



## CENTRE OF EXCELLENCE IT4INNOVATIONS

division of the University of Ostrava  
Institute for Research and Applications of Fuzzy Modeling

---

# A Graded Approach to Cardinal Theory of Finite Fuzzy Sets, Part II: Fuzzy Cardinality Measures and Their Relationship to Graded Equipollence

Michal Holčapek

Research report No. 163

2012

*Submitted/to appear:*

Information Sciences

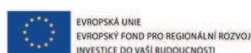
*Supported by:*

This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070).

Centre of Excellence IT4Innovations, division of the University of Ostrava  
Institute for Research and Applications of Fuzzy Modeling  
30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-59-7091401 fax: +420-59-6120478

e-mail: Michal.Holcapek@osu.cz



# A Graded Approach to Cardinal Theory of Finite Fuzzy Sets, Part II: Fuzzy Cardinality Measures and Their Relationship to Graded Equipollence <sup>☆</sup>

Michal Holčapek

*Centre of Excellence IT4Innovations  
division of University of Ostrava  
Institute for Research and Applications of Fuzzy Modeling  
30. dubna 22, 701 03 Ostrava 1, Czech Republic  
michal.holcapek@osu.cz*

---

## Abstract

In this article, we propose an axiomatic system for fuzzy “cardinality” measures (referred to as fuzzy  $c$ -measures for short) assigning to each finite fuzzy set a generalized cardinal that expresses the number of elements that the fuzzy set contains. The system generalizes an axiomatic system introduced by J. Casanovas and J. Torrens (2003). We show that each fuzzy  $c$ -measure is determined by two appropriate homomorphisms between the reducts of residuated-dually residuated (rdr-)lattices. For linearly ordered rdr-lattices, we prove that each fuzzy  $c$ -measure is a product of a non-decreasing and a non-increasing fuzzy  $c$ -measure, which indicates that there is a close relation between fuzzy  $c$ -measures and FGCount, FLCCount and FECCount provided by L.A. Zadeh (1983) and generalized by M. Wygralak (2001). Finally, the relationship of fuzzy  $c$ -measures to graded equipollence introduced in the first part of this contribution is analyzed.

*Key words:* residuated-dually residuated lattice, generalized cardinals, graded equipollence, cardinality of finite fuzzy sets, fuzzy  $c$ -measures

---

## 1. Introduction

In fuzzy mathematics, the cardinality of a fuzzy set is a measure of the “number of elements belonging to the fuzzy set”. Analogous to the cardinal theory of sets, there are two approaches to the cardinality of fuzzy sets: one approach that is based on the concept of equipollence (equipotency, equinumerosity) and the other approach that uses a set of numbers or a fuzzy generalization of these numbers.

---

<sup>☆</sup>This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070) and also partially supported by the Grant IAA108270901 of the GA AV ČR.

The first approach has been discussed in the Part I of this contribution devoted to the graded equipollence. The second approach can be divided to *scalar* and *fuzzy* depending on the type of objects used to express the power of the fuzzy sets.

Because the scalar approach is not the subject of this paper, let us say only that a single number (usually a non-negative real number) is assigned to each finite fuzzy set. Most of the definitions of scalar cardinality are included in the axiomatic system proposed by M. Wygralak in [22] (see also [5, 23]).

In the fuzzy approach, the cardinality of the fuzzy sets is usually defined as a mapping that assigns to each fuzzy set an appropriate fuzzy set in a universe of cardinal numbers. Such fuzzy sets are usually referred to as *generalized cardinals* (see, e.g., [7, 18, 19, 20]), and they are assumed to be convex. Let us recall that a fuzzy set  $A : x \rightarrow [0, 1]$  is said to be *convex* in  $x$  equipped with an ordering  $\leq$  if

$$A(a) \wedge A(c) \leq A(b)$$

holds for all  $a, b, c \in x$  such that  $a \leq b \leq c$ . From the practice, one may see that the cardinality of finite fuzzy sets plays a central role. The generalized cardinals for finite fuzzy sets are usually convex fuzzy sets in the set of natural numbers. The first definition of the cardinality of finite fuzzy sets that uses generalized natural numbers was proposed by L.A. Zadeh in [25]. When L.A. Zadeh developed a computational approach to fuzzy quantifiers in natural language,<sup>1</sup> he introduced in [24] three types of finite fuzzy set cardinality expressions, namely *FGCount*, *FLCount*, and *FECCount*. Note that

$$\text{FGCount}(A)(k) = \bigvee \{a \mid |A_a| \geq k\}$$

expresses the degree to which  $A$  contains *at least*  $k$  elements. The dual variant

$$\text{FLCount}(A)(k) = 1 - \text{FGCount}(A)(k + 1)$$

determines the degree to which  $A$  has *at most*  $k$  elements. The degree to which  $A$  has *exactly*  $k$  elements is expressed by

$$\text{FECCount}(A)(k) = \text{FGCount}(A)(k) \wedge \text{FLCount}(A)(k).$$

Further approaches to the definition of the cardinality for finite fuzzy sets can be found in [6, 7, 15, 16]. The *FGCount*, *FLCount* and *FECCount* were generalized, and a cardinal theory of finite fuzzy sets over triangular norms was developed by M. Wygralak in [22] (see also [23]). Note that all three types of fuzzy cardinalities were reasonably designed, but no unifying axiomatic system, which would follow the scalar approach, was provided. An attempt to unify many of the common definitions under an axiomatic system was proposed by J. Casanovas and J. Torrens in [3]. The axiomatic system defined an infinite class of

---

<sup>1</sup>For example, *for nearly all* is a fuzzy quantifier in “For nearly all students visited the lecture”, *few* in “Few single women stayed at home”, or *approximately half* in “Approximately half questions were not answered”.

fuzzy cardinalities for finite fuzzy sets, which included the original definitions of FGCount, FLCCount, and FECCount. Nevertheless, the axiomatic system was designed to only deal with the operations of the minimum and the maximum in the unit interval, and thus, Wygralak's approach to the fuzzy cardinality over triangular norms cannot be included in this system. An extension of FGCount for fuzzy sets with the membership degrees in a linearly ordered lattice was proposed by P. Lubczonok in [13]. An approach to the non-convex cardinality of fuzzy sets was proposed in [4].

