $e_1 = e_1 e_2 = e_2$ for two identity elements). In what follows, the identity element will be always denoted by e. Note that, in contrast to exactly one identity element, an MI-monoid can have an arbitrary number of pseudoidentity elements (including no pseudoidentity element, i.e. $E \setminus \{e\} = \emptyset$). One may imagine that a grater number of pseudoidentities in G means more "fuzziness" in elements from G.

Definition 2. An MI-monoid G is said to be *good*, if the following condition is satisfied:

(M6) $\forall x \in G, \forall a \in E : x \circ a \in E$ then $x \in E$.

If G is a good MI-monoid, then we need not suppose $e \in E$ in (M2), but it suffices to put $e \in G$. In fact, by (M2) we have $ae = a$ for any $a \in E$. Hence, $e \in E$ follows from (M6).

Remark 1. Trivially, each monoid G is an MI-monoid with $E = \{e\}$. Moreover, one can simply check that $(G, \circ, E)$ is an MI-monoid if and only if $(G, \circ)$ is a monoid and $(E, \circ)$ is a commutative monoid such that $E \subseteq G$ and (M4) and (M5) are satisfied. Thus an MI-monoid is a suitable combination of two monoids, where the second one is, moreover, abelian.

Example 1. *In [1], we have shown that the set of extensional fuzzy numbers endowed by $+$ and $\cdot$ forms good abelian additive and multiplicative MI-monoids, respectively. Further, the set of real intervals endowed by the standard interval operations[1] of $+$ and $\cdot$ naturally forms non-good abelian additive and multiplicative MI-monoids, respectively. Note that using a suitable restriction of the set of pseudoidentities we may also define a good MI-monoid of real intervals.*

A standard approach to the arithmetic of fuzzy numbers by the arithmetic of real intervals is demonstrated in the following example (cf. [2],[4]).

Example 2. *Let G be the MI-monoid of real intervals (see Example 3 in [1]) and H be the set of all mappings $g : [0,1] \to G$ provided*

$$\begin{aligned} g(\alpha) &\subseteq g(\beta) \quad \text{for all } \alpha, \beta \in [0,1],\ \alpha \geq \beta, \\ \bigcup_{\beta < \alpha} g(\beta) &= g(\alpha) \quad \text{for all } \alpha \in]0,1]. \end{aligned} \tag{1}$$

One can see that g represents a continuous and convex fuzzy set in $\mathbb{R}$ with a finite support (a fuzzy number). More precisely, if $g \in H$, then

$$A_g(x) = \bigvee_{\substack{\alpha \in [0,1], \\ x \in g(\alpha)}} \alpha$$

is a continuous and convex fuzzy set $A_g : \mathbb{R} \to [0,1]$. Put E^0 the set of all g such that $0 \in g(\alpha)$ for some $\alpha \in [0,1]$ and define pointwise $g + g'$ using the addition of real intervals. Then $(H, +, E^0)$ is an abelian additive MI-monoid of fuzzy numbers. Analogously, put E^1 the set of all g such that $1 \in g(\alpha)$ for some $\alpha \in [0,1]$ and define pointwise $g \cdot g'$ using the multiplication of real intervals. Then $(H, \cdot, E^1)$ is an abelian multiplicative MI-monoid of fuzzy numbers.[2]

As it has been discussed in [1], in practice, fuzzy numbers are often represented by several parameters to simplify their arithmetic ([4],[8]). It should be noted that the simplified arithmetic is only an approximation of the standard arithmetic of fuzzy numbers mentioned in the previous example.

Example 3. *Let T be the set of all trapezoidal fuzzy numbers, i.e., fuzzy numbers given in a parametric form $A = \langle a_A, b_A, c_A, d_A \rangle$, where $a_A, b_A, c_A, d_A \in \mathbb{R}$ with $a_A \leq b_A \leq c_A \leq d_A$, and defined as*

$$A(x) = \begin{cases} 0, & x < a_A \text{ or } d_A \leq x; \\ (x - a_A)/(b_A - a_A), & a_A \leq x < b_A; \\ 1, & b_A \leq x < c_A; \\ (d_A - x)/(d_A - c_A), & c_A \leq x < d_A. \end{cases}$$

Note that, although, we use the term "trapezoidal", the set of these fuzzy numbers contains also triangular fuzzy numbers (i.e. $b_A = c_A$), or real intervals (i.e., $a_A = b_A$ and $c_A = d_A$) including one element intervals (singletons representing the real numbers). Define the addition on T by

$$\begin{aligned} A + B = \langle a_A, b_A, c_A, d_A \rangle + \langle a_B, b_B, c_B, d_B \rangle = \\ \langle a_A + a_B, b_A + b_B, c_A + c_B, d_A + d_B \rangle. \end{aligned} \tag{2}$$

Put E^0 the set of all trapezoidal fuzzy numbers A for which $0 \in (a_A, d_A)$, i.e., A is a pseudoidentity element, if $A(0) > 0$. Clearly, $\langle 0,0,0,0 \rangle$ is the identity element. One may simply check that $(T, +, E^0)$ is a non-good abelian MI-monoid. Further, define the multiplication on T by

$$\begin{aligned} A \cdot B = &\langle a_A, b_A, c_A, d_A \rangle \cdot \langle a_B, b_B, c_B, d_B \rangle = \\ &\langle a_{A\cdot B}, b_{A\cdot B}, c_{A\cdot B}, d_{A\cdot B} \rangle, \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_{A\cdot B} &= \min(a_A a_B, a_A d_B, d_A a_B, d_A d_B) \\ b_{A\cdot B} &= \min(b_A b_B, b_A c_B, c_A b_B, c_A c_B), \\ c_{A\cdot B} &= \max(b_A b_B, b_A c_B, c_A b_B, c_A c_B), \\ d_{A\cdot B} &= \min(a_A a_B, a_A d_B, d_A a_B, d_A d_B). \end{aligned}$$

Put E^1 the set of all trapezoidal fuzzy numbers for which $1 \in (a_A, d_A)$, i.e., A is a pseudoidentity element, if $A(1) > 0$. Clearly, $\langle 1,1,1,1 \rangle$ is the identity element. Again, one may

[1]For details, we refer to [6],[7].

