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A Graded Approach to Cardinal Theory of Finite Fuzzy Sets, Part I: Graded Equipollence ☆

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Abstract

In this article, we propose a fuzzy class relation assigning to each pair of finite fuzzy sets a degree to which they are equipollent, which indicates that they have the same number of elements. The concepts of fuzzy sets and fuzzy classes in the class of all countable sets are introduced, and several standard relations and constructions, such as the fuzzy power set and exponentiation, are defined. A functional approach to the cardinal theory of finite fuzzy sets based on graded equipollence is shown, and a relation to generalized cardinals and Wygralak's cardinal theory of finite fuzzy sets defined over triangular norms is demonstrated.

Key words: residuated-dually residuated lattice, finite fuzzy sets, generalized cardinal, graded equipollence, fuzzy cardinality

1. Introduction

In the classical set theory, we can recognize two approaches to the cardinality of sets. The first one is a functional approach that uses one-to-one correspondences between sets to compare their sizes. More precisely, we say that two sets a and b are *equipollent* (*equipotent, equivalent, bijective* or have *the same cardinality*) and write $a \sim b$ if there exists a one-to-one mapping of a onto b (see, e.g., [11, 12]). The relation "to be equipollent" is an equivalence on the class of all sets and is called *equipollence* (or *equipotence, equinumerosity* etc.). The second approach is based on the concept of cardinal number expressing the power of a set. A cardinal number of a, denoted by $|a|$, is usually defined as

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the class of all sets equipollent to a or by the initial ordinal number of this class (supposing the axiom of choice).[1]

The two approaches to the cardinality for fuzzy sets mentioned above have also been studied in the fuzzy set theory. Nevertheless, the attention of the fuzzy audience to a cardinal theory for fuzzy sets is not significant in comparison with the attention paid to cardinal theory of sets, although we have to agree with M. Wygralak, who wrote in [24]:

> If one likes to introduce the notion of cardinality for fuzzy sets, the main difficulty and difference in comparison with sets lies in the graduation of membership of the elements of a universe **M** in a fuzzy set. Consequently, counting and cardinal calculus under fuzziness become a challenging task which is generally more advanced and complicated than in the case of sets. Motivations for undertaking that task follow from both mathematical theory and multiple applications. Cardinality seems to be one of the most fascinating and enigmatic mathematical aspects of fuzzy sets.

The equipollence of (finite) fuzzy sets has been investigated primarily by S. Gottwald [5, 6] and M. Wygralak [21, 22, 24, 25] (see also [14]). S. Gottwald proposed a graded approach to the equipollence of fuzzy sets defined using the uniqueness of fuzzy mappings in his set theory for fuzzy sets of higher level. The equivalence classes form *fuzzy cardinals*. Note that fuzzy mappings are defined here as special fuzzy relations. Because this approach is purely theoretical, further substantial research in this direction has not been realized yet. An approach to the non-graded equipollence of fuzzy sets was proposed by M. Wygralak [21] (see also [22]).[2] The equipollence of fuzzy sets is defined by the following equality based on α-cuts of fuzzy sets

$$\bigvee\{\alpha \in [0,1] \mid |A_\alpha| \geq i\} = \bigvee\{\alpha \in [0,1] \mid |B_\alpha| \geq i\} \tag{1}$$

that has to be satisfied for any cardinal number i. One may see that this definition becomes rather trivial for finite fuzzy sets, i.e., fuzzy sets with finite supports, and leads to a mapping between universes under which one fuzzy set is an image of the second fuzzy set. In [24] (see also [25] and [4]), M. Wygralak developed the cardinal theory of finite fuzzy sets defined over triangular norms. To ensure the consistency of the cardinal theory, he introduced an appropriate concept of equipollence of finite fuzzy sets that modifies (1) and respects the requirements of the application of triangular norms. Additionally, a graded generalization of equipollence suggesting that fuzzy sets have approximately the same number of elements has been noted by M. Wygralak in [21, 22, 24, 25], but

[1] In this case, when cardinal numbers are expressed as classes, one has to admit classes as objects of the considered theory. For example, this definition is not admissible in Zermelo-Fraenkel set theory with the axiom of choice, but it does work in type theory.

[2] Specifically, M. Wygralak defined the equipollence of vaguely defined objects, which includes fuzzy sets.

substantial development of cardinal theory based on this type of equipollence has not been realized yet. In [10] (see also [8], [9]), we proposed a new approach to the equipollence of fuzzy sets over a universe of sets (e.g., a universe of all finite or countable sets, all sets, or a Grothendieck universe). Analogously to Gottwald's approach, a graded equipollence is considered, where the degrees of being equipollent are obtained more simply than in Gottwald's approach. More precisely, this approach is based on a graded one-to-one correspondence between fuzzy sets, where the crisp mappings between universes are considered.

The aim of this first article is to develop a cardinal theory of finite fuzzy sets based on the concept of graded equipollence.The definition of graded equipollence for finite fuzzy sets generalizes the definition proposed in [10] (see also [8, 9]) in the sense of the use of the multiplication of a residuated lattice as an alternative operation to the infimum. We shall show that the graded equipollence with respect to $\odot \in \{\wedge, \otimes\}$ is a fuzzy similarity class relation on the class of all finite fuzzy sets, and well-known statements of the cardinal theory of sets (including the Cantor-Bernstein theorem and the Cantor theorem stating the different cardinalities for sets and their power sets) can mostly be proved in a graded design, where if-then formulas are replaced by the inequalities between the degrees in which the antecedent and consequent are satisfied. For example, the classical formula

$$\text{if } a \sim b \ \text{ and } \ c \sim d \ \text{ then } \ a \times c \sim b \times d,$$

is expressed by the inequality

$$[A \sim^{\odot} B] \otimes [C \sim^{\odot} D] \leq [A \times C \sim^{\odot} B \times D],$$

where $[A \sim^{\odot} B]$ expresses the degree to which fuzzy sets A and B are equipollent.[3] An interesting question is whether the classes of equipollent fuzzy sets that are evidently proper may be represented by appropriate fuzzy sets, that means by generalized cardinals. We shall prove that a non-graded correspondence between generalized cardinals and "extended classes" of equipollent fuzzy sets exists if one supposes that an algebraic structure for membership degrees is linearly ordered and satisfies the conditional cancellation law (see Theorem 7.8). A full representation of classes of all equipollent fuzzy sets by generalized cardinals exists only if one restricts ourselves to the case of $\sim^{\wedge}$ (see Theorem 7.9).

This paper is structured as follows. In the next section, we propose a residuated-dually residuated lattice as an algebraical structure of membership degrees of fuzzy sets that is motivated by requirements of the second part of our contribution devoted to fuzzy c-measures and their relation to graded equipollence [7]. A theory of fuzzy sets over countable universes of discourse is described in Section 3. The concept of generalized cardinal is introduced in Section 4. The

[3] More formally, one may write $A \sim^{\odot} B \;\&\; C \sim^{\odot} D \Rightarrow A \times C \sim^{\odot} B \times D$, where the symbols of $\sim^{\odot}$, &, and $\Rightarrow$ are interpreted as the graded equipollence, multiplication $\otimes$, and residuum $\rightarrow$, respectively. Nevertheless, for the purpose of this contribution, we use the form of inequalities.

graded equipollence of finite fuzzy sets is proposed, and some of their properties are analyzed in Section 5. In Section 6, the functional approach to cardinal theory based on graded equipollence is elaborated. A relation between the functional approach to cardinality and generalized cardinals is analyzed in Section 7. The last section concludes.

2. Algebraic Structures of Membership Degrees of Fuzzy Sets

In our contribution on a graded approach to the cardinality theory of finite fuzzy sets, the membership degrees of fuzzy sets are interpreted in a residuated lattice that is extended by an adjoint pair of dual operations. The extension of residuated lattices by new operations is motivated by requirements of the second part, where fuzzy c-measures are developed (see Definition 3.1 and a subsequent comment). Throughout this article, except for Subsection 7.2, one should be satisfied with a residuated lattice that is a reduct of the following more general algebra.

We shall say that an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, \oplus, \ominus, \bot, \top \rangle$ with six binary operations and two constants is a *residuated-dually residuated lattice* (shortly, an rdr-lattice) if

(i) $\langle L, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice, where $\bot$ is the least element and $\top$ is the greatest element of L,

(ii) $\langle L, \otimes, \top \rangle$ and $\langle L, \oplus, \bot \rangle$ are commutative monoids,

(iii) the pairs $\langle \otimes, \to \rangle$ and $\langle \oplus, \ominus \rangle$ form adjoint pairs, i.e.,

$$\alpha \leq \beta \to \gamma \quad \text{if and only if} \quad \alpha \otimes \beta \leq \gamma, \tag{2}$$

$$\alpha \leq \beta \oplus \gamma \quad \text{if and only if} \quad \alpha \ominus \beta \leq \gamma \tag{3}$$

hold for each $\alpha, \beta, \gamma \in L$ ($\leq$ denotes the corresponding lattice ordering).

The operations $\otimes$, $\to$, $\oplus$ and $\ominus$ are called *multiplication*, *residuum*, *addition* and *difference*, respectively.[4] We shall say that an rdr-lattice is *complete* (*linearly ordered*), if $\langle L, \wedge, \vee, \bot, \top \rangle$ is a complete (linearly ordered) lattice. An rdr-lattice satisfies the *prelinearity* axiom if $(\alpha \to \beta) \vee (\beta \to \alpha) = \top$ holds for all $\alpha, \beta \in L$.[5] Note that to achieve the aim of this paper, we shall sometimes use the rdr-lattices satisfying the prelinearity axiom to ensure the finite distributivity of $\wedge$ over $\vee$, i.e., $\alpha \wedge \bigvee_{i=1}^{n} \beta_i = \bigvee_{i=1}^{n} (\alpha \wedge \beta_i)$.[6].

[4] Notice that if the adjoint pair $\langle \oplus, \ominus \rangle$ (or $\langle \otimes, \to \rangle$) is forgotten in the rdr-lattice $\mathbf{L}$, we obtain the *residuated lattice* (or *dually residuated lattice*).

[5] Note that one may simply reformulate the concept of prelinearity for the dual adjoint pair $(\oplus, \ominus)$. Because we do not deal with such dual concepts, we omit their definitions here.

[6] See Theorems 2.37 and 2.39 in [1].

Example 2.1 Let T and S be a left continuous t-norm and a right continuous t-conorm, respectively; we define $\rightarrow_T$ and $\ominus_S$ by

$$\alpha \rightarrow_T \beta = \bigvee\{\gamma \in [0,1] \mid T(\alpha,\gamma) \leq \beta\},$$
$$\alpha \ominus_S \beta = \bigwedge\{\gamma \in [0,1] \mid S(\beta,\gamma) \geq \alpha\}.$$

Then, the algebra $\mathbf{L} = \langle [0,1], \min, \max, T, \rightarrow_T, S, \ominus_S, 0, 1\rangle$ is a complete rdr-lattice. If T and S are the product (or the Łukasiewicz conjunction) and the probabilistic sum (or the Łukasiewicz disjunction), respectively, then we shall use $\mathbf{L}_{\mathrm{P}}$ ($\mathbf{L}_{\mathrm{Ł}}$) to denote the Goguen (Łukasiewicz) rdr-lattice.[7]

Example 2.2 Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \bot, \top\rangle$ be an MV-algebra,[8] i.e., a residuated lattice satisfying the prelinearity axiom (i.e., $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha) = \top$ holds for all $\alpha, \beta \in L$) and the law of double negation (i.e., $(\alpha \rightarrow \bot) \rightarrow \bot = \alpha$ holds for all $\alpha \in L$). Let us put $\neg\alpha = \alpha \rightarrow \bot$ and define $\alpha \oplus \beta = \neg(\neg\alpha \otimes \neg\beta)$ and $\alpha \ominus \beta = \alpha \otimes \neg\beta$. Then, $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus, \bot, \top\rangle$ is an rdr-lattice. Obviously, if $\mathbf{L}$ is a complete MV-algebra, then $\mathbf{L}$ is a complete rdr-lattice.

Although the following example will not be explicitly used for a demonstration of the proposed theory, we want to illustrate a somewhat different structure for membership degrees from the common ones presented in the previous examples.

Example 2.3 Let $[0,\infty]$ be the set of non-negative real numbers extended by infinity. Let us define $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$ and

$$\alpha \oplus \beta = \begin{cases} \alpha + \beta, & \alpha, \beta \in [0,\infty), \\ \infty, & \text{otherwise} \end{cases} \qquad \alpha \ominus \beta = \begin{cases} 0 \vee (\alpha - \beta), & \alpha, \beta \in [0,\infty), \\ 0, & \beta = \infty, \\ \infty, & \text{otherwise,} \end{cases}$$

$$\alpha \otimes \beta = \frac{1}{\frac{1}{\alpha} \oplus \frac{1}{\beta}} \qquad \alpha \rightarrow \beta = \frac{1}{\frac{1}{\beta} \ominus \frac{1}{\alpha}},$$

for any $\alpha, \beta \in [0,\infty]$. Then, one can simply check that $\mathbf{L} = \langle [0,\infty], \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus, 0, \infty\rangle$ is a complete rdr-lattice (of the *non-negative real numbers*).

Let us define the following additional operations for any $\alpha, \beta \in L$ and set

[7] Note that the reduct of $\mathbf{L}$ omitting the dually adjoint pair $(S, \ominus_S)$ is a complete residuated lattice. In particular, the reduct of $\mathbf{L}_{\mathrm{P}}$ ($\mathbf{L}_{\mathrm{Ł}}$) is called the product or Goguen (Łukasiewicz) algebra (see, e.g., [1, 18]).

[8] For further information, we refer to [1, 2, 18].

$\{\alpha_i \mid i \in I\}$ of elements from L over a countable (possibly empty) index set I:

$$\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha), \qquad (\textit{biresiduum})$$
$$\neg\alpha = \alpha \to \bot, \qquad (\textit{negation})$$
$$\bigotimes_{i\in I} \alpha_i = \begin{cases} \top, & I = \emptyset, \\ \bigwedge_{K\in\mathfrak{Fin}(I)} \bigotimes_{i\in K} \alpha_i, & \text{otherwise}, \end{cases} \qquad (\textit{countable multiplication})$$
$$\bigoplus_{i\in I} \alpha_i = \begin{cases} \bot, & I = \emptyset, \\ \bigvee_{K\in\mathfrak{Fin}(I)} \bigoplus_{i\in K} \alpha_i, & \text{otherwise}, \end{cases} \qquad (\textit{countable addition})$$

where $\mathfrak{Fin}(I)$ denotes the set of all finite subsets of I. If $|I| = n$ is a finite index set and $\alpha_i = \alpha$ for any $i \in I$, then we will write $\alpha^n = \bigotimes_{i\in I} \alpha_i$ or $n\alpha = \bigoplus_{i\in I} \alpha_i$.

To integrate some alternative constructions based on the operations of $\wedge$ and $\otimes$, we shall use the common symbol $\odot$ (or $\bigodot$). Analogously, we shall use $\overline{\odot}$ (or $\overline{\bigodot}$) to denote one of the operations, $\vee$ or $\oplus$.

In the following proposition, we recall several important properties of the biresiduum that holds in each (complete) residuated lattice. Analogously, one could reformulate these properties for the absolute difference that is the dual operation to the biresiduum, but we do not need it explicitly in this contribution; therefore, we omit them here. According to the definition, all equalities and inequalities also hold in each (complete) rdr-lattice.

Proposition 2.1 (see [1, 18]) *Let* $\mathbf{L}$ *be a residuated lattice. Then, the following items hold for arbitrary* $\alpha, \beta, \gamma, \delta \in L$*:*

$$\alpha \leftrightarrow \alpha = \top, \tag{4}$$
$$\alpha \leftrightarrow \beta = \beta \leftrightarrow \alpha, \tag{5}$$
$$(\alpha \leftrightarrow \beta) \otimes (\beta \leftrightarrow \gamma) \le \alpha \leftrightarrow \gamma, \tag{6}$$
$$(\alpha \leftrightarrow \beta) \otimes (\gamma \leftrightarrow \delta) \le (\alpha \otimes \gamma) \leftrightarrow (\beta \otimes \delta), \tag{7}$$
$$(\alpha \leftrightarrow \beta) \otimes (\gamma \leftrightarrow \delta) \le (\alpha \to \gamma) \leftrightarrow (\beta \to \delta), \tag{8}$$
$$(\alpha \leftrightarrow \beta) \wedge (\gamma \leftrightarrow \delta) \le (\alpha \wedge \gamma) \leftrightarrow (\beta \wedge \delta), \tag{9}$$
$$(\alpha \leftrightarrow \beta) \wedge (\gamma \leftrightarrow \delta) \le (\alpha \vee \gamma) \leftrightarrow (\beta \vee \delta). \tag{10}$$

Moreover, let $\mathbf{L}$ *be a complete residuated lattice. Then, the following items hold for arbitrary sets* $\{\alpha_i \mid i \in I\}$, $\{\beta_i \mid i \in I\}$ *of elements from* L *over an arbitrary set of indices* I*:*

$$\alpha \otimes \bigvee_{i\in I} \beta_i = \bigvee_{i\in I} (\alpha \otimes \beta_i) \tag{11}$$
$$\bigwedge_{i\in I} (\alpha_i \leftrightarrow \beta_i) \le (\bigwedge_{i\in I} \alpha_i) \leftrightarrow (\bigwedge_{i\in I} \beta_i), \tag{12}$$
$$\bigwedge_{i\in I} (\alpha_i \leftrightarrow \beta_i) \le (\bigvee_{i\in I} \alpha_i) \leftrightarrow (\bigvee_{i\in I} \beta_i). \tag{13}$$

3. Fuzzy Sets in $\mathfrak{Count}$

Intuitively, finite fuzzy sets are fuzzy sets apparently defined in finite universes of discourse. To express their cardinalities, one may imagine something like fuzzy sets defined on the set $\mathbb{N}$ of natural numbers (with 0) as suitable candidates for this purpose. This consideration seems to be correct and has been verified in [24, 25, 26]. Because $\mathbb{N}$ is denumerable and because one of our goals is to describe the power of finite fuzzy sets by fuzzy sets in $\mathbb{N}$, it is advantageous to use the proper class $\mathfrak{Count}$ of all countable sets as a framework of our theory. Let us denote by $\mathfrak{Fin}$ the class of all finite sets. Note that both classes are examples of universes of sets proposed in [10, 8] and that a set theory can be built inside these classes.

Now, let us suppose that a complete rdr-lattice $\mathbf{L}$ is given. A fuzzy set in $\mathfrak{Count}$ is defined as follows.

Definition 3.1 A mapping $A : x \to L$ is called a fuzzy set in $\mathfrak{Count}$ if x is a set in $\mathfrak{Count}$.

One may note that fuzzy sets can be defined on different universes and thus the empty mapping $\emptyset : \emptyset \to L$ is permitted. Let us show some advantages of this non-standard approach to fuzzy sets assuming a class of universes instead of a one fixed universe. First, one may deal with fuzzy sets of "fresh apples", "tall men", or "speed cars". Then, we do not see a substantial reason to express all three fuzzy sets in a one fixed universe containing apples, men and cars to compare their cardinality. Second, a fixed universe for a cardinal theory of finite fuzzy sets does not allow us to compare the cardinalities of fuzzy sets and power fuzzy sets (see Definition 3.10) because the power fuzzy sets are evidently defined on different universes from the original ones.

In this contribution, we shall use $\mathfrak{Fcount}$ to denote the proper class of all fuzzy sets in $\mathfrak{Count}$. Although the fuzzy sets in $\mathfrak{Fcount}$ may be referred to "countable" fuzzy sets, we shall not use this notation, and will instead use only fuzzy sets. Notice that each fuzzy set (as a relation expressing the mapping) or pair of fuzzy sets belongs to $\mathfrak{Count}$, i.e., $\mathfrak{Fcount} \subseteq \mathfrak{Count}$ or $\mathfrak{Fcount} \times \mathfrak{Fcount} \subseteq \mathfrak{Count}$, respectively.

