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# MI-algebras: a new framework for arithmetics of (extensional) fuzzy numbers

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This paper is a tribute dedicated to the memory of Milan Mareš.

**Abstract**

Distinct so far existing arithmetics of fuzzy numbers, usually stemming from the Zadeh's extensional principle, do not preserve some of the important properties of the standard arithmetics of real numbers. Although one cannot expect that a generalization of the standard arithmetic will preserve precisely all its properties, the most important properties should be preserved at least in a weakened form.

We present a novel framework of arithmetics of extensional fuzzy numbers that preserves more or less all the important (algebraic) properties of the arithmetic of real numbers and thus, seems to be an important seed for further investigations on this topic. The suggested approach to arithmetics of extensional fuzzy numbers is demonstrated on many examples and besides the algebraic properties, it is also shown that it carries some desirable practical properties.

The investigation leads to novel algebraic structures – MI-algebras (MI-monoids, MI-groups, MI-fields) – that abstract the discussed properties. The main idea of these structures is based on a set of "pseudoidentities" that complements the only commonly used identity element in a monoid/group structure. These pseudoidentities are elements that generalize the identity-like behavior and that allow us to weaken the standard form of algebraic properties that are highly desirable to be preserved at leats in the weakened form. Appropriate properties of suggested MI-structures are introduced and demonstrated again on a plenty of practical examples.

*Keywords:* Fuzzy numbers, Arithmetics of fuzzy numbers, MI-monoid, MI-group, MI-field

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## 1. Introduction

### *1.1. Motivation*

Since fuzzy numbers were designed in order to generalize standard real numbers, it is unquestionable that arithmetics of fuzzy numbers should generalize the standard one as well. The importance of fuzzy numbers, i.e., fuzzy sets representing vaguely defined real numbers, has been published many times [5, 14]. Thus, developing working arithmetics of fuzzy numbers is obviously of the same of importance. However, despite a tremendous job has been done in this field [5, 14, 6, 7], up to our best knowledge, so far existing arithmetics of fuzzy numbers do not carry even some very important algebraic properties that are naturally valid for the arithmetic of real numbers.

By these very important algebraic properties we basically mean the following three properties

$$a + (-a) = 0, \tag{1}$$

$$a \cdot a^{-1} = 1, \tag{2}$$

$$(a + b) \cdot c = a \cdot c + b \cdot c, \tag{3}$$

i.e., an existence of an identity element for both summation and multiplication and the distributivity law, that are naturally valid for any $a, b, c \in \mathbb{R}$.

As we know, none of the properties (1)-(3) generally holds for $a, b, c \in \mathcal{F}(\mathbb{R})$ where $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy sets on real numbers. And neither restrictions to certain subsets of $\mathcal{F}(\mathbb{R})$ such as to R-L fuzzy numbers [18] or to L-U fuzzy numbers [23] helps sufficiently in order to obtain a kind of valid variant of the three properties recalled above.

In our opinion, the main problem is that fuzzy sets, although defined on the universe of real numbers and restricted by distinct technical conditions (e.g., upper semi-continuity of R-L fuzzy numbers), are still fuzzy sets. That is, they are generalizations of classical sets, not generalizations of real numbers that could be considered as vague representations of real numbers. Moreover, one can easily see that the classical (crisp) sets equipped with the arithmetical operations based on the Zadeh's extension principle do not preserve the properties of the arithmetic of real numbers. And thus, we may hardly expect that its generalization to fuzzy sets would again gain these lost properties. Let us recall that the fact that fuzzy numbers are rather fuzzified version of crisp sets or more precisely intervals, was noted by several authors, see e.g. [15, 19, 8].

We present a novel approach to arithmetics of fuzzy numbers that gathers several basic principles that are highly desirable such as:

- we restrict our considerations to fuzzy numbers that are clearly conceptually created from real numbers because arithmetic operations work on numbers but not on crisp sets;
- we do not stick to the Zadeh's extensional principle as the only and basic principle from which our approach should stem;

- we capture an intuitive nature of arithmetics with vague numbers;
- and finally, we lower the required computational costs.

Besides the implementation of the above listed principles, the main goal is to introduce a "fully working" (in a broad sense) scheme for arithmetics of fuzzy numbers that may be inherited for basically all purposes were fuzzy numbers are naturally used.

## 2. Preliminaries

### *2.1. Conceptual approach - basic notions*

As foreshadowed above, we do not find all fuzzy sets that are usually called fuzzy numbers to be appropriate candidates being dealt as summands or multipliers in the generalized arithmetic operations. Once more, we recall and stress that arithmetic operations do not generally work on crisp sets [8] and thus, if fuzzy sets naturally extend crisp sets, one may hardly expect obtaining a working arithmetic on fuzzy sets. Therefore, the suggested arithmetic of fuzzy numbers will be designed for fuzzy sets that are natural and mainly conceptual generalizations of real numbers, i.e., appropriate models of vaguely understood real numbers.

If a fuzzy set $a \in \mathcal{F}(\mathbb{R})$ is supposed to model the meaning, e.g. of an amount "about five", naturally it should be constructed using the real number 5 and a tolerance of values close to 5. The closeness may be easily modelled within the fuzzy framework by the use of the so called similarity relation [11] (also fuzzy equivalence or fuzzy equality [12]) that generalizes the crisp equality. With respect to the goal of our paper, we restrict most of the definitions introduced in the latter to the universe of real numbers.

**Definition 2.1.** Let $*$ be a left-continuous t-norm. Binary fuzzy relation on the set of real numbers $S : \mathbb{R}^2 \to [0,1]$ is called $*$-*similarity* if the following holds

$$\begin{aligned} S(x,x) &= 1, \\ S(x,y) &= S(y,x), \\ S(x,y) * S(y,z) &\leq S(x,z), \end{aligned}$$

for all $x, y, z \in \mathbb{R}$. Furthermore, we say that $S$ is *shift-invariant* if for all $x, y, h \in \mathbb{R}$ it holds

$$S(x,y) = S(x+h, y+h).$$

Similarity relation generalizes the classical equality and thus, it is very useful in definitions of further notions that again generalize the classical ones. For example, extensionality is a standard notion from the set theory that may be easily generalized for fuzzy sets with help of the use of the similarity relation, see e.g. [2, 3].

**Definition 2.2.** [1] Let $S$ be a $*$-similarity relation on $\mathbb{R}$. A fuzzy set $a \in \mathcal{F}(\mathbb{R})$ is said to be *extensional* w.r.t. $S$ if

$$a(x) * S(x, y) \le a(y),$$

holds for any $x, y \in \mathbb{R}$.

Extensional fuzzy sets compose an important class of fuzzy sets that represent collections of objects having the same property if they are close to each other. Although not each fuzzy set is extensional, for each fuzzy set we may define its extensional hull that is an extensional fuzzy superset of the given fuzzy set.

**Definition 2.3.** Let $S$ be a $*$-similarity relation on $\mathbb{R}$ and $a$ be a fuzzy set on $\mathbb{R}$. A fuzzy set $\widehat{a} \in \mathcal{F}(\mathbb{R})$ is an *extensional hull* of $a$ if it is its least fuzzy superset that is extensional w.r.t. $S$.

**Theorem 2.1.** *[11] Let $S$ be a $*$-similarity relation on $\mathbb{R}$ and $a$ be a fuzzy set on $\mathbb{R}$. The extensional hull of $a$ is a fuzzy set $\widehat{a} \in \mathcal{F}(\mathbb{R})$ that is given as follows*

$$\widehat{a}(x) = \bigvee_{y \in \mathbb{R}} a(y) * S(x, y). \tag{4}$$

Since the crisp set is a special case of a fuzzy set, we may easily determine an extensional hull of a crisp set as well. A special case of crisp sets is a singleton which can serve as a representative of a real number.

**Definition 2.4.** *A fuzzy set $\tilde{x} \in \mathbb{R}$ is called* singleton *if there exists $x \in \mathbb{R}$ for which $\tilde{x}(x) = 1$ and $\tilde{x}(y) = 0$ for any $y \in \mathbb{R}$, $y \ne x$.*

Thus, if we consider a real number $x \in \mathbb{R}$ represented by a singleton $\tilde{x} \in \mathcal{F}(\mathbb{R})$ we may construct an extensional hull of $x$ with respect to a given similarity relation $S$. This notion is usually called *fuzzy point* and its use has been found out beneficial in many investigations [11, 12, 10].

Indeed, fuzzy point is in our opinion conceptually the genuine representation of the vaguely defined real number since it is composed as an extensional hull of the given real number. In other words, it expresses a real number with its neighborhood, i.e., with its class of (fuzzy) equivalence given by the similarity relation $S$. Thus, we precisely meet the original goal of modelling expression such as “about five” by dealing with crisp number 5 and a tolerance of close values (given by $S$). Because a fuzzy point could be generally constructed from an arbitrary point in an arbitrary space and not necessarily from a real number, for our purpose we will again rather restrict our focus to the universe of discourse $\mathbb{R}$ and rename it as *extensional fuzzy number*.

**Definition 2.5.** [10] Let $S$ be a $*$-similarity relation on $\mathbb{R}$, $x \in \mathbb{R}$ and $\tilde{x} \in \mathcal{F}(\mathbb{R})$ be a singleton. The *extensional fuzzy number (fuzzy point)* $x_S \in \mathcal{F}(\mathbb{R})$ is the extensional hull of $\tilde{x}$ with respect to $S$.

For the sake of simplicity and clarity in our notation, we will not distinguish between $x \in \mathbb{R}$ and a singleton $\tilde{x}$ that attains normality at $x$ and freely talk about an extensional hull of a real number $x$.

**Lemma 2.2.** *The extensional hull of $x \in \mathbb{R}$ w.r.t. a $*$-similarity relation $S$ on $\mathbb{R}$ may be expressed as follows*

$$x_S(y) = S(x, y), \quad y \in \mathbb{R}.$$

*Proof.* By the direct use of (4). □

Despite Lemma 2.2 is trivial, it is very important because it shows that dealing with extensional fuzzy numbers is as easy as dealing with chosen similarity relations evaluated in the given real numbers.

*2.2. Elementary but working example*

Let us restrict our focus on fuzzy numbers that are extensional w.r.t. a single fixed similarity relation $S$. We will get a very simple but yet working example. Consider a similarity relation $S$ on $\mathbb{R}$ and the set of all fuzzy numbers extensional w.r.t. $S$:

$$\mathcal{F}_S(\mathbb{R}) = \{x_S \mid x \in \mathbb{R}\}.$$

Let us define operations $+$ and $\cdot$ on $\mathcal{F}_S(\mathbb{R})$ as follows:

$$x_S + y_S = (x + y)_S,$$
$$x_S \cdot y_S = (xy)_S,$$

for any $x_S, y_S \in \mathcal{F}_S(\mathbb{R})$. Moreover, both operations are obviously commutative. Further, let us define the identity elements for both operations as follows:

$$\mathbf{0} = 0_S = S(0, -), \quad \mathbf{1} = 1_S = S(1, -).$$

Indeed, one may easily check that

$$x_S + \mathbf{0} = x_S + 0_S = (x + 0)_S = x_S,$$
$$x_S \cdot \mathbf{1} = x_S \cdot 1_S = (x \cdot 1)_S = x_S$$

holds for arbitrary $x_S \in \mathcal{F}_S(\mathbb{R})$. Finally, let us define the inverse elements with respect to both operations as follows:

$$-(x_S) = (-x)_S, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}),$$
$$(x_S)^{-1} = (1/x)_S, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}) \setminus \{\mathbf{0}\}.$$

Again, one can easily check that

$$x_S + (-x_S) = \mathbf{0}, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}),$$
$$x_S \cdot (x_S)^{-1} = \mathbf{1}, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}) \setminus \{\mathbf{0}\}.$$

Please note that the identity elements are not singletons. This is the main difference to the usual arithmetics of fuzzy numbers based on the Zadeh's extensional principle whose calculus directly enforces that only singletons $\tilde{0}, \tilde{1}$ may serve as identity elements. However, this is precisely the problem of such arithmetics because it is not possible to determine inverse elements $-a, a^{-1}$ that would yield

$$a + (-a) = \tilde{0}, \quad \forall\, a \in \mathcal{F}(\mathbb{R}),$$
$$a \cdot (a)^{-1} = \tilde{1}, \quad \forall\, a \in \mathcal{F}(\mathbb{R}) \smallsetminus \{\tilde{0}\}.$$

Let us stress that the fact that $\mathbf{0}, \mathbf{1}$ are not singletons is again very natural. If we deal with vaguely defined numbers, identity elements are necessarily elements of the set of vaguely defined numbers too and thus, one may hardly expect these identity elements to be precise. So, we may conclude that in the case of arithmetics of based on the extensional principle, the fact that calculus enforces the identity elements to be singletons $\tilde{0}, \tilde{1}$ is "guilty" for the non-existence of inverse elements in such a calculus.

Furthermore, one may easily check that in the above introduced motivation example, associativity as well as both distributive laws:

$$(x_S + y_S) \cdot z_S = (x_S \cdot z_S) + (y_S \cdot z_S),$$
$$z_S \cdot (x_S + y_S) = (z_S \cdot x_S) + (z_S \cdot y_S)$$

hold. Therefore, the algebraic structure $(\mathcal{F}_S(\mathbb{R}), +, \cdot, \mathbf{0}, \mathbf{1})$ is a *field*, i.e., the same algebraic structure that is formed by $(\mathbb{R}, +, \cdot, 0, 1)$.