[2]It is well known that the both presented arithmetical operations are equivalent to that obtained by the Zadeh's extension principle.

verify that $(T, \cdot, E^1)$ is a non-good abelian multiplicative MI-monoid.

Example 4. *In [9] (see also [8]), a parametric representation of fuzzy numbers using monotonic interpolation is provided and, for instance, the trapezoidal fuzzy numbers give an example of such parametric representation. Let*

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = 1 \tag{4}$$

be real numbers for a finite decomposition of the unit interval. Without detailed comment, considering the differentiable case, an LU-fuzzy number A is represented by the following system of vectors

$$A = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\ldots,N}, \tag{5}$$

with the data

$$u_0^- \le u_1^- \le \cdots \le u_N^- \le u_N^+ \le u_{N-1}^+ \le \cdots \le u_0^+$$

and the slopes

$$\delta u_i^- \ge 0 \quad and \quad \delta u_i^+ \le 0.$$

Denote L the set of all LU-fuzzy numbers in the form (5) for a fixed finite decomposition of $[0,1]$ determined by (4). Define the addition on L by (α_i is omitted here for simplicity)

$$(u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\ldots,N} + (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,\ldots,N} = (u_i^- + v_i^-, \delta u_i^- + \delta v_i^-, u_i^+ + v_i^+, \delta u_i^+ + \delta v_i^+)_{i=0,\ldots,N}$$

Put E^0 the set of all LU-fuzzy numbers of L such that $0 \in [u_i^-, u_i^+]$ for some $i \in \{0, \ldots, N\}$. By the results in [8], $(L, +, E^0)$ is a non-good abelian additive MI-monoid.

Further, define the multiplication on L by

$$(u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\ldots,N} \cdot (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,\ldots,N} = ((uv)_i^-, \delta(uv)_i^-, (uv)_i^+, \delta(uv)_i^+)_{i=0,\ldots,N},$$

where

$$\begin{aligned}(uv)_i^- &= \min(u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+),\\ (uv)_i^+ &= \max(u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+)\end{aligned}$$

and

$$\begin{aligned}(uv)_i^- &= \delta u_i^{p_i^-} v_i^{q_i^-} + u_i^{p_i^-} \delta v_i^{q_i^-}\\ (uv)_i^+ &= \delta u_i^{p_i^+} v_i^{q_i^+} + u_i^{p_i^+} \delta v_i^{q_i^+}\end{aligned}$$

with (p_i^-, q_i^-) being the pair associated to the combination of superscripts + and - giving the minimum $(uv)_i^-$ and (p_i^+, q_i^+) being the pair associated to the combination of + and - giving the maximum $(uv)_i^+$. Put E^1 the set of all LU-fuzzy numbers from L such that $1 \in [u_i^-, u_i^+]$ for some $i \in \{0, \ldots, N\}$. By the results in [8], $(L, \cdot, E^1)$ is a non-good abelian multiplicative MI-monoid.

Remark 2. Although, the MI-monoids of fuzzy numbers in Examples 2, 3 and 4 are defined to be non-good, one may simply redefine E^0 and E^1 in such way that the goodness will be satisfied. For instance, redefining E^0 in Example 3 by $E^0 = \{\langle a, 0, 0, d\rangle \mid a, d \in \mathbb{R}, a \le 0 \le d\}$, we obtain a good MI-monoid of trapezoidal fuzzy numbers. Nevertheless, it should be noted that this redefinition seems to be artificial in contrast to the definition of the good MI-monoid of extensional fuzzy numbers proposed in [1]. Moreover, a natural definition of MI-field of trapezoidal fuzzy numbers needs to suppose the non-good abelian additive MI-monoid (consider Example 19).

Let us show two examples of MI-monoids that are not purely fuzzy and generalize the interval arithmetic discussed in Example 3 in [1].

Example 5. *Let M be the set of all convex subsets of $\mathbb{R}^n$ and $\mathbf{a} + \mathbf{b}$ denote the addition of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Let $\oplus$ be the Minkowski addition, i.e.,*

$$A \oplus B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$

It is easy to see that $\oplus$ is commutative, associative and the set containing only the null vector $\mathbf{0}$ is the identity element. Put E^0 the set of all sets in M that contains the null vector. Then (M3) and (M5) are trivially satisfied. Since the sets from M are convex, then $A \oplus A \in E^0$ implies the existence of $\mathbf{a}, \mathbf{b} \in A$ for which $\mathbf{a} + \mathbf{b} = 0$, whence $\mathbf{b} = -\mathbf{a}$. From the convexity of A, we obtain $\mathbf{0} \in A$ by $0.5\mathbf{a} + 0.5(-\mathbf{a}) = \mathbf{0}$ and $A \in E^0$. Thus, (M4) is satisfied and $(M, \oplus, E^0)$ is an abelian additive MI-monoid of convex sets in $\mathbb{R}^n$.[3]

Example 6. *Let $\mathbb{R}_+^n$ denote the set of all positive real vectors and $\mathbf{a} \cdot \mathbf{b}$ be the product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^n$ defined pointwise (analogously to the addition of vectors). For simplicity, we standardly use $\mathbf{a} \cdot \mathbf{b} = \mathbf{ab}$. A subset A of $\mathbb{R}_+^n$ is said to be* product convex, *if for any $\mathbf{a}, \mathbf{b} \in A$ and $\lambda \in [0,1]$ we have $\mathbf{a}^\lambda \mathbf{b}^{1-\lambda} \in A$. Let N be the set of all convex subsets of $\mathbb{R}^n$ with respect to $\cdot$ and + be the "Minkowski" multiplication defined by*

$$A \odot B = \{\mathbf{ab} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$