In [10], we introduced an axiomatic system for fuzzy cardinality measures (referred to as fuzzy  $c$ -measures for short) that, in some sense, unifies the three mentioned approaches to the cardinality of finite fuzzy sets. In this paper, we slightly modify the original approach from [10] with respect to the purpose of this contribution and elaborate fuzzy  $c$ -measures in details. It should be noted that we use rather fuzzy  $c$ -measures as opposed to fuzzy cardinalities, because not all fuzzy  $c$ -measures may be considered as a true cardinality of finite fuzzy sets. The proposed system generalizes the axiomatic system provided by J. Casasnovas and J. Torrens in [3], and a lot of the fuzzy cardinalities based on the triangular norms suggested by Wygralak in [22, 23] can be introduced using this system (see Corollary 3.11). Moreover, we do not restrict ourselves to the unit interval here and consider a general structure called a residuated-dually residuated lattice (rdr-lattice for short) defined in [10] (see also [9, 11]). Analogous to the representations of scalar cardinalities and fuzzy cardinalities using appropriate mappings provided in [21, 3], we prove that each fuzzy  $c$ -measure can be represented by two special homomorphisms between the reducts of an rdr-lattice. An interesting question is whether and how fuzzy  $c$ -measures are related to the graded equipollence proposed in the first portion of this work. A possible answer is given here as well.

The paper is structured as follows. The next section provides the necessary preliminaries to understand the text of this paper. Section 3 is devoted to the fuzzy  $c$ -measures, which include the axiomatic system, the basic properties, the representation theorem, the characterizations of the fuzzy  $c$ -measures, and a discussion concerning the valuation property. In Section 4, we show several relations between the graded equipollence of finite fuzzy sets and the similarity of the generalized cardinals obtained using the fuzzy  $c$ -measures, and we demonstrate that, to some fuzzy  $c$ -measures, a corresponding non-graded equipollence can be established. The last section concludes the work.

## 2. Preliminaries

In this section, we would like to summarize the basic notions and results without any details from the Part I that are significant to understand the text and make this paper more compact. A reader acquainted with the contents of the first portion of this work may omit this preliminary and use it only for references to results applied in the proofs.

As a structure for the membership degrees of fuzzy sets, we assume an rdr-lattice  $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus, \perp, \top \rangle$ , i.e., a (bounded) residuated lattice ex-

tended by a dual adjoint pair of operations  $(\oplus, \ominus)$ , i.e.,  $\alpha \leq \beta \oplus \gamma$  if and only if  $\alpha \ominus \beta \leq \gamma$ . The least (greatest) element is denoted by  $\perp$  ( $\top$ ). An rdr-lattice is said to be *linearly ordered* if the reduct  $\langle L, \wedge, \vee, \perp, \top \rangle$  is a linearly ordered lattice. Further, an rdr-lattice satisfies the *prelinearity* axiom if  $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha) = \top$  for all  $\alpha, \beta \in L$ . A simple example of a linearly ordered rdr-lattice is an algebra derived by a left continuous t-norm  $T$  and a right continuous t-conorm  $S$  or an MV-algebra (see Ex. 2.1 and Ex. 2.2 in [9]). We shall say that an rdr-lattice satisfies the *conditional cancellation law* for  $\otimes$  (for  $\oplus$ ) if  $\alpha \otimes \beta = \alpha \otimes \gamma > \perp$  ( $\alpha \oplus \beta = \alpha \oplus \gamma < \top$ ) implies  $\beta = \gamma$  for any  $\alpha, \beta, \gamma \in L$ . For example, we can consider linearly ordered MV-algebras (see Ex. 2.2 in [9] and Theorem 2.45 in [1]) or rdr-lattices determined by left-continuous t-norms and right-continuous t-conorms (see Ex. 2.1. in [9] and Remark 2.19 in [12] or Section 1.4 in [23]).

A framework for fuzzy sets is the class of all countable sets denoted by **Count**. A subclass of **Count** contains all finite sets is denoted by **Fin**. A mapping  $A : z \rightarrow L$  is a *fuzzy set (fuzzy class)* in **Count** if  $z \in \mathbf{Count}$  ( $z \subseteq \mathbf{Count}$ ). The class of all fuzzy sets is denoted by **Fcount**. The set  $z = \text{Dom}(A)$  is called a *universe of discourse* of  $A$ . The set  $\{x \in z \mid A(x) > \perp\}$  is called a *support* of  $A$  and is denoted by  $\text{Supp}(A)$ . The empty mapping  $\emptyset : \emptyset \rightarrow L$  is called the *empty fuzzy set*, and a fuzzy set  $A$  is called a *singleton* if  $\text{Dom}(A)$  contains only one element. A singleton is denoted by  $\{\alpha/x\}$ , where  $\alpha \in L$  and  $x \in \mathbf{Count}$ . The fuzzy set  $A$  is called *crisp* if  $A(x) \in \{\perp, \top\}$  for any  $x \in z$ . Note that the empty fuzzy set is crisp, since the presumption is trivially satisfied. The fuzzy sets  $A$  and  $B$  are said to be *the same* (symbolically  $A = B$ ) if  $\text{Dom}(A) = \text{Dom}(B)$  and  $A(x) = B(x)$  for any  $x \in \text{Dom}(A)$  and to be *equivalent* (symbolically  $A \equiv B$ ) if  $\text{Supp}(A) = \text{Supp}(B)$  and  $A(x) = B(x)$  for any  $x \in \text{Supp}(A)$ . The class of all fuzzy sets equivalent to  $A$  is denoted by  $\text{cls}(A)$ . The fuzzy sets  $A$  and  $B$  are *equivalent* in the degree  $\alpha$  (symbolically,  $[A \approx B] = \alpha$ ) if