Let us summarize several well-known notions of fuzzy set theory and suppose that $A : z \to L$ is a fuzzy set from $\mathfrak{Fcount}$. The set $z = \mathrm{Dom}(A)$ is called a *universe of discourse of* A (briefly, *universe of* A), $\mathrm{Supp}(A) = \{x \in z \mid A(x) > \bot\}$ is called a *support of fuzzy set* A, $\mathrm{Ker}(A) = \{x \in z \mid A(x) = \top\}$ is called a *kernel of fuzzy set* A, and $A_\alpha = \{x \in z \mid A(x) \geq \alpha\}$ is called an *α-cut of fuzzy set* A. The fuzzy set A is said to be *crisp* and refers to a *crisp set* if $A(x) \in \{\bot, \top\}$ for any $x \in z$. One may see that the empty mapping as a fuzzy set is crisp because the presumption is trivially satisfied. If a fuzzy set A has a finite universe, then we shall use the following simple notation

$$A = \{\alpha_1/x_1, \ldots, \alpha_n/x_n\},$$

where $\mathrm{Dom}(A) = \{x_1, \ldots, x_n\}$ and $\alpha_i \in L$ for any $i = 1, \ldots, n$. Let $f : z \to z'$ be a mapping and $A : z \to L$ be a fuzzy set in $\mathfrak{Count}$. We shall use $f^{\rightarrow}(A)$

to denote the *image of A under f* defined by Zadeh's extension principle (see [16, 17]), i.e.,

$$f^{\rightarrow}(A)(y) = \bigvee_{x \in z;\, f(x)=y} A(x) \tag{14}$$

for any $y \in z'$. Let $A, B \in \mathfrak{Fcount}$. We shall say that A is *a fuzzy subset* of B and denote it by $A \subseteq B$ if $A(x) \leq B(x)$ holds for any $x \in \mathrm{Supp}(A)$. Obviously, $\subseteq$ is a partial ordering on $\mathfrak{Fcount}$.

In the literature (see, e.g., [3, 16, 17]), a fuzzy set assigning $\perp$ to each element of its universe is usually referred to the empty fuzzy set, and a fuzzy set assigning $\alpha > \perp$ to only one element of its universe to a singleton (fuzzy set). In our theory, we shall use a different interpretation of the empty fuzzy set and singleton in $\mathfrak{Count}$ as follows.

Definition 3.2 The empty mapping $\emptyset : \emptyset \to L$ is called the *empty fuzzy set.* A fuzzy set A is called a *singleton* if $\mathrm{Dom}(A)$ contains only one element.

In this paper, we shall use $\{\alpha/x\}$ to denote a singleton, where $x \in \mathfrak{Count}$ and $\alpha \in L$.

Remark 3.1 One may be surprised to find that $\{\perp/x\}$ is considered to be a singleton because it is totally beyond a natural conception of a singleton. While this is true, in further sections, we shall intensively deal with these types of objects, and it would be unnecessarily complicated to distinguish a singleton having $\alpha > \perp$ and something like a singleton for $\alpha = \perp$. Therefore, we use a common concept that covers both cases.

In classical set theory, two sets are the same if they have the same elements. In our theory, we define the analogical notion as follows.

Definition 3.3 We shall say that fuzzy sets A and B are *the same* (symbolically, $A = B$) if $\mathrm{Dom}(A) = \mathrm{Dom}(B)$ and $A(x) = B(x)$ for any $x \in \mathrm{Dom}(A)$.

An essential predicate in our theory is a binary relation that extends the concept of being the same fuzzy sets and states that two fuzzy sets are the same except for elements having the zero membership degree. This concept enables us to deal with a class of fuzzy sets that practically represents one fuzzy set in different universes.

Definition 3.4 We shall say that fuzzy sets A and B are *equivalent* (symbolically, $A \equiv B$) if $\mathrm{Supp}(A) = \mathrm{Supp}(B)$ and $A(x) = B(x)$ for any $x \in \mathrm{Supp}(A)$.

We shall use $\mathrm{cls}(A)$ to denote the class of all fuzzy sets from $\mathfrak{Fcount}$ that are equivalent with A.

Remark 3.2 Let $A \in \mathfrak{Fcount}$ and $x \in \mathfrak{Count}$. Then, $A' : \mathrm{Dom}(A) \cup x \to L$ defined by $A'(z) = A(z)$ for any $z \in \mathrm{Dom}(A)$ and $A'(z) = \perp$ for any $z \in x \setminus \mathrm{Dom}(A)$ belongs to $\mathrm{cls}(A)$. Hence, $\mathrm{cls}(A)$ is a proper class.

Example 3.3 Obviously, we have $\emptyset \equiv \{0/a, 0/b\}$ or $\{0.9, /a\} \equiv \{0.9/a, 0/b\}$ and $\{0/a, 0/b\} \in \mathrm{cls}(\emptyset)$.

The following statement shows an equivalent definition of "being equivalent fuzzy sets". Let $A \in \mathfrak{Fcount}$ and $x \subseteq \mathrm{Dom}(A)$. We shall use $A \upharpoonright x$ to denote the restriction of A to x.

Proposition 3.1 *Let $A, B \in \mathfrak{Fcount}$. Then, $A \equiv B$ if and only if there exists $x \in \mathfrak{Count}$ such that* $\mathrm{Supp}(A) \cup \mathrm{Supp}(B) \subseteq x$ *and* $A \upharpoonright x = B \upharpoonright x$.

Proof. Let $A, B \in \mathfrak{Fcount}$.

($\Rightarrow$) If $A \equiv B$, then it is sufficient to put $x = \mathrm{Supp}(A)$.

($\Leftarrow$) Let us suppose that there exists $x \in \mathfrak{Count}$ such that $\mathrm{Supp}(A) \cup \mathrm{Supp}(B) \subseteq x$ and $A \upharpoonright x = B \upharpoonright x$. Hence, we have $\mathrm{Supp}(A) = \mathrm{Supp}(A \upharpoonright x) = \mathrm{Supp}(B \upharpoonright x) = \mathrm{Supp}(B)$ and $A(z) = (A \upharpoonright x)(z) = (B \upharpoonright x)(z) = B(z)$ for any $z \in \mathrm{Supp}(A)$, i.e., $A \equiv B$. □

A graded generalization of the relation to be equivalent may be given as follows.

Definition 3.5 We shall say that fuzzy sets A and B are *the equivalent* in the degree α (symbolically, $[A \approx B] = \alpha$) if

$$\alpha = \bigwedge_{x \in \mathrm{Dom}(A) \cup \mathrm{Dom}(B)} (A'(x) \leftrightarrow B'(x)), \tag{15}$$

holds for $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ with $\mathrm{Dom}(A') = \mathrm{Dom}(B') = \mathrm{Dom}(A) \cup \mathrm{Dom}(B)$.

One may simply check that $[A \approx B] = [A' \approx B']$ holds for any $A' \in \mathrm{cls}(A)$ and $B' \in \mathrm{cls}(B)$ and, hence, that the previous definition is correct. Moreover, $[A \approx B] = \top$ if and only if $A \equiv B$.

Let us introduce several basic operations with fuzzy sets. The definitions are from [9], where arbitrary fuzzy sets were investigated. Recall that, to each $A \in \mathfrak{Fcount}$ and $x \in \mathfrak{Count}$, there exists $A' \equiv A$ such that $\mathrm{Dom}(A') = \mathrm{Dom}(A) \cup x$ (see Remark 3.2).

Definition 3.6 Let $A, B \in \mathfrak{Fcount}$, $x = \mathrm{Dom}(A) \cup \mathrm{Dom}(B)$ and $A' \equiv A$, $B' \equiv B$ such that $\mathrm{Dom}(A') = \mathrm{Dom}(B') = x$. Then,

- *the union* of A and B is a mapping $A \cup B : x \to L$ defined by

$$(A \cup B)(a) = A'(a) \vee B'(a) \tag{16}$$

  for any $a \in x$,

- *the intersection* of A and B is a mapping $A \cap B : x \to L$ defined by

$$(A \cap B)(a) = A'(a) \wedge B'(a) \tag{17}$$

  for any $a \in x$,

- *the difference* of A and B is a mapping $A \setminus B : x \to L$ defined by

$$A \setminus B(a) = A'(a) \otimes (B'(a) \to \bot) = A'(a) \otimes \neg B'(a) \tag{18}$$

for any $a \in x$.

Remark 3.4 It is easy to see that $A \setminus \emptyset = A$, $\emptyset \setminus \emptyset = \emptyset$ and $\emptyset \setminus A = \emptyset$.

Definition 3.7 Let $A, B \in \mathcal{F}$, $x = \mathrm{Dom}(A) \times \mathrm{Dom}(B)$ and $y = \mathrm{Dom}(A) \sqcup \mathrm{Dom}(B)$ (the disjoint union). Then,

- *the product* of A, B is a mapping $A \times B : x \to L$ defined by

$$(A \times B)(a, b) = A(a) \wedge B(b) \tag{19}$$

  for any $(a, b) \in x$,

- *the strong product* of A, B is a mapping $A \otimes B : x \to L$ defined by

$$(A \otimes B)(a, b) = A(a) \otimes B(b) \tag{20}$$

  for any $(a, b) \in x$,

- *the disjoint union* of A, B is a mapping $A \sqcup B : y \to L$ defined by

$$(A \sqcup B)(a, i) = \begin{cases} A(a, i), & \text{if } i = 1, \\ B(a, i), & \text{if } i = 2, \end{cases} \tag{21}$$

  for any $(a, i) \in y$.

Definition 3.8 A fuzzy set $\overline{A} = \mathrm{Dom}(A) \setminus A$ is called the *complement of* A.

Remark 3.5 By the definition, we obtain $\overline{\emptyset} = \emptyset$, but $\overline{A} \notin \mathrm{cls}(\emptyset)$ for any $A \in \mathrm{cls}(\emptyset)$ such that $A \neq \emptyset$.

The following theorem shows that the relation "being equivalent" is a congruence for all mentioned operations except the complement. Obviously, the last operation is strongly dependent on its universe of discourse (cf. Theorem 4.3. in [8]).

Theorem 3.2 *Let* $A, B \in \mathfrak{Fcount}$ *and* $\circledast \in \{\cap, \cup, \setminus, \times, \otimes, \sqcup\}$. *If* $C \equiv A$ *and* $D \equiv B$, *then* $C \circledast D \equiv A \circledast B$.

Proof. We shall prove only the case $A \setminus B$. The remaining cases can be proved analogously.

Let $A, B, C, D \in \mathfrak{Fcount}$ and $A \equiv C$ and $B \equiv D$. We have to prove $A \setminus B \equiv C \setminus D$. One may see that it trivially holds for $A \equiv \emptyset$. Let $A \not\equiv \emptyset$. Then,

$\mathrm{Dom}(A) \cup \mathrm{Dom}(B) \neq \emptyset$. Put $x = \mathrm{Supp}(A) \cap \mathrm{Supp}(B)$, $y = \mathrm{Supp}(A) \setminus \mathrm{Supp}(B)$ and $z = x \cup y$. Clearly, $z = \mathrm{Supp}(A) = \mathrm{Supp}(C)$ and

$$\mathrm{Supp}(A \setminus B) \cup \mathrm{Supp}(C \setminus D) \subseteq z \subseteq \mathrm{Dom}(A \setminus B) \cap \mathrm{Dom}(C \setminus D).$$

Due to Proposition 3.1, it suffices to show that $(A \setminus B) \upharpoonright z = (C \setminus D) \upharpoonright z$. Since $A \equiv C$, then $A(a) = C(a)$ for any $a \in z$. Since $B \equiv D$, then $B(a) = D(a)$ for any $a \in x$. One may simply check that

$$((A \setminus B) \upharpoonright z)(a) = A(a) = C(a) = ((C \setminus D) \upharpoonright z)(a).$$

for any $a \in y$ and

$$((A \setminus B) \upharpoonright z)(a) = A(a) \otimes \neg B(a) = C(a) \otimes \neg D(a) = ((C \setminus D) \upharpoonright z)(a)$$

for any $a \in x$, which concludes the proof. □

EXAMPLE 3.6 Let $\mathbf{L}$ be the Łukasiewicz algebra and $A = \{1/a, 0.4/b\}$ and $B = \{0.6/a, 0.2/c\}$. Then, we have

$$\begin{aligned} A \cup B &= \{1/a, 0.4/b, 0.2/c\}, \\ A \cap B &= \{0.6/a, 0/b, 0/c\}, \\ A \setminus B &= \{0.4/a, 0.4/b, 0/c\}, \\ A \times B &= \{0.6/(a,a), 0.2/(a,c), 0.4/(b,a), 0.2/(b,c)\}, \\ A \otimes B &= \{0.6/(a,a), 0.2/(a,c), 0/(b,a), 0/(b,c)\}, \\ A \sqcup B &= \{1/(a,1), 0.4/(b,1), 0.6/(a,2), 0.2/(c,2)\}, \\ \overline{A} &= \{0/a, 0.6/b\}. \end{aligned}$$

Now, we can introduce the concept of finite fuzzy set. We propose the following definition based on the finiteness of universes of discourse (cf. [22, 25]).

DEFINITION 3.9 We shall say that a fuzzy set A from $\mathfrak{Fcount}$ is *finite* if there exists $A' \in \mathrm{cls}(A)$ such that $\mathrm{Dom}(A')$ is a finite set.

We shall use $\mathfrak{Ffin}$ to denote the class of all finite fuzzy sets in $\mathfrak{Count}$. Obviously, A is a finite fuzzy set if and only if $\mathrm{Supp}(A)$ is a finite set and the arbitrary operation introduced above applied on finite fuzzy sets gives finite fuzzy sets.

It is not easy to say what the power set means for fuzzy sets. We propose the following simple definition that straightforwardly generalizes the classical approach to the concept of power set. Moreover, this definition respects our framework of finite fuzzy sets (cf. Definition 3.14 in [9]). We shall use χ_x to denote the characteristic function of a set x.

DEFINITION 3.10 Let $A \in \mathfrak{Fcount}$ and $x = \{y \mid y \subseteq \mathrm{Dom}(A)\}$. Then, a fuzzy set $\mathbf{P}(A) : x \to L$ defined by

$$\mathbf{P}(A)(y) = \bigwedge_{z \in \mathrm{Dom}(A)} (\chi_y(z) \to A(z)) \tag{22}$$

is called a *fuzzy power set of A*.

It is easy to see that $\mathbf{P}(A)(y) = \bot$ whenever $y \setminus \mathrm{Supp}(A) \neq \emptyset$. In fact, if there exists $z \in y$ for which $A(z) = \bot$, then $\chi_y(z) \to A(z) = \top \to \bot = \bot$. Hence, the fuzzy power set of a finite fuzzy set is a finite fuzzy set. It should be noted that $\mathbf{P}(A)(y)$ can be interpreted as the degree in which the set y is a fuzzy subset of A (cf. Definition 3.8 in [8]). A simple consequence of the definition is the following expression

$$\mathbf{P}(A)(y) = \bigwedge_{z \in y} A(z), \tag{23}$$

as $\top \to A(z) = A(z)$ and $\bot \to A(z) = \top$. The following statement shows that fuzzy power sets preserve the equivalence of fuzzy sets (cf. Lemma 3.9 in [9]).

PROPOSITION 3.3 *If $A' \equiv A$, then $\mathbf{P}(A') \equiv \mathbf{P}(A)$.*

PROOF. Let $A, A' \in \mathfrak{Fcount}$ such that $A' \equiv A$. If $y \in \mathrm{Supp}(\mathbf{P}(A'))$, then $\mathbf{P}(A')(y) > \bot$ and $y \subseteq \mathrm{Supp}(A') = \mathrm{Supp}(A)$, whence $\mathbf{P}(A)(y) > \bot$ and $\mathrm{Supp}(\mathbf{P}(A')) \subseteq \mathrm{Supp}(\mathbf{P}(A))$. Analogously, one may check the opposite inclusion, which implies that $\mathrm{Supp}(\mathbf{P}(A')) = \mathrm{Supp}(\mathbf{P}(A))$. By (23), we have

$$\mathbf{P}(A)(y) = \bigwedge_{z \in y} A(z) = \bigwedge_{z \in y} A'(z) = \mathbf{P}(A')(y)$$

for any $y \subseteq \mathrm{Supp}(A)$, whence $\mathbf{P}(A)(y) = \mathbf{P}(A')(y)$ for any $y \in \mathrm{Supp}(\mathbf{P}(A))$, and thus $\mathbf{P}(A) \equiv \mathbf{P}(A')$. □

EXAMPLE 3.7 Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz algebra and $A = \{1/a, 0.4/b\}$. Then,

$$\mathbf{P}(A) = \{1/\emptyset, 1/\{a\}, 0.4/\{b\}, 0.4/\{a, b\}\}.$$

Finally, we introduce the concept of exponentiation for fuzzy sets. Recall that if x, y are sets, then the exponentiation x^y is the set of all mappings of y to x. We propose the following definition, which generalizes the classical one.

DEFINITION 3.11 Let $A, B \in \mathfrak{Fcount}$ and put $x = \mathrm{Dom}(A)$ and $y = \mathrm{Dom}(B)$. Then, a fuzzy set $B^A : y^x \to L$ defined by

$$B^A(f) = \bigwedge_{z \in x} (A(z) \to B(f(z))) \tag{24}$$

is called an *exponentiation* of A to B.

One can see that if A, B are finite fuzzy sets with finite universes of discourse, then the exponentiation of A to B is a finite fuzzy set. Moreover, the presumption on the finiteness of universes of discourse cannot be omitted here.

EXAMPLE 3.8 Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz algebra and $A = \{1/a, 0.4/b\}$, $B = \{0.6/a, 0.2/c\}$. Obviously, the domain of B^A is the set of all mappings of $\{a, b\}$ to $\{a, c\}$. For example, if $f \in \mathrm{Dom}(B^A)$ is defined as $f(a) = f(b) = a$, then

$$B^A(f) = (1 \to 0.6) \wedge (0.4 \to 0.6) = 0.6 \wedge \top = 0.6.$$

Remark 3.9 It is easy to see that the exponentiation depends on the choice of universes of fuzzy sets (analogously to the operation of complement). There are several possibilities to ensure the independency on the choice of equivalent fuzzy sets (e.g., putting $x = \mathrm{Supp}(A)$ and $y = \mathrm{Supp}(B)$, but then a problem arises when we want to prove a graded relation in Theorem 6.8 discussed in Subsection 6.

Remark 3.10 The usage of the operation of residuum in the definition is motivated by the fact that we want to construct a function f of A to B such that, roughly speaking, $x \in A$ implies $f(x) \in B$, which is naturaly modeled by $A(x) \to B(f(x))$. Moreover, this allows us to show a very important one-to-one correspondence between hom-sets[9]

$$hom(A \otimes B, C) \cong hom(A, C^B), \tag{25}$$

which generalizes the one-to-one correspondence $hom(x \times y, z) \cong hom(x, z^y)$ (see Theorem 6.9). Perhaps this fact could justify the proposed definition of exponentiation of fuzzy sets.

In the end of this subsection, we would like to introduce the concept of a fuzzy class in $\mathfrak{Count}$. Although the fuzzy classes are not major objects of our theory, we shall use them to describe several concepts that are not fuzzy sets and belong to $\mathfrak{Count}$ in the following sense.

Definition 3.12 A mapping $A : x \to L$ is called a fuzzy class in $\mathfrak{Count}$ if $x \subseteq \mathfrak{Count}$.

One may see that each fuzzy set is a fuzzy class because $x \in \mathfrak{Count}$ implies $\mathbf{P}(x) \in \mathfrak{Count}$, whence $x \subseteq \mathfrak{Count}$. Obviously, the opposite implication is not true, and we obtain *proper* fuzzy classes.

4. Generalized cardinals over $\mathbb{N}$

To express the power of a finite fuzzy set, we need to introduce a concept of generalized cardinal over the set $\mathbb{N}$ of all natural numbers with 0. A reasonable condition for this concept is a "convexity" of fuzzy sets (see, e.g., [22, 25]). The convexity of fuzzy sets may be defined over linearly ordered universes as follows. Let us suppose that an rdr-lattice $\mathbf{L}$ is given and recall that $\odot \in \{\wedge, \otimes\}$.

Definition 4.1 Let x be a set in $\mathfrak{Count}$ equipped with a linear ordering $\leq$. We shall say that $A : x \to L$ is a *$\odot$-convex fuzzy set* if $A(a) \odot A(c) \leq A(b)$ holds for any $a, b, c \in x$ with $a \leq b \leq c$.

Definition 4.2 *A generalized cardinal* A is a $\odot$-convex fuzzy set $A : \mathbb{N} \to L$.

[9] A hom-set $hom(x, y)$ is the set of all mappings of x to y.