Obviously, there is a one-to-one correspondence between the structures for $\mathcal{F}_S(\mathbb{R})$ and for $\mathbb{R}$ and the example is too simplistic in order to capture the needs of the intended calculus of fuzzy numbers. However, this structure that naturally grown up from fuzzy numbers extensional with respect to a single fixed similarity relation serves as a motivation example that shows that:

- the arithmetic based on extensional fuzzy numbers is intuitive, easy to compute and very natural;
- the so far often met problem of the non-existence of inverse elements is only a secondary consequence of the crispness of identity elements;
- the suggested arithmetical operations based on extensional fuzzy numbers enforce that the identity elements are also vaguely defined and thus, allows the existence of inverse elements;

which are unquestionably encouraging observations.

Let us finally point out that on one hand, computations according to the above suggested arithmetic of extensional fuzzy numbers are intuitive since they follow the idea of computing "about five" plus "about three" is something "about (five plus three)". This is in our opinion much closer to the human style of computing with imprecise numbers than computing with $\alpha$-cuts or parameters of functions that express the imprecise numbers. On the other hand,

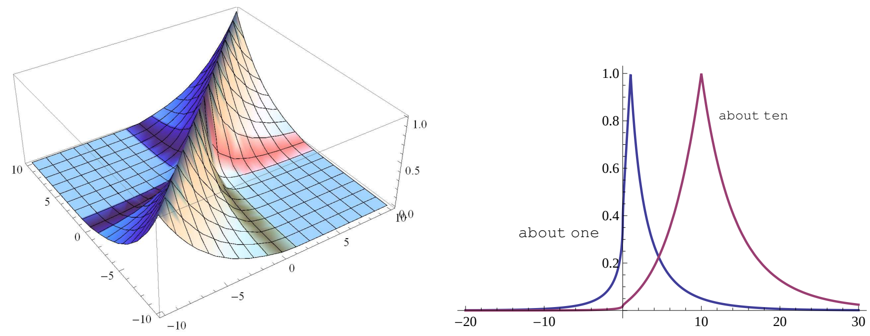


Figure 1: Non-shift invariant product type similarity relation $S_f$ (p=0.6) and extensional fuzzy numbers "about one" and "about ten" with respect to $S_f$.

in the suggested arithmetic the tolerance expressed by "about" is always the same, not only in both summands but also in the computed result and this may seem very unnatural in many cases. Let this serve as a motivation for the latter section.

**Remark 2.1.** *In the above text we have mentioned that in the simple motivation example there is a problem of the only measure of tolerance which might be rather unnatural. However, this is meant in such a way that we deal with the only similarity relation that expresses the tolerance which does not necessarily mean that we tolerate the same imprecision or vagueness over the whole universe of reals. This would be truth only in case of a shift-invariant similarity $S$ but one may easily construct a similarity $S$ that is not shift-invariant. For example, if we define $f(x) = \mathrm{sgn}(x)|x|^p$, $p > 0$ and $p \neq 1$, then*

$$S_f(x, y) = e^{-|f(x)-f(y)|}$$

*is a $*$-similarity relation $\mathbb{R}$, where $*$ is the product t-norm, but it is not a shift-invariant similarity. As one can see from Fig. 1, such similarity allows to take into account the scale of compared values, e.g., $S(1, 2) = 0.597$, but $S(10, 11) = 0.791$, which is very natural. Thus, even the simple example with a single similarity captures a sort of flexibility that is desirable.*

## 3. Arithmetics of extensional fuzzy numbers

The drawback of the above introduced example of the arithmetic of fuzzy numbers that are extensional w.r.t. a single similarity relation is obvious – the only tolerance or vagueness measure. However, there is a natural way out of it – employing a whole system of similarity relations into the suggested arithmetic. For this purpose, we need to define the transitive closure of similarity relations. Recall that if the intersection of $*$-similarity relations is again a $*$-similarity relation.

**Definition 3.1.** [4] Let $*$ be a left-continuous t-norm and let $S$ be a reflexive and symmetric binary fuzzy relation on $\mathbb{R}$. Then, the binary fuzzy relation on $\mathbb{R}$ denoted by $\mathrm{cl}(S)$ is the *transitive closure* of $S$ if it is its least superset that is a $*$-similarity relation, i.e.,

$$\mathrm{cl}(S) = \bigcap\{T \mid T \text{ is } *\text{-similarity and } S \subseteq T\}$$

where $\bigcap$ stands for the standard intersection of fuzzy sets.

Obviously, if $S$ is a $*$-similarity relation then $\mathrm{cl}(S) = S$. In the other case the transitive closure may be determined as follows.

**Lemma 3.1.** *[4] Let $*$ be a left-continuous t-norm and let $S$ be a reflexive and symmetric binary fuzzy relation on $\mathbb{R}$. Then,*

$$\mathrm{cl}(S) = \bigcup_{n=1}^{\infty} S^{(n)},$$

*where $\bigcup$ stands for the standard union of fuzzy sets and*

$$S^{(1)}(x,y) = S(x,y),$$
$$S^{(n)}(x,y) = \bigvee_{z\in\mathbb{R}} \left(S^{(1)}(x,z) * S^{(n-1)}(z,y)\right), \quad n \geq 2$$

*for $x, y \in \mathbb{R}$*

Finally, we may consider the transitive closure of the union of two given similarity relations to define a union in the set of extensional $*$-similarity relations:

$$S \uplus T = \mathrm{cl}(S \cup T).$$

The following lemma summarizes the important properties of $\uplus$ used in the verification of properties of arithmetical operations on extensional fuzzy numbers.

**Lemma 3.2.** *The operation $\uplus$ is commutative, associative and preserves the shift-invariance of $*$-similarity relations.*

*Proof.* Obviously, $\uplus$ is commutative. Let $\mathrm{cls}(S)$ denote the set of all superset of $S$ that are $*$-similarity relations. By Definition 3.1, we have $\mathrm{cl}(S) = \bigcap\ \mathrm{cls}(S)$. Let $R, S, T$ be $*$-similarity relations. To prove the associativity of $\uplus$ it suffices to check the following equality:

$$\mathrm{cls}(R \uplus (S \uplus T)) = \mathrm{cls}((R \uplus S) \uplus T). \tag{5}$$

If $H \in \mathrm{cls}(R \uplus (S \uplus T))$, then $R \cup (S \uplus T) \subseteq H$ which implies $R \cup S \cup T \subseteq H$. Since $R \cup S \subseteq H$, then $R \uplus S \subseteq H$, whence $(R \uplus S) \uplus T \subseteq H$. Thus, $H \in \mathrm{cls}((R \uplus S) \uplus T)$ and $\mathrm{cls}(R \uplus (S \uplus T)) \subseteq \mathrm{cls}((R \uplus S) \uplus T)$. Analogously, one may check the opposite inclusion and (5) is proved.

Let $S, T$ be shift-invariant $*$-similarity relations and put $H = S \cup T$. Obviously, $H$ is reflexive, symmetric and shift-invariant. We will check that the construction of the transitive closure introduced in Lemma 3.1 preserves the shift-invariance of relations. By induction, $H^{(1)} = H$ is shift-invariant and suppose that the same holds for $H^{(n)}$. We will prove that $H^{(n+1)}$ is shift-invariant. Let $x, y, h \in \mathbb{R}$. Then,

$$H^{(n+1)}(x,y) = \bigvee_{t\in\mathbb{R}} \left(H^{(1)}(x,t) * H^{(n)}(t,y)\right) =$$
$$\bigvee_{t\in\mathbb{R}} \left(H^{(1)}(x+h,t+h) * H^{(n)}(t+h,y+h)\right) =$$
$$\bigvee_{s\in\mathbb{R}} \left(H^{(1)}(x+h,s) * H^{(n)}(s,y+h)\right) = H^{(n+1)}(x+h,y+h).$$

Thus, $H^{(n)}$ is shift-invariant for any $n = 1, 2, \ldots$ Hence,

$$\mathrm{cl}(H)(x+h,y+h) = \bigcup_{i=1}^{\infty} H^{(n)}(x+h,y+h) = \bigcup_{i=1}^{\infty} H^{(n)}(x,y) = \mathrm{cl}(H)(x,y),$$

which concludes the proof. □

It should be noted that the intersection of shift-invariant $*$-similarity relations is again a shift-invariant $*$-similarity relation. Let $\mathcal{S}$ be a set of $*$-similarity relations and denote

$$S^* = \bigcap \mathcal{S}.$$

For the rest of the paper, without explicit mentioning, we will consider only such systems of similarity relations $\mathcal{S}$ for which the following two conditions hold:

- $S^* \in \mathcal{S}$,
- $S \cup T \in \mathcal{S}$ for arbitrary $S, T \in \mathcal{S}$.

The next natural step is to extend the motivating example of a functional arithmetic of fuzzy numbers that are extensional w.r.t a similarity relation to arithmetics of fuzzy numbers that are extensional w.r.t. a similarity from a given system of similarity relations $\mathcal{S}$. Such extension would allow us to deal with distinct “measures of vagueness or tolerance” form the given system.

Let us denote the set of all fuzzy numbers that are extensional w.r.t. a similarity from a given system of similarity relations $\mathcal{S}$ by $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$:

$$\mathcal{F}_{\mathcal{S}}(\mathbb{R}) = \{x_S \mid x \in \mathbb{R} \text{ and } S \in \mathcal{S}\}$$

and define operations + and $\cdot$ on $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$ as follows:

$$x_S + y_T = (x+y)_{S\cup T}, \quad x_S \cdot y_T = (x \cdot y)_{S\cup T}, \quad S, T \in \mathcal{S}.$$

Furthermore, let the identity elements are defined as follows:

$$\mathbf{0} = 0_{S^*} = S^*(0, -), \quad \mathbf{1} = 1_{S^*} = S^*(1, -).$$

Indeed, one can check that

$$x_S + \mathbf{0} = x_S + 0_{S^*} = (x+0)_{S\cup S^*} = x_S,$$
$$x_S \cdot \mathbf{1} = x_S \cdot 1_{S^*} = (x \cdot 1)_{S\cup S^*} = x_S$$

and moreover, with help of Lemma 3.2, the obtained arithmetic will be associative and distributive, i.e., the following distributive laws:

$$(x_R + y_S) \cdot z_T = (x_R \cdot z_T) + (y_S \cdot z_T),$$
$$z_T \cdot (x_R + y_S) = (z_T \cdot x_R) + (z_T \cdot y_S)$$

will hold for any $R, S, T \in \mathcal{S}$.

If we approach the inverse elements similarly to the elementary motivation example, i.e., if

$$-(x_S) = (-x)_S, \quad \forall\, x_S \in \mathcal{F}_{\mathcal{S}}(\mathbb{R}),$$
$$(x_S)^{-1} = (1/x)_S, \quad \forall\, x_S \in \mathcal{F}_{\mathcal{S}}(\mathbb{R}) \setminus \{0_S \mid S \in \mathcal{S}\}$$

we will encounter the following problems:

$$x_S + (-x_S) = 0_S \neq 0_{S^*} = \mathbf{0},$$
$$x_S \cdot (x_S)^{-1} = 1_S \neq 1_{S^*} = \mathbf{1},$$

for all $S \in \mathcal{S}$ such that $S \neq S^*$. So, the inverse elements do not work as the classical inverse element and do not give us the identity elements $\mathbf{0}$ and $\mathbf{1}$, respectively. However, at least the obtained elements $0_S, 1_S$ are constructed with help of $x = 0$ and $x = 1$, respectively, and thus, they differ from the identity elements $\mathbf{0}$ and $\mathbf{1}$ only in the used measure of vagueness given by the similarity relation $S$.

In other words, $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, \cdot, \mathbf{0}, \mathbf{1})$ is not a field but it is an algebraic structure that is very close to a field. The question, what kind of algebraic structure it is, is partially answered in the next section. One can see that considering a set of identity elements instead of a single identity for each operations is the most natural way out of this problem.