$$\alpha = \bigwedge_{x \in \text{Dom}(A) \cup \text{Dom}(B)} (A'(x) \leftrightarrow B'(x)), \quad (1)$$

holds for  $A' \in \text{cls}(A)$ ,  $B' \in \text{cls}(B)$  with  $\text{Dom}(A') = \text{Dom}(B') = \text{Dom}(A) \cup \text{Dom}(B)$ . A fuzzy set  $A$  is said to be a *fuzzy subset* of a fuzzy set  $A$  if  $A(x) \leq B(x)$  for any  $x \in \text{Supp}(A)$ . Let  $A, B \in \mathbf{Fcount}$ ,  $x = \text{Dom}(A) \cup \text{Dom}(B)$  and  $A \equiv A'$  and  $B \equiv B'$  be such that  $\text{Dom}(A') = \text{Dom}(B') = x$ . Then, the *union* of  $A$  and  $B$  is a mapping  $A \cup B : x \rightarrow L$  defined by  $(A \cup B)(a) = A'(a) \vee B'(a)$ , and the *intersection* of  $A$  and  $B$  is a mapping  $A \cap B : x \rightarrow L$  defined by  $(A \cap B)(a) = A'(a) \wedge B'(a)$  for any  $a \in x$ . A fuzzy set  $A$  is said to be *finite* if  $\text{Supp}(A) \in \mathbf{Fin}$ . The class of all finite fuzzy sets is denoted by **Fin**.

Let  $x$  be a set equipped with an ordering relation  $\leq$ . A fuzzy set  $A : x \rightarrow L$  is a  $\odot$ -convex fuzzy set if  $A(a) \odot A(c) \leq A(b)$  holds for any  $a, b, c \in x$  with  $a \leq b \leq c$ . A *generalized cardinal* is a  $\odot$ -convex fuzzy set with the set of natural numbers  $\mathbb{N}$  as its universe of discourse. The set of all generalized cardinals is denoted by  $\mathfrak{N}$ , and the operation of addition on  $\mathfrak{N}$  is defined, by Zadeh's

extension principle, as

$$(A + B)(i) = \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} (A(k) \odot B(l)). \quad (2)$$

Define  $\mathbf{0} : \mathbb{N} \rightarrow L$  by  $\mathbf{0}(k) = \top$ , if  $k = 0$ , and  $\mathbf{0}(k) = \perp$ , otherwise. Using Theorem 4.1 in [9], the triplet  $\mathfrak{N} = (\mathfrak{N}, +, \mathbf{0})$  is an abelian monoid, where the prelinearity of the rdr-lattice  $\mathbf{L}$  must be supposed for  $\odot = \wedge$ .

Let  $A, B \in \mathfrak{F}\mathbf{in}$  and  $|\text{Dom}(A)| = |\text{Dom}(B)|$ , i.e., they have the same cardinality. The set  $\text{Perm}(A, B)$  contains all one-to-one mappings of  $\text{Dom}(A)$  onto  $\text{Dom}(B)$ . If  $f \in \text{Perm}(A, B)$ , then

$$[A \sim_f^\odot B] = \bigodot_{x \in \text{Dom}(A)} (A(x) \leftrightarrow B(f(x))) \quad (3)$$

gives the degree to which  $f$  is a *one-to-one mapping of A onto B* with respect to  $\odot$ . The degree to which the fuzzy sets  $A$  and  $B$  are *equipollent* with respect to  $\odot$  is denoted by  $[A \sim^\odot B]$  and can be derived by the following theorem.

**THEOREM 2.1** [9] *Let  $A, B \in \mathfrak{F}\mathbf{in}$ . Then,*

$$[A \sim^\odot B] = \bigvee_{f \in \text{Perm}(C, D)} [C \sim_f^\odot D] \quad (4)$$

for any  $C \in \text{cls}(A)$  and  $D \in \text{cls}(B)$  such that  $|\text{Dom}(C)| = |\text{Dom}(D)| = m$ .

To compute the degree to which  $A$  and  $B$  have the same cardinality, it is sufficient to find a pair of equivalent fuzzy sets to  $A$  and  $B$  defined over finite universes of discourse. The correctness of the definition of “to be equipollent” is provided by the following corollaries.

**COROLLARY 2.2** [9] *If  $A \in \mathfrak{F}\mathbf{in}$ , then  $[A \sim^\odot B] = \top$  for all  $B \in \text{cls}(A)$ .*

**COROLLARY 2.3** [9] *Let  $A, B \in \mathfrak{F}\mathbf{in}$ ,  $C \in \text{cls}(A)$ , and  $D \in \text{cls}(B)$ . Then,  $[A \sim^\odot B] = [C \sim^\odot D]$ .*

The fuzzy class relation  $\sim^\odot$  is a similarity relation (precisely  $\otimes$ -similarity) on the class  $\mathfrak{F}\mathbf{in}$  (Theorem 5.6 in [9]).

Let  $A \in \mathfrak{F}\mathbf{in}$  be a finite fuzzy set and define a fuzzy class  $p_A^\odot : \mathbb{N} \times \mathfrak{F}\mathbf{in} \rightarrow L$  by

$$p_A^\odot(i, y) = \begin{cases} \bigvee_{z \subseteq y, |z|=i} \bigodot_{x \in z} A(x), & \text{if } y \subseteq \text{Dom}(A); \\ \perp, & \text{otherwise,} \end{cases} \quad (5)$$

for any  $(i, y) \in \mathbb{N} \times \mathfrak{F}\mathbf{in}$ . The following proposition summarizes the basic properties of  $p_A^\odot$ .