One can see that generalized cardinals need not be finite fuzzy sets. We shall use $\mathfrak{N}$ to denote the set of all generalized cardinals. To be able to manipulate with generalized cardinals, we establish, by Zadeh's extension principle, the operation of addition on $\mathfrak{N}$ as

$$(A+B)(i) = \bigvee_{\substack{k,l\in\mathbb{N}\\ k+l=i}} (A(k)\odot B(l)). \tag{26}$$

Let us define $\mathbf{0} : \mathbb{N} \to L$ by $\mathbf{0}(k) = \top$, if $k = 0$, and $\mathbf{0}(k) = \bot$, otherwise.

THEOREM 4.1 *The triplet $\mathfrak{N} = (\mathfrak{N}, +, \mathbf{0})$ is an abelian monoid, where the prelinearity of the rdr-lattice $\mathbf{L}$ is assumed for $\odot = \wedge$.*

PROOF. Let $+$ be defined by (26). First, we shall prove that the addition of two generalized cardinals is a generalized cardinal. Let $A, B \in \mathfrak{N}$. Let $k, l, m \in \mathbb{N}$ such that $k \le l \le m$ and put $C = A + B$. Then,

$$C(k)\odot C(m) =$$

$$\Bigg(\bigvee_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2=k}} (A(k_1)\odot B(k_2))\Bigg)\odot\Bigg(\bigvee_{\substack{m_1,m_2\in\mathbb{N}\\ m_1+m_2=m}} (A(m_1)\odot B(m_2))\Bigg) =$$

$$\bigvee_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2=k}}\ \bigvee_{\substack{m_1,m_2\in\mathbb{N}\\ m_1+m_2=m}} ((A(k_1)\odot A(m_1))\odot(B(k_2)\odot B(m_2))).$$

Note that the second equality above holds in each rdr-lattice for $\odot = \otimes$ and in each rdr-lattice satisfying the prelinearity axiom for $\odot = \wedge$.[10] Now, let us show that to each $k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 = k$ and $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 = m$ there exist $l_1, l_2 \in \mathbb{N}$ with $l_1 + l_2 = l$ such that l_i lies between k_i and m_i, $i = 1, 2$. We can distinguish the three following cases:

Case 1. $k_1 \le m_1$ and $k_2 \le m_2$;

if $l \le k_1+m_2$, then put $l_1 = k_1$ and $l_2 = l-l_1$; $k_1+k_2 \le l_1+l_2 \le k_1+m_2$ implies $k_2 \le l_2 \le m_2$;

If $l > k_1 + m_2$, then put $l_2 = m_2$ and $l_1 = l - l_2$; $k_1 + m_2 < l_1 + l_2 \le m_1 + m_2$ implies $k_1 < l_1 \le m_1$.

Case 2. $m_1 \le k_1$ and $k_2 \le m_2$; put $l_1 = k_1$ and $l_2 = l - l_1$; $k_1 + k_2 \le l_1 + l_2 \le m_1 + m_2$ implies $k_2 \le l_2 \le m_1 - k_1 + m_2 \le m_2$ because $m_1 - k_1 \le 0$ according to the presumption;

Case 3. $k_1 \le m_1$ and $m_2 \le k_2$; put $l_2 = k_2$ and $l_1 = l - l_2$; $k_1 + k_2 \le l_1 + l_2 \le m_1 + m_2$ implies $k_1 \le l_1 \le m_1 + m_2 - k_1 \le m_1$ because $m_2 - k_2 \le 0$ according to the presumption;

[10] Note that the completeness of rdr-lattice need not be supposed because the supremum is applied on a finite set.

By the $\odot$-convexity of A and B, we have

$$C(k) \odot C(m) = \bigvee_{\substack{k_1,k_2 \in \mathbb{N} \\ k_1+k_2=k}} \bigvee_{\substack{m_1,m_2 \in \mathbb{N} \\ m_1+m_2=m}} ((A(k_1) \odot A(m_1)) \odot (B(k_2) \odot B(m_2))) \leq$$
$$\bigvee_{\substack{l_1,l_2 \in \mathbb{N} \\ l_1+l_2=l}} ((A(l_1) \odot B(l_1)) = C(l)$$

and C is $\odot$-convex. Hence, the addition is defined correctly. The commutativity of $+$ immediately follows from the commutativity of $\odot$ and $+$ (for natural numbers). Further, we have

$$((A+B)+C)(i) = \bigvee_{\substack{l,k \in \mathbb{N} \\ l+k=i}} (\bigvee_{\substack{l_1,l_2 \in \mathbb{N} \\ l_1+l_2=l}} (A(l_1) \odot B(l_2)) \odot C(k) =$$
$$\bigvee_{\substack{l,k \in \mathbb{N} \\ l+k=i}} \bigvee_{\substack{l_1,l_2 \in \mathbb{N} \\ l_1+l_2=l}} (A(l_1) \odot B(l_2) \odot C(k)) =$$
$$\bigvee_{\substack{l_1,l_2,k \in \mathbb{N} \\ l_1+l_2+k=i}} (A(l_1) \odot B(l_2) \odot C(k)) = (A+B+C)(i)$$

for any $i \in \mathbb{N}$. Analogously, we can check that $((A+B)+C)(i) = (A+B+C)(i)$ holds for any $i \in \mathbb{N}$, which implies the associativity of $+$. Finally, we have

$$(A+\mathbf{0})(i) = \bigvee_{\substack{k,l \in \mathbb{N} \\ k+l=i}} (A(k) \odot \mathbf{0}(l)) = A(i) \odot \mathbf{0}(0) = A(i) \odot \top = A(i),$$

whence $\mathbf{0}$ is the neutral (zero) element. Thus, $\mathfrak{N} = (\mathfrak{N}, +, \mathbf{0})$ is an abelian monoid. □

5. Graded equipollence of finite fuzzy sets

5.1. Definition

In [8], we introduced the concept of graded equipollence using the degrees in which a one-to-one mapping between sets is a one-to-one mapping between fuzzy sets. This idea practically generalizes one of Cantor's approaches to the cardinality of sets, namely, that two sets have the same cardinality (are equipollent) if there exists a one-to-one mapping of one set onto the second one. Thus, to check the same cardinality of two sets, one needs to construct a one-to-one correspondence between them. However, in the fuzzy case, the situation is more complicated by the membership degrees, and intuitively, not all one-to-one correspondences between universes of fuzzy sets are appropriate to assert that fuzzy sets have the same cardinality. Moreover, if two fuzzy sets are very similar (consider $[A \approx B] = 0.999$) but there is no one-to-one correspondence between them (consider $f^{\rightarrow}(A) = B$), then it seems to be advantageous to say that these sets have approximately the same cardinality. Thus, the graded equipollence gives a degree in which two fuzzy sets have approximately the same cardinality, and this degree is derived from degrees in which one may construct one-to-one correspondences between fuzzy sets.

Let us start with the concept of one-to-one mapping between fuzzy sets in a degree. Because we deal with countable sets, we can use both $\wedge$ and $\otimes$ (generally $\odot$) in the definition, which is a difference from Definition 4.1. in [8], where only $\wedge$ is considered. The usage of $\otimes$ in the definition gives an alternative approach to the graded equipollence proposed in [8], and it is mainly applied in a comparison between the functional approach and the approach based on fuzzy c-measures to the cardinal theory for finite fuzzy sets presented in the second part of our contribution.

Definition 5.1 Let $A, B \in \mathfrak{Fin}$, $x, y \in \mathfrak{Count}$ and $f : x \to y$ be a one-to-one mapping of x onto y in $\mathfrak{Count}$. We shall say that f is a *one-to-one mapping of A onto B in the degree α with respect to* $\odot$ if $\mathrm{Supp}(A) \subseteq x \subseteq \mathrm{Dom}(A)$ and $\mathrm{Supp}(B) \subseteq y \subseteq \mathrm{Dom}(B)$ and

$$\alpha = \bigodot_{z \in x} (A(z) \leftrightarrow B(f(z))). \tag{27}$$

We shall write $[A \sim_f^{\odot} B] = \alpha$ if f is a one-to-one mapping of A onto B in the degree α with respect to $\odot$. Before we introduce the definition of being equipollent in the degree α, let us denote $\mathrm{Bij}(A, B)$ as the set of all one-to-one correspondences between A and B. Obviously, the set $\mathrm{Bij}(A, B)$ can be characterized as follows.

Proposition 5.1 *Let $A, B \in \mathfrak{Fin}$. The mapping $f : x \to y$ belongs to the set* $\mathrm{Bij}(A, B)$ *if*

(i) f is a one-to-one mapping of x onto y and $x, y \in \mathfrak{Count}$,

(ii) $\mathrm{Supp}(A) \subseteq x \subseteq \mathrm{Dom}(A)$*, and*

(iii) $\mathrm{Supp}(B) \subseteq y \subseteq \mathrm{Dom}(B)$.

Proof. Obvious. □

The following statement is an immediate consequence of the definition of the degree of being a one-to-one mapping.

Proposition 5.2 *Let $A, B \in \mathfrak{Fin}$ and $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$. Then, $[A \sim_f^{\odot} B] = [C \sim_f^{\odot} D]$ for any $f \in \mathrm{Bij}(A, B) \cap \mathrm{Bij}(C, D)$.*

A consequence of $\bot \to \bot = \top$ is the following useful proposition.

Proposition 5.3 *Let $A, B \in \mathfrak{Fin}$ and $f \in \mathrm{Bij}(A, B)$. Then, there exists $f' \in \mathrm{Bij}(A, B)$ such that $\mathrm{Dom}(f') \in \mathfrak{Fin}$ and $[A \sim_f^{\odot} B] = [A \sim_{f'}^{\odot} B]$.*

In [8] (see also [10]), we introduced a degree in which two fuzzy sets in a universe of sets are equipollent. Because the universes of sets may contain fuzzy sets of arbitrary size (finite, denumerable, infinite), we used degrees of one-to-one correspondences between fuzzy sets computed by the operation of infimum in the definition. The following definition is analogous to that in [8, 10], only we restrict ourselves to the finite case and add the operation of the multiplication to make the theory of equipollence much richer.

DEFINITION 5.2 Let $A, B \in \mathfrak{Ffin}$. We shall say that A *is equipollent with* B (or A *has the same cardinality as* B) *in the degree* α *with respect to* $\odot$ if there exist fuzzy sets $C \in \mathrm{cls}(A)$ and $D \in \mathrm{cls}(B)$ such that

$$\alpha = \bigvee_{f \in \mathrm{Bij}(C,D)} [C \sim_f^{\odot} D] \tag{28}$$

and, for each $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ and $f \in \mathrm{Bij}(A', B')$, there is $[A' \sim_f^{\odot} B'] \leq \alpha$.

Let $A, B \in \mathfrak{Ffin}$ such that $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)|$. We shall use $\mathrm{Perm}(A, B)$ to denote the set of all $f \in \mathrm{Bij}(A, B)$ such that $\mathrm{Dom}(f) = \mathrm{Dom}(A)$ and $\mathrm{Ran}(f) = \mathrm{Dom}(B)$.[11] The following theorem shows how to find the degree of equipollence α between A and B.

THEOREM 5.4 *Let* $A, B \in \mathfrak{Ffin}$. *Then,*

$$[A \sim^{\odot} B] = \bigvee_{f \in \mathrm{Perm}(C,D)} [C \sim_f^{\odot} D]. \tag{29}$$

for any $C \in \mathrm{cls}(A)$ *and* $D \in \mathrm{cls}(B)$ *such that* $|\mathrm{Dom}(C)| = |\mathrm{Dom}(D)| = m$.

PROOF. If $A, B \in \mathrm{cls}(\emptyset)$, then it suffices to put $C = D = \emptyset$.

Let us suppose that $A \notin \mathrm{cls}(\emptyset)$ or $B \notin \mathrm{cls}(\emptyset)$ and $A' \in \mathrm{cls}(A)$ and $B' \in \mathrm{cls}(B)$. Since $\bot \leftrightarrow \bot = \top$, then to find the degree of equipollence for A' and B', we may restrict ourselves to all one-to-one mappings $f \in \mathrm{Bij}(A', B')$ for which there is no $x \in \mathrm{Dom}(f)$ such that $A'(x) = B'(f(x)) = \bot$. Denote the set of all such one-to-one mappings by $\mathrm{Bij}^*(A', B')$. Since $A, B \in \mathfrak{Ffin}$, then $\mathrm{Dom}(f)$ is a finite set for all $f \in \mathrm{Bij}^*(A', B')$.[12] Obviously, each $f \in \mathrm{Bij}^*(A', B')$ can be represented by a $2 \times m$ matrix

$$\xi_f = \begin{pmatrix} \gamma_1 & \cdots & \gamma_m \\ \delta_1 & \cdots & \delta_m \end{pmatrix}, \tag{30}$$

[11]Although f is not a permutation on a universe in general, we use the denotation Perm, because to each pair of universes of fuzzy sets A and B we can define a common universe and fuzzy sets A' and B' equivalent to A and B, respectively, such that each mapping from $\mathrm{Perm}(A', B')$ is a permutation on this common universe.

[12]Note that $|\mathrm{Dom}(f)| \leq |\mathrm{Supp}(A)| + |\mathrm{Supp}(B)|$.

where for each pair (γ_i, δ_i) there is a unique $x \in \mathrm{Dom}(f)$ such that $A(x) = \gamma_i$ and $B(f(x)) = \delta_i$.[13] If ξ_f is a $2 \times m$ matrix, then we shall write $|\xi_f| = m$. Hence, the degree in which a mapping f is a one-to-one mapping between A' and B' may be expressed as

$$[A' \sim_f^{\odot} B'] = [\xi_f] = \bigodot_{i=1,\dots,|\xi_f|} \gamma_i \leftrightarrow \delta_i. \tag{32}$$

First, we shall prove the following simple but important claim.

Claim 1 *To each ξ_f, $f \in \mathrm{Bij}^*(A', B')$, such that $\gamma_j = \delta_i = \bot$ for some $1 \le i, j \le |\xi_f|$, there exists $g \in \mathrm{Bij}^*(A', B')$ such that $|\xi_g| = |\xi_f| - 1$ and $[\xi_f] \le [\xi_g]$.*

Note that, by the presumption, we have $|\xi_f| \ge 2$. If $i = j$, then it suffices to consider $g \in \mathrm{Bij}(A', B')$ such that the pair $(\gamma_i, \delta_i) = (\bot, \bot)$ is omitted in ξ_g. Let $i \neq j$. By (6) of Proposition 2.1:

$$(\gamma_i \leftrightarrow \bot) \odot (\bot \leftrightarrow \delta_j) \le \gamma_i \leftrightarrow \delta_j \tag{33}$$

for $\odot \in \{\wedge, \otimes\}$. Hence, we can simply determine $g \in \mathrm{Bij}(A', B')$ such that

$$\xi_g = \begin{pmatrix} \gamma_1 & \cdots & \gamma_i & \cdots & \gamma_m \\ \delta_1 & \cdots & \delta_j & \cdots & \delta_m \end{pmatrix}, \tag{34}$$

where the pair $(\gamma_j, \delta_i) = (\bot, \bot)$ is omitted in (34). Clearly, $|\xi_g| = |\xi_f| - 1$ and $[\xi_g] \ge [\xi_f]$ follows from (33).

A simple consequence of a finite procedure based on the first claim is the second claim.

Claim 2 *For any $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ and $f \in \mathrm{Bij}^*(A', B')$, there exists $g \in \mathrm{Bij}^*(A', B')$ such that the first or the second row of ξ_g do not contain $\bot$ and $[\xi_g] \ge [\xi_f]$. Moreover, $\mathrm{Dom}(g) = \mathrm{Supp}(A)$ or $\mathrm{Ran}(g) = \mathrm{Supp}(B)$.*

Further, let us suppose that $n = |\mathrm{Supp}(A)| \ge |\mathrm{Supp}(B)|$ (the opposite inequality can be proved analogously) and consider $E \in \mathrm{cls}(A)$ and an arbitrary $F \in \mathrm{cls}(B)$ such that $\mathrm{Dom}(E) = \mathrm{Supp}(A)$ and $|\mathrm{Dom}(F)| = n$ (the existence of F is obvious). Since A is finite and $\mathrm{Dom}(E) = \mathrm{Supp}(A)$, then $\mathrm{Bij}(E, F) = \mathrm{Bij}^*(E, F) = \mathrm{Perm}(E, F)$. Put

$$\alpha = \bigvee_{f \in \mathrm{Perm}(E,F)} [E \sim_f^{\odot} F]. \tag{35}$$

[13] Note that f may be represented in more than one way. Clearly, if π is a permutation on the set $\{1, \dots, m\}$, then the matrix

$$\xi'_f = \begin{pmatrix} \gamma_{\pi(1)} & \cdots & \gamma_{\pi(i)} \\ \delta_{\pi(1)} & \cdots & \delta_{\pi(i)} \end{pmatrix}, \tag{31}$$

again represents the mapping f. Nevertheless, this fact is not important in our consideration.

By the presumption on E and F, we have

$$\xi_f = \begin{pmatrix} \gamma_1 & \cdots & \gamma_m \\ \delta_1 & \cdots & \delta_m \end{pmatrix}, \tag{36}$$

for any $f \in \mathrm{Perm}(E,F)$, where $\gamma_i \neq \bot$ for all $i = 1,\dots,m$. Obviously, to each $g \in \mathrm{Perm}(E,F)$, there is a permutation π on $\{1,\dots,m\}$, and g may be represented by

$$\xi_g = \begin{pmatrix} \gamma_1 & \cdots & \gamma_m \\ \delta_{\pi(1)} & \cdots & \delta_{\pi(m)} \end{pmatrix}. \tag{37}$$

If $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ and $f \in \mathrm{Bij}^*(A',B')$, then by the second claim there exists $h \in \mathrm{Bij}^*(A',B')$ which representation ξ_h has the form (37) for a suitable permutation π and $[\xi_h] \geq [\xi_f]$. Hence, there exists $g \in \mathrm{Perm}(E,F)$ with $\xi_g = \xi_h$, which implies

$$\alpha \geq [E \sim_g^{\odot} F] = [\xi_g] = [\xi_h] \geq [\xi_f] = [A' \sim_f^{\odot} B'].$$

Thus, $[A \sim^{\odot} B] = \alpha$.

Finally, let $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$, $|\mathrm{Dom}(C)| = |\mathrm{Dom}(D)| = m$ and $f \in \mathrm{Perm}(C,D)$. Put $z = \{f(x) \mid x \in \mathrm{Supp}(A)\}$, $E = C \upharpoonright \mathrm{Supp}(A)$ and $F = D \upharpoonright z$. Obviously, each $g \in \mathrm{Perm}(E,F)$ can be extended to $g' \in \mathrm{Perm}(C,D)$ (i.e., $g' \upharpoonright \mathrm{Supp}(E) = g$) and $[E \sim_g^{\odot} F] = [C \sim_{g'}^{\odot} D]$. Hence, by (35), we obtain

$$\alpha \leq \bigvee_{f \in \mathrm{Perm}(C,D)} [C \sim_f^{\odot} D].$$

The equality in (29) immediately follows from the fact that $\alpha = [A \sim^{\odot} B]$. □

The following is a straightforward consequence.

COROLLARY 5.5 *If* $A \in \mathfrak{Fin}$, *then* $[A \sim^{\odot} B] = \top$ *for all* $B \in \mathrm{cls}(A)$.

5.2. Graded equipollence as a fuzzy class relation on $\mathfrak{Fin}$

As we have mentioned above, the class $\mathfrak{Fin} \times \mathfrak{Fin}$ is a subclass of $\mathfrak{Count}$. Hence, we can define the following "fuzzy class" (precisely "fuzzy class relation") in $\mathfrak{Count}$, which is analogous to the equivalence relation "to be equipollent" on the class of all (finite) sets (for details about fuzzy classes in a universe of sets, we refer to [8, 10]).

THEOREM 5.6 *The fuzzy class relation* $\sim^{\odot}\colon \mathfrak{Fin} \times \mathfrak{Fin} \to L$ *is a similarity relation (*$\otimes$*-equivalence) on the class* $\mathfrak{Fin}$*, i.e.,*

(i) $[A \sim^{\odot} A] = \top$,

(ii) $[A \sim^{\odot} B] = [B \sim^{\odot} A]$,

(iii) $[A \sim^{\odot} B] \otimes [B \sim^{\odot} C] \leq [A \sim^{\odot} C]$,

which holds for arbitrary fuzzy sets $A, B, C \in \mathfrak{Ffin}$.

Proof. Let $A, B, C \in \mathfrak{Ffin}$.