**Example 3.1.** *Let $*$ be the Łukasiewicz t-norm [13, 20] and consider the following system of embedded shift-invariant $*$-similarities:*

$$\mathcal{S} = \{S_p \mid p \in [a, b] \text{ and } S_p(x, y) = (1 - p|x - y|) \vee 0\},$$

*where $0 < a \leq b$. Since $S_p \cup S_{p'} = S_p$ for $p \leq p'$ and since all similarities of $\mathcal{S}$ are transitive, it also holds that*

$$S_p \uplus S_{p'} = S_p, \text{ for } p \leq p'.$$

*Therefore,*

$$x_{S_p} + y_{S_{p'}} = (x + y)_{S_{p''}}, \quad x_{S_p} \cdot y_{S_{p'}} = (x \cdot y)_{S_{p''}}, \quad p'' = \min\{p, p'\}.$$

*Finally, $S^* = \bigcap \mathcal{S} = S_b$ and, therefore,*

$$\mathbf{0} = 0_{S_b} = S_b(0, -), \quad \mathbf{1} = 1_{S_b} = S_b(1, -)$$

*where $S_b(x, y) = (1 - b|x - y|) \vee 0$. On Fig. 2, the summation and multiplication of "about 2" and "about 7" are depicted. One can see that the resulted fuzzy sets "about 9" and "about 14" keep the fuzziness of "about 7", because "about 7" contains the fuzziness of "about 2" (i.e., $S_{0.1} \subseteq S_{0.5}$).*

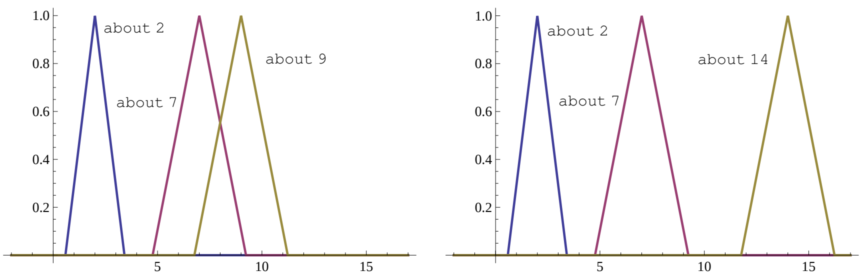


Figure 2: Summation and multiplication of extensional fuzzy numbers "about 2" ($S_{0.5}$) and "about 7"($S_{0.1}$).

Example 3.1 describes the arithmetic of triangular shaped fuzzy numbers that are very common. It should be stressed that in comparison to often used arithmetics of such fuzzy numbers that stem from the Zadeh's extensional principle, i.e., that are $\alpha$-cut based arithmetics, there are at least three unquestionable advantages:

- the obtained triangular fuzzy numbers do not get inevitably wide;
- the triangular shape is preserved even in the case of the multiplication or the division, which is not only intuitive but also highly desirable (the fuzzy number is still a vaguely defined number with some tolerance of the vagueness);
- the computational efforts are very low since computing the central point is computationally as cheap as in the case of real numbers and the only possibly expensive step – the computation of the transitive closure of two involved similarities – is in this case as cheap as the comparison of $p$ and $p'$.

One may argue, that the last argument in favor of the suggested approach is not that significant since the standard approach based on the Zadeh's extensional principle is very efficient if it is not applied in the $\alpha$-cut based scheme but with the use of parameters of the triangular fuzzy numbers. This is true but only in the case of addition and this particular shape of fuzzy sets i.e., this particular $\mathcal{S}$. Let us consider another still rather simple example but yet computationally expensive in the case of the standard Zadeh's extensional principle based approach (especially for the multiplication). This example of arithmetic is still as computationally cheap as the one given in Example 3.1, if we use the suggested approach based on extensional fuzzy numbers.

**Example 3.2.** *Let $*$ be the product t-norm [13, 20] and consider the following system of embedded shift-invariant $*$-similarities:*

$$\mathcal{S} = \{S_p \mid p \in [a,b] \text{ and } S_p(x,y) = e^{-p|x-y|},$$

*where* $0 < a \leq b$. *Again, since* $S_p \cup S_{p'} = S_p$ *for* $p \leq p'$ *and since all similarities of* $\mathcal{S}$ *are transitive, it also holds that*

$$S_p \cup S_{p'} = S_p, \text{ for } p \leq p'$$

*and thus, as in the previous example:*

$$x_{S_p} + y_{S_{p'}} = (x+y)_{S_{p''}}, \quad x_{S_p} \cdot y_{S_{p'}} = (x \cdot y)_{S_{p''}}, \quad p'' = \min\{p, p'\}.$$

*Finally,* $S^* = \bigcap \mathcal{S} = S_b$ *and therefore,*

$$\mathbf{0} = 0_{S_b} = S_b(0, -), \quad \mathbf{1} = 1_{S_b} = S_b(1, -)$$

*where* $S_b(x, y) = e^{-b|x-y|}$.

Example 3.2 describes the arithmetic that deals with the Gaussian-shaped[1] fuzzy sets. Again, as in the case of the arithmetic described in Example 3.1, the computation is even for the multiplication (and hence division too) computationally as cheap as in the case of real numbers and the only possibly expensive step – the computation of the transitive closure of two involved similarities – is in this case as cheap as the comparison of $p$ and $p'$. Again, it can be stated that the resulting fuzzy sets do not get inevitably wide.

**Remark 3.1.** *Note that since all* $S_p \in \mathcal{S}$ *from Examples 3.1 and 3.2 are shift-invariant, it holds that* $S_p(1, 2) = S_{p'}(10001, 10002)$. *However, the flexibility is not lost at all, the flexibility is provided by the variety of similarity relations that are at disposal in the system* $\mathcal{S}$ *and thus, one can easily model a similarity of small values by an* $S_p$ *for some high* $p$ *and a similarity of high values by an* $S_{p'}$ *for some small* $p'$.

## 4. MI-monoids

*4.1. Main definition and examples*

In the above sections, we have introduced a novel approach to arithmetics of extensional fuzzy numbers that seems to be well working from an algebraic point of view, i.e., to be close to the field structure. However, one big drawback seems to be present, that is that for $S \neq S^*$ the following holds:

$$x_S + (-x_S) = 0_S \neq \mathbf{0}.$$

Let us focus on the properties of such $0_S$. We may see that

$$0_S = -0_S,$$
$$x_S + x_S = 0_S \text{ then } x_S = -x_S \text{ and thus } x_S = 0_S$$

[1]We are fully aware of the fact that (due to their unlimited support) Gaussian-shaped fuzzy sets do not fit into most of the schemes of fuzzy numbers. However, we do not see any theoretical nor conceptual reason for that exclusion and we are convinced that this restriction is only of a technical nature. On the contrary, in our opinion such fuzzy sets perfectly fit into the scheme that considers computing with vaguely defined numbers.

which are exactly the properties carried by $\mathbf{0} = 0_{S^*}$. In other words, $0_S$ is a zero-like element. The same can be shown about $1_S$ w.r.t. the multiplication operation.

Therefore, we naturally introduce an algebraic structure that contains a set of identity elements. The introduced structure will be called MI-monoid, where MI stands for *Many Identities*.

**Definition 4.1.** A triplet $(G, \circ, E)$ is said to be an *MI-monoid* if $E$ is a non-empty subset of $G$ and $\circ$ is a binary operation on $G$ such that for any $x, y, z \in G$ and $a, b \in E$ the following conditions are satisfied:

(M1) $x \circ (y \circ z) = (x \circ y) \circ z$,

(M2) $\exists e \in E, \forall x \in G : x \circ e = e \circ x = x$,

(M3) $a \circ b \in E$,

(M4) $x \circ x = a$ then $x \in E$,

(M5) $x \circ a = a \circ x$.

Elements from $E$ are called *pseudoidentity elements* (*pseudoidentities*, for short) and a pseudoidentity $e$ satisfying (M2) is called a *(strong) identity element*. An MI-monoid $G$ is said to be *abelian* or *commutative* if $x \circ y = y \circ x$ holds for any $x, y \in G$.

Standardly, we write $G = (G, \circ, E)$ and $x \circ y = xy$. Clearly, for the abelian MI-monoids the axiom (M5) may be omitted. One can easily verify that each MI-monoid possesses a unique identity element $e \in E$, i.e., just one pseudoidentity satisfies (M2). In what follows, the identity element will be always denoted by $e$. Note that, in contrast to exactly one identity element, an MI-monoid can have an arbitrary number of pseudoidentity elements (including no pseudoidentity element except the strong identity, i.e., $E \setminus \{e\} = \varnothing$). A greater number of pseudoidentities in $G$ could be interpreted as more "fuzziness" in elements from $G$.

**Definition 4.2.** An MI-monoid $G$ is said to be *good* if the following condition is satisfied:

(M6) $\forall x \in G, \forall a \in E : x \circ a \in E$ then $x \in E$.

If $G$ is a good MI-monoid, then we need not suppose $e \in E$ in (M2), but it suffices to consider $e \in G$. In fact, by (M2), we have $ae = a$ for any $a \in E$. Hence, $e \in E$ follows from (M6).

**Remark 4.1.** *Obviously, each monoid $G$ is an MI-monoid with $E = \{e\}$. Moreover, one can simply check that $(G, \circ, E)$ is an MI-monoid if and only if $(G, \circ)$ is a monoid and $(E, \circ)$ is a commutative monoid such that $E \subseteq G$ and (M4) and (M5) are satisfied. Thus, an MI-monoid is a suitable combination of two monoids, where the second one is, moreover, abelian.*

One can check that $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}})$ where $E^0_{\mathcal{S}} = \{0_S \mid S \in \mathcal{S}\}$ is an MI-monoid with the strong identity element $0_{S^*}$ and similarly also $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), \cdot, E^1_{\mathcal{S}})$ where $E^1_{\mathcal{S}} = \{1_S \mid S \in \mathcal{S}\}$ is an MI-monoid with the strong identity element $1_{S^*}$.

Moreover, it can be shown that both examples of MI-monoids of extensional fuzzy numbers $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}})$ and $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), \cdot, E^1_{\mathcal{S}})$ are also good MI-monoids. However, is there a non-trivial example of an MI-monoid that is not good? The answer is positive and we provide readers with two very natural examples of such MI-monoids.

**Example 4.1.** *Let* $G = \{[a,b] \mid a,b \in \mathbb{R}, a \le b\}$ *be the set of all real intervals,* $E^0 = \{[a,b] \mid a,b \in \mathbb{R}, a \le b$ *and* $0 \in [a,b]\}$ *and the operation* $+$ *on* $G$ *be defined as follows:*

$$[a,b] + [c,d] = [a+c, b+d].$$

*Then, the strong identity element is* $[0,0]$*, and* $(G, +, E^0)$ *is an additive abelian MI-monoid of real intervals that is not good because, e.g.,*

$$[-5,5] + [1,2] = [-4,7]$$

*but* $[1,2] \notin E$*.*

*Similarly, let* $G$ *be defined as above,* $E^1$ *denote the set of all intervals containing 1 and the operation* $\cdot$ *be defined as follows*

$$[a,b] \cdot [c,d] = [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)].$$

*Then, the strong identity element is* $[1,1]$*, and* $(G, \cdot, E^1)$ *is an abelian multiplicative MI-monoid that is not good because, e.g.,*

$$[0.2,2] \cdot [2,4] = [0.4,8]$$

*but* $[2,4] \notin E$*.*

Of course, both examples of non-good MI-monoids can be enhanced by a redefinition of the sets of pseudoidentity elements in such a way that they contain only intervals symmetrically distributed around 0 or 1.

Suitability of the MI-monoid structure may be demonstrated also by the fact that it includes an algebraic framework for the standard approach to the arithmetic of fuzzy numbers (cf. [5, 14]). This fact is demonstrated in the following example.

**Example 4.2.** *Let* $G$ *be the MI-monoid of real intervals (see Example 4.1) and* $H$ *be the set of all mappings* $g : [0,1] \to G$ *provided*

$$\begin{aligned} g(1) &\ne \emptyset, \\ g(\alpha) &\subseteq g(\beta) \quad \text{for all } \alpha,\beta \in [0,1],\ \alpha \ge \beta, \\ \bigcup_{\substack{\beta \in [0,1] \\ \alpha < \beta}} g(\beta) &= g(\alpha) \quad \text{for all } \alpha \in [0,1). \end{aligned} \tag{6}$$

*One can see that g represents an upper semicontinuous and convex fuzzy set in* $\mathbb{R}$ *with a finite support* $g(0)$ *(a fuzzy number). More precisely, if* $g \in H$*, then*

$$A_g(x) = \bigvee_{\substack{\alpha\in[0,1],\\ x\in g(\alpha)}} \alpha$$

*is an upper semicontinuous and convex fuzzy set* $A_g : \mathbb{R} \to [0,1]$*. Put* $E^0$ *the set of all* $g$ *such that* $0 \in g(0)$ *and define pointwise* $g + g'$ *using the addition of real intervals. Then,* $(H, +, E^0)$ *is an abelian additive MI-monoid of fuzzy numbers. Analogously, put* $E^1$ *the set of all* $g$ *such that* $1 \in g(0)$ *and define pointwise* $g \cdot g'$ *using the multiplication of real intervals. Then,* $(H, \cdot, E^1)$ *is an abelian multiplicative MI-monoid of fuzzy numbers.*[2]

In practice, fuzzy numbers are often represented by several parameters in order to simplify their notation and mainly their arithmetic [14, 22]. It should be noted that the simplified arithmetic is only an approximation of the standard arithmetic of fuzzy numbers mentioned in the previous example.