PROPOSITION 2.4 [9] Let  $A \in \mathfrak{Ffin}$  and  $y \in \mathfrak{Fin}$  such that  $y \subseteq \text{Dom}(A)$ . Then, for any  $i, j \in \mathbb{N}$ , we have

- (i)  $p_A^\circ(i, y) \leq p_A^\circ(j, y)$ , if  $i \geq j$ ,
- (ii)  $p_A^\circ(i, y) = \perp$ , if  $i > |y|$ ,
- (iii)  $p_A^\circ(0, y) = \top$ ,
- (iv)  $p_A^\circ(1, y) = \bigvee_{x \in y} A(x)$ ,
- (v)  $p_A^\circ(0, \emptyset) = \top$  and  $p_A^\circ(i, \emptyset) = \perp$  for any  $i > 0$ , and
- (vi)  $p_A^\circ(-, y)$  is a non-increasing generalized cardinal.

PROPOSITION 2.5 [9] Let  $\mathbf{L}$  be linearly ordered. Then,  $p_A^\circ(i, y) = \bigotimes_{k=0}^i p_A^\wedge(k, y)$  for any  $A \in \mathfrak{Ffin}$  and  $(i, y) \in \mathbb{N} \times \mathfrak{Fin}$ .

Let  $\mathbf{L}$  be linearly ordered and  $A \in \mathfrak{Ffin}$ . A set  $s \in \mathfrak{Fin}$  such that  $s \subseteq \text{Supp}(A)$  is said to be a *substantial segment* of  $A$  with respect to  $\odot$  if

- (i)  $\odot_{x \in s} A(x) > \perp$ ,
- (ii) if  $z \in \mathfrak{Fin}$  such that  $s \subset z \subseteq \text{Supp}(A)$ , then  $\odot_{x \in z} A(x) = \perp$ ,
- (iii) if  $y \subseteq \text{Supp}(A)$  satisfies (i) and (ii), then there exists a one-to-one mapping  $f : y \rightarrow s$  such that  $A(x) \leq A(f(x))$  for any  $x \in y$ .

The fuzzy set  $A^s = A \upharpoonright s$  is called a *fuzzy substantial segment* of  $A$  with respect to  $\odot$  (see Definition 7.1. and Remark 7.3. in [9]). Recall that  $A \upharpoonright s$  is the restriction of  $A$  to  $s$ .

An important theorem in our investigation of the relation between the non-graded equipollence<sup>2</sup> and a fuzzy c-measure that provides non-increasing generalized cardinals is the following.

THEOREM 2.6 [9] Let  $\mathbf{L}$  be linearly ordered and, for  $\odot = \otimes$ , satisfy the conditional cancellation law for  $\otimes$ . Let  $A, B \in \mathfrak{Ffin}$  and  $y, y' \in \mathfrak{Fin}$  such that  $s \subseteq y \subseteq \text{Dom}(A)$  and  $t \subseteq y' \subseteq \text{Dom}(B)$ , where  $s$  and  $t$  are substantial segments of  $A$  and  $B$  with respect to  $\odot$ , respectively. Then,  $[A^s \sim^\odot B^t] = \top$  if and only if  $p_A^\circ(-, y) = p_B^\circ(-, y')$ .

The following theorem plays a significant role in the derivation of a relation between the graded equipollence and a fuzzy c-measure.

THEOREM 2.7 [9] Let  $\mathbf{L}$  be linearly ordered,  $A, B \in \mathfrak{Ffin}$  and  $y, y' \in \mathfrak{Fin}$  such that  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$  and  $\text{Supp}(B) \subseteq y' \subseteq \text{Dom}(B)$ . Then,

$$[A \sim^\wedge B] = [p_A^\wedge(-, y) \approx p_B^\wedge(-, y')]. \quad (6)$$

---

<sup>2</sup>This means that we consider only the situation when fuzzy sets are or are not equipollent, and two fuzzy sets  $A$  and  $B$  are equipollent if  $[A \sim^\odot B] = \top$ .

Let  $A \in \mathfrak{F}\mathbf{in}$  be a finite fuzzy set and define a fuzzy class  $p_A^{\bar{\circ}} : \mathbb{N} \times \mathfrak{F}\mathbf{in} \rightarrow L$  by

$$p_A^{\bar{\circ}}(i, y) = \begin{cases} \bigwedge_{z \subseteq y, |z|=i} \bar{\circ}_{x \in z} A(x), & \text{if } y \subseteq \text{Dom}(A); \\ \perp, & \text{otherwise,} \end{cases} \quad (7)$$

for any  $(i, y) \in \mathbb{N} \times \mathfrak{F}\mathbf{in}$ . It should be noted that  $p_A^{\bar{\circ}}$  is designed to be a dual to  $p_A^{\circ}$ . The basic properties of  $p_A^{\bar{\circ}}$  used in the text are mentioned in the following proposition.

**PROPOSITION 2.8** [9] *Let  $A \in \mathfrak{F}\mathbf{in}$  and  $y \in \mathfrak{F}\mathbf{in}$  such that  $y \subseteq \text{Dom}(A)$ . Then, for any  $i, j \in \mathbb{N}$ , we have*

- (i)  $p_A^{\bar{\circ}}(i, y) \geq p_A^{\bar{\circ}}(j, y)$ , if  $i \geq j$ ,
- (ii)  $p_A^{\bar{\circ}}(i, y) = \top$ , if  $i > |y|$ ,
- (iii)  $p_A^{\bar{\circ}}(0, y) = \perp$ ,
- (iv)  $p_A^{\bar{\circ}}(1, y) = \bigwedge_{x \in y} A(x)$ ,
- (v)  $p_A^{\bar{\circ}}(0, \emptyset) = \perp$  and  $p_A^{\bar{\circ}}(i, \emptyset) = \top$  for any  $i > 0$ , and
- (vi)  $p_A^{\bar{\circ}}(-, y)$  is a non-decreasing generalized cardinal.