(i) According to Corollary 5.5, the fuzzy class relation $\sim^{\odot}$ is reflexive.

(ii) If $D \in \text{cls}(A)$, $E \in \text{cls}(B)$ such that $|\text{Dom}(D)| = \text{Dom}(E)| = m$ and $f \in \text{Perm}(D, E)$, then obviously $f^{-1} \in \text{Perm}(E, D)$. One can simply verify that $[D \sim_f E] = [E \sim_{f^{-1}} D]$. Hence, by Theorem 5.4, we obtain

$$[A \sim^{\odot} B] = \bigvee_{f \in \text{Perm}(D,E)} [D \sim^{\odot}_f E] = \bigvee_{g \in \text{Perm}(E,D)} [E \sim^{\odot}_g D] = [B \sim^{\odot} A],$$

and $\sim^{\odot}$ is symmetric.

(iii) Without loss of generality, let D and E be the same fuzzy sets as in (ii) and suppose $F \in \text{cls}(C)$ such that $|\text{Dom}(F)| = m$. According to Theorem 5.4 and by (11) of Proposition 2.1, we have

$$[A \sim^{\odot} B] \otimes [B \sim^{\odot} C] = \Big(\bigvee_{f \in \text{Perm}(D,E)} [D \sim^{\odot}_f E]\Big) \otimes \Big(\bigvee_{g \in \text{Perm}(E,F)} [E \sim^{\odot}_g F]\Big) = \bigvee_{f \in \text{Perm}(D,E)} \bigvee_{g \in \text{Perm}(E,F)} [D \sim^{\odot}_f E] \otimes [E \sim^{\odot}_g F],$$

where

$$[D \sim^{\odot}_f E] \otimes [E \sim^{\odot}_g F] = \bigodot_{x \in \text{Dom}(D)} (D(x) \leftrightarrow E(f(x))) \otimes \bigodot_{y \in \text{Dom}(E)} (E(y) \leftrightarrow F(g(y))) \leq \bigodot_{x \in \text{Dom}(D)} \bigodot_{y \in \text{Dom}(E)} \big((D(x) \leftrightarrow E(f(x))) \otimes (E(y) \leftrightarrow F(g(y)))\big) \leq \bigodot_{x \in \text{Dom}(D)} \big((D(x) \leftrightarrow E(f(x))) \otimes (E(f(x)) \leftrightarrow F(g(f(x))))\big) \leq \bigodot_{x \in \text{Dom}(D)} (D(x) \leftrightarrow F(g \circ f(x))) = [D \sim^{\odot}_{g \circ f} F].$$

Hence, we obtain

$$[A \sim^{\odot} B] \otimes [B \sim^{\odot} C] = \bigvee_{f \in \text{Perm}(D,E)} \bigvee_{g \in \text{Perm}(E,F)} [D \sim^{\odot}_f E] \otimes [E \sim^{\odot}_g F] \leq \bigvee_{f \in \text{Perm}(D,E)} \bigvee_{g \in \text{Perm}(E,F)} [D \sim^{\odot}_{g \circ f} F] \leq \bigvee_{h \in \text{Perm}(D,F)} [D \sim^{\odot}_h F] = [A \sim^{\odot} C],$$

and the fuzzy class relation $\sim^{\odot}$ is transitive. □

DEFINITION 5.3 The fuzzy class relation $\sim^{\odot}$ is called a *graded equipollence* of finite fuzzy sets.

The following corollary shows that our definition of graded equipollence is correct, which means that it does not depend on the choice of equivalent fuzzy sets.

COROLLARY 5.7 *Let* $A, B \in \mathfrak{Fin}$ *and* $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$. *Then,* $[A \sim^{\odot} B] = [C \sim^{\odot} D]$.

PROOF. Using Corollary 5.5, we have $[A \sim^{\odot} C] = [B \sim^{\odot} D] = \top$. From the symmetry and transitivity of $\sim^{\odot}$, we have

$$[A \sim^{\odot} B] \leq [C \sim^{\odot} A] \otimes [A \sim^{\odot} B] \otimes [B \sim^{\odot} D] \leq [C \sim^{\odot} D].$$

Because the opposite inequality is also true, we obtain the desired equality. □

A generalization of the previous corollary is the statement enabling us to introduce classes of equipollent fuzzy sets, i.e., classes in which each pair of fuzzy sets is equipollent in the degree $\top$.

COROLLARY 5.8 *Let* $A, B \in \mathfrak{Fin}$ *and* $[C \sim^{\odot} A] = [D \sim^{\odot} B] = \top$. *Then,* $[A \sim^{\odot} B] = [C \sim^{\odot} D]$.

6. A functional approach to cardinal theory based on graded equipollence

As we have mentioned in the Introduction, we may distinguish two approaches to cardinality theory - functional and based on cardinal numbers. In this section, we shall investigate some of the familiar relations studied in the classical cardinal theory by the relation of equipollence of sets (restricted, of course, to the finite case) that are transformed to graded reasoning, i.e., when $\perp$ (false) and $\top$ (true) are replaced by a wider scale of truth values.

Recall that we write $a \sim b$ to denote that the sets a and b are equipollent. Let us start with a discussion on the Cantor-Bernstein theorem, which is among the most familiar theorems in set theory. One of its forms states that if a, b, c, d are sets such that $b \subseteq a$ and $d \subseteq b$ and $a \sim d$ and $b \sim c$, then $a \sim c$. A graded version of this theorem could be written as follows (cf. Theorem 4.7. in [25]):

GRADED CANTOR-BERNSTEIN THEOREM: *Let* $A, B, C, D \in \mathfrak{Fin}$ *such that* $B \subseteq A$ *and* $D \subseteq C$. *Then,*

$$[A \sim^{\odot} D] \odot [C \sim^{\odot} B] \leq [A \sim^{\odot} C]. \tag{38}$$

Unfortunately, we cannot prove this GCB theorem in such generality, and its truthfulness is an open problem for us. Here, we only show a weaker version of the GCB theorem as follows and then its truthfulness under some special conditions.

Theorem 6.1 *Let* $A, B, C, D \in \mathfrak{Fin}$ *such that* $B \subseteq A$, $D \subseteq C$ *and* $\mathrm{Dom}(A) = \mathrm{Dom}(B)$, $\mathrm{Dom}(C) = \mathrm{Dom}(D)$ *and* $|\mathrm{Dom}(A)| = |\mathrm{Dom}(C)| = m$. *Then,*

$$[A \sim_f^{\odot} D] \odot [B \sim_f^{\odot} C] \leq [A \sim_f^{\odot} C] \tag{39}$$

holds for any $f \in \mathrm{Perm}(A, C)$.

Proof. Let $A, B, C, D \in \mathfrak{Fin}$ satisfy the presumption of the theorem. Using (10) of Proposition 2.1 and $\alpha \odot \beta \leq \alpha \wedge \beta$ for any $\alpha, \beta \in L$, we have

$$[A \sim_f^{\odot} D] \odot [B \sim_f^{\odot} C] =$$
$$\Big(\bigodot_{x \in \mathrm{Dom}(A)} (A(x) \leftrightarrow D(f(x))) \Big) \odot \Big(\bigodot_{x \in \mathrm{Dom}(B)} (B(x) \leftrightarrow C(f(x))) \Big) \leq$$
$$\bigodot_{x \in \mathrm{Dom}(A)} \big((A(x) \leftrightarrow D(f(x))) \wedge (B(x) \leftrightarrow C(f(x))) \big) \leq$$
$$\bigodot_{x \in \mathrm{Dom}(A)} \big(((A(x) \vee B(x)) \leftrightarrow (D(f(x)) \vee C(f(x))) \big) = [A \sim_f^{\odot} C],$$

and the proof is finished. □

Remark 6.1 One may simply see that the inequality (39) does not imply (38) applying the supremum over all mappings from $\mathrm{Perm}(A, C)$.

Remark 6.2 Let us note that if one applies (9) instead of (10) in the previous proof, then $[A \sim_f^{\odot} D] \odot [B \sim_f^{\odot} C] \leq [B \sim_f^{\odot} D]$ holds. Thus, we obtain a more general form of (39)

$$[A \sim_f^{\odot} D] \odot [B \sim_f^{\odot} C] \leq [A \sim_f^{\odot} C] \wedge [B \sim_f^{\odot} D]. \tag{40}$$

Theorem 6.2 *Let* $\mathbf{L}$ *be a linearly ordered rdr-lattice and* $A, B, C, D \in \mathfrak{Fin}$ *such that* $B \subseteq A$ *and* $D \subseteq C$. *Then, we*

$$[A \sim^{\wedge} D] \wedge [C \sim^{\wedge} B] \leq [A \sim^{\wedge} C]. \tag{41}$$

Proof. This will be proved using Theorems 6.1 and 7.9 on page 39. □

Remark 6.3 Analogously to Remark 6.2, if $\mathbf{L}$ is a linearly ordered rdr-lattice, then we can write (41) in a more general form

$$[A \sim^{\wedge} D] \wedge [C \sim^{\wedge} B] \leq [A \sim^{\wedge} C] \wedge [B \sim^{\wedge} D]. \tag{42}$$

A consequence of Theorem 6.2 and Remark 6.2 is the following form of the graded Cantor-Bernstein theorem (cf. Corollary 4.8 in [25]), which generalizes a more commonly used version of the Cantor-Bernstein theorem.

Corollary 6.3 *Let* $\mathbf{L}$ *be a linearly ordered rdr-lattice and* $A, B, C \in \mathfrak{Fin}$ *such that* $A \subseteq B \subseteq C$. *Then,*

$$[A \sim^{\wedge} C] \leq [A \sim^{\wedge} B] \wedge [B \sim^{\wedge} C]. \tag{43}$$

Proof. Let $\mathbf{L}$ be a linearly ordered rdr-lattice and $A \subseteq B \subseteq C$. By Remark 6.3, we simply obtain

$$[A \sim^{\wedge} C] \odot [B \sim^{\wedge} B] = [A \sim^{\wedge} C] \leq [A \sim^{\wedge} B] \wedge [B \sim^{\wedge} C],$$

where $[B \sim^{\wedge} B] = \top$ by the reflexivity of $\sim^{\wedge}$. □

Let a, b, c, d be sets such that $a \sim c$ and $b \sim d$. Then, it is well-known that

$$\begin{aligned} &a \cup b \sim c \cup d, \quad \text{whenever } a \cap b = \emptyset \text{ and } c \cap d = \emptyset,\\ &a \times b \sim c \times d,\\ &a \sqcup b \sim c \sqcup d. \end{aligned}$$

These equalities guarantee the uniqueness of $+$ and $\cdot$ (i.e., the independency of the choice of representatives), which could be defined on the class of equipollent sets. The following theorem shows the graded versions of these and two further statements.

Theorem 6.4 *Let* $A, B, C, D \in \mathfrak{Fin}$. *Then,*

(i) $[A \sim^{\odot} B] \leq [\overline{A} \sim^{\odot} \overline{B}]$

(ii) $[A \sim^{\odot} B] \otimes [C \sim^{\odot} D] \leq [A \otimes C \sim^{\odot} B \otimes D]$,

(iii) $[A \sim^{\odot} B] \otimes [C \sim^{\odot} D] \leq [A \times C \sim^{\odot} B \times D]$,

(iv) if $\mathrm{Supp}(A) \cap \mathrm{Supp}(B) = \mathrm{Supp}(C) \cap \mathrm{Supp}(D) = \emptyset$, *then*

$$[A \sim^{\odot} C] \otimes [B \sim^{\odot} D] \leq [A \cup B \sim^{\odot} C \cup D],$$

(v) $[A \sim^{\odot} B] \otimes [C \sim^{\odot} D] \leq [A \sqcup C \sim^{\odot} B \sqcup D]$.

Proof. Let $A, B, C, D \in \mathfrak{Fin}$. Without loss of generality, let us suppose that the domains of A, B, C, D have the same cardinality m.

(i) A simple consequence of (8) of Proposition 2.1 is the inequality $a \leftrightarrow b \leq (a \to \bot) \leftrightarrow (b \to \bot)$. By the definition of the complement, we have $\mathrm{Dom}(A) = \mathrm{Dom}(\overline{A})$. Hence, we obtain

$$[A \sim^{\odot} B] = \bigvee_{f \in \mathrm{Perm}(A,B)} \bigodot_{x \in \mathrm{Dom}(A)} (A(x) \leftrightarrow B(f(x))) \leq$$
$$\bigvee_{f \in \mathrm{Perm}(A,B)} \bigodot_{x \in \mathrm{Dom}(\overline{A})} ((A(x) \to \bot) \leftrightarrow (B(f(x)) \to \bot)) = [\overline{A} \sim^{\odot} \overline{B}],$$

and (i) is proved.

(ii) From (7) and (11) of Proposition 2.1 and $(\bigodot_{i\in I}\alpha_i)\otimes(\bigodot_{i\in I}\beta_i)\leq\bigodot_{i\in I}(\alpha_i\otimes\beta_i)$ for a finite index set I, we have

$$
\begin{aligned}
&[A\sim^{\odot}B]\otimes[C\sim^{\odot}D]=\Big(\bigvee_{f\in\mathrm{Perm}(A,B)}\bigodot_{x\in\mathrm{Dom}(A)}\big(A(x)\leftrightarrow B(f(x))\big)\Big)\otimes\\
&\Big(\bigvee_{g\in\mathrm{Perm}(C,D)}\bigodot_{y\in\mathrm{Dom}(C)}\big(C(y)\leftrightarrow D(g(y))\big)\Big)\leq\\
&\bigvee_{(f,g)\in\mathrm{Perm}(A\otimes C,B\otimes D)}\bigodot_{(x,y)\in\mathrm{Dom}(A\otimes C)}\big((A(x)\leftrightarrow B(f(x)))\otimes(C(y)\leftrightarrow D(g(y)))\big)\leq\\
&\bigvee_{(f,g)\in\mathrm{Perm}(A\otimes C,B\otimes D)}\bigodot_{(x,y)\in\mathrm{Dom}(A\otimes C)}((A(x)\otimes C(y))\leftrightarrow(B(f(x))\otimes D(g(y))))=\\
&\bigvee_{(f,g)\in\mathrm{Perm}(A\otimes C,B\otimes D)}\bigodot_{(x,y)\in\mathrm{Dom}(A\otimes C)}((A\otimes C)(x,y)\leftrightarrow(B\otimes D)(f(x),g(y)))\leq\\
&\bigvee_{h\in\mathrm{Perm}(A\otimes C,B\otimes D)}\bigodot_{z\in\mathrm{Dom}(A\otimes C)}((A\otimes C)(z)\leftrightarrow(B\otimes D)(h(z)))=\\
&[A\otimes C\sim^{\odot}B\otimes D],
\end{aligned}
$$

and hence, (ii) is true.

(iii) This can be proved analogously using $(\alpha\leftrightarrow\beta)\otimes(\gamma\leftrightarrow\delta)\leq(\alpha\wedge\gamma)\leftrightarrow(\beta\wedge\delta)$ derived from (9) and $\alpha\otimes\beta\leq\alpha\wedge\beta$.

(iv) This follows from $(\alpha\leftrightarrow\beta)\otimes(\gamma\leftrightarrow\delta)\leq(\alpha\vee\gamma)\leftrightarrow(\beta\vee\delta)$ derived from (10) and $\alpha\otimes\beta\leq\alpha\wedge\beta$ and the fact that, for any $f\in\mathrm{Perm}(A,B)$ and $g\in\mathrm{Perm}(C,D)$, the mapping $f\cup g:\mathrm{Dom}(A)\cup\mathrm{Dom}(B)\to\mathrm{Dom}(C)\cup\mathrm{Dom}(D)$ defined by

$$
(f\cup g)(x)=\begin{cases} f(x), & x\in\mathrm{Dom}(A),\\ g(x), & x\in\mathrm{Dom}(B)\end{cases}
$$

belongs to $\mathrm{Perm}(A\cup B,C\cup D)$.

(v) Put $x=\mathrm{Dom}(A)\times\{1\}$ and $y=\mathrm{Dom}(C)\times\{2\}$ and define $A'(t,1)=A(t)$ and $C'(s,2)=C(s)$ for any $(t,1)\in x$ and $(s,2)\in y$. Analogously, let us establish B' and D'. One may simply check that $[A\sim^{\odot}A']=[B\sim^{\odot}B']=[C\sim^{\odot}C']=[D\sim^{\odot}D']=\top$. By Corollary 5.8, we obtain

$$[A\sim^{\odot}B]=[A'\sim^{\odot}B']\text{ and }[C\sim^{\odot}D]=[C'\sim^{\odot}D'].$$

Moreover, we have $\mathrm{Supp}(A')\cap\mathrm{Supp}(C')=\mathrm{Supp}(B')\cap\mathrm{Supp}(D')=\emptyset$. Since $A\sqcup C=A'\cup C'$ and $B\sqcup D=B'\cup D'$, then the desired inequality immediately follows from (iv). □

Under the presumption of the distributivity of $\wedge$ over $\vee$, we may state stronger inequalities for $\odot=\wedge$.

THEOREM 6.5 *Let* $\mathbf{L}$ *satisfy the prelinearity and* $A, B, C, D \in \mathfrak{Fin}$*. Then,*

(i) $[A \sim^{\wedge} B] \wedge [C \sim^{\wedge} D] \leq [A \otimes C \sim^{\wedge} B \otimes D]$,

(ii) $[A \sim^{\wedge} B] \wedge [C \sim^{\wedge} D] \leq [A \times C \sim^{\wedge} B \times D]$,

(iii) $[A \sim^{\wedge} B] \wedge [C \sim^{\wedge} D] \leq [A \sqcup C \sim^{\wedge} B \sqcup D]$,

(iv) if $\mathrm{Supp}(A) \cap \mathrm{Supp}(B) = \mathrm{Supp}(C) \cap \mathrm{Supp}(D) = \emptyset$*, then*

$$[A \sim^{\wedge} C] \wedge [B \sim^{\wedge} D] \leq [A \cup B \sim^{\wedge} C \cup D].$$

PROOF. This may be proved analogously as above, where the distributivity of $\wedge$ over $\vee$ is applied. □

If a, b are sets and $a \sim b$, then $\mathbf{P}(a) \sim \mathbf{P}(b)$, where $\mathbf{P}(a)$ and $\mathbf{P}(b)$ denote the power sets of a and b, respectively. The following theorem is a graded version of this classical statement with a restriction on the operation $\wedge$ used in the computation of degree to which $\mathbf{P}(A)$ and $\mathbf{P}(B)$ are equipollent. Let us note the difficulty that enforces the mentioned restriction. The problem arises when we want to prove

$$[A \sim_f^{\otimes} B] \leq [\mathbf{P}(A) \sim_{\hat{f}}^{\otimes} \mathbf{P}(B)],$$

which is equivalent to the verification of the following inequality

$$\bigotimes_{x \in \mathrm{Dom}(A)} (A(x) \leftrightarrow B(f(x))) \leq \bigotimes_{y \in \mathrm{Dom}(\mathbf{P}(A))} (\mathbf{P}(A)(y) \leftrightarrow \mathbf{P}(B)(\hat{f}(y))),$$

where $\hat{f}$ is an extension of f to the mapping of $\mathrm{Dom}(\mathbf{P}(A))$ onto $\mathrm{Dom}(\mathbf{P}(B))$. One may see that the number of elements in domains of power fuzzy sets significantly grows with respect to an increasing number of elements in domains of original fuzzy sets. Because $\otimes$ is often decreasing in both arguments, the right side of the above inequality tends to $\bot$ (e.g., consider the Łukasiewicz t-norm that is archimedean [15], i.e., for any $\alpha \in [0, 1]$, there is $n \in \mathbb{N}$ such that $\alpha^n = 0$) for fuzzy sets with a greater number of element in their domains. However, in this case, there is a problem to ensure that the left side of the inequality tends to $\bot$ in general.