**Example 4.3.** *Let* $T$ *be the set of all trapezoidal fuzzy numbers, i.e., fuzzy numbers given in a parametric form* $A = \langle a_A, b_A, c_A, d_A\rangle$*, where* $a_A, b_A, c_A, d_A \in \mathbb{R}$ *with* $a_A \le b_A \le c_A \le d_A$*, and defined as*

$$A(x) = \begin{cases} 0, & x < a_A \text{ or } d_A < x; \\ (x - a_A)/(b_A - a_A), & a_A \le x < b_A; \\ 1, & b_A \le x \le c_A; \\ (d_A - x)/(d_A - c_A), & c_A < x \le d_A. \end{cases}$$

*Note that, although, we use the term "trapezoidal", the set of these fuzzy numbers contains also triangular fuzzy numbers (i.e.,* $b_A = c_A$*), or real intervals (i.e.,* $a_A = b_A$ *and* $c_A = d_A$*) including one element intervals (singletons representing the real numbers). Define the addition on* $T$ *by*

$$\begin{aligned} A + B = \langle a_A, b_A, c_A, d_A\rangle + \langle a_B, b_B, c_B, d_B\rangle = \\ \langle a_A + a_B, b_A + b_B, c_A + c_B, d_A + d_B\rangle. \end{aligned}$$

*Put* $E^0$ *the set of all trapezoidal fuzzy numbers* $A$ *for which* $0 \in (a_A, d_A)$*, i.e.,* $A$ *is a pseudoidentity element if* $A(0) > 0$*. Clearly,* $\langle 0,0,0,0\rangle$ *is the identity element. One may simply check that* $(T, +, E^0)$ *is a non-good abelian additive MI-monoid. Further, define the multiplication on* $T$ *by*

$$\begin{aligned} A \cdot B = &\langle a_A, b_A, c_A, d_A\rangle \cdot \langle a_B, b_B, c_B, d_B\rangle = \\ &\langle a_{A\cdot B}, b_{A\cdot B}, c_{A\cdot B}, d_{A\cdot B}\rangle, \end{aligned}$$

[2] It is well known that the both presented arithmetical operations are equivalent to that obtained by the Zadeh's extension principle.

*where*

$$\begin{aligned}
a_{A\cdot B} &= \min(a_A a_B, a_A d_B, d_A a_B, d_A d_B)\\
b_{A\cdot B} &= \min(b_A b_B, b_A c_B, c_A b_B, c_A c_B),\\
c_{A\cdot B} &= \max(b_A b_B, b_A c_B, c_A b_B, c_A c_B),\\
d_{A\cdot B} &= \max(a_A a_B, a_A d_B, d_A a_B, d_A d_B).
\end{aligned}$$

*Put* $E^1$ *the set of all trapezoidal fuzzy numbers for which* $1 \in (a_A, d_A)$*, i.e.,* $A$ *is a pseudoidentity element if* $A(1) > 0$*. Clearly,* $\langle 1,1,1,1\rangle$ *is the identity element. Again, one may verify that* $(T, \cdot, E^1)$ *is a non-good abelian multiplicative MI-monoid.*

A recent stream in the development of the fuzzy number theory is closely related with the parametric representation of fuzzy numbers using monotonic interpolation introduced by Guerra and Stefanini in [7]. The advantage of this type of parametric representation consists in the additional parameters characterizing more complex shapes of fuzzy numbers. These more complex functions give a possibility to better approximate the fuzzy numbers from data and, furthermore, what is very important, a simple arithmetic based on the manipulations with parameters can be introduced for them.

**Example 4.4.** *In [7] (see also [22]), a parametric representation of fuzzy numbers using monotonic interpolation is provided and, for instance, the trapezoidal fuzzy numbers give an example of such parametric representation. Let*

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = 1 \tag{7}$$

*be real numbers for a finite decomposition of the unit interval. Without detailed comment, considering the differentiable case, an* $LU$*-fuzzy number* $A$ *is represented by the following system of vectors*

$$A = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\ldots,N}, \tag{8}$$

*with the data*

$$u_0^- \le u_1^- \le \cdots \le u_N^- \le u_N^+ \le u_{N-1}^+ \le \cdots \le u_0^+$$

*and the slopes*

$$\delta u_i^- \ge 0 \quad and \quad \delta u_i^+ \le 0.$$

*Denote* $L$ *the set of all* $LU$*-fuzzy numbers in the form (8) for a fixed finite decomposition of* $[0,1]$ *determined by (7). Define the addition on* $L$ *by (*$\alpha_i$ *is omitted here for simplicity)*

$$\begin{aligned}
(u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\ldots,N} + (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,\ldots,N} =\\
(u_i^- + v_i^-, \delta u_i^- + \delta v_i^-, u_i^+ + v_i^+, \delta u_i^+ + \delta v_i^+)_{i=0,\ldots,N}
\end{aligned}$$

*Put $E^0$ the set of all LU-fuzzy numbers of $L$ such that $0 \in [u_0^-, u_0^+]$. By the results in [22], $(L, +, E^0)$ is a non-good abelian additive MI-monoid.*

*Further, define the multiplication on $L$ by*

$$(u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\dots,N} \cdot (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,\dots,N} = ((uv)_i^-, \delta(uv)_i^-, (uv)_i^+, \delta(uv)_i^+)_{i=0,\dots,N},$$

*where*

$$(uv)_i^- = \min(u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+),$$
$$(uv)_i^+ = \max(u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+)$$

*and*

$$(uv)_i^- = \delta u_i^{p_i^-} v_i^{q_i^-} + u_i^{p_i^-} \delta v_i^{q_i^-}$$
$$(uv)_i^+ = \delta u_i^{p_i^+} v_i^{q_i^+} + u_i^{p_i^+} \delta v_i^{q_i^+}$$

*with $(p_i^-, q_i^-)$ being the pair associated to the combination of superscripts + and - giving the minimum $(uv)_i^-$ and $(p_i^+, q_i^+)$ being the pair associated to the combination of + and - giving the maximum $(uv)_i^+$. Put $E^1$ the set of all LU-fuzzy numbers from $L$ such that $1 \in [u_0^-, u_0^+]$. By the results in [22], $(L, \cdot, E^1)$ is a non-good abelian multiplicative MI-monoid.*

Although Examples 4.2, 4.3 and 4.4 present MI-monoids of fuzzy numbers that are not good, one may easily redefine $E^0$ and $E^1$ in such a way that the modified MI-monoids will gain this property. For instance, redefining $E^0$ in Example 4.3 by

$$E^0 = \{\langle a, 0, 0, d\rangle \mid a, d \in \mathbb{R}, a \le 0 \le d\},$$

we obtain a good additive MI-monoid of trapezoidal fuzzy numbers.[3] In the definition of a field, the multiplicative group is defined on the support of the additive group, where the zero element is omitted. This idea is also kept in our definition of MI-field (see Definition 6.1). From this perspective, the presumption of goodness of the additive MI-monoid of trapezoidal fuzzy numbers is not justified because not all zero-like elements, i.e., the elements to which we cannot define the inverse elements with respect to the multiplication, are contained in $E^0$. Practically, the presumption of goodness in this case obstructs a natural definition of MI-field of trapezoidal fuzzy numbers.

At the end of this section, we would like to show two examples of MI-monoids that are not purely fuzzy and generalize the interval arithmetic discussed in Example 4.1.

[3]In fact, if $A = \langle a, 0, 0, d\rangle$ and $B = \langle a', b', c', d'\rangle$ such that $A + B \in E^0$, then $0 + b' = 0$ and $0 + c' = 0$, which implies $b' = c' = 0$, and $B \in E^0$.

**Example 4.5.** *Let $M$ be the set of all convex subsets of $\mathbb{R}^n$ and $\mathbf{a}+\mathbf{b}$ denote the addition of two vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$. Let $\oplus$ be the Minkowski addition, i.e.,*

$$A\oplus B=\{\mathbf{a}+\mathbf{b}\mid \mathbf{a}\in A,\ \mathbf{b}\in B\}.$$

*It is easy to see that $\oplus$ is commutative, associative and the set containing only the null vector $\mathbf{0}$ is the identity element. Put $E^0$ the set of all sets in $M$ that contains the null vector. Then, (M3) and (M5) are trivially satisfied. Since the sets from $M$ are convex, then $A\oplus A\in E^0$ implies the existence of $\mathbf{a},\mathbf{b}\in A$ for which $\mathbf{a}+\mathbf{b}=0$, whence $\mathbf{b}=-\mathbf{a}$. From the convexity of $A$, we obtain $\mathbf{0}\in A$ by $0.5\mathbf{a}+0.5(-\mathbf{a})=\mathbf{0}$ and $A\in E^0$. Thus, (M4) is satisfied, and $(M,\oplus,E^0)$ is an abelian additive MI-monoid of convex sets in $\mathbb{R}^n$.*[4]

**Example 4.6.** *Let $\mathbb{R}^n_{>0}$ denote the set of all positive real vectors and $\mathbf{a}\cdot\mathbf{b}$ be the product of vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n_{>0}$ defined pointwise (analogously to the addition of vectors). For simplicity, we standardly use $\mathbf{a}\cdot\mathbf{b}=\mathbf{ab}$. A subset $A$ of $\mathbb{R}^n_{>0}$ is said to be* product convex, *if, for any $\mathbf{a},\mathbf{b}\in A$ and $\lambda\in[0,1]$, we have $\mathbf{a}^\lambda\mathbf{b}^{1-\lambda}\in A$. Let $N$ be the set of all product convex subsets of $\mathbb{R}^n_{>0}$ and $\odot$ be the "Minkowski" multiplication defined by*

$$A\odot B=\{\mathbf{ab}\mid \mathbf{a}\in A,\ \mathbf{b}\in B\}.$$

*Let us show that $A\odot B$ is product convex. If $\mathbf{x},\mathbf{y}\in A\odot B$ and $\lambda\in[0,1]$, then there exist $\mathbf{a},\mathbf{c}\in A$ and $\mathbf{b},\mathbf{d}\in B$ such that $\mathbf{x}=\mathbf{ab}$ and $\mathbf{y}=\mathbf{cd}$, and $\mathbf{x}^\lambda\mathbf{y}^{1-\lambda}=(\mathbf{ab})^\lambda(\mathbf{cd})^{1-\lambda}=(\mathbf{a}^\lambda\mathbf{c}^{1-\lambda})(\mathbf{b}^\lambda\mathbf{d}^{1-\lambda})$. Since $\mathbf{a}^\lambda\mathbf{c}^{1-\lambda}\in A$ and $\mathbf{b}^\lambda\mathbf{d}^{1-\lambda}\in B$, then $\mathbf{x}^\lambda\mathbf{y}^{1-\lambda}\in A\odot B$, and, therefore, $A\odot B$ is product convex. It is easy to see that $\odot$ is commutative, associative, and the set containing only the vector $\mathbf{1}=(1,\dots,1)$ is the identity element. Put $E^1$ the set of all sets in $N$ that contains $\mathbf{1}$. Again, (M3) and (M5) are trivially satisfied. If $A\odot A\in E^1$, then there exist $\mathbf{a},\mathbf{b}\in A$ for which $\mathbf{ab}=\mathbf{1}$. Hence, we obtain $\mathbf{b}=\mathbf{a}^{-1}$. Since $A$ is product convex, then $\mathbf{a}^{0.5}(\mathbf{a}^{-1})^{1-0.5}=\mathbf{a}^{0.5}\mathbf{a}^{-0.5}=\mathbf{1}$, whence $\mathbf{1}\in A$. Thus, $(N,\odot,E^1)$ is an abelian multiplicative MI-monoid of product convex sets in $\mathbb{R}^n_{>0}$.*

One may see that the concept of MI-monoid may serve as a basic algebraic structure for various types of objects and operations defined on them (intervals, sets, fuzzy sets and their arithmetics). This motivates us to develop a theory of MI-monoids and also to introduce further types of MI-algebras.

**Remark 4.2.** *It should be noted that the axiomatic system for MI-monoids is too general to precisely specify what is and what is not an appropriate set of pseudoidentities. As a demonstration of curious MI-monoids, we can consider $(G,+,G)$, where $(G,+)$ is an abelian monoid, or $(G,+,E^0)$, where $G$ is the set of all real intervals, $+$ is defined in Example 4.1, and $E^0=\{[a,a]\mid a\in\mathbb{R}\}$ is the set of all "degenerate" real intervals. In the second example, $E^0$ represents $\mathbb{R}$ and*

[4]Note that the Minkowski addition is used to define the operation of *dilation* which plays a central role in mathematical morphology (see e.g. [21]).

*does not delimit something, which one understands as the zero-like elements. Later it will be clear that many of such curious MI-monoids cannot be extended to MI-groups and thus, eventually to MI-fields.*

*4.2. Homomorphisms and MI-submonoids*

Essential to the study of MI-monoids and further algebraic objects stemming from the MI-monoids are the functions that preserve the given algebraic structure. We provide the following definition that naturally generalizes the homomorphism of monoids.

**Definition 4.3.** Let $G$ and $H$ be MI-monoids. A mapping $f : G \to H$ is a *homomorphism* of MI-monoids provided

(HM1) $f(xy) = f(x)f(y)$ for all $x, y \in G$,

(HM2) $f(e_G) = e_H$,

(HM3) $f(x) \in E_H$ for all $x \in E_G$,

where $e_G$ and $e_H$ ($E_G$ and $E_H$) denote the strong identity (the set of all pseudoidentities) of $G$ and $H$, respectively. If $f$ is injective, $f$ is said to be a *monomorphism*. If $f$ is surjective, $f$ is said to be an *epimorphism*. If $f$ is bijective and $f(E_G) = \{f(a) \mid a \in E_G\} = E_H$, $f$ is said to be an *isomorphism*. In this case $G$ and $H$ are said to be *isomorphic* (written $G \cong H$). A homomorphism $f : G \to G$ is called an *endomorphism* and an isomorphism $f : G \to G$ is called an *automorphism*.