**PROPOSITION 2.9** [9] *Let  $\mathbf{L}$  be linearly ordered. Then,  $p_A^{\oplus}(i, y) = \bigoplus_{k=0}^i p_A^{\vee}(k, y)$  for any  $A \in \mathfrak{F}\mathbf{in}$  and  $(i, y) \in \mathbb{N} \times \mathfrak{F}\mathbf{in}$ .*

Let  $\mathbf{L}$  be linearly ordered and  $A \in \mathfrak{F}\mathbf{in}$ . A set  $s \in \mathfrak{F}\mathbf{in}$  such that  $s \subseteq \text{Supp}(A)$  is said to be a *substantial segment* of  $A$  with respect to  $\bar{\circ}$  if

- (i)  $\bar{\circ}_{x \in s} A(x) < \top$ ,
- (ii) if  $z \in \mathfrak{F}\mathbf{in}$  such that  $s \subset z \subseteq \text{Supp}(A)$ , then  $\bar{\circ}_{x \in z} A(x) = \top$ ,
- (iii) if  $y \subseteq \text{Supp}(A)$  satisfies (i) and (ii), then there exists a one-to-one mapping  $f : y \rightarrow s$  such that  $A(x) \geq A(f(x))$  for any  $x \in y$ .

The fuzzy set  $A^s = A \upharpoonright s$  is called a *fuzzy substantial segment* of  $A$  with respect to  $\bar{\circ}$  (see Definition 7.2. and Remark 7.10. in [9]).

The following statement will be used to prove a relation between the non-graded equipollence and a fuzzy c-measure that provides non-decreasing generalized cardinals.

**THEOREM 2.10** [9] *Let  $\mathbf{L}$  be linearly ordered and, for  $\bar{\circ} = \oplus$ , satisfy the conditional cancellation law for  $\oplus$ . Let  $A, B \in \mathfrak{F}\mathbf{in}$  and  $y, y' \in \mathfrak{F}\mathbf{in}$  such that  $s \subseteq y \subseteq \text{Supp}(A)$  and  $t \subseteq y' \subseteq \text{Supp}(B)$ , where  $s$  and  $t$  are substantial segments of  $A$  and  $B$  with respect to  $\bar{\circ}$ , respectively. Then,  $[A^s \sim^{\bar{\circ}} B^t] = \top$  if and only if  $p_A^{\bar{\circ}}(-, y) = p_B^{\bar{\circ}}(-, y')$ .*



### 3. Fuzzy c-measures

In this section, we propose an axiomatic system for fuzzy measures that can be understood as a natural generalization of the cardinality measure of finite sets. The system is designed in such a way that the axiomatic system for fuzzy cardinalities provided by J. Casasnovas and J. Torrens in [3] is a special case where the unit interval and the operation of the minimum and the maximum are assumed.<sup>3</sup> Moreover, we obtain a wider class of models that can serve as fuzzy measures that indicate the number of elements in a finite fuzzy set.

#### 3.1. Axiomatic system

In the following text, we shall suppose that  $\mathbf{L}$  is an rdr-lattice that satisfies the prelinearity axiom if  $\odot = \wedge$ .

DEFINITION 3.1 A class mapping  $\mathfrak{C} : \mathfrak{F}\mathbf{fin} \rightarrow \mathfrak{N}$  is a *fuzzy cardinality (c-)measure of finite fuzzy sets* with respect to  $\odot$  if, for arbitrary  $A, B \in \mathfrak{F}\mathbf{fin}$ , it satisfies the following axioms

- (C1) if  $\text{Supp}(A) \cap \text{Supp}(B) = \emptyset$ , then  $\mathfrak{C}(A \cup B) = \mathfrak{C}(A) + \mathfrak{C}(B)$ ,
- (C2) if  $i, j \in \mathbb{N}$ ,  $i > |\text{Supp}(A)|$ , and  $j > |\text{Supp}(B)|$ , then  $\mathfrak{C}(A)(i) = \mathfrak{C}(B)(j)$ ,
- (C3) if  $A$  is a crisp set, then  $\mathfrak{C}(A)$  is a crisp set and  $\mathfrak{C}(A)(|\text{Supp}(A)|) = \top$ ,
- (C4) if  $\alpha \in L$ ,  $x, y \in \mathbf{Count}$  and  $i \in \mathbb{N}$ , then  $\mathfrak{C}(\{\alpha/x\})(i) = \mathfrak{C}(\{\alpha/y\})(i)$ , and
- (C5) if  $\alpha, \beta \in L$  and  $x \in \mathbf{Count}$ , then

$$\begin{aligned}\mathfrak{C}(\{\alpha \bar{\odot} \beta/x\})(0) &= \mathfrak{C}(\{\alpha/x\})(0) \odot \mathfrak{C}(\{\beta/x\})(0), \\ \mathfrak{C}(\{\alpha \odot \beta/x\})(1) &= \mathfrak{C}(\{\alpha/x\})(1) \odot \mathfrak{C}(\{\beta/x\})(1).\end{aligned}$$

The axioms (C1)-(C5) are called *the additivity, the variability, the consistency, the singleton independency, the preservation of non-existence and existence*, respectively. In the following text, for the sake of simplicity, we shall often use only “c-measure” instead of “fuzzy c-measure of finite fuzzy sets”, and we shall omit “with respect to  $\odot$ ” when no confusion is possible.

Before we discuss the proposed axioms, let us note that  $\mathfrak{C}$  is a measure in the following sense. We know that the inclusion relation  $\subseteq$  is a partial ordering on  $\mathfrak{F}\mathbf{fin}$ . Let us define another partial ordering on  $\mathfrak{F}\mathbf{fin}$  as follows:  $A \sqsubseteq B$  if there exists  $x \in \mathfrak{F}\mathbf{in}$  such that  $B \upharpoonright x \equiv A$ , i.e.,  $B \upharpoonright x$  and  $A$  are equivalent fuzzy sets.<sup>4</sup> One may simply check that  $A \sqsubseteq B$  implies  $A' \sqsubseteq B'$  for any  $A' \in \text{cls}(A)$  and  $B' \in \text{cls}(B)$ , and thus,  $\sqsubseteq$  is correctly established. Let  $(\mathfrak{N}, +)$  be equipped

<sup>3</sup>In our conception, it means that the rdr-lattice is the corresponding extension of the Gödel algebra on  $[0, 1]$ .