Let us show that $A \odot B$ is product convex. If $\mathbf{x}, \mathbf{y} \in A \odot B$ and $\lambda \in [0,1]$, then there exist $\mathbf{a}, \mathbf{c} \in A$ and $\mathbf{b}, \mathbf{d} \in B$ such that $\mathbf{x} = \mathbf{ab}$ and $\mathbf{y} = \mathbf{cd}$. Then $\mathbf{x}^\lambda \mathbf{y}^{1-\lambda} = (\mathbf{ab})^\lambda (\mathbf{cd})^{1-\lambda} = (\mathbf{a}^\lambda \mathbf{c}^{1-\lambda})(\mathbf{b}^\lambda \mathbf{d}^{1-\lambda})$. Since $\mathbf{a}^\lambda \mathbf{c}^{1-\lambda} \in A$ and $\mathbf{b}^\lambda \mathbf{d}^{1-\lambda} \in B$, then $\mathbf{x}^\lambda \mathbf{y}^{1-\lambda} \in A$ and $A \odot B$ is product convex. It is easy to see that $\odot$ is commutative, associative and the set containing only the vector $\mathbf{1} = (1, \ldots, 1)$ is the identity element. Put E^1 the set of all sets in N that contains $\mathbf{1}$. Again, (M3) and (M5) are trivially satisfied. If $A \odot A \in E^1$, then there exist $\mathbf{a}, \mathbf{b} \in A$ for which $\mathbf{ab} = \mathbf{1}$. Hence, we obtain $\mathbf{b} = \mathbf{a}^{-1}$. Since A is product convex, then $\mathbf{a}^{0.5}(\mathbf{a}^{-1})^{1-0.5} = \mathbf{a}^{0.5}\mathbf{a}^{-0.5} = \mathbf{1}$, whence $\mathbf{1} \in A$. Thus, $(N, \odot, E^1)$ is an abelian multiplicative MI-monoid of product convex sets in $\mathbb{R}_+^n$.

So, one may see that the concept of MI-monoid may serve as a basic algebraic structure for various types of objects and operations defined on them (intervals, sets, fuzzy sets and their arithmetics). This motivates us to develop a theory of MI-monoids and also to introduce further types of MI-algebras.

[3] Note that the Minkovski addition is used to define the operation of *dilation* which plays a central role in mathematical morphology (see e.g. [5]).

Let us start with the concept of MI-submonoid of an MI-monoid.

Definition 3. Let $G = (G, \circ, E)$ be an MI-monoid, $H \subseteq G$ and $F \subseteq E$ be non-empty subsets. If H is itself MI-monoid under the product of G, then $H = (H, \circ, F)$ is said to be an *MI-submonoid* of G. This is denoted by $H \leq G$. An MI-submonoid H of G is said to be *canonical*, if $F = H \cap E$.

The MI-submonoid $(G, \circ, E)$ and $(\{e\}, \circ, \{e\})$ of an MI-monoid $(G, \circ, E)$ are called *trivial*.

Example 7. *The real intervals with endpoints in the set of integers, where the pseudoidentities are defined analogously as for real intervals, form a canonical (additive and multiplicative) MI-submonoid of the MI-monoid of real intervals.*

Example 8. *The triangle fuzzy numbers with the pseudoidentities defined as symmetric fuzzy sets around the zero, i.e.,* $A(x) = A(-x)$ *holds for any* $x \in \mathbb{R}$*, form a non-canonical (additive and multiplicative) MI-submonoid of the MI-monoid of trapezoidal fuzzy numbers.*

The following proposition shows the necessary and sufficient condition under which subsets of G specify an MI-submonoid of G.

Proposition 1. *Let* $H \subseteq G$ *and* $F \subseteq E$ *be non-empty subsets of an MI-monoid G. Then* $H \leq G$ *if and only if for all* $x, y \in H$ *and* $a, b \in F$

(i) $xy \in H$,
(ii) $e \in F$,
(iii) $ab \in F$,
(iv) $xx = a$ *then* $x \in F$.

For the canonical MI-submonoids, the previous a bit complicated proposition may be significantly simplified. Moreover, the form of the following statement is nearly identical to that for monoids (see [10]).

Proposition 2. *Let* $H \subseteq G$ *be a non-empty subset of an MI-monoid* $G = (G, \circ, E)$*. Then* $H = (H, \circ, H \cap E)$ *is an MI-submonoid of G if and only if* $xy \in H$ *for all* $x, y \in H$ *and* $e \in H$.

Example 9. *Let* $(\mathcal{F}_\mathcal{S}(\mathbb{R}), +, E^0_\mathcal{S})$ *be an abelian additive MI-monoid of extensional fuzzy numbers (see [1]) and* $\mathcal{T} \subseteq \mathcal{S}$ *be a subsystem of* $\mathcal{S}$*, i.e.* $S^* = \bigcap \mathcal{S} \in \mathcal{T}$ *and* $\mathrm{cl}(S \cup S') \in \mathcal{T}$ *for any* $S, S' \in \mathcal{T}$*. Using the previous theorem, one may simply check that* $(\mathcal{F}_\mathcal{T}(\mathbb{R}), +, E^0_\mathcal{T})$*, where clearly* $\mathcal{F}_\mathcal{T}(\mathbb{R}) \subseteq \mathcal{F}_\mathcal{S}(\mathbb{R})$ *and* $E^0_\mathcal{T} = E^0_\mathcal{S} \cap \mathcal{F}_\mathcal{T}(\mathbb{R})$*, is a canonical MI-submonoid of* $\mathcal{F}_\mathcal{S}(\mathbb{R})$.