Clearly, the composition of two homomorphisms (monomorphisms, epimorphisms, etc.) is again a homomorphism (monomorphism, epimorphism, etc.).

**Example 4.7.** *Let $(M, \oplus, E^0)$ and $(N, \odot, E^1)$ be the MI-monoids of convex and product convex sets introduced in Examples 4.5 and 4.6, respectively. Then, $f : N \to M$ given by $f(A) = \{\ln \mathbf{a} \mid \mathbf{a} \in A\}$, where $\ln \mathbf{a}$ is defined pointwise, is an isomorphism of the MI-monoids $N$ and $M$. In fact, we have $f(\mathbf{ab}) = \ln \mathbf{ab} = \ln \mathbf{a} + \ln \mathbf{b} = f(\mathbf{a}) + f(\mathbf{b})$, $f(\mathbf{1}) = \ln \mathbf{1} = \mathbf{0}$, and if $\mathbf{1} \in A$, then $\mathbf{0} \in f(A)$. Let $A, B \in N$. To check that $f(A)$ is a convex set, let $\mathbf{a}, \mathbf{b} \in f(A)$ and $\lambda \in [0,1]$. By the definition of $f$, there are $\mathbf{c}, \mathbf{d} \in A$ such that $f(\mathbf{c}) = \mathbf{a}$ and $f(\mathbf{d}) = \mathbf{b}$. Then,*

$$\lambda \mathbf{a} + (1-\lambda)\mathbf{b} = \lambda \ln \mathbf{c} + (1-\lambda) \ln \mathbf{d} =$$
$$\ln(\mathbf{c}^{\lambda}\mathbf{d}^{1-\lambda}) = f(\mathbf{c}^{\lambda}\mathbf{d}^{1-\lambda}).$$

*Since $\mathbf{c}^{\lambda}\mathbf{d}^{1-\lambda} \in A$, then $\lambda \mathbf{a} + (1-\lambda)\mathbf{b} \in f(A)$ and $f(A)$ is convex. Further, we have*

$$f(A \odot B) = \{\ln(\mathbf{ab}) \mid \mathbf{a} \in A,\ \mathbf{b} \in B\} =$$
$$\{\ln \mathbf{a} + \ln \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\} = f(A) \oplus f(B),$$

*whence (HM1) is satisfied. Obviously, $f(\{\mathbf{1}\}) = \{\ln \mathbf{1}\} = \{\mathbf{0}\}$ and if $A \in E^1$, then $\mathbf{0} \in f(A)$ and thus $f(A) \in E^0$. Hence, $f$ is a homomorphism of $N$ to $M$.*

*If $A \neq B$, then there is $\mathbf{a} \in A \setminus B$. Since the natural logarithm is a bijection of the set of positive real numbers to the set of real numbers, we obtain $f(\mathbf{a}) \notin B$, and $f$ is an injection of $N$ to $M$. Let $B \in M$. Put $A = \{e^{\mathbf{b}} \mid \mathbf{b} \in B\}$, where again $e^{\mathbf{b}}$ is defined pointwise. Then, $f(A) = \{\ln e^{\mathbf{b}} \mid \mathbf{b} \in B\} = \{\mathbf{b} \mid \mathbf{b} \in B\} = B$, whence $f$ is surjective. Finally, if $f(A) = B$ and $\mathbf{0} \in B$, then necessary $\mathbf{1} \in A$ and $f(E^1) = E^0$.*

Let $(G, \circ)$ be a monoid and $H \subseteq G$ be non-empty subset. If $H$ contains the identity element of $G$ and the restriction of $\circ$ to $H$ is also a binary operation on $H$, then $(H, \circ)$ is a submonoid of $G$. The following definition provides a generalization of the concept of submonoids for MI-monoids.

**Definition 4.4.** Let $G = (G, \circ, E)$ be an MI-monoid, $H \subseteq G$ and $F \subseteq E$ be non-empty subsets. If $H$ contains the strong identity of $G$ and is itself an MI-monoid under the product of $G$, then $H = (H, \circ, F)$ is said to be an *MI-submonoid* of $G$. This is denoted by $H \leq G$.

The MI-submonoids $(G, \circ, E)$, $(E, \circ, E)$, and $(\{e\}, \circ, \{e\})$ of an MI-monoid $(G, \circ, E)$ are called *trivial*.

**Example 4.8.** *Real intervals with endpoints in the set of integers equipped with the common a, where the pseudoidentities are defined analogously as for real intervals, form an (additive and multiplicative) MI-submonoid of the MI-monoid of real intervals (see Example 4.1).*

**Example 4.9.** *Triangle fuzzy numbers equipped with the addition (multiplication), where the pseudoidentities are defined to be symmetrically distributed around zero (one), i.e., $A(x) = A(-x)$ ($A(x) = A(1/x)$) holds for any $x \in \mathbb{R}$, form an additive (multiplicative) MI-submonoid of the MI-monoid of trapezoidal fuzzy numbers (see Example 4.3).*

The following proposition shows the necessary and sufficient condition under which subsets of $G$ specify an MI-submonoid of $G$.

**Proposition 4.1.** *Let $H \subseteq G$ and $F \subseteq E$ be non-empty subsets of an MI-monoid $G$. Then $H \leq G$ if and only if, for all $x, y \in H$ and $a, b \in F$, the following holds:*

*(i)* $xy \in H$,

*(ii)* $e \in F$,

*(iii)* $ab \in F$,

*(iv) if $xx = a$ then $x \in F$.*

*Proof.* ($\Rightarrow$) Trivial. ($\Leftarrow$) Obviously, the conditions (M1) and (M5) are satisfied, since the product on $H$ is the restriction of the product of $G$. The conditions (M2)-(M4) are equivalent to (i)-(iii), respectively. □

An MI-submonoid $H$ of an MI-monoid $G$ is defined by a suitable pair of subsets $H \subseteq G$ and $F \subseteq E$. If we set $F = H \cap E$ we obtain a special case of MI-submonoid that is determined only by a subset $H$ of a set $G$.

**Definition 4.5.** An MI-submonoid $(H, \circ, F)$ of an MI-monoid $(G, \circ, E)$ is said to be *canonical* if $F = H \cap E$.

The following proposition shows the necessary and sufficient condition for $H \subseteq G$ to define a canonical MI-submonoid, which is significantly simpler than that provided in Proposition 4.1. Moreover, its form is analogous to the standard one for monoids, see [9].

**Proposition 4.2.** *Let $G = (G, \circ, E)$ be an MI-monoid and $H \subseteq G$ be a non-empty subset. Then, $H = (H, \circ, H \cap E)$ is an MI-submonoid of $G$ if and only if $e \in H$ and $xy \in H$ for all $x, y \in H$.*

*Proof.* $(\Rightarrow)$ Trivial. $(\Leftarrow)$ We prove that (iii) and (iv) of Proposition 4.1 is true. If $a, b \in F = H \cap E$, then $ab \in H$ and $ab \in E$. Hence, we obtain $ab \in F$ and (iii) is satisfied. If $x \in H$ and $a \in F$ such that $xx = a$, then $x \in E$ by (M4). Hence, $x \in H \cap E = F$ and (iv) is true. □

One can see that the MI-submonoids of real intervals from Example 4.8 are canonical, but the MI-submonoids of trapezoidal fuzzy numbers from Example 4.9 are not canonical, since the set of pseudoidentities is restricted to the set of symmetrically distributed around the strong identity.

**Example 4.10.** *Let $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}})$ be the above introduced abelian additive MI-monoid of fuzzy numbers that are extensional w.r.t. similarities from a system of similarity relations $\mathcal{S}$. Let $\mathcal{T} \subseteq \mathcal{S}$ be a subsystem of $\mathcal{S}$, i.e., let $S^* = \bigcap \mathcal{S} \in \mathcal{T}$ and $S \cup S' \in \mathcal{T}$ for any $S, S' \in \mathcal{T}$. Using the previous proposition, one may simply check that $(\mathcal{F}_{\mathcal{T}}(\mathbb{R}), +, E^0_{\mathcal{T}})$, where clearly $\mathcal{F}_{\mathcal{T}}(\mathbb{R}) \subseteq \mathcal{F}_{\mathcal{S}}(\mathbb{R})$ and $E^0_{\mathcal{T}} = E^0_{\mathcal{S}} \cap \mathcal{F}_{\mathcal{T}}(\mathbb{R})$, is a canonical MI-submonoid of $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$.*

**Example 4.11.** *Let $(H, \cdot, E^1)$ be the multiplicative MI-monoid and $E^0$ be the set of pseudoidentities of the additive MI-monoid $(H, +, E^0)$ of fuzzy numbers from Example 4.2. Then, $(H \setminus E^0, \cdot, E^1 \setminus E^0)$ is an MI-submonoid (of "non-zero fuzzy numbers") of $(H, \cdot, E^1)$.*

**Example 4.12.** *Let $\mathbb{R}^n_{\geq 0}$ denote the set of all non-negative real vectors and $M_{\geq 0}$ be the set of all (non-negative) convex sets in $\mathbb{R}^n_{\geq 0}$. In Example 4.5, we defined an abelian MI-monoid $M = (M, \oplus, E^0)$, where $\oplus$ is the Minkowski addition. Since $\mathbf{0} \in M_{\geq 0}$ and $A \oplus B \in M_{\geq 0}$ for any $A, B \in M_{\geq 0}$, then, by the previous proposition, $(M_{\geq 0}, \oplus, E^0 \cap M_{\geq 0})$ is a canonical MI-submonoid of $M$.*

If we deal with pseudoidentities, then it is reasonable to introduce a relation $\sim$ on $G$ that is defined as follows. Two elements $x, y \in G$ are in a relation $\sim$:

$$x \sim y \quad \text{if and only if} \quad ax = by \tag{9}$$

holds for some $a, b \in E$, see [16]. Note that $x \sim y$ if and only if $xa = yb$ which holds from (M5).

**Proposition 4.3.** *The relation $\sim$ is a congruence on $G$.*

*Proof.* Obviously, $x \sim x$ and $x \sim y = y \sim x$ are trivially satisfied. Let $x \sim y$ and $y \sim z$. Then $ax = by$ and $cy = dz$ for some $a, b, c, d \in E$. Hence, we have

$$(ca)x = c(ax) = c(by) = (cb)y$$
$$(bd)z = b(dz) = b(cy) = (bc)y = (cb)y,$$

where (M1) and (M5) are applied, and thus $(ca)x = (bd)z$. Since $ca, bd \in E$, then $x \sim z$ and $\sim$ is an equivalence relation on $G$.

Now, let us prove that $\sim$ is compatible with the product. If $x \sim x'$ and $y \sim y'$, then $ax = bx'$ and $cy = dy'$ for suitable $a, b, c, d \in E$. Hence, $(ac)(xy) = (ax)(cy) = (bx')(dy') = (bd)(x'y')$. Since $ac, bd \in E$, then $xy \sim x'y'$. □

**Remark 4.3.** *One can see that $G/\sim$ is a quotient MI-monoid of an MI-monoid $G$ by $\sim$.*

## 5. MI-groups

*5.1. Main definition and examples*

The concept of MI-monoid is just a first step that allows us to deal with more identity-like elements but the next natural step is to introduce inverse elements and thus, a structure that naturally generalizes a group. In this section, we aim this goal and continue in the development of a theory of MI-algebras. We define an important extension of a group to an MI-group. We stress the word "important" because fuzzy numbers (as well as real intervals, convex sets etc.) equipped with an operation do not keep the group structure, in contrast to the monoid structure. Moreover, the groups and their theory play a prominent role among algebraic structures and thus, the same role for MI-groups is expected.

Since the definition of an MI-group is a bit complicated to imitate the important properties of groups, we will start with the definition of the inversion on an MI-monoid. Recall that, in any group, to each $x \in G$ there exists an inverse $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e$. A simple consequence of the group definition is the fact that each element of a group possesses precisely one inverse element, $(xy)^{-1} = y^{-1}x^{-1}$ and $(x^{-1})^{-1} = x$. Let $(G, \circ)$ be a group and define a dual group as $G^{\mathrm{op}} = (G, \circ^{\mathrm{op}})$, where $x \circ^{\mathrm{op}} y = y \circ x$. Then it is easy to see that $f(x) = x^{-1}$ defines an isomorphism $f : G \to G^{\mathrm{op}}$ satisfying the following properties:

$$f(x) \circ x = x \circ f(x) = e \text{ and } f(f(x)) = x.$$

Clearly it is sufficient to consider $x \circ f(x) = e$ if the group $G$ is abelian. Moreover, each monoid $G$ is a group if and only if there exists an isomorphism $f : G \to G^{\mathrm{op}}$ having the properties mentioned above. This isomorphism $f$ is unique.[5]

[5] In fact, if $f, g : G \to G^{\mathrm{op}}$ are two isomorphisms satisfying the mentioned properties, then, for any $x \in G$, $f(x) \circ x = g(x) \circ x = e$ implies $f(x) = (f(x) \circ x) \circ f(x) = (g(x) \circ x) \circ f(x) = g(x) \circ (x \circ f(x)) = g(x)$, whence $f(x) = g(x)$ for any $x \in G$, and thus $f$ is unique. Note that the property $f(f(x)) = x$ is a consequence of the fact that $f$ is an isomorphism which satisfies $f(x) \circ x = x \circ f(x) = e$.