<sup>4</sup>For the definition of equivalence of fuzzy sets, see Definition 3.4 in [9] or Preliminaries here.

with the partial ordering  $\leq$  defined as follows:  $A \leq B$  if there exists  $C \in \mathfrak{N}$  such that  $A + C = B$ .<sup>5</sup> Then, it is easy to prove that

$$A \sqsubseteq B \quad \text{then} \quad \mathfrak{C}(A) \leq \mathfrak{C}(B). \quad (8)$$

In fact, let us suppose  $A = B \upharpoonright x$ .<sup>6</sup> Then, defining  $z = \text{Dom}(B) \setminus x$  and  $B' = B \upharpoonright z$ , we obviously obtain  $\text{Supp}(A) \cap \text{Supp}(B') = \emptyset$  and  $B = A \cup B'$ . The desired inequality follows from the additivity of  $\mathfrak{C}$ . Thus,  $\mathfrak{C}$  is a monotone mapping. Furthermore, all images of fuzzy sets of  $\mathfrak{Ffin}$  under  $\mathfrak{C}$  form a subset of  $\mathfrak{N}$ , where  $\mathfrak{C}(\emptyset)$  is the least element with respect to  $\leq$ .<sup>7</sup> In this sense,  $\mathfrak{C}$  may be regarded as a fuzzy measure (in a general conception) satisfying the additivity axiom provided in (C1).

Now, let us comment on the proposed axioms. The first three axioms are adopted from [3] and their motivation is as follows. The additivity of the c-measures is the standard property of the cardinality of sets. Note that we choose the intersection of the supports for the characterization of disjointness of the fuzzy sets, because non-linear residuated lattices can be used in general. In this case, we can determine two fuzzy sets with the same support that differ from the empty fuzzy set, but their intersection is a fuzzy set equivalent to the empty fuzzy set. In such cases, it is difficult to ensure additivity. One can see that the intersection of fuzzy sets can be used for linearly ordered residuated lattices. The variability states that the c-measures of fuzzy sets are not influenced by the elements that do not belong to the supports. A consequence of this axiom is the correctness of our axiomatic system, which means that the equivalent finite fuzzy sets have the same c-measure. The axiom of consistency ensures the fact that the c-measures are extensions of the cardinality measure (for the finite sets). The singleton independency ensures that a form of elements does not influence the value of a c-measure. Finally, the membership degree  $\mathfrak{C}(\{A(x)/x\})(0)$  expresses the degree to which  $\{A(x)/x\}$  may be considered as the empty set  $\{\perp/x\}$ , i.e.,  $x$  does not belong to  $A$ . Analogously, the membership degree  $\mathfrak{C}(\{A(x)/x\})(1)$  determines the degree to which  $\{A(x)/x\}$  may be considered as the singleton  $\{\top/x\}$ , i.e.,  $x$  belongs to  $A$ . Because  $\alpha \bar{\odot} \beta \geq \alpha \vee \beta$ , then one could deduce that the singletons  $\{\alpha/x\}$  and  $\{\beta/x\}$  are closer to the empty set  $\{\perp/x\}$  than the singleton  $\{\alpha \bar{\odot} \beta/x\}$ . From the closeness of the singletons to the empty set, one could conclude that  $\mathfrak{C}(\{\alpha \bar{\odot} \beta/x\})(0) \leq \mathfrak{C}(\{\alpha/x\})(0)$  and  $\mathfrak{C}(\{\alpha \bar{\odot} \beta/x\})(0) \leq$

<sup>5</sup>Note that  $\leq$  is well established, i.e., if  $A + C = B$  and  $B + D = A$ , then  $A = B$ . In fact, one could prove that if  $A + C + D = A$ , then  $C + D = \mathbf{0}$  or  $C + D = \chi_{\mathbb{N}}$ , where  $\mathbf{0}$  is the neutral element in the monoid  $\mathfrak{N}$  and  $\chi_{\mathbb{N}}$  is the characteristic function of  $\mathbb{N}$ . If  $C + D = \mathbf{0}$ , then  $C = D = \mathbf{0}$ . Hence,  $A + \mathbf{0} = B$  implies  $A = B$ . If  $C + D = \chi_{\mathbb{N}}$ , then one can check that at least one of the elements  $C$  and  $D$  is equal to  $\chi_{\mathbb{N}}$ . If  $C = \chi_{\mathbb{N}}$ , then  $A + \chi_{\mathbb{N}} = A$  and  $A + \chi_{\mathbb{N}} = B$  imply  $A = B$ . If  $D = \chi_{\mathbb{N}}$ , then we can use the analogous equations  $B + \chi_{\mathbb{N}} = B$  and  $B + \chi_{\mathbb{N}} = A$ , which imply  $A = B$ .

<sup>6</sup>Using Corollary 3.2 provided later, we can use such restriction without any impact on the result.

<sup>7</sup>Note that  $\emptyset$  is the least element in  $\mathfrak{N}$  with respect to  $\leq$ , and  $0 < \mathfrak{C}(\emptyset)$  for any c-measure  $\mathfrak{C}$  (see Proposition 3.1).

$\mathfrak{C}(\{\beta/x\})(0)$ , which could be written as

$$\mathfrak{C}(\{\alpha \bar{\odot} \beta/x\})(0) \leq \mathfrak{C}(\{\alpha/x\})(0) \wedge \mathfrak{C}(\{\beta/x\})(0).$$

The “strong” relation in axiom (C5) stems from the previous inequality, where we prefer the usage of the operation  $\bar{\odot}$ , since it better corresponds to  $\bar{\odot}$  than  $\wedge$ , and the equality that enables us a deeper investigation of the c-measures properties. Analogously, one could deduce the second “strong” relation in axiom (C5).