Example 10. *Let* $(H, \cdot, E^1)$ *be the multiplicative MI-monoid and* E^0 *be the set of pseudoidentities of the additive MI-monoid* $(H, +, E^0)$ *of fuzzy numbers from Example 2. Then* $(H \setminus E^0, \cdot, E^1 \setminus E^0)$ *is an MI-submonoid (of "non-zero fuzzy numbers") of* $(H, \cdot, E^1)$.

Example 11. *Let* M_+ *be the set of all (non-negative) convex sets in* $\mathbb{R}^n_+ \cup \{\mathbf{0}\}$*. In Example 5, we defined an abelian MI-monoid* $M = (M, \oplus, E^0)$*, where* $\oplus$ *is the Minkowski addition. Since* $\mathbf{0} \in M^+$ *and* $A \oplus B \in M^+$ *for any* $A, B \in M^+$*, then, by the previous proposition,* $(M^+, \oplus, E^0 \cap M^+)$ *is a canonical MI-submonoid of* M.

A homomorphism of MI-monoids is proposed as follows.

Definition 4. Let G and H be MI-monoids. A mapping $f : G \to H$ is a *homomorphism of MI-monoids* provided

(HM1) $f(xy) = f(x)f(y)$ for all $x, y \in G$,
(HM2) $f(e_G) = e_H$,
(HM3) $f(a) \in E_H$ for all $a \in E_G$,

where e_G and e_H (E_G and E_H) denote the identity elements (the sets consist of the identity elements and the pseudoidentities) of G and H, respectively. If f is injective, f is said to be a *monomorphism*. If f is surjective, f is said to be an *epimorphism*. If f is bijective and $f(E_G) = \{f(a) \mid a \in E_G\} = E_H$, f is said to be an *isomorphism*. In this case G and H are said to be *isomorphic* (written $G \cong H$). A homomorphism $f : G \to G$ is called an *endomorphism* and an isomorphism $f : G \to G$ is called an *automorphism*.

One may simply check that the composition of two homomorphisms (monomorphisms, epimorphisms, etc.) is again a homomorphism (monomorphism, epimorphism, etc.).

Example 12. *Let* $(M, \oplus, E^0)$ *and* $(N, \odot, E^1)$ *be the MI-monoids of convex and product convex sets introduced in Examples 5 and 6, respectively. Then* $f : N \to M$ *given by* $f(A) = \{\ln \mathbf{a} \mid \mathbf{a} \in A\}$*, where* $\ln \mathbf{a}$ *is defined pointwise, is an isomorphism of the MI-monoids* N *and* M*. In fact, we have* $f(\mathbf{ab}) = \ln \mathbf{ab} = \ln \mathbf{a} + \ln \mathbf{b} = f(\mathbf{a}) + f(\mathbf{b})$*,* $f(\mathbf{1}) = \ln \mathbf{1} = \mathbf{0}$ *and if* $\mathbf{1} \in A$*, then* $\mathbf{0} \in f(A)$*. Let* $A, B \in N$*. To check that* $f(A)$ *is a convex set, let* $\mathbf{a}, \mathbf{b} \in f(A)$ *and* $\lambda \in [0, 1]$*. By the definition of* f*, there are* $\mathbf{c}, \mathbf{d} \in A$ *such that* $f(\mathbf{c}) = \mathbf{a}$ *and* $f(\mathbf{d}) = \mathbf{b}$*. Then*

$$\lambda \mathbf{a} + (1 - \lambda)\mathbf{b} = \lambda \ln \mathbf{c} + (1 - \lambda) \ln \mathbf{d} = \ln(\mathbf{c}^\lambda \mathbf{d}^{1-\lambda}) = f(\mathbf{c}^\lambda \mathbf{d}^{1-\lambda}).$$

Since $\mathbf{c}^\lambda \mathbf{d}^{1-\lambda} \in A$*, then* $\lambda \mathbf{a} + (1 - \lambda)\mathbf{b} \in f(A)$ *and* $f(A)$ *is convex. Further, we have*

$$f(A \odot B) = \{\ln(\mathbf{ab}) \mid \mathbf{a} \in A,\ \mathbf{b} \in B\} = \{\ln \mathbf{a} + \ln \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\} = f(A) \oplus f(B),$$

whence (HM1) is satisfied. Obviously, $f(\{\mathbf{1}\}) = \{\ln \mathbf{1}\} = \{\mathbf{0}\}$ *and if* $A \in E^1$*, then* $\mathbf{0} \in f(A)$ *and thus* $f(A) \in E^0$*. Hence,* f *is a homomorphism of* N *to* M*. If* $A \neq B$*, then there is* $\mathbf{a} \in A \setminus B$*. Since the natural logarithm is a bijection of the set of positive real numbers to the set of real numbers, we obtain* $f(\mathbf{a}) \notin B$ *and* f *is an injection of* N *to* M*. Let* $B \in M$*. Put* $A = \{e^{\mathbf{b}} \mid \mathbf{b} \in B\}$*, where again* $e^{\mathbf{b}}$ *is defined pointwise. Then* $f(A) = \{\ln e^{\mathbf{b}} \mid \mathbf{b} \in B\} = \{\mathbf{b} \mid \mathbf{b} \in B\} = B$*, whence* f *is surjective. Finally, if* $f(A) = B$ *and* $\mathbf{0} \in B$*, then necessary* $\mathbf{1} \in A$ *and* $f(E^1) = E^0$.

If we deal with pseudoidentities then it is reasonable to introduce a relation $\sim$ on G defined as follows:

$$x \sim y \quad \text{if and only if} \quad ax = by \tag{6}$$

holds for some $a, b \in E$ (cf. [3]). Note that $x \sim y$ if and only if $xa = ya$ which holds from (M5).

Lemma 1. *The relation $\sim$ is a congruence on G.*

One can see that $G\backslash\sim$ is a quotient MI-monoid of an MI-monoid G by $\sim$.