**Definition 5.1.** Let $G = (G, \circ, E)$ be an MI-monoid. A *dual* MI-monoid to $G$ is $G^{\mathrm{op}} = (G, \circ^{\mathrm{op}}, E)$, where $\circ^{\mathrm{op}}$ is given by $x \circ^{\mathrm{op}} y = y \circ x$.

**Remark 5.1.** *Clearly, $G = G^{op}$ if $G$ is an abelian MI-monoid.*

**Definition 5.2.** An MI-monoidal isomorphism $f$ of $G$ onto $G^{\mathrm{op}}$ that satisfies

(G1) $f(x) \circ x \in E$,

(G2) $f(x) \circ x = x \circ f(x)$,

(G3) $f(f(x)) = x$

for any $x \in G$ is called an *f-inversion* (inversion, for short) in $G$.

One can see that (G1) is the only one difference between the isomorphism that defines inverse element of a group and the isomorphism for an $MI$-pregroup. Nevertheless, as we have mentioned above, the pseudoidentity elements concentrate the fuzziness in the elements and it would be contra-intuitive to require the satisfaction of the law $f(x)x = e$. One can imagine this fact as the impossibility to destroy the fuzziness in elements. On the other hand, we have $f(x)x \sim e$, i.e., $f(x)x$ differs from $e$ up to a pseudoidentity (up to the fuzziness).

It should be noted that (G3) is redundant for $E = \{e\}$. In fact, using (G1) and (G2), one can simply check that $f(x)x = yx$ implies $f(x) = y$. Putting $f(f(x)) = y$, we obtain $f(f(x))f(x) = e = yf(x)$, which implies $f(f(x)) = y = x$. On the other hand, (G3) is crucial for $E \neq \{e\}$. In fact, if $f(f(x)) = y$ and thus $f(f(x))f(x) = a = yf(x)$ for some $a \in E$, then there is no argument to suppose that also $xf(x) = a$, which could imply $y = x$ (see Definition 5.4 below).

Obviously, if $f = 1_G$, where $1_G$ denotes the identity homomorphism of MI-monoids, then (G2) and (G3) are trivially satisfied. Nevertheless, from $f(x)x = xx = a$ for some $a \in E$, we obtain $x \in E$ (by (M4)), and hence $G = E$. Thus, the inversion given by $f = 1_G$ can be defined on MI-monoids that contain just the pseudoidentities elements.

One could notice that $f(f(x)x) = f(x)x$. Hence, there are elements in $G$ for which $f(x) = x$. This motivates to introduce the following type of elements.

**Definition 5.3.** Let $G$ be an MI-monoid and $f$ be an inversion in $G$. An element $x \in G$ with $f(x) = x$ is said to be *$f$-symmetric in $G$*. The set of all $f$-symmetric elements in $G$ is denoted by $S_G^f$.

**Proposition 5.1.** *If $x \in G$ is $f$-symmetric, then $x \in E$ ,i.e., $S_G^f \subseteq E$.*

*Proof.* Let $x$ be $f$-symmetric in $G$. Then $f(x)x = xx = a$ for some $a \in E$. Due to (M4), we obtain $x \in E$. □

In the classical definition of group, the unique $f$-symmetric element is $e$ ($f$ is the inversion in $G$), i.e., $e = e^{-1} = f(e)$. The following statement shows that $f$-symmetric elements in $G$ are closed under the product of $G$.

**Proposition 5.2.** *If $x, y \in G$ are f-symmetric, then $xy$ is also f-symmetric.*

*Proof.* Let $x, y \in G$ be $f$-symmetric. Then $f(xy) = f(y)f(x) = yx = xy$, where the last equality follows from the fact that $x, y \in E$ according to Proposition 5.1. □

In contrast to the unique inverse which exists to each element in the groups, we can suppose the existence of more than one inversion in MI-monoids. The following proposition shows the relation between two inversions in an MI-monoid.

**Proposition 5.3.** *If $f$ and $g$ are inversions in $G$, then, for any $x \in G$, there exist $s \in S_G^f$ and $t \in S_G^g$ such that $f(x)t = g(x)s$, i.e., $f(x) \sim g(x)$.*

*Proof.* According to the definition of the inverse in $G$, we have $f(f(x)) = x = g(g(x))$, and if $s \in S_G^f$ and $t \in S_G^g$, then $f(s) = s \in S_G^f$ and $g(t) = t \in S_G^g$. From (G2) and Proposition 5.1, we have $f(x)x = xf(x) \in S_G^f$. Put $f(x)x = s$. Then,

$$g(s) = g(f(x)x) = g(x)g(f(x)) \in E$$

(see (HM3) of Definition 4.3) and

$$xg(s) = xg(x)g(f(x)).$$

Put $t = xg(x)$. Clearly, $t \in S_G^g$. From $xg(s) = tg(f(x))$ and applying (M5) and (G3), we obtain

$$g(x)s = sg(x) = g(xg(s)) = g(tg(f(x))) = f(x)g(t) = f(x)t,$$

which concludes the proof. □

Thus, one can see that two inversions are different up to symmetric pseudoidentities. If $S_G^f = S_G^g = \{e\}$, then $f$ and $g$ coincides and there is the unique inverse in $G$.

Obviously, an MI-group should be an MI-monoid equipped with an f-inversion. Nevertheless, this is not sufficient for a correct definition of MI-monoid, because the well-known and important property of groups, which is called the cancellation law, is not ensured here. Therefore, we propose the following definition of MI-groups, where the presumption on the cancellation law is included.

**Definition 5.4.** *A quadruplet $G = (G, \circ, E, f)$ is said to be* MI-group *if $(G, \circ, E)$ is an MI-monoid, $f$ is an inversion in $(G, \circ, E)$ and the following cancellation law holds:*

*(G4) if $xz = yz$ for $x, y, z \in G$ then $x = y$.*

*An MI-group $G$ is said to be* good *if $G$ is a good MI-monoid.*

Let us note that (G4) may be replaced by the following equivalent condition (G4'): If $xs = ys$ for $x, y \in G$ and $s \in S_G$ then $x = y$. In fact, clearly (G4) implies (G4'). Suppose that (G4') holds. If $xz = yz$ for $x, y, z \in G$, then $x(zz^{-1}) = y(zz^{-1})$, where $zz^{-1} \in S_G$. By (G4'), we obtain $x = y$ and (G4) is satisfied. A simple consequence of (G4') and (M5) is the fact that if $zx = zy$ for $x, y, z \in G$ then $x = y$.

Let us show several examples of MI-groups based on the MI-monoids introduced the previous section.

**Example 5.1.** *The algebraic structure for the summation of extensional fuzzy numbers* $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}}, f^0)$ *with inverse elements defined as* $f^0(x_S) = (-x)_S$ *is a good abelian additive MI-group.*

*The algebraic structure for the multiplication of extensional fuzzy numbers* $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}) \smallsetminus E^0_{\mathcal{S}}, \cdot, E^1_{\mathcal{S}}, f^1)$ *with inverse elements defined as* $f^1(x_S) = (1/x)_S$ *is a good abelian multiplicative MI-group.*

*In both examples, all pseudoidentity elements are symmetric, i.e.,* $E = S^f_G$.

**Example 5.2.** *Let us consider examples of structures with real intervals from Example 4.1. For the additive MI-monoid, we may define inverse elements as* $f^0([a,b]) = [-b,-a]$ *and we obtain an abelian MI-group that is not good.*

*For the multiplicative structure, we first omit* $E^0$ *elements from* $G$ *in order to obtain reduced the structure* $(G \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0)$. *Then, we may define inverse elements as* $f^1([a,b]) = [1/b, 1/a]$, *and we obtain an abelian MI-group that is not good.*

*It should be noted that, in these examples, not all pseudoidentities are symmetric, i.e.,* $S^f_G \subset E$. *Indeed, check e.g.* $[-3,2] \in E^0$ *and its inversion in the additive structure* $[-2,3] \in E^0$ *but* $[-3,2] \neq [-2,3]$.

**Example 5.3.** *Let* $H$ *be the abelian additive MI-monoid of fuzzy numbers from Example 4.2 and put* $f^0(g)(\alpha) = -g(\alpha)$ *for any* $g \in H$ *and* $\alpha \in [0,1]$, *where* $-g(\alpha) = [-g(\alpha)^+, -g(\alpha)^-]$ *for* $g(\alpha) = [g(\alpha)^-, g(\alpha)^+]$. *It is easy to see that* $f^0(g) \in H$, *and* $f^0$ *is an inversion in* $H$. *Hence,* $(H, +, E^0, f^0)$ *is an abelian additive MI-group.*

*Let* $(H \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0)$ *be the multiplicative MI-monoid from Example 4.11 and put* $f^1(g)(\alpha) = g(\alpha)^{-1}$ *for any* $g \in H \smallsetminus E^0$ *and* $\alpha \in [0,1]$, *where* $g(\alpha)^{-1} = [1/g(\alpha)^+, 1/g(\alpha)^-]$ *for* $g(\alpha) = [g(\alpha)^-, g(\alpha)^+]$. *Then,* $(H \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0, f^1)$ *is an abelian multiplicative MI-group.*

**Example 5.4.** *Let* $L$ *be the additive MI-monoid from Example 4.4 and put* $f^0(u) = (-u_i^+, -\delta u_i^+, -u_i^-, -\delta u_i^-)_{i=1,\dots,N}$ *for any* $u \in L$. *Then,* $(L, +, E^0, f^0)$ *is an abelian additive MI-group.*

*One may check that* $(L \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0)$ *is an MI-submonoid of* $(L, \cdot, E^1)$. *If we define*

$$f^1(u) = (1/u_i^+, -1/(u_i^+)^2 \delta u_i^+, 1/u_i^-, -1/(u_i^-)^2 \delta u_i^-)_{i=1,\dots,N}$$

*for any* $u \in L \smallsetminus E^0$, *then* $(L \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0, f^1)$ *is an abelian multiplicative MI-group.*

**Example 5.5.** *Let* $M$ *be the additive MI-monoid from Example 4.5 and put* $f^0(A) = -A = \{-a \mid a \in A\}$ *for any* $A \in M$. *Then,* $(M, \oplus, E^0, f^0)$ *is an abelian* *additive MI-group of convex sets in* $\mathbb{R}^n$.[6] *Analogously, one may define an abelian multiplicative MI-group* $N$ *of product convex sets.*

**Convention:** Where no confusion can arise, for the sake of readability of the text, we will omit $f$ in $(G, \circ, E, f)$ and standardly write $x^{-1}$ instead of $f(x)$ for the inversion in $G$. Moreover, we will use $S_G$ instead of $S_G^f$ to denote the set of all f-symmetric elements.[7]

*5.2. Homomorphisms and MI-subgroups*

A natural generalization of homomorphism of groups under which we can study the important properties of MI-groups is as follows.

**Definition 5.5.** Let $G$ and $H$ be MI-groups. A mapping $f : G \to H$ is a *homomorphism of MI-groups* if

(HG1) $f(xy) = f(x)f(y)$ for all $x, y \in G$,

(HG2) $f(x) \in E_H$ for all $x \in E_G$,

(HG3) $f(x^{-1}) = (f(x))^{-1}$ for all $x \in G$,

where $E_G$ and $E_H$ denote the set of all pseudoidentities of $G$ and $H$, respectively.

All concepts as monomorphism, epimorphism etc. of MI-groups have the same meaning as for MI-monoids. Moreover, if $f : G \to H$ and $g : H \to P$, then $g \circ f$ is a homomorphism of $G$ to $P$. In fact, if $x \in G$, then $f(x^{-1}) = f(x)^{-1}$ and $g(f(x)^{-1}) = g(f(x))^{-1}$. Hence, we obtain $g(f(x^{-1})) = g(f(x))^{-1}$ and (HG3) is satisfied. Similarly, one can prove the remaining conditions.

**Example 5.6.** *The isomorphism* $f$ *of the MI-monoids* $N$ *and* $M$ *defined in Example 4.7 is also an isomorphism of the MI-groups* $N$ *and* $M$ *(see Example 5.5).*

The following proposition states that at most one MI-group up to isomorphism can be defined over a given MI-monoid.

**Proposition 5.4.** *Let* $G_f = (G, \circ, E, f)$ *and* $G_g = (G, \circ, E, g)$ *be MI-groups, where* $f$ *and* $g$ *are two inversions in the MI-monoid* $(G, \circ, E)$. *Then* $h(x) = g(f(x))$ *for any* $x \in G$ *defines an isomorphism of* $G_f$ *and* $G_g$.

*Proof.* First, we will show that $h$ is a homomorphism of MI-groups. If $x, y \in G$, then

$$h(xy) = g(f(xy)) = g(f(y)f(x)) = g(f(x))g(f(y)) = h(x)h(y)$$

[6] Note that $-A$ is called *reflection of A* and it is used to define *erosion* in the mathematical morphology (see e.g. [21]).