THEOREM 6.6 *Let* $A, B \in \mathfrak{Fin}$*. Then,*

$$[A \sim^{\odot} B] \leq [\mathbf{P}(A) \sim^{\wedge} \mathbf{P}(B)]. \tag{44}$$

PROOF. Due to Proposition 3.3, let us suppose that $A, B \in \mathfrak{Fin}$ such that $\mathrm{Dom}(A) = \mathrm{Dom}(B)$. If $f \in \mathrm{Perm}(A, B)$, then we may define $\hat{f} : \mathrm{Dom}(\mathbf{P}(A)) \to \mathrm{Dom}(\mathbf{P}(B))$ by

$$\hat{f}(y) = \{f(x) \mid x \in y\} \tag{45}$$

for any $y \in \mathrm{Dom}(\mathbf{P}(A))$. One may simply check that $\hat{f} \in \mathrm{Perm}(\mathbf{P}(A), \mathbf{P}(B))$. Let $\hat{f} \in \mathrm{Perm}(\mathbf{P}(A), \mathbf{P}(B))$. By (23), we have

$$[\mathbf{P}(A) \sim^{\odot} \mathbf{P}(B)] \geq [\mathbf{P}(A) \sim^{\odot}_{\hat{f}} \mathbf{P}(B)] =$$

$$\bigwedge_{y \in \mathrm{Dom}(\mathbf{P}(A))} \big(\mathbf{P}(A)(y) \leftrightarrow \mathbf{P}(B)(\hat{f}(y))\big) =$$

$$\bigwedge_{y \in \mathrm{Dom}(\mathbf{P}(A))} \Big((\bigwedge_{x \in y} A(x)) \leftrightarrow (\bigwedge_{z \in f(y)} B(z)) \Big) \geq$$

$$\bigwedge_{y \in \mathrm{Dom}(\mathbf{P}(A))} \bigwedge_{x \in y} (A(x) \leftrightarrow B(f(x))) \geq \bigwedge_{y \in \mathrm{Dom}(\mathbf{P}(A))} \bigodot_{x \in y} (A(x) \leftrightarrow B(f(x))) \geq$$

$$\bigwedge_{y \in \mathrm{Dom}(\mathbf{P}(A))} \bigodot_{x \in \mathrm{Dom}(A)} (A(x) \leftrightarrow B(f(x))) = [A \sim^{\odot}_{f} B].$$

Since this inequality holds for any $f \in \mathrm{Perm}(A, B)$, then we obtain

$$[\mathbf{P}(A) \sim^{\odot} \mathbf{P}(B)] \geq \bigvee_{f \in \mathrm{Perm}(A,B)} [A \sim^{\odot}_{f} B] = [A \sim^{\odot} B],$$

and the proof is finished. □

One of the significant Cantor theorems states that a is not equipollent with its power, i.e., $a \not\sim \mathbf{P}(a)$. The following is a generalization of this statement for finite fuzzy sets, stating that A and $\mathbf{P}(A)$ cannot have the same number of elements.

Theorem 6.7 *Let $A \in \mathfrak{Fin}$. Then, $[A \sim^{\odot} \mathbf{P}(A)] < \top$.*

Proof. Let $A, C \in \mathfrak{Fin}$ such that $C \equiv A$ and $|\mathrm{Dom}(C)| = |\mathrm{Dom}(\mathbf{P}(A))|$. Since $C \equiv A$, then $\mathrm{Supp}(C) \subset \mathrm{Supp}(\mathbf{P}(A))$, and thus, for any $f \in \mathrm{Perm}(C, \mathbf{P}(A))$, there is $x \in \mathrm{Dom}(C) \setminus \mathrm{Supp}(C)$ such that $C(x) \leftrightarrow \mathbf{P}(A)(f(x)) < \top$. Hence, $[C \sim^{\odot}_{f} \mathbf{P}(A)] < \top$ holds for any $f \in \mathrm{Perm}(C, \mathbf{P}(A))$. Since $\mathrm{Perm}(C, \mathbf{P}(A))$ is a finite set, then also $[C \sim^{\odot} \mathbf{P}(A)] < \top$, and the statement is a consequence of Corollary 5.7. □

Example 6.4 If $A = \{1/a, 0.4/b\}$ and $\mathbf{P}(A) = \{1/\emptyset, 1/\{a\}, 0.4/\{b\}, 0.4/\{a, b\}\}$ are from Example 3.7, then with $C = \{1/a, 0.4/b, 0/c, 0/d\}$, one may simply check that

$$[A \sim^{\wedge} \mathbf{P}(A)] = [C \sim^{\wedge} \mathbf{P}(A)] = (1 \leftrightarrow 1) \wedge (1 \leftrightarrow 0.4) \wedge (0.4 \leftrightarrow 0) \wedge (0.4 \leftrightarrow 0) =$$
$$1 \wedge 0.4 \wedge 0.6 \wedge 0.6 = 0.4.$$

Hence, we have $0 < [A \sim^{\wedge} \mathbf{P}(A)] < 1$. On the other hand, we obtain $[A \sim^{\otimes} \mathbf{P}(A)] = 0$.

If a, b, c, d are sets such that $a \sim c$ and $b \sim d$, then $b^a \sim d^c$. The following theorem is a graded version of this statement with analogous restriction on $\wedge$ discussed for power fuzzy sets. Moreover, due to Remark 3.9, we can compare only fuzzy sets with the same number of elements (the exponentiation is dependent on the choice of universes). Finally, we suppose the finiteness of universes of fuzzy sets to ensure that the exponentiations are also finite fuzzy sets.

THEOREM 6.8 *Let* $A, B, C, D \in \mathfrak{Ffin}$ *such that* $|\mathrm{Dom}(A)| = |\mathrm{Dom}(C)| = m$ *and* $|\mathrm{Dom}(B)| = |\mathrm{Dom}(D)| = n$*. Then,*

$$[A \sim^{\odot} C] \otimes [B \sim^{\odot} D] \leq [B^A \sim^{\wedge} C^D]. \tag{46}$$

PROOF. Let $A, B, C, D \in \mathfrak{Ffin}$ satisfy the presumption of the theorem and $f \in \mathrm{Perm}(A, C)$, $g \in \mathrm{Perm}(B, D)$. Define a mapping $r_{fg} : \mathrm{Dom}(B^A) \to \mathrm{Dom}(D^C)$ by

$$r_{fg}(h) = g \circ h \circ f^{-1}, \tag{47}$$

where f^{-1} denotes the inverse mapping to f. We shall prove that r_{fg} is a one-to-one mapping of $\mathrm{Dom}(B^A)$ onto $\mathrm{Dom}(D^C)$. By the definition, the domain of h is $\mathrm{Dom}(A)$. Since f^{-1} is a one-to-one mapping of $\mathrm{Dom}(C)$ onto $\mathrm{Dom}(A)$, we obtain that the domain of $r_{fg}(h)$ is $\mathrm{Dom}(C)$. Hence, r_{fg} is correctly defined. If $r_{fg}(h) = r_{fg}(h')$ for $h, h' \in \mathrm{Dom}(B^A)$, then $g \circ h \circ f^{-1} = g \circ h' \circ f^{-1}$, which implies

$$h = (g^{-1} \circ g) \circ h \circ f^{-1} \circ f = (g^{-1} \circ g) \circ h' \circ f^{-1} \circ f = h',$$

where g^{-1} denotes the inverse mapping to g and r_{fg} is a one-to-one mapping. If $h' \in \mathrm{Dom}(D^C)$, then $h = g^{-1} \circ h' \circ f$ belongs to $\mathrm{Dom}(B^A)$ and $r_{fg}(h) = h'$. Hence, r_{fg} is onto, and thus $r_{fg} \in \mathrm{Perm}(B^A, D^C)$. Then, using (8) of Proposition 2.1, we have

$$\begin{aligned}
&[A \sim^{\odot}_f C] \otimes [B \sim^{\odot}_f D] = \\
&\Big(\bigodot_{x \in \mathrm{Dom}(A)} (A(x) \leftrightarrow C(f(x))) \Big) \otimes \Big(\bigodot_{y \in \mathrm{Dom}(B)} (B(y) \leftrightarrow C(g(y))) \Big) \leq \\
&\bigwedge_{x \in \mathrm{Dom}(A)} \big((A(x) \leftrightarrow C(f(x))) \otimes (B(h(x)) \leftrightarrow D(g \circ h(x))) \big) \leq \\
&\bigwedge_{x \in \mathrm{Dom}(A)} \big((A(x) \to B(h(x))) \leftrightarrow (C(f(x)) \to D(g \circ h(x))) \big) \leq \\
&\Big(\bigwedge_{x \in \mathrm{Dom}(A)} (A(x) \to B(h(x))) \Big) \leftrightarrow \Big(\bigwedge_{y \in \mathrm{Dom}(C)} (C(y) \to D(g \circ h \circ f^{-1}(y))) \Big) = \\
&B^A(h) \leftrightarrow D^C(r_{fg}(h)).
\end{aligned}$$

Since this inequality holds for any $h \in \mathrm{Dom}(B^A)$, then

$$[A \sim_f^{\odot} C] \otimes [B \sim_g^{\odot} D] \leq \bigwedge_{h \in \mathrm{Dom}(B^A)} (B^A(h) \leftrightarrow D^C(r_{fg}(h))) = [B^A \sim_{r_{fg}}^{\wedge} D^C].$$

Hence, we have

$$[A \sim^{\odot} C] \otimes [B \sim^{\odot} D] = \Big(\bigvee_{f \in \mathrm{Perm}(A,C)} [A \sim_f^{\odot} C] \Big) \otimes \Big(\bigvee_{g \in \mathrm{Perm}(B,D)} [B \sim_g^{\odot} D] \Big) =$$

$$\bigvee_{f \in \mathrm{Perm}(A,C)} \bigvee_{g \in \mathrm{Perm}(B,D)} \big([A \sim_f^{\odot} C] \otimes [B \sim_g^{\odot} D]\big) \leq$$

$$\bigvee_{f \in \mathrm{Perm}(A,C)} \bigvee_{g \in \mathrm{Perm}(B,D)} [B^A \sim_{r_{fg}}^{\wedge} D^C] \leq$$

$$\bigvee_{t \in \mathrm{Dom}(B^A)} [B^A \sim_t^{\wedge} D^C] = [B^A \sim^{\wedge} D^C],$$

and the proof is finished. □

Theorem 6.9 *Let $A, B, C \in \mathfrak{Ffin}$ such that their universes are finite. Then, $[C^{A \otimes B} \sim^{\odot} (C^B)^A] = \top$.*

Proof. Let $A, B, C \in \mathfrak{Fcount}$ and put $x = \mathrm{Dom}(A)$, $y = \mathrm{Dom}(B)$ and $z = \mathrm{Dom}(C)$. Recall that $(A \otimes B)(a, b) = A(a) \otimes B(b)$ for any $(a, b) \in x \times y$. Let $f : z^{x \times y} \to (z^y)^x$ be a one-to-one mapping defined by $f(h) = h_x$ for any $h : x \times y \to z$, where $h_x : x \to z^y$ defined by $h_x(a)(b) = h(a, b)$ for any $a \in x$ and $b \in y$. Then, for any $h \in z^{x \times y}$, we have

$$C^{A \otimes B}(h) = \bigwedge_{(a,b) \in x \times y} (A \otimes B)(a, b) \to C(h(a, b)) =$$

$$\bigwedge_{(a,b) \in x \times y} ((A(a) \otimes B(b)) \to C(h(a, b)) =$$

$$\bigwedge_{(a,b) \in x \times y} (A(a) \to (B(b) \to C(h(a, b)))) =$$

$$\bigwedge_{a \in x} \bigwedge_{b \in y} (A(a) \to (B(b) \to C(h(a, b)))) =$$

$$\bigwedge_{a \in x} (A(a) \to \bigwedge_{b \in y} (B(b) \to C(h_x(a)(b)))) =$$

$$\bigwedge_{a \in x} (A(a) \to C^B(h_x(a))) = (C^B)^A(h_x).$$

Since $f \in \mathrm{Perm}(C^{A\otimes B}, (C^B)^A)$, then

$$[C^{A\otimes B} \sim^{\odot} (C^B)^A] \geq [C^{A\otimes B} \sim^{\odot}_f (C^B)^A] =$$
$$\bigodot_{h\in z^{x\times y}} (C^{A\otimes B}(h) \leftrightarrow (C^B)^A(f(h))) =$$
$$\bigodot_{h\in z^{x\times y}} (C^{A\otimes B}(h) \leftrightarrow (C^B)^A(h_x)) = \bigodot_{h\in z^{x\times y}} \top = \top,$$

which concludes the proof. □

Remark 6.5 Note that an analogous relation to $\mathbf{P}(a) \sim \mathbf{2}^a$ cannot be proved for fuzzy sets.

One may see that a functional approach to the cardinal theory of finite fuzzy sets based on the concept of graded equipollence with respect to $\odot = \wedge$ or $\odot = \otimes$ can be further developed. Even if we admit proper (fuzzy) classes as objects of our theory, we may introduce the concept of cardinal as a class of equipollent fuzzy sets in the degree $\top$ and write the proposed theory in the language of cardinals. For example, if $|A|$ and $|B|$ denote cardinals, and we define the graded equivalence $\sim^{\odot}$ on the class of all cardinals by $|A| \sim^{\odot} |B| = [A \sim^{\odot} B]$ (the correctness of this definition follows from Corollary 5.8), then, for example, it holds

$$|A| \sim^{\odot} |B| \otimes |C| \sim^{\odot} |D| \leq |A \times B| \sim^{\odot} |C \times D|.$$

However, a more interesting question is whether there exist objects that could represent the mentioned classes of equipollent fuzzy sets. This question is discussed in the following section.

7. Graded equipollence and generalized cardinals

In this section, we shall define special objects that, under some presumptions, represent the classes of fuzzy sets equipollent in the degree $\top$. Practically, we shall follow the idea of many authors investigating fuzzy cardinality, including Zadeh [26], Ralescu [19] and, in particular, Wygralak [21, 22, 23, 25], who developed the entire cardinal theory for finite and infinite fuzzy sets.

7.1. Non-increasing generalized cardinals

Let $A \in \mathfrak{Ffin}$ be a finite fuzzy set and define a fuzzy class $p^{\odot}_A : \mathbb{N} \times \mathfrak{Fin} \to L$ by

$$p^{\odot}_A(i, y) = \begin{cases} \bigvee_{z\subseteq y, |z|=i} \bigodot_{x\in z} A(x), & \text{if } y \subseteq \mathrm{Dom}(A); \\ \bot, & \text{otherwise,} \end{cases} \tag{48}$$

for any $i \in \mathbb{N}$. Note that we divide the definition of $p^{\odot}_A$ into two parts to ensure its correctness. In fact, if $y \nsubseteq \mathrm{Dom}(A)$ and $z \subseteq y$ such that $|z| = i$ and, for instance, $z \cap \mathrm{Dom}(A) = \emptyset$, then $\bigodot_{x\in z} A(x)$ does not make sense because $A(x)$ is not defined for any $x \in z$. Therefore, we define $p^{\odot}_A(i, y) = \bot$ for $y \nsubseteq \mathrm{Dom}(A)$.

The following proposition summarizes some of the properties of $p_A(-, y)$.

PROPOSITION 7.1 *Let* $A \in \mathfrak{Ffin}$ *and* $y \in \mathfrak{Fin}$ *such that* $y \subseteq \mathrm{Dom}(A)$. *Then, for any* $i, j \in \mathbb{N}$, *we have*

(i) $p^{\odot}_A(i, y) \leq p^{\odot}_A(j, y)$, *if* $i \geq j$,

(ii) $p^{\odot}_A(i, y) = \bot$, *if* $i > |y|$,

(iii) $p^{\odot}_A(0, y) = \top$,

(iv) $p^{\odot}_A(1, y) = \bigvee_{x \in y} A(x)$,

(v) $p^{\odot}_A(0, \emptyset) = \top$ *and* $p^{\odot}_A(i, \emptyset) = \bot$ *for any* $i > 0$,

(vi) $p^{\odot}_A(-, y)$ *is a non-increasing generalized cardinal.*

PROOF. (i) Let $A \in \mathfrak{Ffin}$, $y \in \mathfrak{Fin}$ and $i, j \in \mathbb{N}$. If $i \geq j$ such that there exists $z \subseteq y$ with $|z| = i$, then there exists $z' \subset z$ with $|z'| = j$ and

$$\bigodot_{x \in z} A(x) \leq \bigodot_{x \in z'} A(x).$$

Hence, we obtain $p^{\odot}_A(i, y) \leq p^{\odot}_A(j, y)$. If there is no subset $z \subseteq y$ with $|z| = i$, then $p^{\odot}_A(i, y) = \bigvee \emptyset = \bot$. Hence, trivially, $p^{\odot}_A(i, y) \leq p^{\odot}_A(j, y)$.

(ii) If $i > |y|$, then there is no $z \subseteq y$ with $|z| = i$, and as above, we obtain $p^{\odot}_A(i, y) = \bigvee \emptyset = \bot$.

(iii) This immediately follows from $\bigodot_{x \in \emptyset} A(x) = \top$.[14]

(iv) This follows from

$$p^{\odot}_A(1, y) = \bigvee_{\substack{z \subseteq y, \\ |z| = 1}} \bigodot_{x \in z} A(x) = \bigvee_{x \in y} A(x).$$

(v) This immediately follows from (ii) and (iii).

(vi) If $i \leq j \leq k$, then $p^{\odot}_A(k, y) \leq p^{\odot}_A(j, y)$ by (i), whence also $p^{\odot}_A(i, y) \odot p^{\odot}_A(k, y) \leq p^{\odot}_A(j, y)$. □

PROPOSITION 7.2 *Let* $\mathbf{L}$ *be linearly ordered. Then,*

$$p^{\otimes}_A(i, y) = \bigotimes_{k=0}^{i} p^{\wedge}_A(k, y) \tag{49}$$

holds for any $A \in \mathfrak{Ffin}$ *and* $(i, y) \in \mathbb{N} \times \mathfrak{Fin}$.

PROOF. Let $A \in \mathfrak{Ffin}$ and $y \in \mathfrak{Fin}$. If $y \not\subseteq \mathrm{Dom}(A)$, then the equality is trivially true by the definition. Let $y \subseteq \mathrm{Dom}(A)$. If $y = \emptyset$, then again, the equality is satisfied by (v). Let $y = \{x_1, \ldots, x_m\}$. Due to the linearity of $\mathbf{L}$,

[14] Note that $\bigwedge \emptyset = \top$ holds from the infimum and $\bigotimes \emptyset = \top$ follows from the definition of the countable multiplication.

for each $1 < i \leq m$ there exists $x_{j_i} \in y$ such that $p^{\wedge}_A(i,y) = A(x_{j_i})$. Thus, we trivially have $p^{\wedge}_A(0,y) = p^{\otimes}_A(0,y) = \top$ and $p^{\wedge}_A(i,y) = p^{\otimes}_A(i,y) = \bot$ for $i > m$, which implies (58). Since $A(x_{j_k}) \geq A(x_{j_{k+1}})$ holds for any $1 \leq k < m-1$, then

$$p^{\otimes}_A(i,y) = \bigvee_{\substack{z \subseteq y \\ |z|=i}} \bigotimes_{x \in z} A(x) = \bigotimes_{k=1}^{i} A(x_{j_k}) = \bigotimes_{k=0}^{i} p^{\wedge}_A(k,y),$$

holds for any $1 \leq i \leq m$. □

Example 7.1 Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz rdr-lattice and

$$A = \{0.5/a, 0.8/b, 0.1/c, 0.4/d, 0/e\}.$$

Put $y = \mathrm{Dom}(A)$. Then,

$$p^{\wedge}_A(-,y) = \{1/0, 0.8/1, 0.5/2, 0.4/3, 0.1/4, 0/5, 0/6, \dots\},$$
$$p^{\otimes}_A(-,y) = \{1/0, 0.8/1, 0.3/2, 0/3, 0/4, \dots\}.$$

It should be noted that we obtain the same result for any y such that $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$ (see Proposition 7.4 introduced later).

Remark 7.2 If $L = [0,1]$, $\odot = \wedge$ and $y = \mathrm{Supp}(A)$ (or equivalently $y = \mathrm{Dom}(A)$), then the generalized cardinal number in the sense of FGCount is usually denoted by a vector

$$([A]_0, [A]_1, \dots, [A]_n, (0)), \tag{50}$$

where $[A]_i = p^{\wedge}_A(i,y)$ and $n = |\mathrm{Supp}(A)|$ (see Section 2.5 in [25] for a survey of approaches to cardinalities of fuzzy sets). If we consider $\otimes$ to be a continuous t-norm and define $[A]_i = p^{\otimes}_A(i,y)$, then the vector in (50) precisely defines cardinals for the generalized FGCounts, which has been proposed by Wygralak in [24] (see [25]).

The following proposition shows the relation between equivalent fuzzy sets and the equality generalized cardinals derived from the fuzzy classes under a suitable choice of set y.