Definition 5. An MI-monoid G is said to be *unlimited in fuzziness*, if $G\backslash\sim\, = (\{[e]\}, \circ, \{[e]\})$. An MI-monoid that is not unlimited is called *limited*.

Trivial examples of unlimited MI-monoids are that with $G = (G, \circ, G)$, i.e., G contains only the neutral element and eventually pseudoidentities. Indeed, if $a, b \in G$, then $ab = ba$ implies $a \sim b$. A useful lemma stating a sufficient condition for being unlimited in fuzziness is the following one.

Lemma 2. *If for any $x \in G$ there is $a \in E$ such that $xa \in E$, then $(G, \circ, E)$ is unlimited in fuzziness.*

Using this lemma, one may simply check that the MI-monoids of real intervals, fuzzy numbers, trapezoidal fuzzy numbers, representations of fuzzy numbers using monotonic interpolation or convex and product convex sets are unlimited in fuzziness. The MI-monoids of extensional fuzzy numbers are examples of limited in fuzziness MI-monoids . Recall that extensional fuzzy numbers form a good MI-monoid (see [1]). A relation between "to be good" and "to be unlimited fuzziness" is given in the following lemma.

Lemma 3. *If G is unlimited in fuzziness with $E \subset G$, then G is not good.*

III. MI-GROUPS

In this section, we will continue in our developing of a theory of MI-algebras and we will define an important extension of groups to MI-groups. We say "important" because the fuzzy numbers (real intervals, convex sets) endowed by an operation do not keep the group structure, in contrast to the monoid structure. Moreover, the groups and their theory play a prominent role among algebraic structures and the same is to be expected also for MI-groups.

Since the definition of MI-groups is a bit complicated to imitate the important properties of groups, we will start with a simpler structure called MI-pregroup. Let $G = (G, \circ)$ be a group and $G^{\mathrm{op}} = (G, \circ^{\mathrm{op}})$ denote a dual group to G, where $x \circ^{op} y = y \circ x$ for any $x, y \in G$.[4] Recall that to each $x \in G$ there exists an inverse element $x^{-1} \in G$ such that $x \circ x^{-1} = x^{-1} \circ x = e$. A simple consequence of the group definition is the fact that each element of G has exactly one inverse element, $(x \circ y)^{-1} = y^{-1} \circ x^{-1} = x^{-1} \circ^{op} y^{-1}$ and $(x^{-1})^{-1} = x$. If we define $h : G \to G$ by $h(x) = x^{-1}$, then h is clearly an isomorphism of G onto G^{op} that possesses the following properties:

$$h(x)x = xh(x) = e,\ \ h(e) = e \text{ and } h(h(x)) = x$$

[4]Obviously, $(G^{\mathrm{op}})^{\mathrm{op}} = G$. If G is abelian, then $G = G^{\mathrm{op}}$.

describing the properties of inverse elements. One can prove that a monoid G is a group if and only if there exists the unique isomorphism $h : G \to G^{\mathrm{op}}$ having the properties mentioned above. This assertion motivates us to introduce an MI-pregroup as follows. Analogously to a dual group G^{op}, we define a dual MI-monoid $G^{\mathrm{op}} = (G, \circ^{\mathrm{op}}, E)$ to an MI-monoid $G = (G, \circ, E)$.

Definition 6. An MI-monoid G is said to be an *MI-pregroup*, if there exists an MI-monoidal isomorphism h of G onto G^{op} that satisfies

(G1) $h(x)x \in E$,
(G2) $h(x)x = xh(x)$,
(G3) $h(h(x)) = x$

for any $x \in G$. The isomorphism h is called an *inversion* in G and $h(x) = x^h$ is called an *h-inverse* element to x.

One can see that (G1) provides the difference between the isomorphism that defines inverse elements in a group and the isomorphism for an MI-pregroup. Nevertheless, as we have mentioned above, pseudoidentities concentrate a fuzziness in elements of a structure and it seems to be contra-intuitive to require the satisfaction of the law $h(x)x = e$ (cf. Subsection 3.4 in [3]). One can imagine this fact as the impossibility to destroy the fuzziness in elements. On the other hand, we have $h(x)x \sim e$, i.e., $h(x)x$ differs from e up to a pseudo neutral element (up to a fuzziness).

Obviously, if h is the identity mapping, i.e., $h = \mathrm{id}_G$, then (G2) and (G3) are trivially satisfied. Nevertheless, from $h(x)x = xx = a$ for some $a \in E$, we obtain $x \in E$ (by (M4)) and hence $G = E$. Thus the identity mapping defines an inversion in G only if G consists of only pseudoidentities. It should be noted that $h(h(x)x) = h(x)x$, i.e., $h(y) = y$ for $y = h(x)x$. An element $x \in G$ with $h(x) = x$ for an isomorphism h of G onto G^{op} is said to be *h-symmetric in G*. The set of all h-symmetric elements in G is denoted by S_G^h.

Lemma 4. *If $x \in G$ is h-symmetric, then $x \in E$,i.e., $S_G^h \subseteq E$.*

In the classical definition of group, the unique h-symmetric element is e, i.e., $e = e^{-1} = e^h$. The following statement shows that h-symmetric elements in G are closed under the product of G.

Lemma 5. *If $x, y \in G$ are h-symmetric, then xy is also h-symmetric.*

In contrast to exactly one inversion to each element in groups, we can consider more than one inversion in MI-pregroups. The following lemma shows a relation between two inversions.

Lemma 6. *If h and k are inversions in G, then for any $x \in G$ there exist $a \in S_G^h$ and $b \in S_G^k$ such that $x^h a = y^k b$, i.e., $x^h \sim y^k$.*

Hence, one can see that two inversions of a single element are different up to symmetric pseudoidentities. If $S_G^h = S_G^k = \{e\}$, then clearly $h = k$ and there is a unique inverse in an MI-pregroup G.