First, let us prove the correctness of the proposed definition. This correctness means that the value of  $\mathfrak{C}$  for  $A$  does not depend on the choice of the fuzzy set from the class  $\text{cls}(A)$  of all equivalent finite fuzzy sets (see also Definition 3.4 in [9]). Recall that  $\mathbf{0}$  is the neutral element in the monoid  $\mathfrak{N}$  and the fuzzy set (characteristic function) of natural numbers is denoted by  $\chi_{\mathbb{N}}$ , i.e.,  $\chi_{\mathbb{N}}(n) = \top$  for any  $n \in \mathbb{N}$ .

**PROPOSITION 3.1** *Let  $\mathfrak{C}$  be a c-measure. Then, either  $\mathfrak{C}(A) = \mathbf{0}$  for all  $A \in \text{cls}(\emptyset)$  or  $\mathfrak{C}(A) = \chi_{\mathbb{N}}$  for all  $A \in \text{cls}(\emptyset)$ .*

**PROOF.** Since  $A \in \text{cls}(\emptyset)$ , then  $A$  is crisp and  $\text{Supp}(A) = \emptyset$ . According to (C3), we obtain  $\mathfrak{C}(A)(i) \in \{\perp, \top\}$  for each  $i \in \mathbb{N}$  and  $\mathfrak{C}(A)(0) = \top$ . As a consequence of (C2),  $\mathfrak{C}(A)(i) = \mathfrak{C}(A)(j)$  for all  $i, j > 0$ . Therefore, either  $\mathfrak{C}(A)(i) = \perp$  for all  $i > 0$ , or  $\mathfrak{C}(A)(i) = \top$  for all  $i > 0$ , and thus,  $\mathfrak{C}(A) = \mathbf{0}$ , or  $\mathfrak{C}(A) = \chi_{\mathbb{N}}$ .

If  $A' \in \text{cls}(\emptyset)$ , then  $\mathfrak{C}(A)(0) = \mathfrak{C}(A')(0)$ , and, by (C2), we have  $\mathfrak{C}(A)(i) = \mathfrak{C}(A')(i)$  for all  $i > 0$ . Hence, we obtain  $\mathfrak{C}(A) = \mathfrak{C}(A')$ , which concludes the proof.  $\square$

**PROPOSITION 3.2** *Let  $A \in \mathfrak{F}\text{fin}$ . Then,  $\mathfrak{C}(A) = \mathfrak{C}(B)$  for all  $B \in \text{cls}(A)$ .*

**PROOF.** Let  $A \in \mathfrak{F}\text{fin}$  and define  $C = A \upharpoonright \text{Supp}(A)$  and  $D = A \upharpoonright (\text{Dom}(A) \setminus \text{Supp}(A))$ . Obviously,  $C \equiv A$ ,  $D \equiv \emptyset$ ,  $A = C \cup D$  and  $\text{Supp}(C) \cap \text{Supp}(D) = \emptyset$ . Using the additivity property, we obtain

$$\mathfrak{C}(A) = \mathfrak{C}(C \cup D) = \mathfrak{C}(C) + \mathfrak{C}(D) = \mathfrak{C}(C) + \mathfrak{C}(\emptyset) = \mathfrak{C}(C \cup \emptyset) = \mathfrak{C}(C),$$

where  $\mathfrak{C}(D) = \mathfrak{C}(\emptyset)$  is the consequence of Proposition 3.1. Let  $B \in \text{cls}(A)$ . Then, by the definition of the equivalence of fuzzy sets,  $C = B \upharpoonright \text{Supp}(B)$ , and analogous to  $A$ , one can check that  $\mathfrak{C}(B) = \mathfrak{C}(C)$ . Hence, we obtain  $\mathfrak{C}(A) = \mathfrak{C}(B)$ .  $\square$

**COROLLARY 3.3** *Let  $A \in \mathfrak{F}\text{fin}$ . Then, either  $\mathfrak{C}(A)(i) = \perp$  for all  $i \in \mathbb{N}$  such that  $i > |\text{Supp}(A)|$  or  $\mathfrak{C}(A)(i) = \top$  for all  $i \in \mathbb{N}$  such that  $i > |\text{Supp}(A)|$ .*

**PROOF.** Using (C2), we obtain  $\mathfrak{C}(A)(i) = \mathfrak{C}(\emptyset)(i)$  for any  $i > |\text{Supp}(A)|$ . Now, it is sufficient to use Proposition 3.1.  $\square$

The following examples show the FGCount, FLCCount and FECCount expressed as c-measures of finite fuzzy sets for  $\odot = \wedge$  (cf., [15, 3, 22, 23]). Let us suppose that the structure of the membership degrees is the Łukasiewicz rdr-lattice  $\mathbf{L}_L$  on  $[0, 1]$ ; recall that  $\alpha \otimes \beta = \max(0, \alpha + \beta - 1)$  and  $\alpha \rightarrow \beta = 1 - \alpha + \beta$ . For the sake of simplicity, we shall restrict ourselves to fuzzy sets with non-empty and finite universe. For the remaining fuzzy sets, we apply Proposition 3.2.