Proposition 7.3 *If $A, B \in \mathfrak{Ffin}$, $A \equiv B$ and $y \in \mathfrak{Fin}$ such that $y \subseteq \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, then $p^{\odot}_A(-,y) = p^{\odot}_B(-,y)$.*

Proof. Since $B \in \mathrm{cls}(A)$ and $y \subseteq \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, then $A \restriction y = B \restriction y$. Hence, we have

$$p^{\odot}_A(i,y) = \bigvee_{\substack{z \subseteq y \\ |z|=i}} \bigodot_{x \in z} A(x) = \bigvee_{\substack{z \subseteq y \\ |z|=i}} \bigodot_{x \in z} A \restriction y(x) =$$
$$\bigvee_{\substack{z \subseteq y \\ |z|=i}} \bigodot_{x \in z} B \restriction y(x) = \bigvee_{\substack{z \subseteq y \\ |z|=i}} \bigodot_{x \in z} B(x) = p^{\odot}_B(i,y). \quad \square$$

A simple but useful consequence of the previous proposition is as follows.

PROPOSITION 7.4 *If* $A, B \in \mathfrak{F}\mathfrak{fin}$, $A \equiv B$ *and* $y, y' \in \mathfrak{Fin}$ *such that* $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$ *and* $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$, *then* $p_A^{\odot}(-, y) = p_B^{\odot}(-, y')$.

PROOF. By Proposition 7.3, it suffices to prove that $p_A^{\odot}(-, \mathrm{Supp}(A)) = p_A^{\odot}(-, y)$ for any $y \in \mathfrak{F}\mathfrak{fin}$ such that $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$. Let $i > |\mathrm{Supp}(A)|$. Then, we simply obtain $p_A^{\odot}(i, \mathrm{Supp}(A)) = \bot = p_A^{\odot}(i, y)$. If $i \leq |\mathrm{Supp}(A)|$, then the substantial subsets of y in the derivation of $p_A^{\odot}(i, y)$ are only subsets of $\mathrm{Supp}(A)$. Hence, we again obtain $p_A^{\odot}(i, \mathrm{Supp}(A)) = p_A^{\odot}(i, y)$. □

The following statement gives an interesting relation between the equipollence of finite fuzzy sets and the equivalence of generalized cardinals derived from them.

THEOREM 7.5 *Let* $A, B \in \mathfrak{F}\mathfrak{fin}$ *and* $y, y' \in \mathfrak{Fin}$ *such that* $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$ *and* $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$. *Then,*

$$[A \sim^{\odot} B] \leq [p_A^{\odot}(-, y) \approx p_B^{\odot}(-, y')]. \tag{51}$$

PROOF. Without loss of generality, let us suppose that $A, B \in \mathfrak{F}\mathfrak{fin}$ are fuzzy sets such that $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$ and put $y = \mathrm{Dom}(A)$ and $y' = \mathrm{Dom}(B)$; otherwise, we may chose equivalent fuzzy sets and use Corollary 5.7 and Proposition 7.4. If $y = \emptyset$, then (51) is a simple consequence of (v) of Proposition 7.1. Let $y \neq \emptyset$. By the definition of $\approx$, it is sufficient to show that

$$[A \sim^{\odot} B] \leq p_A^{\odot}(i, y) \leftrightarrow p_B^{\odot}(i, y') \tag{52}$$

for any $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$ and $f \in \mathrm{Perm}(A, B)$. To avoid the cases with the empty index set, one may see from (ii) and (iii) of Proposition 7.1 that (52) is trivially satisfied for $i > m$ and $i = 0$, respectively. For $0 < i \leq m$, we have

$$p_A^{\odot}(i, y) \leftrightarrow p_B^{\odot}(i, y') = \Bigg(\bigvee_{\substack{z \subseteq y \\ |z| = i}} \bigodot_{x \in z} A(x) \Bigg) \leftrightarrow \Bigg(\bigvee_{\substack{z' \subseteq y' \\ |z'| = i}} \bigodot_{x' \in z'} B(x') \Bigg) \geq$$

$$\bigwedge_{\substack{z \subseteq y \\ |z| = i}} \Bigg(\bigodot_{x \in z} A(x) \leftrightarrow \bigodot_{x' \in f(z)} B(x') \Bigg) \geq \bigwedge_{\substack{z \subseteq y \\ |z| = i}} \bigodot_{x \in z} (A(x) \leftrightarrow B(f(x))) \geq$$

$$\bigwedge_{z \subseteq y} \bigodot_{x \in z} (A(x) \leftrightarrow B(f(x))) = \bigodot_{x \in y} (A(x) \leftrightarrow B(f(x))) = [A \sim_f^{\odot} B],$$

where we applied the monotony of $\odot$ from which

$$\bigodot_{x \in y} (A(x) \leftrightarrow B(f(x))) \leq \bigodot_{x \in z} (A(x) \leftrightarrow B(f(x)))$$

holds for any $z \subseteq y$. Hence, we obtain

$$p_A^{\odot}(i, y) \leftrightarrow p_B^{\odot}(i, y') \geq \bigvee_{f \in \mathrm{Perm}(A,B)} [A \sim_f^{\odot} B] = [A \sim^{\odot} B],$$

which concludes the proof. □

As a result of this theorem, if A and B are equipollent in the degree $\top$, then generalized cardinals assigned to them by the fuzzy classes $p_A^{\odot}$ and $p_B^{\odot}$ for suitable $y, y' \in \mathfrak{Fin}$ coincide. Unfortunately, the reverse inequality is not true, even if one supposes that $p_A^{\odot}(-, y) = p_B^{\odot}(-, y')$, as the following simple example demonstrates.

Example 7.3 Let us consider the Łukasiewicz rdr-lattice and fuzzy sets

$$\begin{aligned} A &= \{1/a, 0.3/b, 0.1/c\} \\ B &= \{0.9/e, 0.2/f\} \\ C &= \{1/g, 0.3/h\}. \end{aligned}$$

Then, we have

$$\begin{aligned} p_A^{\otimes}(-, x) &= \{1/0, 1/1, 0.3/2, 0/3, \dots\}, \\ p_B^{\otimes}(-, y) &= \{1/0, 0.9/1, 0.1/2, 0/3, \dots\}, \\ p_C^{\otimes}(-, z) &= \{1/0, 1/1, 0.3/2, 0/3, \dots\}, \end{aligned}$$

where $x = \{a, b, c\}$, $y = \{e, f\}$ and $z = \{g, h\}$. Then, one may simply derive

$$\begin{aligned} &[A \sim^{\otimes} B] = 0.7 = 0.9 \otimes 0.9 \otimes 0.9 \leq 1 \wedge 0.9 \wedge 0.8 = 0.8 = [p_A^{\otimes}(-, x) \approx p_A^{\otimes}(-, y)], \\ &[A \sim^{\otimes} C] = 1 \otimes 1 \otimes 0.9 \leq 1 \wedge 1 \wedge 1 = 1 = [p_A^{\otimes}(-, x) \approx p_C^{\otimes}(-, z)]. \end{aligned}$$

It is easy to see that the failure in the second case is clearly caused by the fact that the Łukasiewicz multiplication is nilpotent and this allows $p_A^{\otimes}(-, x) = p_C^{\otimes}(-, z)$, although A and C are not equipollent in the degree $\top$. Thus, we can state that the generalized cardinals with respect to $\otimes$ are less sensitive for comparison of the cardinality of fuzzy sets than the graded equipollence.

We shall say that an rdr-lattice satisfies the *conditional cancellation law for* $\otimes$ if $\alpha \otimes \beta = \alpha \otimes \gamma > \bot$ implies $\beta = \gamma$ for any $\alpha, \beta, \gamma \in L$.[15] As examples, we may consider linearly ordered MV-algebras (see Example 2.2 here and Theorem 2.45 in [1]) or rdr-lattices determined by left-continuous t-norms and right-continuous t-conorms, where the left-continuous t-norms are Archimedian (see Example 2.1 here and Remark 2.19 in [15] or Section 1.4 in [25]).

[15] Note that the term "conditional cancellation law" is used from [15], and it is different from a standard concept of "cancellation law" for residuated lattices or "cancellative" residuated lattices (see, e.g., [13]).

Definition 7.1 Let $\mathbf{L}$ be linearly ordered and $A \in \mathfrak{Fin}$. A set $s \in \mathfrak{Fin}$ such that $s \subseteq \mathrm{Supp}(A)$ is said to be a *substantial segment* of A with respect to $\odot$ if

(i) $\bigodot_{x\in s} A(x) > \bot$,

(ii) if $z \in \mathfrak{Fin}$ such that $s \subset z \subseteq \mathrm{Supp}(A)$, then $\bigodot_{x\in z} A(x) = \bot$,

(iii) if $y \subseteq \mathrm{Supp}(A)$ satisfies (i) and (ii), then there exists a one-to-one mapping $f : y \to s$ such that $A(x) \leq A(f(x))$ for any $x \in y$.

The fuzzy set $A^s = A \upharpoonright s$ is called a *fuzzy substantial segment* of A with respect to $\odot$.

In the following part, we shall omit "with respect to $\odot$", and we shall say only a (fuzzy) substantial segment of A if we speak about (fuzzy) substantial segments for the both cases of $\odot$ and no confusion can appear.

Theorem 7.6 *Let* $\mathbf{L}$ *be linearly ordered. Then, for each* $A \in \mathfrak{Fin}$, *there exists its substantial segment. If* s, t *are substantial segments of* A, *then* $[A^s \sim^{\odot} A^t] = \top$.

Proof. Let $A \in \mathfrak{Fin}$. If $A \equiv \emptyset$, then one can simply prove that $s = \emptyset$ and $s(A) = \emptyset$. If $A \not\equiv \emptyset$, then, using the linearity of $\mathbf{L}$, we can consider $z = \{x_1, \dots, x_m\} = \mathrm{Supp}(A)$ such that $A(x_i) \geq A(x_{i+1})$ for any $i = 1, \dots, m-1$. Let $s = \{x_1, x_2, \dots, x_k\} \subseteq z$ such that $\alpha = A(x_1) \odot A(x_2) \odot \cdots \odot A(x_k) > \bot$ and $\alpha \odot A(x_{k+1}) = \bot$. Clearly, s satisfies (i) and (ii). Let $y \in \mathfrak{Fin}$, $y \subseteq \mathrm{Supp}(A)$, satisfy (i) and (ii). If $|y| > m$, then, by the definition of s, there exists a one-to-one mapping $g : s \to y$ such that $A(x) \geq A(g(x))$ for any $x \in s$. Hence, we simply obtain $\bigodot_{x\in y} A(x) = \bot$, which is a contradiction with the presumption on y. Let $y = \{y_1, \dots, y_l\}$ such that $A(y_i) \geq A(y_{i+1})$ for any $i = 1, \dots, l$. Then, $f : y \to s$ defined by $f(y_i) = x_i$, $i = 1, \dots, l$, is the desired mapping for which (iii) is satisfied.

If s, t are two substantial segments of A, then there are two one-to-one mappings $f : t \to s$ and $g : s \to t$ for which $A(x) \leq A(f(x))$ for any $x \in t$ and $A(x) \leq A(g(x))$ for any $x \in s$. Putting $h = g \circ f$ we obtain a permutation on t with $A(x) \leq A(f(x)) \leq A(h(x))$. One can see that $A(x) < A(h(x))$ for some $x \in t$ gives a contradiction with the presumption on h. Hence, $A(x) = A(h(x))$ which implies $A^t(x) = A(x) = A(f(x)) = A^s(f(x))$ for any $x \in t$, and thus $[A^s \sim^{\odot} A^t] = \top$. □

Example 7.4 let us suppose the Łukasiewicz rdr-lattice and consider $A = \{1/a, 0.5/b, 0.5/c\}$, then $s = \{a, b\}$ and $t = \{a, c\}$ are two different substantial segments of A with respect to $\otimes$, and $A^s \not\equiv A^t$, but $[A^s \sim^{\otimes} A^t] = \top$ (consider the mapping $f(a) = a$ and $f(b) = c$).

Remark 7.5 One can see that the substantial segment of A with respect to $\wedge$ is the support of A, i.e., $s = \mathrm{Supp}(A)$, and $s(A) \equiv A$. The same need not be true for $\otimes$, nevertheless, if $A \equiv B$ and s is a substantial segment of

A with respect to $\otimes$, then s is also a substantial segment of B with respect to $\otimes$. This trivially follows from the fact that $s \subseteq \mathrm{Supp}(A)$ in general, and $A \upharpoonright \mathrm{Supp}(A) = B \upharpoonright \mathrm{Supp}(B)$.

PROPOSITION 7.7 *Let* $\mathbf{L}$ *be linearly ordered,* $A \in \mathfrak{Ffin}$ *and* $y \in \mathfrak{Fin}$ *such that* $s \subseteq y \subseteq \mathrm{Dom}(A)$*, where* s *is a substantial segment of* A*. Then,* $p^{\odot}_{A}(-, s) = p^{\odot}_{A}(-, y)$*.*

PROOF. By Proposition 7.4, this is true for $\odot = \wedge$ because $s = \mathrm{Supp}(A)$. Let us consider $\odot = \otimes$. From the definition of the substantial segment s with respect to $\otimes$, one can see that $p^{\wedge}_{A}(i, s) = p^{\wedge}_{A}(i, y)$ for any $i \leq |s|$. By Proposition 7.2, we simply obtain $p^{\otimes}_{A}(i, s) = p^{\otimes}_{A}(i, y)$ for any $i \leq |s|$. Moreover, if $|s| < i \leq |y|$, then clearly for each set $z \subseteq y$ with $|z| = i$, there is $z' \subseteq y$ such that $s \subset z'$, $|z'| = i$ and $\alpha = \bigotimes_{x \in z'} A(x) \geq \bigotimes_{x \in z} A(x)$. However, $\alpha = \bot$ by (ii) of Definition 7.1, whence $p^{\otimes}(i, y) = \bot = p^{\otimes}(i, s)$. The equality for $i > |y|$ follows from (ii) of Proposition 7.1. □

Now, we can give a weaker inverse implication to that proposed in Theorem 7.5, where we restrict ourselves to the equipollence in the degree $\top$ between the fuzzy substantial segments of fuzzy sets.

THEOREM 7.8 *Let* $\mathbf{L}$ *be linearly ordered and, for* $\odot = \otimes$*, satisfy the conditional cancellation law for* $\otimes$*. Let* $A, B \in \mathfrak{Ffin}$ *and* $y, y' \in \mathfrak{Fin}$ *such that* $s \subseteq y \subseteq \mathrm{Dom}(A)$ *and* $t \subseteq y' \subseteq \mathrm{Dom}(B)$*, where* s *and* t *are substantial segments of* A *and* B *with respect to* $\odot$*, respectively. Then,* $[A^s \sim^{\odot} B^t] = \top$ *if and only if* $p^{\odot}_{A}(-, y) = p^{\odot}_{B}(-, y')$*.*

PROOF. Let $A, B \in \mathfrak{Ffin}$. Obviously, the statement is true for all $A \in \mathrm{cls}(\emptyset)$. Let us suppose that $A \notin \mathrm{cls}(\emptyset)$, s, t denote substantial segments of A, B with respect to $\odot$, respectively, and $y, y' \in \mathfrak{Fin}$ satisfy the presumption of the theorem.

($\Rightarrow$) If $[A^s \sim^{\odot} B^t] = \top$, then there exists a one-to-one mapping f of s onto t such that $A^s(x) = B^t(f(x))$. In fact, it is easy to see that $\mathrm{Supp}(A^s) = \mathrm{Dom}(A^s) = s$ and similarly for B^t. If $|s| > |t|$, then to find the degree of equipollence, it is sufficient to consider $C \equiv B^t$ for which $|s| = |\mathrm{Dom}(C)|$. Since the sets $\mathrm{Perm}(A^s, C)$ and $\mathrm{Supp}(A^s)$ are finite, then there exists $f \in \mathrm{Perm}(A^s, C)$ such that $A^s(x) \leftrightarrow C(f(x)) = \top$ for any $x \in s$. Then, however, there exists $x \in s$ such that $A^s(x) \leftrightarrow C(f(x)) = A^s(x) \leftrightarrow \bot$, which implies $A^s(x) = \bot$, and this is a contradiction with $s = \mathrm{Supp}(A^s)$. Analogously, we obtain a contradiction by supposing $|s| < |t|$. Hence, we obtain $|s| = |t|$, and the existence of f follows from the finiteness of the set $\mathrm{Perm}(A^s, B^t)$.

Now, it suffices to prove that $p^{\odot}_{A}(-, y) = p^{\odot}_{A^s}(-, s)$; the rest immediately follows from Theorem 7.5. According to the definition of a substantial segment of A, we trivially have $p^{\wedge}_{A}(-, s) = p^{\wedge}_{A^s}(-, s)$, where s is a substantial segment of A with respect to $\odot$ (see the construction of s in the proof of Theorem 7.6). By Proposition 7.2, we obtain $p^{\odot}_{A}(-, s) = p^{\odot}_{A^s}(-, s)$. The equality $p^{\odot}_{A}(-, y) = p^{\odot}_{A}(-, s) = p^{\odot}_{A^s}(-, s)$ immediately follows from Proposition 7.7, which concludes the proof of this implication.

(⇐) Let $p_A^{\odot}(-,y) = p_B^{\odot}(-,y')$. If $\odot = \wedge$, then, by Proposition 7.7, we have $p_A^{\wedge}(-,s) = p_B^{\wedge}(-,t)$, where s,t are substantial segments with respect to $\wedge$. Note that $|s| = |t|$; otherwise, $p_A^{\wedge}(-,s) \neq p_B^{\wedge}(-,t)$. Denote $\mathbf{m} = \{1,\dots,m\}$, where $m = |s|$. Since $\mathbf{L}$ is linearly ordered, one may simply check that $[A^s \sim^{\wedge} p_A^{\wedge}(-,s) \upharpoonright \mathbf{m}] = \top$ and $[B^t \sim^{\wedge} p_B^{\wedge}(-,t) \upharpoonright \mathbf{m}] = \top$. The statement follows from the transitivity of $\sim^{\wedge}$ and $p_A^{\wedge}(-,s) \upharpoonright \mathbf{m} = p_B^{\wedge}(-,t) \upharpoonright \mathbf{m}$ (see Corollary 5.5).

If $\odot = \otimes$, then it is sufficient to prove $p_A^{\wedge}(-,s) = p_B^{\wedge}(-,t)$, where s,t are substantial segments with respect to $\otimes$; the rest could be performed analogously to the case $\odot = \wedge$, where one can apply the fact that $[A^s \sim^{\wedge} B^t] = \top$ if and only if $[A^s \sim^{\otimes} B^t] = \top$. Trivially, we have $p_A^{\wedge}(0,s) = p_B^{\wedge}(0,t)$ and $p_A^{\wedge}(1,s) = p_B^{\wedge}(1,t)$. If $|s| = 1$, then $p_A^{\wedge}(i,s) = p_B^{\wedge}(i,t)$, $i > 1$, follows from (ii) of Proposition 7.1. Let $|s| > 1$. Since $\mathbf{L}$ satisfies the conditional cancellation law for $\otimes$ and $p_A^{\wedge}(1,s) = p_B^{\wedge}(1,t)$, then

$$p_A^{\otimes}(2,s) = p_A^{\wedge}(1,s) \otimes p_A^{\wedge}(2,s) = p_B^{\wedge}(1,t) \otimes p_B^{\wedge}(2,t) = p_B^{\otimes}(2,t) > \bot$$

implies that $p_A^{\wedge}(2,s) = p_B^{\wedge}(2,t)$. Repeating this procedure, we obtain $p_A^{\wedge}(i,s) = p_B^{\wedge}(i,t)$ for all $i \leq |s|$. Hence, $p_A^{\wedge}(-,s) = p_B^{\wedge}(-,t)$ (using (ii) of Proposition 7.1), and the implication is proved. □

It is easy to see that the statement above cannot be generalized to the equality

$$[A \upharpoonright y_A \sim^{\odot} B \upharpoonright y_B] = [p_A^{\odot}(-,y) \approx p_B^{\odot}(-,y')], \tag{53}$$

as the following example shows.