Now, let us consider the trivial fact that holds in all groups: $ae = be$ then $a = b$. Replacing the identity element e by h-symmetric elements $s \in S^h_G$, we obtain a natural condition for MI-pregroups to be MI-groups.

Definition 7. An MI-pregroup is said to be an *MI-group*, if the following cancellation property is held for any inversion h in G:

(G4) if $xa = ya$ for $x, y \in G$ and $a \in S^h_G$ then $x = y$.

An MI-group G is said to be *good*, if G is a good MI-monoid.

To avoid some technical problems with a possible existence of more than one inversion in MI-groups, in the next part, we deal with an MI-group $(G, \circ, E, h)$ restricted to one chosen inversion h in $(G, \circ, E)$. In this case, the remaining inversions in $(G, \circ, E)$ are forgotten.[5] Usually, if we deal with more than one MI-group, we use the more precise denotation $G = (G, \cdot, E_G, h_G)$. On the other hand, we write only S_G instead of S^h_G, since only one inversion is considered. Let us show several examples of MI-groups (restricted to a unique inversion).

Example 13. *In [1], we have shown that the abelian additive and multiplicative MI-monoids of extensional fuzzy numbers and real intervals are abelian additive and multiplicative MI-groups, respectively.*

Example 14. *Let H be the abelian additive MI-monoid of fuzzy numbers from Example 2 and put $h(g)(\alpha) = -g(\alpha)$ for any $g \in H$ and $\alpha \in [0, 1]$, where $-g(\alpha) = [-g(\alpha)^+, -g(\alpha)^-]$ for $g(\alpha) = [g(\alpha)^-, g(\alpha)^+]$ (see Example 5 in [1]). It is easy to see that $h(g) \in H$ and h is an inversion in H for which $h(g) = h(g')$ implies $g = g'$ for any $g, g' \in S_H$. Hence, $(H, +, E^0, h)$ is an abelian additive MI-group.*

Let $(H \setminus E^0, \cdot, E^1 \setminus E^0)$ be the multiplicative MI-monoid from Example 10 and put $h(g)(\alpha) = g(\alpha)^{-1}$ for any $g \in H \setminus E^0$ and $\alpha \in [0, 1]$, where $g(\alpha)^{-1} = [1/g(\alpha)^+, 1/g(\alpha)^-]$ for $g(\alpha) = [g(\alpha)^-, g(\alpha)^+]$ (see Example 5 in [1]). Then $(H \setminus E^0, \cdot, E^1 \setminus E^0, h)$ is an abelian multiplicative MI-group.

Example 15. *Let L be the additive MI-monoid from Example 4 and put $h(u) = (-u^+_i, -\delta u^+_i, -u^-_i, -\delta u^-_i)_{i=1,\dots,N}$ for any $u \in L$. Then $(L, +, E^0, h)$ is an abelian additive MI-group.*

One may check that $(L \setminus E^0, \cdot, E^1 \setminus E^0)$ is an MI-submonoid of $(L, \cdot, E^1)$. If we define

$$h(u) = (1/u^+_i, -1/(u^+_i)^2 \delta u^+_i, 1/u^-_i, -1/(u^-_i)^2 \delta u^-_i)_{i=1,\dots,N}$$

for any $u \in L \setminus E^0$, then $(L \setminus E^0, \cdot, E^1 \setminus E^0, h)$ is an abelian multiplicative MI-group.

Example 16. *Let M be the additive MI-monoid from Example 5 and put $h(A) = -A = \{-a \mid a \in A\}$ for any $A \in M$. Then $(M, \oplus, E^0, h)$ is an abelian additive MI-group of convex sets in $\mathbb{R}^n$.[6] Analogously, one may define an abelian multiplicative MI-group N of product convex sets.*

In what follows the restriction of a mapping h to a set $H \subseteq G$ is denoted by $h \restriction H$. The concept of MI-subgroup of an MI-group is naturally defined as follows.

Definition 8. Let $G = (G, \circ, E_G, h_G)$ be an MI-group, $H \subseteq G$ and $E_H \subseteq E_G$ be non-empty sets and $h_H = h_G \restriction H$. If $H = (H, \circ, E_H, h_H)$ is itself MI-group under the product of G, then H is said to be an *MI-subgroup* of G. This is denoted by $H \leq G$. An MI-subgroup H of G is said to be *canonical*, if $E_H = H \cap E_G$.

Example 17. *Let G be an MI-group. Clearly, G and $(\{e\}, \circ, \{e\}, \mathrm{id}_{\{e\}})$ are trivial MI-subgroup of G. Moreover, $(S^h_G, \circ, S^h_G, \mathrm{id}_{S^h_G})$, where $\circ$ is the restriction of the product of G to S^h_G, is an MI-subgroup of G. Note that this MI-subgroup plays an analogous role as the trivial group $(\{e\}, \circ)$ in the classical group theory (see e.g. Theorem 1) and the elements of S_G perfectly simulate the properties of the identity element in a group, i.e., the symmetric pseudoidentities are very close to the identity element in a group.*[7]

The following proposition shows the necessary and sufficient condition under which subsets of G specify an MI-subgroup of G.

Proposition 3. *Let $H \subseteq G$ and $F \subseteq E$ be non-empty subsets of an MI-group of G. Then $H \leq G$ if and only if for all $x, y \in H$ and $a, b \in F$*

(i) $xy^{h_G} \in H$,
(ii) $e_G \in F$,
(iii) $ab \in F$,
(iv) $xx = a$ *then* $x \in F$,
(v) $a^{h_G} \in F$,
(vi) $x^{h_G} x \in F$.

Again, the choice of F is the cause of many conditions that have to be verified to declare H to be an MI-subgroup of G. Let us show the necessary and sufficient conditions for canonical MI-subgroups (cf. [10]).