[7] Note that if two MI-groups are defined on the same MI-monoid, then both MI-groups are isomorphic (see Example 5.4). Therefore, the presumption of a particular inversion in the MI-group is not so important for the development of the MI-group theory.

and (HG1) is proved. If $a \in E$, then $h(a) = g(f(a))$. Since $f(a) \in E$, then also $g(f(a)) \in E$ and (HG2) holds. Since $f(f(x)) = x$ by (G3), then $h(f(x)) = g(f(f(x))) = g(x)$, whence (HG3) is fulfilled. One may simply check that $h$ is a bijective mapping and $h(E) \subseteq E$. If $b \in E$, then, putting $a = h(g(b))$, we obtain $a \in E$ and $h(a) = b$. Hence, $h(E) = E$, and $h$ is an isomorphism of MI-groups. □

**Proposition 5.5.** *Let $f : G \to H$ be a homomorphism of $MI$-groups. Then*

*(i)* $f(S_G) \subseteq S_H$.

*(ii)* $f(e_G) = e_H$.

*Proof.* (i) Since $s = s^{-1} \in S_G$, then $f(s) = f(s^{-1}) = f(s)^{-1}$, whence $f(s) \in S_H$.

(ii) If $x \in G$, then $f(xe_G) = f(e_G)f(x) = e_H f(x)$. By the cancellation law, which holds in $H$, we obtain $f(e_G) = e_H$. □

A very important concept in the group theory is the kernel of a homomorphism. For MI-groups, we propose the following definition.

**Definition 5.6.** Let $f : G \to H$ be a homomorphism of $MI$-groups. The *$MI$-kernel* of $f$, denoted by MI-Ker $f$, is the set $\{x \in G \mid f(x) \in S_H\}$.

A characterization of monomorphism and isomorphism of MI-groups is provided in the following theorem. Denote $1_G$ the identity homomorphism of MI-groups.

**Theorem 5.6.** *Let $f : G \to H$ be a homomorphism of $MI$-groups. Then,*

*(i) $f$ is a monomorphism if and only if* MI-Ker $f = S_G$ *and $f$ restricted to $S_G$ is an injective mapping,*

*(ii) $f$ is an isomorphism if and only if there is a homomorphism $f^{-1} : H \to G$ such that $ff^{-1} = 1_H$ and $f^{-1}f = 1_G$.*

*Proof.* (i) Let $f$ be a monomorphism. Obviously, $f$ restricted to $S_G$ is an injective mapping. We will prove that MI-Ker $f = S_G$. Since $f(S_G) \subseteq S_H$, then $S_G \subseteq$ MI-Ker $f$. If $x \in$ MI-Ker $f$, then $f(x) \in S_H$. Therefore, $f(x) = f(x)^{-1} = f(x^{-1})$, which implies $x = x^{-1}$, since $f$ is a monomorphism. Hence, $x \in S_G$ and MI-Ker $f = S_G$.

Let MI-Ker $f = S_G$ and $f$ restricted to $S_G$ is injective. If $f(x) = f(y)$ for $x, y \in G$, then $f(xy^{-1}) = f(x)f(y)^{-1} = f(y)f(y)^{-1} = s$ for some $s \in S_H$. Since MI-Ker $f = S_G$, then there exists $t \in S_G$ such that $t = xy^{-1}$. Put $r = y^{-1}y \in S_G$. Then, $xr = xy^{-1}y = ty = yt$ (using (M5)), which implies $f(x)f(r) = f(xr) = f(yt) = f(y)f(t)$. Since $f(x) = f(y)$, we obtain $f(r) = f(t)$ by the cancellation law. Using the presumption on the injectivity of $f$ restricted to $S_G$, we obtain $r = t$. Hence, $xr = yt = yr$, where $r \in S_G$, and, by the cancellation law, we obtain $x = y$, which implies the injectivity of $f$.

(ii) Let $f : G \to H$ be an isomorphism and define $f^{-1} : H \to G$ in such a way that $ff^{-1} = 1_H$ and simultaneously $f^{-1}f = 1_G$ hold. We will prove that $f^{-1}$ is a homomorphism. If $x, y \in H$, then $f(f^{-1}(xy)) = xy$ and $f(f^{-1}(x)f^{-1}(y)) = f(f^{-1}(x))f(f^{-1}(y)) = xy$. Hence, $f(f^{-1}(xy)) = f(f^{-1}(x)f^{-1}(y))$. Since $f$ is bijective, then $f^{-1}(xy) = f^{-1}(x)f^{-1}(y)$ and (HG1) is satisfied. Since $f$ is an isomorphism, then $f(E_G) = E_H$, whence $f^{-1}(E_H) = E_G$ and (HG2) holds. If $x \in H$, then $f(f^{-1}(x^{-1})) = x^{-1}$ and $f((f^{-1}(x))^{-1}) = (ff^{-1}(x))^{-1} = x^{-1}$. Hence, $f(f^{-1}(x^{-1})) = f((f^{-1}(x))^{-1})$. Since $f$ is bijective, then $f^{-1}(x^{-1}) = (f^{-1}(x))^{-1}$ and (HG3) holds. If $x, y \in S_H$ and $f^{-1}(x) = f^{-1}(y)$, then $ff^{-1}(x) = ff^{-1}(y)$, whence $x = y$. Thus, $f^{-1}$ is a homomorphism.

If $f^{-1} : H \to G$ is a homomorphism such that $ff^{-1} = 1_H$ and $f^{-1}f = 1_G$, then $f$ is a bijection. Now, by the definition of isomorphism of MI-groups, it is sufficient to prove that $f(E_G) = E_H$. Using (i) of Proposition 5.5, we have $f(E_G) \subseteq E_H = ff^{-1}(E_H) \subseteq f(E_G)$, which implies $f(E_G) = E_H$, and thus $f$ is an isomorphism. □

The concept of MI-subgroup of an MI-group is naturally defined as follows.

**Definition 5.7.** Let $G = (G, \circ, E)$ be an MI-group, $H \subseteq G$ and $F \subseteq E$ be non-empty sets. If $H = (H, \circ, F)$ is itself MI-group under the product and the inversion of $G$, then $H$ is said to be an *MI-subgroup* of $G$. This is denoted by $H \leq G$.

**Example 5.7.** *Let $G$ be an MI-group. Clearly, $G$ and $(\{e\}, \circ, \{e\}, 1_{\{e\}})$ are the trivial MI-subgroups of $G$. Moreover, $(S_G, \circ, S_G, 1_{S_G})$ is an MI-subgroup. This MI-subgroup can be understood as a generalization of the trivial group $(\{e\}, \circ, \{e\})$, because the symmetric elements perfectly simulate the properties of the identity elements as one could notice above.*

The following proposition shows the necessary and sufficient condition under which subsets of $G$ and $F$ specify an MI-subgroup of $G$.

**Proposition 5.7.** *Let $H \subseteq G$ and $F \subseteq E$ be non-empty subsets of an MI-subgroup of $G$. Then $H \leq G$ if and only if, for all $x, y \in H$ and $a, b \in F$, the following holds:*

*(i)* $xy^{-1} \in H$,

*(ii)* $e \in F$,

*(iii)* $ab \in F$,

*(iv)* $xx = a$ *then* $x \in F$,

*(v)* $a^{-1} \in F$,

*(vi)* $x^{-1}x \in F$.

*Proof.* $(\Rightarrow)$ Trivial. $(\Leftarrow)$ Since $e \in F$, then, by (i), $ex^{-1} = x^{-1} \in H$ for any $x \in H$. Moreover, $(x^{-1})^{-1} = x$ is preserved by the restriction of the inversion to $H$. If $x, y \in H$, then $y^{-1} \in H$ and $x(y^{-1})^{-1} = xy \in H$. Hence and from (ii)-(iv), we obtain that $H$ is an MI-submonoid of $G$ (see Proposition 4.1). Let $f : H \to H^{\mathrm{op}}$

be given by $f(x) = x^{-1}$. We will prove that $f$ is the inversion in $H$.[8] Clearly, the satisfaction of (HM1) and (HM2) by $f$ immediately follows from the fact that $f(x) = x^{-1}$ is the restriction of the inversion in $G$. Moreover, due to (v), $f$ also satisfies (HM3), and thus $f$ is a monoidal homomorphism. The injectivity of $f$ is preserved by the inversion in $G$ and the surjectivity is a simple consequence of $f(f(x)) = (x^{-1})^{-1} = x$. Using (v), we simply obtain $f(F) = F$, and thus $f$ is a monoidal isomorphism. The satisfaction of $(G2)$ and $(G3)$ by $f$ is preserved by the inversion in $G$. By (vi), $f$ satisfies (G1), and $f$ is an inversion in $H$. Thus, the restriction of the inversion in $G$ to $H$ is again an inversion. If $x, y, z \in H$ such that $xz = yz$, then this equality is also true in $G$. Hence, $x = y$, which implies (G4), and $H$ is an MI-subgroup of $G$. □

Analogously to the definition of canonical MI-submonoid, we can define a canonical MI-subgroup, which is determined only by one subset.

**Definition 5.8.** An MI-subgroup $(H, \circ, F)$ of an MI-group $(G, \circ, E)$ is said to be *canonical* if $F = H \cap E$.

The following proposition shows the necessary and sufficient condition for $H \subseteq G$ to define a canonical MI-submonoid, which is significantly simpler than that provided in Proposition 5.7 (cf. [9]).

**Proposition 5.8.** *Let $G = (G, \circ, E)$ be an MI-group and $H \subseteq G$ be a non-empty subset. Then, $H = (H, \circ, H \cap E)$ is an MI-subgroup of $G$ if and only if $e \in H$ and $xy^{-1} \in H$ for all $x, y \in H$.*

*Proof.* $(\Rightarrow)$ Trivial. $(\Leftarrow)$ We prove (ii)-(vi) of Proposition 5.7. If $x \in H$, then $x^{-1} = ex^{-1} \in H$. Hence, $xy = x(y^{-1})^{-1} \in H$ for any $x, y \in H$. Put $F = H \cap E$. Obviously, $e \in F$ and (ii) is true. If $a, b \in F$, then $ab \in H$ and also $ab \in E$. Hence, $ab \in F$ and (iii) is proved. If $xx = a$ for $x \in H$ and $a \in F$, then $x \in E$ and thus $x \in F$. Hence, (iv) is true. If $a \in F$, then $a \in E$ and $a^{-1} \in E$. Since also $a^{-1} \in H$, then $a^{-1} \in F$ and (v) is proved. If $x \in H$, then $xx^{-1} \in H$ and also $xx^{-1} \in E$. Hence, $xx^{-1} \in F$ and (vi) is satisfied. □

Let us show several examples of MI-subgroups of MI-groups introduced in the previous subsection.

**Example 5.8.** *Let $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}}, f^0)$ be the abelian additive MI-group of fuzzy numbers that are extensional w.r.t. a similarity from a system of similarity relations $\mathcal{S}$ introduced in Example 5.1. Then, $(\mathcal{F}_{\mathcal{T}}(\mathbb{R}), +, E^0_{\mathcal{T}}, f^0)$, where $(\mathcal{F}_{\mathcal{T}}(\mathbb{R}), +, E^0_{\mathcal{T}})$ is the canonical MI-submonoid defined in Example 4.10 and $f^0$ denotes the restriction of the inversion of $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$ to $\mathcal{F}_{\mathcal{T}}(\mathbb{R})$,[9] is a canonical MI-subgroup of $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$.*

[8] Note that, although the restriction of an inversion to an MI-submonoid $H$ preserves many of the properties of the original inversion, it need not be an inversion in $H$ in general.

[9] Analogously to the restriction of + to $\mathcal{F}_{\mathcal{T}}(\mathbb{R})$, we use the same denotation for the inversion $f^0$ in $\mathcal{F}_{\mathcal{T}}(\mathbb{R})$ and believe that no confusion can arise.

**Example 5.9.** *Let* $(H \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0, f^1)$ *be the abelian multiplicative MI-group of non-zero fuzzy numbers defined in Example 5.3. Define the set of all positive fuzzy numbers as*

$$H_{>0} = \{g \mid g \in H \smallsetminus E^0 \ \& \ g(0) \subseteq (0, \infty)\}.$$

*Then, one may simply check that* $(H_{>0}, \cdot, S_{H \smallsetminus E^0}, f^1)$*, where* $f^1$ *denotes the restriction of the inversion of* $H \smallsetminus E^0$ *to* $H_{>0}$ *and* $S_{H \smallsetminus E^0}$ *is the set of all* $f^1$*-symmetric elements in* $H \smallsetminus E^0$*, is an MI-subgroup of all positive fuzzy numbers of* $H \smallsetminus E^0$*, which is not canonical.*

**Example 5.10.** *Let* $\mathbb{Z}$ *be the set of integers and* $M_{\mathbb{Z}}$ *be the set of all convex sets in* $\mathbb{Z}^n$*. In Example 5.5, we defined the abelian MI-group* $(M, \oplus, E^0, f^0)$*. It is easy to see that* $(M_{\mathbb{Z}}, \oplus, E^0 \cap M_{\mathbb{Z}}, f^0)$ *is a canonical MI-subgroup of* $M$*.*

## 6. MI-fields

An original motivation of our analysis of properties of arithmetical operations with fuzzy numbers was to develop working arithmetics of fuzzy numbers. Our idea was to introduce a structure of fuzzy numbers with appropriate arithmetical operations that is very close to the field of real numbers. The essential step in this concept was a generalization of the group structure leading to the notion of an MI-group. Now, we are ready to introduce a generalization of the field structure.