Example 7.6 Let us consider the Łukasiewicz rdr-lattice and fuzzy sets

$$A = \{1/a, 0.3/b, 0.1/c\}$$
$$B = \{0.7/e, 0.2/f\}$$

Then, we have $s = \{a,b\}$ and $t = \{e\}$ (note that $0.7 \otimes 0.2 = 0$, whence $f \notin t$) and

$$p_A^{\otimes}(-,x) = \{1/0, 1/1, 0.3/2, 0/3, \dots\},$$
$$p_B^{\otimes}(-,y) = \{1/0, 0.7/1, 0.1/2, 0/3, \dots\},$$

where $x = \{a,b,c\}$, $y = \{e,f\}$. By a simple computation, we obtain

$$[A^s \sim^{\otimes} B^t] = 0.7 \otimes 0.7 = 0.4 < 0.7 =$$
$$1 \wedge 0.7 \wedge 0.8 \wedge 1 \wedge \dots = [p_A^{\otimes}(-,x) \approx p_B^{\otimes}(-,y)].$$

Remark 7.7 In Wygralak's approach to the generalized FGCounts, the appropriate equipollence can be defined by the graded equipollence of fuzzy substantial segments of fuzzy sets in the degree $\top$. More precisely, fuzzy sets A and B are equipollent (denoted by $A \sim B$) in Wygralak's approach if and only if $[A^s \sim^{\odot} B^t] = \top$, where s, t are substantial segments of A, B, respectively. Thus, the class of all equipollent fuzzy sets in Wygralak's approach is wider and evidently contains fuzzy sets that are not equipollent in the degree $\top$ in our approach. Nevertheless, Wygralak showed in [24] and [25] that nearly all common constructions with fuzzy sets are saved with respect to $\sim$, and the class of all equipollent fuzzy sets with A may be represented by the generalized cardinal $p_A^{\odot}(-, s)$. In contrast, the approach based on the equipollence $\sim$ cannot be understood, in essence, as a functional approach in general. An exception to this is the case $\odot = \wedge$, as will be shown below. Thus, a penalty for new constructions based on more general operations $\otimes$ is a rather non-functional approach to the concept of equipollence of fuzzy sets.

As we have noted above, the inequality (51) cannot be changed to the equality in general, and hence, $p_A^{\odot}(-, y)$ as a generalized cardinal cannot fully represent the class of all equipollent fuzzy sets. To achieve this goal, we have to consider a wider class based on $\sim$ that allows us to develop only a non-graded cardinal theory of fuzzy sets discussed in Remark 7.7. Nevertheless, if we restrict ourselves to the case $\odot = \wedge$ and suppose $\mathbf{L}$ to be linearly ordered, then a generalized cardinal $p_A^{\wedge}(-, s)$ (or, equivalently, $p_A^{\wedge}(-, y)$ for any $\mathrm{Supp}(A) = s \subseteq y \subseteq \mathrm{Dom}(A)$) fully represents the class of all equipollent fuzzy sets with A in the degree $\top$, as shown in the following statement.

THEOREM 7.9 *Let* $\mathbf{L}$ *be linearly ordered,* $A, B \in \mathfrak{Ffin}$ *and* $y, y' \in \mathfrak{Fin}$ *such that* $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$ *and* $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$. *Then,*

$$[A \sim^{\wedge} B] = [p_A^{\wedge}(-, y) \approx p_B^{\wedge}(-, y')]. \tag{54}$$

PROOF. Let $\mathbf{L}$ be linearly ordered, $A, B \in \mathfrak{Ffin}$ and $y, y' \in \mathfrak{Fin}$ such that $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$ and $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$. Due to Theorem 7.5, it suffices to prove that

$$[A \sim^{\wedge} B] \geq [p_A^{\wedge}(-, y) \approx p_B^{\wedge}(-, y')]. \tag{55}$$

Analogously to the proof of Theorem 7.5, we may suppose that $m = |\mathrm{Dom}(A)| = |\mathrm{Dom}(B)|$ and put $y = \mathrm{Dom}(A)$ and $y' = \mathrm{Dom}(B)$. If $y = \emptyset$, then $A = B = \emptyset$, and the inequality (55) is trivially true.

Let $y \neq \emptyset$. From the linearity of $\mathbf{L}$ and the finiteness of domains of A and B, there exist two finite sequences $\{u_i\}_{i=1}^m$ and $\{v_i\}_{i=1}^m$ of elements of y and y', respectively, such that $A(u_i) \geq A(u_j)$ and $B(v_i) \geq B(v_j)$ whenever $i \geq j$. One can simply check that

$$p_A^{\wedge}(i, y) = A(u_i) \quad \text{and} \quad p_B^{\wedge}(i, y') = B(v_i)$$

for any $i = 1, \ldots, m$. Recall that by (ii) and (iii) of Proposition 7.1, we have $p_{A^{\wedge}}(i, y) = p_{B^{\wedge}}(i, y')$ for any $i = 0$ and $i > m$. Now, let us define $g(u_i) = v_i$ for

$i = 1, \dots, m$. Since $g \in \mathrm{Perm}(A, B)$, then

$$[A \sim^{\wedge} B] = \bigvee_{f \in \mathrm{Perm}(A,B)} [A \sim^{\wedge}_{f} B] =$$

$$\bigvee_{f \in \mathrm{Perm}(A,B)} \bigwedge_{x \in y} (A(x) \leftrightarrow B(f(x))) \geq \bigwedge_{x \in y} (A(x) \leftrightarrow B(g(x))) =$$

$$\bigwedge_{i=1}^{m} (A(u_i) \leftrightarrow B(g(u_i))) = \bigwedge_{i=1}^{m} (A(u_i) \leftrightarrow B(v_i)) =$$

$$\bigwedge_{i=1}^{m} (p^{\wedge}_{A}(i, y) \leftrightarrow p^{\wedge}_{B}(i, y')) = \bigwedge_{i \in \mathbb{N}} (p^{\wedge}_{A}(i, y) \leftrightarrow p^{\wedge}_{B}(i, y')) = [p^{\wedge}_{A}(-, y) \approx p^{\wedge}_{B}(-, y')],$$

and the inequality is proved. □

A consequence of the previous theorem is the following corollary, showing that under some presumptions on domains of considered finite fuzzy sets there exists a permutation determining the degree to which these fuzzy sets are equipollent.

COROLLARY 7.10 *Let* $\mathbf{L}$ *be linearly ordered and* $A, B \in \mathfrak{Fin}$ *such that* $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$. *Then, there exists* $g \in \mathrm{Perm}(A, B)$ *such that*

$$[A \sim^{\wedge} B] = [g^{\rightarrow}(A) \approx B]. \tag{56}$$

PROOF. Let $\mathbf{L}$ be linearly ordered and $A, B \in \mathfrak{Fin}$ such that $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$. If $m = 0$, then $A = B = \emptyset$, $g^{\rightarrow} = \emptyset$ and $g^{\rightarrow}(\emptyset) = \emptyset$. Hence, (56) is trivially satisfied.

Let $m > 0$. Put $y = \mathrm{Dom}(A)$ and $y' = \mathrm{Dom}(B)$. By the proof of Theorem 7.9, there exists a one-to-one mapping g of y onto y' such that

$$[A \sim^{\wedge} B] = [A \sim^{\wedge}_{g} B] = [p^{\wedge}_{A}(-, y) \approx p^{\wedge}_{B}(-, y')].$$

Since g is a one-to-one correspondence between y and y', then

$$g^{\rightarrow}(A)(z) = \bigvee_{\substack{x \in y \\ g(x) = z}} A(x) = A(g^{-1}(z)).$$

Clearly, $\mathrm{Dom}(g^{\rightarrow}(A)) = \mathrm{Dom}(B)$, and hence, we have

$$[A \sim^{\wedge} B] = [A \sim^{\wedge}_{g} B] = \bigwedge_{x \in y} (A(x) \leftrightarrow B(g(x))) =$$

$$\bigwedge_{z \in y'} (A(g^{-1}(z)) \leftrightarrow B(z)) = \bigwedge_{z \in y'} (g^{\rightarrow}(A)(z) \leftrightarrow B(z)) = [g^{\rightarrow}(A) \approx B].$$

□

Example 7.8 Let $\mathbf{L}$ be the Łukasiewicz rdr-lattice and $y = \{c_1 \dots, c_5\}$ be a set of criteria. Let us suppose that the following fuzzy sets

$$A = \{0.3/c_1, 0.8/c_2, 0.4/c_3, 0/c_4, 0/c_5\}$$
$$B = \{0/c_1, 0.6/c_2, 0.9/c_3, 0/c_4, 0.7/c_5\}$$

express the satisfaction of all criteria by two applicants, and our goal is to determine whether the numbers of satisfied criteria are more or less the same for both applicants. According to (48), we can establish

$$p^{\wedge}_A(-, y) = \{1/0, 0.8/1, 0.4/2, 0.3/3, 0/4, 0/5, \dots\},$$
$$p^{\wedge}_B(-, y) = \{1/0, 0.9/1, 0.7/2, 0.6/3, 0/4, 0/5, \dots\},$$

and using (7.9) of Theorem 7.9, we may simply derive

$$[A \sim^{\wedge} B] = \bigwedge_{i \in N} (p^{\wedge}_A(i, y) \leftrightarrow p^{\wedge}_B(i, y)) = 0.7.$$

Hence, we can conclude that both applicants have more or less the same number of satisfied criteria.

Now, we may return to the proof of Theorem 6.2 stated on page 22. One should note that we use generalized cardinals in a non-standard manner in comparison with the common proof of Cantor-Bernstein theorem to simplify the derivation of the relation among the degrees to which fuzzy sets are equipollent.

Proof (Graded Cantor-Bernstein theorem). Let $\mathbf{L}$ be linearly ordered and $A, B, C, D \in \mathfrak{Fin}$ such that $B \subseteq A$ and $D \subseteq C$. Without loss of generality, we may suppose that $y = \mathrm{Dom}(A) = \mathrm{Dom}(B)$, $y' = \mathrm{Dom}(C) = \mathrm{Dom}(D)$ and $|y| = |y'| = m$; otherwise, we may choose equivalent fuzzy sets and use Corollary 5.5, Theorem 5.6 and Proposition 7.4. If $y = \emptyset$, then all fuzzy sets are the empty fuzzy set and the statement is trivially fulfilled.

Let $y \neq \emptyset$, and put $\mathbf{m} = \{1, \dots, m\}$ and $\mathrm{id}_{\mathbf{m}}$, the identity mapping of $\mathbf{m}$ onto $\mathbf{m}$. One may simply check

$$[p^{\wedge}_A(-, y) \approx p^{\wedge}_D(y, -)] = [p^{\wedge}_A(-, y) \upharpoonright \mathbf{m} \sim^{\wedge}_{\mathrm{id}_{\mathbf{m}}} p^{\wedge}_D(-, y') \upharpoonright \mathbf{m}]$$

and check analogously for B, C. Since $B \subseteq A$ and $D \subseteq C$, then, by the definition of $p^{\wedge}_{-}$, we have $p^{\wedge}_B(-, y) \upharpoonright \mathbf{m} \subseteq p^{\wedge}_A(-, y) \upharpoonright \mathbf{m}$ and $p^{\wedge}_D(-, y) \upharpoonright \mathbf{m} \subseteq p^{\wedge}_C(-, y) \upharpoonright \mathbf{m}$. Hence, by Theorems 6.1 and 7.9, we have

$$[A \sim^{\wedge} D] \wedge [B \sim^{\wedge} C] =$$
$$[p^{\wedge}_A(-, y) \upharpoonright \mathbf{m} \sim^{\wedge}_{\mathrm{id}_{\mathbf{m}}} p^{\wedge}_D(-, y') \upharpoonright \mathbf{m}] \wedge [p^{\wedge}_B(-, y) \upharpoonright \mathbf{m} \sim^{\wedge}_{\mathrm{id}_{\mathbf{m}}} p^{\wedge}_D(-, y') \upharpoonright \mathbf{m}] \leq$$
$$[p^{\wedge}_A(-, y) \upharpoonright \mathbf{m} \sim^{\wedge}_{\mathrm{id}_{\mathbf{m}}} p^{\wedge}_C(-, y') \upharpoonright \mathbf{m}] = [A \sim^{\wedge} C],$$

and the proof is finished. □

7.2. Non-decreasing generalized cardinals

In this subsection, we shall introduce a dual notion to $p_A^{\odot}(i,y)$ for the operation $\overline{\odot} \in \{\vee, \oplus\}$ that can also serve as a representative of equipollent fuzzy sets to degree $\top$ and will be important in the second part of our contribution on a graded approach to cardinality theory of finite fuzzy sets.

Let $A \in \mathfrak{Ffin}$ be a finite fuzzy set and define a fuzzy class $p_A^{\overline{\odot}} : \mathbb{N} \times \mathfrak{Fin} \to L$ by

$$p_A^{\overline{\odot}}(i,y) = \begin{cases} \bigwedge_{z\subseteq y, |z|=i} \overline{\bigodot}_{x\in z} A(x), & \text{if } y \subseteq \mathrm{Dom}(A); \\ \bot, & \text{otherwise.} \end{cases} \tag{57}$$

for any $(i,y) \in \mathbb{N} \times \mathfrak{Fin}$.

Proposition 7.11 *Let $A \in \mathfrak{Ffin}$ and $y \in \mathfrak{Fin}$ such that $y \subseteq \mathrm{Dom}(A)$. Then, for any $i,j \in \mathbb{N}$, we have*

(i) $p_A^{\overline{\odot}}(i,y) \leq p_A^{\overline{\odot}}(j,y)$, *if* $i \leq j$,

(ii) $p_A^{\overline{\odot}}(i,y) = \top$, *if* $i > |y|$,

(iii) $p_A^{\overline{\odot}}(0,y) = \bot$,

(iv) $p_A^{\overline{\odot}}(1,y) = \bigwedge_{x\in y} A(x)$,

(v) $p_A^{\overline{\odot}}(0,\emptyset) = \bot$ *and* $p_A^{\overline{\odot}}(i,\emptyset) = \top$ *for any* $i > 0$,

(vi) $p_A^{\overline{\odot}}(-,y)$ *is a non-decreasing generalized cardinal.*

Proof. This can be performed analogously to the proof of Theorem 7.1. □

Proposition 7.12 *Let $\mathbf{L}$ be linearly ordered. Then,*

$$p_A^{\oplus}(i,y) = \bigoplus_{k=0}^{i} p_A^{\vee}(k,y) \tag{58}$$

holds for any $A \in \mathfrak{Ffin}$ and $(i,y) \in \mathbb{N} \times \mathfrak{Fin}$.

Proof. This can be performed analogously to the proof of Proposition 7.2. □

Example 7.9 Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz rdr-lattice on $[0,1]$. Let A be from Ex. 7.1. Put $y = \mathrm{Dom}(A)$. Then,

$$\begin{aligned} p_A^{\vee}(-,y) &= \{0/0, 0/1, 0.1/2, 0.4/3, 0.5/4, 0.8/5, 1/6, 1/7, \dots\}, \\ p_A^{\oplus}(-,y) &= \{0/0, 0/1, 0.1/2, 0.5/3, 1/4, 1/5, \dots\}. \end{aligned}$$

Obviously, we obtain a different result for $y' = \mathrm{Supp}(A)$. For example, we have $p_A^{\wedge}(0,y') = 0$, $p_A^{\wedge}(1,y') = 0.1$ and $p_A^{\wedge}(2,y') = 0.5$. Thus, the mapping $p_A^{\overline{\odot}}(-,y)$ strongly depends on the choice of the set y.

In contrast to $p_A^{\odot}(-,y)$, the mappings $p_A^{\overline{\odot}}(-,y)$ are not finite but denumerable fuzzy sets, i.e., $|\mathrm{Supp}(p_A^{\overline{\odot}}(-,y))| = \aleph_0$. One could note that an analogous statement to Proposition 7.3 does not hold for $p_A^{\overline{\odot}}$.

Proposition 7.13 *If $A, B \in \mathfrak{Ffin}$, $A \equiv B$ and $y \in \mathfrak{Fin}$ such that $y \subseteq \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, then $p_A^{\overline{\odot}}(-,y) = p_B^{\overline{\odot}}(-,y)$.*

Proof. Since $B \in \mathrm{cls}(A)$ and $y \subseteq \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, then $A \upharpoonright y = B \upharpoonright y$. Hence, we have

$$p_A^{\overline{\odot}}(i,y) = \bigwedge_{\substack{z \subseteq y \\ |z|=i}} \overline{\bigodot_{x \in z}} A(x) = \bigwedge_{\substack{z \subseteq y \\ |z|=i}} \overline{\bigodot_{x \in z}} A \upharpoonright y(x) =$$

$$\bigwedge_{\substack{z \subseteq y \\ |z|=i}} \overline{\bigodot_{x \in z}} B \upharpoonright y(x) = \bigwedge_{\substack{z \subseteq y \\ |z|=i}} \overline{\bigodot_{x \in z}} B(x) = p_B^{\overline{\odot}}(i,y). \quad \square$$

Remark 7.10 The idea of non-decreasing generalized cardinals is closely related to the concept of cardinals in generalized FLCounts proposed in [24] (see also [25]), although it is not as straightforward as in the case of FGCounts. If $L = [0,1]$ and $\odot = \wedge$, then the generalized cardinal in the sense of FLCount is usually defined as a vector (see Section 2.5 in [25])

$$(1 - [A]_1, 1 - [A]_2, \dots, 1 - [A]_n, (1)) =$$
$$(1 - p_A^{\vee}(n,y), 1 - p_A^{\vee}(n-1,y), \dots, 1 - p_A^{\vee}(1,y), (1))$$

where $y = \mathrm{Supp}(A)$ and $n = |y|$. One may see that $[A]_1 = \bigwedge_{x \in y} A(x) = p_A^{\vee}(n,y)$, and similarly, $[A]_2$ is the second greatest membership degree of A that is precisely equal to $p_A^{\vee}(n-1,y)$. Hence, one may see that our replacement is correct.

If we consider $\odot = \otimes$ to be a continuous t-norm and $\nu : [0,1] \to [0,1]$ to be a strict negation,[16] then $\oplus$ defined by

$$\alpha \oplus \beta = \nu^{-1}(\nu(\alpha) \otimes \nu(\beta)) \tag{59}$$

is a continuous t-conorm (see [20]). Hence, the operations $\otimes$ and $\oplus$ determine a complete rdr-lattice (see Example 2.1). In [24], Wygralak proposed the following form of cardinals in the sense of generalized FLCounts

$$(\nu([A]_1) \otimes \cdots \otimes \nu([A]_n), \dots, \nu([A]_{n-1}) \otimes \nu([A]_n), \nu([A]_n), (1)).$$

Since ν is a bijection, then we may rewrite this vector using non-decreasing generalized cardinals as follows:

$$\begin{aligned}(\nu([A]_1 \oplus \cdots \oplus [A]_n), \dots, \nu([A]_{n-1} \oplus [A]_n), \nu([A]_n), (1)) = \\ (\nu(p_A^{\oplus}(n,y)), \dots, \nu(p_A^{\oplus}(2,y)), \nu(p_A^{\oplus}(1,y)), (1)).\end{aligned} \tag{60}$$

[16] Note that a strict negation is a strictly decreasing continuous function $\nu : [0,1] \to [0,1]$ for which $\nu(0) = 1$ and $\nu(1) = 0$ (see Definition 11.3 in [15] or Section 1.2 in [25])

Note that the main reason for the usage of $p_A^{\overline{\odot}}$ as a dual of $p_A^{\odot}$ instead of Wygralak's concept of generalized cardinals was to build up a dual theory of graded equipollence based on dual operations $\oplus$ or $\vee$ and $\ominus$. A further advantage of $p_A^{\overline{\odot}}$ defined over the rdr-lattice of non-negative real numbers (see Example 2.3) is its application to the graded approach to the scalar cardinality of finite fuzzy sets (see [23, 25]. However, it is beyond the scope of this paper.