Proposition 4. *Let $H \subseteq G$ be a non-empty subset of an MI-group G and $h_H = h_G \restriction H$. Then $H = (H, \circ, H \cap E, h_H)$ is an MI-subgroup of G if and only if $e_G \in H$ and $xy^{h_G} \in H$ for all $x, y \in H$.*

Definition 9. Let G and H be MI-groups. A mapping $f : G \to H$ is a *homomorphism of MI-groups* provided

(HG1) $f(xy) = f(x)f(y)$ for all $x, y \in G$,
(HG2) $f(a) \in E_H$ for all $a \in E_G$,
(HG3) $f(x^{h_G}) = (f(x))^{h_H}$ for all $x \in G$,
(HG4) $f(a) = f(b)$ for $a, b \in S_G$ then $a = b$,

[5] Later, we will show that if h and k are two inversions in an MI-group $(G, \circ, E)$, then the MI-groups $(G, \circ, E, h)$ and $(G, \circ, E, k)$ are isomorphic (see Example 19).

[6] Note that $-A$ is called *reflection of* A and it is used to define *erosion* in the mathematical morphology (see e.g. [5]).

[7] A reason why we do not consider $E = S^h_G$ is closely related to our approach to the concept of MI-field when we cannot define the inverse elements with respect to the multiplication also for non-symmetric pseudoidentities with respect to addition (i.e., pseudozeros).

where E_G and E_H (h_G and h_H) denote the sets consist of the identity elements and the pseudoidentities (the inversions) of G and H, respectively.

All concepts as monomorphism, epimorphism etc. of MI-groups have the same meaning as for MI-monoids. Moreover, if $f : G \to H$ and $g : H \to P$, then $g \circ f$ is a homomorphism of G to P. In fact, if $a \in G$, then $f(a^{h_G}) = f(a)^{h_H}$ and $g(f(a)^{h_H}) = g(f(a))^{h_P}$. Hence, we obtain $g(f(a^{h_G})) = g(f(a))^{h_P}$ and (HG3) is satisfied. Similarly, one can prove the remaining conditions.

It should be noted that (HG4) is a redundant condition in the classical group theory. However, if an MI-group contains a symmetric pseudoidentity, then (HG4) cannot be proved from (HG1)-(HG3). Nevertheless, (HG4) seems to be essential for our further investigation of MI-groups.

Example 18. *The isomorphism f of the MI-monoids N and M defined in Example 16 is also an isomorphism of MI-groups N and M.*

Example 19. *Let $G_h = (G, \circ, E, h)$ and $G_k = (G, \circ, E, k)$ be MI-groups, where h and k be two inversions in $(G, \circ, E)$. Then $f(x) = (x^h)^k$ for any $x \in G$ defines an isomorphism of G_h and G_k. In fact, $f(xy) = ((xy)^h)^k = (y^h x^h)^k = (x^h)^k (y^h)^k = f(x)f(y)$ and (HG1) is proved. If $a \in E$, then $f(a) = (a^h)^k$. Since $a^h \in E$, then also $f(a) = (a^h)^k \in E$ and (HG2) is true. Since $(x^h)^h = x$ by (G3), then $f(x^h) = ((x^h)^h)^k = x^k$, whence (HG3) is fulfilled. If $f(a) = f(b)$ for $a, b \in S_{G_h}$, then $(a^h)^k = (b^h)^k$ implies $a = b$, because the inversions h and k are the isomorphisms of G onto G^{op}, i.e., (HG4) is also satisfied. One may see that f is a bijective mapping and $f(E) \subseteq E$. If $b \in E$, then putting $a = (b^k)^h$ we obtain $a \in E$ and $f(a) = b$, whence $f(E) = E$ and f is an isomorphism.*

Proposition 5. *Let $f : G \to H$ be a homomorphism of MI-groups. Then*
(i) $f(S_G) \subseteq S_H$.
(ii) $f(e_G) = e_H$.

An important concept for groups is the kernel of a homomorphism $f : G \to H$, i.e., $\operatorname{Ker} f = \{x \in G \mid f(x) = e_H\}$. For the MI-groups, we need a generalization of this concept.

Definition 10. Let $f : G \to H$ be a homomorphism of MI-groups. The *MI-kernel* of f, denoted by MI-Ker f, is the set $\{x \in G \mid f(x) \in S_H\}$. If A is a subset of G, then $f(A) = \{y \in H \mid y = f(x) \text{ for some } x \in A\}$ is the *image of* A. $f(G)$ is called the *image of* f and denoted by $\operatorname{Im} f$. If B is a subset of H, then $f^{-1}(B) = \{x \in G \mid f(x) \in B\}$ is the *inverse image of* B.

A characterization of a monomorphism by MI-Ker and an isomorphism by its inverse homomorphism is provided in the following theorem.

Theorem 1. *Let $f : G \to H$ be a homomorphism of MI-groups. Then*
(i) f is a monomorphism if and only if MI-Ker $f = f(S_G)$,
(ii) f is an isomorphism if and only if there is a homomorphism $f^{-1} : H \to G$ such that $ff^{-1} = \mathrm{id}_G$ and $f^{-1}f = \mathrm{id}_H$.

IV. Full MI-subgroups and Lagrange's Theorem

In this section, we present only a part of MI-group theory devoted to cosets and related notions. Let us start with the concept of a closure of subsets of MI-groups.

Definition 11. Let G be an MI-group and $H \subseteq G$ be a non-empty subset. The set H is said to be *closed* under S_G, if $xa \in H$ for $x \in G$ and $a \in S_G$ implies $x \in H$. The set

$$\widehat{H} = \bigcap \{K \subseteq G \mid K \text{ is closed under } S_G \text{ and } H \subseteq K\}$$

is called a *closure* of H under S_G.

Obviously, $\widehat{H}$ contains all elements that belongs under the product of G with all h_G-symmetric elements to H. Moreover, if H is closed under S_G, then $H = \widehat{H}$.