**Definition 6.1.** An algebra $(G, +, \cdot, E^0, E^1, f^0, f^1)$ is said to be an *MI-field* if

(F1) $(G, +, E^0, f^0)$ is an abelian additive MI-group,

(F2) $(G \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0, f^1)$ is an abelian multiplicative MI-group,

(F3) for any $x, y, z \in G$ the following *distributive law*

$$x(y + z) = xy + xz$$

holds.

One can see that the definition of MI-field is nearly the same as the definition of field, only the abelian additive and multiplicative groups are replaced by the corresponding abelian MI-groups. Analogously to the field, the multiplicative MI-group is defined on the support of the additive group without zero-like elements, since the set $E^0$ specifies the elements to which we cannot define the multiplicative inversion. Let us show two examples of MI-fields.

**Example 6.1.** *For any system of similarity relations* $\mathcal{S}$ *the set of extensional fuzzy numbers equipped with* $+$ *and* $\cdot$ *defined in Section 3 form a structure*

$$(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, \cdot, E^0_{\mathcal{S}}, E^1_{\mathcal{S}}, f^0, f^1)$$

*that is an MI-field.*

**Example 6.2.** *Let* $K = \{(x,d) \mid x \in \mathbb{R},\ d \in I\}$*, where* $I$ *denotes an interval of non-negative real numbers. Note that each pair* $(a,b)$ *represents the interval* $[x-d, x+d]$*. Let us define the operations of addition and multiplication as follows*

$$(x,d) + (y,e) = (x+y, \max(d,e)),$$
$$(x,d) \cdot (y,e) = (x \cdot y, \max(d,e)).$$

*Put*

$$E^0 = \{(x,d) \mid (x,d) \in K,\ |x| \le d\},$$
$$E^1 = \{(x,d) \mid (x,d) \in K,\ |x-1| \le d\}.$$

*One can simply prove that* $(K,+,E^0,f^0)$*, where* $f^0(x,d) = (-x,d)$*, is an abelian additive MI-monoid, and* $(K \smallsetminus E^0, \cdot, E^1 \smallsetminus E^0, f^1)$*, where* $f^1(x,d) = (1/x, b)$ *is an abelian multiplicative MI-monoid (cf., Examples 4.1 and 5.2). Since*

$$((x,d)+(y,e))(z,f) = (x+y, \max(d,e))(z,f) = (xz+yz, \max(d,e,f)) = \\ (xz, \max(d,f)) + (yz, \max(e,f)) = (x,d)(z,f) + (y,e)(z,f),$$

*we obtain that* $(K,+,\cdot,E^0,E^1,f^0,f^1)$ *is an MI-field, which defines an arithmetics on the real intervals. It should be noted that this MI-field can serve as an* $\alpha$*-cut representation of the MI-field* $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$ *defined in the previous example for systems of continuous shift-invariant similarity relations.*[10]

As it has been discussed in Introduction, in general, only the subdistributivity is satisfied for the arithmetical operations for intervals and fuzzy numbers, i.e., $x(y+z) \subseteq xy + xz$. Nevertheless, it is interesting that for any $x, y, z$ we may find a pseudoidentity $a$ such that $a + x(y+z) = xy + xz$.[11] This motivate us to define a weaker algebraic structure to the MI-field.

**Definition 6.2.** An algebra $(G,+,\cdot,E^0,E^1,f^0,f^1)$ is said to be a *weak MI-field*, if (F1) and (F2) hold and further if

(WF3) for any $x,y,z \in G$ there exists $a \in E^0$ such that the following *weak distributive law*

$$a + x(y+z) = xy + xz$$

holds.

[10] Note that if a similarity relation $S$ is shift-invariant, then $S(x,x-y) = S(x,x+y)$. In fact, $S(x,x-y) = S(0,-y) = S(y,0) = S(0,y) = S(x,x+y)$. Then, the extensional fuzzy numbers are symmetrically distributed around the central point $x$ (see Fig. 2 on page 11).

[11] By (9), we may express this equality as $x(y+z) \sim_+ xy + xz$, where $\sim_+$ is the congruence on $G$ with respect to $+$.

**Example 6.3.** *The sets of real intervals, fuzzy numbers, parametric representations equipped with the arithmetical operations defined in Examples 4.1-4.4 form weak MI-fields. Note that $M$ from Example 4.5 does not form a weak MI-field, since $M \setminus E^0$ contains also sets that are not product convex.*

It should be noted that the distribution law is very important for the dealing with real numbers, e.g., mathematical formulas can be simplified or changed for a comparison. Thus, if one wants to use the benefit of the distribution law for his work with fuzzy numbers, he has to consider such types of fuzzy numbers and arithmetics defined on them which leads to an MI-field. A natural class of fuzzy numbers satisfying this presumption consists of the extensional fuzzy numbers which are constructed using similarity relations, and their arithmetics do not stem from the Zadeh's extension principle (see Example 6.1). On the other hand, the common approaches can be well described in the weaker form of MI-field (a generalization of MI-field), which enables to investigate their properties in an abstract form and developed a general theory of fuzzy numbers. It is easy to see that using the concepts introduced for the MI-groups, we may simply define analogous concepts for (weak) MI-fields, e.g., (canonical) MI-subfield, homomorphism of (weak) MI-fields, etc. Nevertheless, such development of MI-field theory is beyond the scope of this contribution and it should be a subject of an individual paper with some theoretical results on the MI-fields. The same holds for the development of the MI-group theory.

## 7. Final remarks

### *7.1. Some reflections on "Computation over fuzzy quantities"*

It has been a well-known fact [16] that if we consider a set of standard (R-L, L-U or most usual definitions) fuzzy numbers $\mathcal{F}(\mathbb{R})$ and operations $+, \cdot$ defined with help of the Zadeh's extensional principle, we get the following two monoids

$$(\mathcal{F}(\mathbb{R}), +, \tilde{0}) \text{ and } (\mathcal{F}(\mathbb{R}), \cdot, \tilde{1}).$$

But due to the singleton identity elements $\tilde{0}$ and $\tilde{1}$ we are not able to determine inverse elements and thus, we cannot proceed further towards a group structure or even towards a field structure. This shows the importance of introducing the MI-monoid, i.e., a structure that generalizes a monoid. Although it is not a problem to get a monoid with the previous approaches based on the extensional principle, the MI-monoid allows us to continue in the development of more advanced structures.

Our work introduced in this paper is mainly motivated by the lack of a field-like structure for the so far existing arithmetics of fuzzy numbers. Up to our best knowledge most of the articles focusing on arithmetics of fuzzy numbers were rather application, technically or computationally oriented than investigating algebraic properties. However, there are interesting and very valuable exceptions. In our opinion, we may highlight one of such exceptions that deserves a special focus among all other authors who also devoted their research to algebraic backgrounds of arithmetics of fuzzy numbers. It is the work of M. Mareš that was published in many articles in late 80s and early 90s and that is mostly collected in the book *"Computation over fuzzy quantities"* [16].

Obviously, although our approach and the above recalled approach to computation over fuzzy quantities stem from completely different roots and thus, lead to different results, there is a non-trivial overlap of both theories.

In [16] we may find out that the author also finds the non-existence of inverse elements as one of two crucial problems. The other one is – not surprisingly – the fact that the distributive law does not hold. And as in our case, also M. Mareš states:

*"it would be absurd to expect that arithmetic manipulation with vague fuzzy numbers will result in a crisp number, either* **0** *or* **1***".*

The way how the above mentioned idea is employed in [16] is little bit different than in our approach. The author first defines so called *y-symmetric fuzzy quantity* $a \in \mathcal{F}(\mathbb{R})$ as a fuzzy set for which the following holds:

$$a(y + x) = a(y - x), \quad x \in \mathbb{R} \tag{10}$$

and denotes the set of $y$-symmetric fuzzy quantities as $\mathbb{S}_y$. Set of symmetric fuzzy quantities is a union of all the $y$-symmetric ones over the whole real domain. Obviously, all the extensional fuzzy sets involved in our approach are symmetric as well, which comes directly from the definition of the similarity relation. The important point in the approach provided in [16] is that the so called *additive equivalence* is defined. Two fuzzy quantities $a, b \in \mathcal{F}(\mathbb{R})$ are said to be *additively equivalent*[12] (denoted by $a \sim_+ b$) if there exist 0-symmetric quantities $s_1, s_2 \in \mathbb{S}_0$ such that

$$a + s_1 \sim_+ b + s_2. \tag{11}$$

In other words, although the approach based on the Zadeh's extensional principle (which is still kept in [16]) does enforce crisp identity elements, the author defines a sort of tolerance of given equalities up to some 0-symmetric fuzzy quantities. And in our approach, these 0-symmetric fuzzy quantities would be elements of $S_G$ and consequently, due to Lemma 5.1, they would be elements of $E^0_{\mathcal{S}}$, i.e., pseudoidentities.

The crucial difference consists in the fact that 0-symmetric elements from (11) may be arbitrarily wide. While in our approach, that does not involve the extensional principle and suggests a completely different calculus, the width (vagueness) of pseudoidentity elements is given by the width (vagueness) of the original elements.

Note, that as our approach uses the similarity relation as a crucial conceptual object, the same may be said about the approach by M. Mareš. The difference lies in the fact that we use it for the conceptual construction of extensional fuzzy

[12] Multiplicative equivalence is defined analogously.

numbers, i.e., to a construction of objects on which we develop the arithmetic. In the case of [16], the author used the similarity relation as a generalization of the classical equality in order to weakened the algebraic properties. Particularly, it can be shown that $\sim_+$ is a $*$-similarity relation for $*$ being the Gödel t-norm. Thus, $a = b$ implies $a \sim_+ b$ and the monoid structure axioms are also valid in their weakened form

$$a + b \sim_+ b + a,$$
$$a + (b + c) \sim_+ (a + b) + c,$$
$$a + \mathbf{0} \sim_+ a.$$

Furthermore, in this weakened form, also the other desirable properties hold:

$$a + s \sim_+ a, \quad s \in \mathbb{S}_0,$$
$$a + (-a) \sim_+ \mathbf{0}$$

and thus, a sort of group-like structure is obtained.

Although the whole calculus is still based on the extensional principle and thus, it is different to the one we suggested, the overlap of crucial ideas is obvious. No matter that our approach provides some unquestionable advantages such as: computational simplicity; limitation of undesirable widening of fuzzy numbers; preserving the pre-given shape of fuzzy numbers; a field-like (MI-field) structure, i.e., the distributivity law and the intuitive way of calculating with extensional fuzzy numbers, we are very grateful for the existence of the work by M. Mareš that was mainly gathered in [16] and then followed e.g., in [17] that motivated us.

### *7.2. Conclusions*

We have recalled standard approaches to arithmetics of fuzzy numbers stemming from the Zadeh's extensional principle and we have recalled the drawbacks of such approaches. We have listed the crucial points for such drawbacks: the fact that fuzzy numbers do not conceptually stem from numbers but from sets and the fact that the Zadeh's extensional principle enforces that only the singletons $\tilde{0}, \tilde{1}$ may serve as the identity elements.

With these observations and motivated by the importance of arithmetics of vaguely defined numbers, we have introduced a novel approach that tries to mimic the arithmetic of real numbers from the very roots. It means that we have started from the conceptually clear definition of an extensional fuzzy number and introduced arithmetic operations on such objects that obeys the human's intuition. Further, we have shown that with this approach we get an algebraic structure that is isomorphic to the field of real numbers.

Motivated by this and by the natural requirement to generalize the structure for more similarity relations that model tolerance measures, we came up with a generalization of such arithmetic for whole (even uncountable) systems of similarity relations. We have demonstrated very promising algebraic properties that are very similar to the highly desirable properties of groups and fields with the only difference, the non-uniqueness of identity elements in such structures. Moreover, we have shown that if the system of similarities is appropriately chosen, e.g., as in case of the embedded systems of shift-invariant similarities, the suggested arithmetic provides us with unquestionable practical advantages including the computational efficiency.

In order to formally describe an algebraic structure for the introduced framework of such arithmetics, we have presented novel many identities algebras, namely MI-monoids, MI-groups and finally MI-fields. These MI-algebras capture the required properties due to a set of (pseudo)identity elements. On many examples, we have shown, that MI-algebras do not formalize only the suggested arithmetic of extensional fuzzy numbers but also many well known arithmetics of intervals, fuzzy numbers or sets of real vectors can be viewed from the MI-algebras perspective.

This fact enables us to investigate abstractly the properties of structures of various types sets expressing the imprecisely defined (not-necessary real) numbers. Moreover, we think that the development of a theory of MI-algebraic structures should be interesting from the theoretical as well as practical point of view. In this paper, we presented only several chosen results of the possible algebraic theory to demonstrate its functionality and a close relation to the standard results from the theory of algebraic structures that serve as an algebraic backgrounds for the arithmetic of reals. It should be noted that the results for MI-algebras cannot be obtained by a trivial reconstruction of known proofs for algebras and these proofs have to be designed by a novel consideration. Thus, the development of a theory of MI-algebras becomes a challenge for our future research.

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