The following theorem states an analogous inequality as in Theorem 7.5, where we have to restrict ourselves to the case $\overline{\odot} = \vee$.[17]

THEOREM 7.14 *Let* $A, B \in \mathfrak{Ffin}$ *and* $y, y' \in \mathfrak{Fin}$ *such that* $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$*,* $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$ *and* $|y| = |y'|$*. Then,*

$$[A \sim^{\odot} B] \leq [p_A^{\vee}(-, y) \approx p_B^{\vee}(-, y')]. \tag{61}$$

PROOF. Let $A, B \in \mathfrak{Ffin}$ and $y, y' \in \mathfrak{Fin}$ such that $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$, $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$ and $|y| = |y'| = m$. Let $C = A \restriction y$ and $D = B \restriction y'$. Note that $[A \sim^{\odot} B] = [C \sim^{\odot} D]$ by Corollary 5.7. If $y = \emptyset$, then (61) follows from (v) of Proposition 7.11. Let $y \neq \emptyset$. By the definition of $\approx$ and the fact that $[C \sim^{\odot} D] \leq [C \sim^{\wedge} D]$, it suffices to show that

$$[C \sim^{\wedge} D] \leq p_A^{\vee}(i, y) \leftrightarrow p_B^{\vee}(i, y') \tag{62}$$

for any $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$. Clearly, the inequality (62) is trivially satisfied for $i = 0$ and $i > m$. For $0 < i \leq m$, we have

$$p_A^{\vee}(i, y) \leftrightarrow p_B^{\vee}(i, y') = \Big(\bigwedge_{\substack{z \subseteq y \\ |z| = i}} \bigvee_{x \in z} A(x) \Big) \leftrightarrow \Big(\bigwedge_{\substack{z' \subseteq y' \\ |z'| = i}} \bigvee_{x' \in z'} B(x') \Big) =$$

$$\Big(\bigwedge_{\substack{z \subseteq y \\ |z| = i}} \bigvee_{x \in z} C(x) \Big) \leftrightarrow \Big(\bigwedge_{\substack{z' \subseteq y' \\ |z'| = i}} \bigvee_{x' \in z'} D(x') \Big) \geq \bigwedge_{\substack{z \subseteq y \\ |z| = i}} \Big(\bigvee_{x \in z} C(x) \leftrightarrow \bigvee_{x' \in f(z)} D(x') \Big) \geq$$

$$\bigwedge_{\substack{z \subseteq y \\ |z| = i}} \bigwedge_{x \in z} \big(C(x) \leftrightarrow D(f(x)) \big) \geq \bigwedge_{x \in z} \big(C(x) \leftrightarrow D(f(x)) \big) = [C \sim_f^{\wedge} D].$$

Hence, we obtain

$$p_A^{\vee}(i, y) \leftrightarrow p_B^{\vee}(i, y') \geq \bigvee_{f \in \mathrm{Perm}(C, D)} [C \sim_f^{\wedge} D] = [C \sim^{\wedge} D],$$

which concludes the proof. □

[17]Note that, generally, $\bigotimes_{i \in I} (\alpha_i \leftrightarrow \beta_i) \leq (\bigoplus_{i \in I} \alpha_i) \leftrightarrow (\bigoplus_{i \in I} \beta_i)$ does not hold in an rdr-lattice. On the other hand, $\bigwedge_{i \in I} (\alpha_i \leftrightarrow \beta_i) \leq (\bigvee_{i \in I} \alpha_i) \leftrightarrow (\bigvee_{i \in I} \beta_i)$ by (13).

We shall say that an rdr-lattice satisfies the *conditional cancellation law for* $\oplus$ if $\alpha \oplus \beta = \alpha \oplus \gamma < \top$ implies $\beta = \gamma$ for any $\alpha, \beta, \gamma \in L$.

DEFINITION 7.2 Let $\mathbf{L}$ be linearly ordered and $A \in \mathfrak{Fin}$. A set $s \in \mathfrak{Fin}$ such that $s \subseteq \mathrm{Supp}(A)$ is said to be a *substantial segment* of A with respect to $\overline{\odot}$ if

(i) $\overline{\bigodot}_{x \in s} A(x) < \top$,

(ii) if $z \in \mathfrak{Fin}$ such that $s \subset z \subseteq \mathrm{Supp}(A)$, then $\overline{\bigodot}_{x \in z} A(x) = \top$,

(iii) if $y \subseteq \mathrm{Supp}(A)$ satisfies (i) and (ii), then there exists a one-to-one mapping $f : y \to s$ such that $A(x) \geq A(f(x))$ for any $x \in y$.

The fuzzy set $A^s = A \restriction s$ is called a *fuzzy substantial segment* of A with respect to $\overline{\odot}$.

In the following part, we shall omit "with respect to $\overline{\odot}$", and we shall say only a (fuzzy) substantial segment of A if we speak about (fuzzy) substantial segments for the both cases of $\overline{\odot}$ and no confusion can appear.

THEOREM 7.15 *Let* $\mathbf{L}$ *be linearly ordered. Then, for each* $A \in \mathfrak{Fin}$*, there exists its substantial segment. If* s, t *are substantial segments of* A*, then* $[A^s \sim^{\odot} A^t] = \top$.

PROOF. This can be performed analogously to the proof of Theorem 7.6, where one can consider $z = \{x_1, \dots, x_m\} = \mathrm{Supp}(A) \setminus \mathrm{Ker}(A)$ such that $A(x_i) \leq A(x_{i+1})$ for any $i = 1, \dots, m-1$, and $s = \{x_1, x_2, \dots, x_k\} \subseteq z$ such that $\alpha = A(x_1) \,\overline{\odot}\, A(x_2) \,\overline{\odot} \cdots \overline{\odot}\, A(x_k) < \top$ and $\alpha \,\overline{\odot}\, A(x_{k+1}) = \top$. □

REMARK 7.11 One can see that the substantial segment with respect to $\vee$ is the set $s = \mathrm{Supp}(A) \setminus \mathrm{Ker}(A)$. In general, $s \subseteq \mathrm{Supp}(A) \setminus \mathrm{Ker}(A)$, where $s = \emptyset$ if it is the substantial segment of $A \in \mathrm{cls}(\emptyset)$.

PROPOSITION 7.16 *Let* $\mathbf{L}$ *be a linearly ordered,* $A \in \mathfrak{Fin}$ *and* $y \in \mathfrak{Fin}$ *such that* $s \subseteq y \subseteq \mathrm{Supp}(A)$*, where* s *is a substantial segment of* A*. Then,* $p^{\overline{\odot}}_A(-, s) = p^{\overline{\odot}}_A(-, y)$.

PROOF. Let $\overline{\odot} = \vee$ and s be a substantial segment with respect to $\vee$. By remark 7.11, one can simply check that $p^{\vee}_A(i, s) = p^{\vee}_A(i, y)$ for any $i \leq |s|$. By (ii) of Proposition 7.11 and the definition of s, we have $p^{\vee}_A(i, s) = p^{\vee}_A(i, y) = \top$ for any $i > |s|$. Hence, we obtain $p^{\vee}_A(-, s) = p^{\vee}_A(-, y)$.

Let $\overline{\odot} = \oplus$ and s be a substantial segment with respect to $\oplus$. From the definition of s, we obtain $p^{\vee}_A(i, s) = p^{\vee}_A(i, y)$ for any $i \leq |s|$. Using Proposition 7.12, we have $p^{\oplus}_A(-, s) = p^{\oplus}_A(-, y)$ for any $i \leq |s|$. Moreover, if $|s| < i \leq |y|$, then clearly for each set $z \subseteq y$ with $|z| = i$, there is $z' \subseteq y$ such that $y_A \subset z'$, $|z'| = i$ and $\alpha = \bigoplus_{x \in z'} A(x) \leq \bigoplus_{x \in z} A(x)$. However, $\alpha = \top$ by (ii) of Definition 7.2, whence $p^{\oplus}_A(i, s) = \top = p^{\oplus}_A(i, y)$. The inequality for $i > |y|$ follows from (ii) of Proposition 7.11. □

Now, we can give an analogous statement to that in Theorem 7.8. Note that $p_A^{\overline{\odot}}(-,y)$ is strongly dependent on y, and thus we have to modify the presumption of the theorem.

THEOREM 7.17 *Let* $\mathbf{L}$ *be linearly ordered and, for* $\overline{\odot} = \oplus$, *satisfy the conditional cancellation law for* $\oplus$. *Let* $A, B \in \mathfrak{Fin}$ *and* $y, y' \in \mathfrak{Fin}$ *such that* $s \subseteq y \subseteq \mathrm{Supp}(A)$ *and* $t \subseteq y' \subseteq \mathrm{Supp}(B)$, *where* s *and* t *are substantial segments of* A *and* B *with respect to* $\overline{\odot}$, *respectively. Then,* $[A^s \sim^{\odot} B^t] = \top$ *if and only if* $p_A^{\overline{\odot}}(-,y) = p_B^{\overline{\odot}}(-,y')$.

PROOF. Obviously, the statement is true for $A \in \mathrm{cls}(\emptyset)$. Let us suppose that $A, B \in \mathfrak{Fin}$ such that $A \notin \mathrm{cls}(\emptyset)$, s, t are substantial segments of A, B with respect to $\overline{\odot}$, respectively, and $y, y' \in \mathfrak{Fin}$ satisfy the presumption of the theorem. Let us suppose that $m = |s|$ and put $s = \{x_1, \dots, x_m\}$ such that $A(x_i) \leq A(x_{i+1})$ for any $i \in \{1, \dots, m-1\}$.

($\Rightarrow$) By analogous arguments used in the proof of Theorem 7.8 (see the part ($\Rightarrow$)), there exists a one-to-one mapping f of s onto t such that $A^s(x_i) = B^t(f(x_i))$ for any $x_i \in s$. We shall show that $p_A^{\overline{\odot}}(-,y) = p_{A^s}^{\overline{\odot}}(-,s)$. Let $\overline{\odot} = \vee$. Then, by the definition of s for $\vee$, we simply obtain $p_A^{\vee}(i,y) = p_{A^s}^{\vee}(i,s) = A^s(x_i)$ for any $i \leq m$. For $i > m$, this equality follows from (ii) of Proposition 7.11 and the fact that $p_A^{\vee}(i,y) = A(x) = \top$ if $m < i \leq |y|$, where $x \in \mathrm{Ker}(A)$. Let $\overline{\odot} = \oplus$. Then, we obtain $p_A^{\oplus}(i,y) = p_{A^s}^{\oplus}(i,s)$ for any $i \leq m$ by the definition of s for $\oplus$ and Proposition 7.12. For $i > m$, we have $p_A^{\oplus}(i,y) = \top$ by (ii) of the definition of s (if $m < i \leq |y|$) and $p_{A^s}^{\oplus}(i,s) = \top$ by (ii) of Proposition 7.11 (for $i > |y|$).

Now, we shall prove that $p_A^{\overline{\odot}}(-,y) = p_B^{\overline{\odot}}(-,y')$. By (iii) of Proposition 7.11, we have $p_A^{\overline{\odot}}(0,y) = p_B^{\overline{\odot}}(0,y')$. Let $i \geq 1$. If $\overline{\odot} = \vee$, then, from the existence of $f : s \to t$ (s, t are for $\vee$) for which $A^s(x_i) = B^t(f(x_i))$, we can derive

$$p_{A^s}^{\vee}(i,s) = A^s(x_i) = B^t(f(x_i)) = p_{B^t}^{\vee}(i,t)$$

for any $0 < i \leq m$. By (ii) of Proposition 7.11 and from the equality $p_A^{\vee}(-,y) = p_{A^s}^{\vee}(-,s)$, we obtain $p_A^{\vee}(-,y) = p_B^{\vee}(-,y')$. If $\overline{\odot} = \oplus$, then, from Proposition 7.12, we can derive (s, t are for $\oplus$)

$$p_{A^s}^{\oplus}(i,s) = \bigoplus_{k=1}^{i} A^s(x_k) = \bigoplus_{k=1}^{i} B^t(f(x_k)) = p_{B^t}^{\oplus}(i,t)$$

for any $0 < i \leq m$. By (ii) of Proposition 7.11 and from the equality $p_A^{\oplus}(-,y) = p_{A^s}^{\oplus}(-,s)$, we obtain $p_A^{\oplus}(-,y) = p_B^{\oplus}(-,y')$, and the implication is proved.

($\Leftarrow$) Let $p_A^{\overline{\odot}}(-,y) = p_B^{\overline{\odot}}(-,y')$. If $\overline{\odot} = \vee$, then, by Proposition 7.16, we have $p_A^{\vee}(-,s) = p_B^{\vee}(-,t)$, where s, t are substantial segments of A, B with respect to $\vee$, respectively. Note that $|s| = |t|$; otherwise, $p_A^{\vee}(-,s) \neq p_B^{\vee}(-,t)$. Denote $\mathbf{m} = \{1, \dots, m\}$ (recall that $m = |s|$). Then, one may simply check that $[A^s \sim^{\odot} p_A^{\vee}(-,s) \upharpoonright \mathbf{m}] = \top$ and $[B^t \sim^{\odot} p_B^{\vee}(-,t) \upharpoonright \mathbf{m}] = \top$. Since $p_A^{\vee}(-,s) \upharpoonright \mathbf{m} = p_B^{\vee}(-,t) \upharpoonright \mathbf{m}$, then $[A^s \sim^{\odot} B^t] = \top$ as a consequence of the transitivity of $\sim^{\odot}$.

If $\overline{\odot} = \oplus$, then it suffices to prove $p^{\vee}_A(-, s) = p^{\vee}_B(-, t)$, where s, t are substantial segments of A, B with respect to $\oplus$, respectively; the rest could be performed analogously to the case $\overline{\odot} = \vee$. Trivially, we have $p^{\vee}_A(0, s) = p^{\vee}_B(0, t)$ and $p^{\vee}_A(1, s) = p^{\vee}_B(1, t)$. If $m = 1$, then $p^{\vee}_A(i, y) = p^{\vee}_B(i, y')$, $i > 1$, follows from (ii) of Proposition 7.11. Let $m > 1$. Since $\mathbf{L}$ satisfies the conditional cancellation law for $\oplus$ and $p^{\vee}_A(1, s) = p^{\vee}_B(1, t)$, then (using Proposition 7.12)

$$p^{\oplus}_A(2, s) = p^{\vee}_A(1, s) \oplus p^{\vee}_A(2, s) = p^{\vee}_B(1, t) \oplus p^{\vee}_B(2, t) = p^{\oplus}_B(2, t) < \top,$$

which implies $p^{\vee}_A(2, s) = p^{\vee}_B(2, t)$. Repeating the procedure, we obtain $p^{\vee}(i, s) = p^{\vee}_B(i, t)$ for any $i \leq m$, whence $p^{\vee}(-, s) = p^{\vee}_B(-, t)$ (using (ii) of Proposition 7.11), and the implication is proved. □

Remark 7.12 In contrast to the non-increasing generalized cardinals discussed in the previous subsection, the result of the previous theorem for the non-decreasing generalized cardinals does not have a close relation to the generalized FLCounts provided by Wygralak in [24, 25] and sketched in Remark 7.10. In fact, a generalized FLCount is defined using a negation ν which is applied on the values of a non-decreasing generalized cardinal $p^{\overline{\odot}}_A(-, y)$ in a reverse direction as could be seen in (60). Hence, the equality $p^{\overline{\odot}}_A(-, y) = p^{\overline{\odot}}_B(-, y')$ need not correspond to the equality of non-decreasing cardinals obtained by a generalized FLCount.

In Theorem 7.9, we have shown that non-increasing generalized cardinals can fully represent the classes of all equipollent fuzzy sets if we restrict ourselves to the linearly ordered rdr-lattice and the case $\odot = \wedge$. The following theorem states that also non-increasing generalized cardinals can serve as the representatives of classes of equipollent fuzzy sets under the mentioned restriction.

Theorem 7.18 *Let* $\mathbf{L}$ *be linearly ordered,* $A, B \in \mathfrak{Ffin}$ *and* $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$ *and* $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$ *such that* $|y| = |y'| = m$. *Then,*

$$[A \sim^{\wedge} B] = [p^{\vee}_A(-, y) \approx p^{\vee}_B(-, y')]. \tag{63}$$

Proof. Let $A, B \in \mathfrak{Ffin}$ and $y, y' \in \mathfrak{Fin}$ such that $\mathrm{Supp}(A) \subseteq y \subseteq \mathrm{Dom}(A)$, $\mathrm{Supp}(B) \subseteq y' \subseteq \mathrm{Dom}(B)$ and $|y| = |y'| = m$. Let $C = A \restriction y$ and $D = B \restriction y'$. Note that $[A \sim^{\odot} B] = [C \sim^{\odot} D]$ by Corollary 5.7. Due to Theorem 7.14, it suffices to prove that

$$[C \sim^{\wedge} D] \geq [p^{\vee}_A(-, y) \approx p^{\vee}_B(-, y')]. \tag{64}$$

If $y = \emptyset$, then $C = D = \emptyset$, and the inequality (64) is trivially true.

Let $y \neq \emptyset$. From the linearity of $\mathbf{L}$ and the finiteness of domains of C and D, there exist two finite sequences $\{u_i\}_{i=1}^m$ and $\{v_i\}_{i=1}^m$ of elements of y and y', respectively, such that $C(u_i) \leq C(u_j)$ and $D(v_i) \leq D(v_j)$ whenever $i \leq j$. One can simply check that

$$p^{\vee}_A(i, y) = C(u_i) \quad \text{and} \quad p^{\vee}_B(i, y') = D(v_i)$$

for any $i = 1, \dots, m$. Recall that, by (ii) and (iii) of Proposition 7.11, we have $p^\vee_A(i, y) = p^\vee_B(i, y')$ for any $i = 0$ and $i > m$. Now, let us define $g(u_i) = v_i$ for $i = 1, \dots, m$. Since $g \in \mathrm{Perm}(C, D)$, then

$$[C \sim^\wedge D] = \bigvee_{f \in \mathrm{Perm}(C,D)} [C \sim^\wedge_f D] =$$

$$\bigvee_{f \in \mathrm{Perm}(C,D)} \bigwedge_{x \in y} (C(x) \leftrightarrow D(f(x))) \geq \bigwedge_{x \in y} (C(x) \leftrightarrow D(g(x))) =$$

$$\bigwedge_{i=1}^{m} (C(u_i) \leftrightarrow D(g(u_i))) = \bigwedge_{i=1}^{m} (C(u_i) \leftrightarrow D(v_i)) =$$

$$\bigwedge_{i=1}^{m} (p^\vee_A(i, y) \leftrightarrow p^\vee_B(i, y')) = \bigwedge_{i \in \mathbb{N}} (p^\vee_A(i, y) \leftrightarrow p^\vee_B(i, y')) = [p^\vee_A(-, y) \approx p^\vee_B(-, y')],$$

and the inequality is proved. □

Example 7.13 Let us consider the task from Ex. 7.8. By (57), we can establish

$$p^\vee_A(-, y) = \{0/0, 0/1, 0/2, 0.3/3, 0.4/4, 0.8/5, 1/6, 1/7, \dots\},$$
$$p^\vee_B(-, y) = \{0/0, 0/1, 0/2, 0.6/3, 0.7/4, 0.9/5, 1/6, 1/7, \dots\},$$

and, using (63) of Theorem 7.18, we may simply compute

$$[A \sim^\wedge B] = \bigwedge_{i \in \mathbb{N}} (p^\vee_A(i, y) \leftrightarrow p^\vee_B(i, y) = 0.7,$$

which is precisely the same result as in Ex. 7.8, where the degree of equipollence of fuzzy sets A and B was derived by the similarity of non-increasing generalized cardinals $p^\wedge_A(-, y)$ and $p^\vee_B(-, y)$.

8. Conclusion

This paper is the first part of our contribution on the graded approach to the cardinality of finite fuzzy sets. Here, we proposed a cardinal theory based on the concept of graded equipollence. Fuzzy sets are designed to be inside the universe (class) of all countable sets, which indicates that each fuzzy set has a countable universe of discourse. Thus, in contrast to the standard approach to fuzzy sets, we do not suppose a fixed universe for all fuzzy sets. A basic theory of fuzzy sets in the universe of all countable sets was introduced, including constructions such as fuzzy power sets and exponentiation. Graded equipollence was introduced as a fuzzy class relation assigning a degree to each pair of fuzzy sets, expressing the fact that these fuzzy sets have approximately the same cardinality (are approximately equipollent). The graded equipollence is derived from degrees in which one-to-one mappings between sets may be considered to be one-to-one correspondences between fuzzy sets. With this concept, a cardinal theory of finite fuzzy sets (the functional approach to cardinality) was developed, and several well-known statements, including the Cantor-Bernstein theorem and the Cantor theorem stating different cardinalities for sets and their power sets, were generalized in a graded design. Finally, the relationship between the graded equipollence and generalized cardinals was shown.

The second part [7] of our contribution will present an axiomatic approach to the cardinality of finite fuzzy sets. A simple axiomatic system for fuzzy cardinal (c-)measures will be introduced to express how many elements finite fuzzy sets have. A relation between the graded equipollence and fuzzy c-measures, i.e., between a functional approach to cardinality and objects designed to express cardinality, will be shown.

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