Proposition 6. *If $H \leq G$, then $\widehat{H} \leq G$, where $\widehat{H} = (\widehat{H}, \circ, \widehat{E_H}, h_{\widehat{H}})$.*

The second concept is a generalization of "congruence modulo". Recall that the set of all h-symmetric elements forms an MI-subgroup in each MI-group. In what follows, we restrict ourselves to the closed MI-subgroups of G under S_G and we do not stress it below.

Definition 12. Let H be a closed MI-subgroup of an MI-group G and $x, y \in G$. We say that x is *right congruent* to y *modulo* H denoted $x \equiv_r y \ (mod\ H)$, if there exists an h_G-symmetric element $a \in H$ such that $xay^{h_G} \in H$. We say that x is *left congruent* to y *modulo* H denoted $x \equiv_l y \ (mod\ H)$, if there exists an h_G-symmetric element $a \in H$ such that $x^{h_G}ay \in H$.

If G is abelian then right and left congruence modulo H coincide. To investigate analogous properties to that for "congruence modulo" in the group theory, we restrict ourselves to the following class of MI-subgroups.

Definition 13. An MI-subgroup H of an MI-group G is said to be *full* and denoted by $H \leq_f G$, if $S_G \subseteq H$ and H is closed under S_G.

Theorem 2. *Let H be a full MI-subgroup of an MI-group G.*
(i) Right (resp. left) congruence modulo H is an equivalence relation on G.
(ii) The equivalence class of $x \in G$ under right (resp. left) congruence modulo H is the closure of $Hx = \{hx \mid h \in H\}$, (resp. $xH = \{xh \mid h \in H\}$) under S_G.
(iii) $|\widehat{Hx}| = |H| = |\widehat{xH}|$.

If $H \leq G$ and $x \in G$, then a *right coset* of H in G is the set $\widehat{Hx}$ and a *left coset* of H in G is the set $\widehat{xH}$. Note that if $S_G = \{e\}$ then $\widehat{Hx} = Hx$.

Definition 14. If $H \leq_f G$, then the *index* of H in G, denoted by $[G : H]$, is the cardinal number of the set of distinct right (resp. left) cosets of H in G.

Definition 15. If G is an MI-group, then the *order* of G, denoted by $|G|$, is the cardinal number of the set G.

The last two theorems are well known in the group theory (cf. [10]).

Theorem 3. *If G is an MI-group and $K \leq_f H \leq_f G$, then*

$$[G : K] = [G : H][H : K].$$

Theorem 4 (*Lagrange*)**.** *If G is a finite MI-group and $H \leq_f G$, then $|H|$ divides $|G|$ and $[G : H] = |G| \setminus |H|$.*

V. MI-FIELDS

An original motivation of our analysis of properties of arithmetical operations with fuzzy numbers was to develop working arithmetics of fuzzy numbers. Our idea was to introduce a structure of fuzzy numbers with appropriate arithmetical operations that is very close to the field of real numbers. The essential step in this conception was a generalization of group structure leading to the notion of MI-group. Now, we are ready to introduce a generalization of the field structure.

Definition 16. An algebra $(G, +, \cdot, E^0, E^1)$ is said to be an *MI-field*, if

(F1) $(G, +, E^0)$ is an abelian additive MI-group,
(F2) $(G \setminus E^0, \cdot, E^1 \setminus E^0)$ is an abelian multiplicative MI-group,
(F3) for any $x, y, z \in G$ the following

$$x(y + z) = xy + xz \quad \text{(distributive law)}$$

holds.

Example 20. *The set of extensional fuzzy numbers (under an arbitrary system of similarity relations) endowed by $+$ and $\cdot$ form an MI-field (see Example 6 in [1]).*

As it has been discussed in Introduction, only the subdistributivity is satisfied for the arithmetical operations for intervals and fuzzy numbers, i.e., $x(y + z) \subseteq xy + xz$. Nevertheless, it is interesting that for any x, y, z we may find a pseudoidentity a such that $a + x(y + z) = xy + xz$.[8] This motivate us to define a weaker algebraic structure to the MI-field.

Definition 17. An algebra $(G, +, \cdot, E^0, E^1)$ is said to be a *weak MI-field*, if (F1), (F2) and

(WF3) for any $x, y, z \in G$ there exists $a \in E^0$ such that the following

$$a + x(y + z) = xy + xz \quad \text{(weak distributive law)}$$

holds.

Example 21. *The sets of real intervals, fuzzy numbers, parametric representations endowed by the arithmetical operations defined in Examples 1-4 form a weak MI-field. Note that M does not form a weak MI-field, since $M \setminus E^0$ contains also sets that are not product convex.*

[8] By (6), we may express this equality as $x(y + z) \sim_+ xy + xz$, where $\sim_+$ is the congruence on G with respect to $+$.

Using the concepts introduced for MI-groups, we may simply define analogous concepts for (weak) MI-fields (e.g. MI-subfield, homomorphism of (weak) MI-fields).

VI. CONCLUSION

In this second part of our contribution on extensional fuzzy numbers and their arithmetics, we presented some of the basic notions related to many identities algebras (namely, MI-monoids, MI-groups and MI-fields) introduced in [1]. We showed that many well known arithmetics for intervals, fuzzy numbers including the proposed extensional fuzzy numbers and sets of real vectors can be viewed from the MI-algebras perspective. This fact enables us to investigate abstractly the properties of structures of various types sets expressing the imprecisely defined (not-necessary real) numbers. Moreover, we think that the development of a theory of MI-algebraic structures should be interesting from the theoretical as well as practical point of view. Here, we presented only several chosen results of the possible algebraic theory to demonstrate its functionality and a close relation to the standard results from the theory of monoids, groups and fields. It should be noted that the results for MI-algebras cannot be obtained by a trivial reconstruction of known proofs for algebras and these proofs have to be designed by a novel consideration. Thus, the development of a theory of MI-algebras becomes a challenge for our future research.

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