EXAMPLE 3.1 The mappings  $\mathfrak{C}_k : \mathfrak{F}\text{in} \rightarrow \mathfrak{N}$ ,  $k = 1, 2, 3$ , defined by

$$\begin{aligned}\mathfrak{C}_1(A)(i) &= \text{FGCount}(A)(i) = \bigvee \{\alpha \mid \alpha \in [0, 1] \ \& \ |A_\alpha| \geq i\}; \\ \mathfrak{C}_2(A)(i) &= \text{FLCCount}(A)(i) = 1 - \mathfrak{C}_1(A)(i + 1); \\ \mathfrak{C}_3(A)(i) &= \text{FECCount}(A)(i) = \mathfrak{C}_1(A)(i) \wedge \mathfrak{C}_2(A)(i),\end{aligned}$$

are c-measures of the finite fuzzy sets with respect to  $\wedge$ . A direct verification of the satisfaction of all the axioms for  $\mathfrak{C}_k$  is a routine that takes over two pages; therefore, we omit it here. Nevertheless, this statement can be also checked indirectly using the representation theorem on page 21 and the theoretical result presented in Corollary 3.12 on page 29. We recommend performing the following simple derivation as a preview to cardinal theory based on c-measures and to return here after a carefully reading the above-mentioned statements. In order to verify that  $\mathfrak{C}_k$  ( $k = 1, 2, 3$ ) is a c-measure with respect to  $\wedge$ , using Theorem 3.7 and Corollary 3.12, it is sufficient to define an appropriate  $\wedge$ -homomorphism  $f$  and an appropriate  $\vee_d$ -homomorphism  $g$  between the reducts of  $\mathbf{L}_L$  (see Definition 3.2). Consider  $f, g, h : [0, 1] \rightarrow [0, 1]$  defined by  $f(\alpha) = \alpha$ ,  $g(\alpha) = 1 - \alpha$ , and  $h(\alpha) = 1$  for any  $\alpha \in [0, 1]$ . Note that  $f$  is the identity  $\wedge$ -homomorphism, and  $h$  is the trivial homomorphism for both cases:  $\wedge$  and  $\vee_d$ . One can simply check that  $f$  is a  $\wedge$ -homomorphism,  $g$  is a  $\vee_d$ -homomorphism, and  $h$  is a  $\wedge$ -homomorphism as well as a  $\vee_d$ -homomorphism of the reducts.<sup>8</sup> Furthermore, it is easy to see that

$$p_A^\wedge(i, y) = \bigvee_{\substack{z \subseteq y \\ |z|=i}} \bigwedge_{x \in z} A(x) = \bigvee \{\alpha \mid \alpha \in [0, 1] \ \& \ |A_\alpha| \geq i\},$$

where  $y = \text{Dom}(A)$ . Finally, a simple consequence of Corollary 3.12 is

$$\begin{aligned}\mathfrak{C}_1(A)(i) &= \mathfrak{C}_{h,f}(A)(i) = f(p_A^\wedge(i, y)) = p_A^\wedge(i, y), \\ \mathfrak{C}_2(A)(i) &= \mathfrak{C}_{g,h}(A)(i) = g(p_A^\wedge(i + 1, y)) = 1 - p_A^\wedge(i + 1, y), \\ \mathfrak{C}_3(A)(i) &= \mathfrak{C}_{g,f}(A)(i) = 1 - p_A^\wedge(i + 1, y) \wedge p_A^\wedge(i, y),\end{aligned}$$

and thus,  $\mathfrak{C}_k$  ( $k = 1, 2, 3$ ) is a c-measure with respect to  $\wedge$ . It should be noted that we need not restrict ourselves to  $f, g$  as defined above, and  $f$  can be defined as an arbitrary order-preserving mapping (i.e.,  $f$  is a  $\wedge$ -homomorphism),

<sup>8</sup>For example,  $g(\alpha \vee \beta) = 1 - (\alpha \vee \beta) = (1 - \alpha) \wedge (1 - \beta) = g(\alpha) \wedge g(\beta)$  and  $g(0) = 1$  (see Definition 3.2).

and  $g$  can be defined as an arbitrary order-reversing mapping (i.e.,  $g$  is a  $\vee_d$ -homomorphism). Therefore,  $\mathfrak{C}$  with respect to  $\wedge$  can be defined by order-preserving and -reversing mappings, which is one of the main result in [3].

Let  $A = \{0.5/a, 0.8/b, 0.1/c, 0.4/d, 0/e\}$ . Then,

$$\mathfrak{C}_1(A) = \{1/0, 0.8/1, 0.5/2, 0.4/3, 0.1/4, 0/5, 0/6, \dots\},$$

$$\mathfrak{C}_2(A) = \{0.2/0, 0.5/1, 0.6/2, 0.9/3, 1/4, 1/5, 1/6, \dots\},$$

$$\mathfrak{C}_3(A) = \{0.2/0, 0.5/1, 0.5/2, 0.4/3, 0.1/4, 0/0, 0/6 \dots\}.$$

Recall that  $\mathfrak{C}_1(A)(i)$  expresses the degree to which it is true that “ $A$  has at least  $i$  elements”, e.g.,  $A$  has at least 3 elements in degree 0.4. Similarly,  $\mathfrak{C}_2(A)(i)$  expresses the degree to which it is true that “ $A$  has at most  $i$  elements”, e.g.,  $A$  has at most 3 elements in degree 0.9. Finally,  $\mathfrak{C}_3(A)(i)$  indicates the degree of the truth that “ $A$  has exactly  $i$  elements”, e.g.,  $A$  has exactly 3 elements in degree 0.4.

Note that  $\mathfrak{C}_k$  ( $k = 1, 2, 3$ ) does not define a c-measure with respect to  $\otimes$  in general. Three c-measures with respect to  $\otimes$  are given in the following example.

EXAMPLE 3.2 Let  $\mathfrak{C}_1$  be the c-measure with respect to  $\wedge$  that was defined above. The complement of  $A$  is a mapping  $\bar{A} : \text{Dom}(A) \rightarrow [0, 1]$  defined by  $\bar{A}(x) = 1 - A(x)$  for all  $x \in \text{Dom}(A)$ .<sup>9</sup> Then,

$$\mathfrak{C}_4(A)(i) = \begin{cases} \top, & i = 0, \\ \mathfrak{C}_4(A)(i-1) \otimes \mathfrak{C}_1(A)(i), & i > 0, \end{cases}$$

$$\mathfrak{C}_5(A)(i) = \mathfrak{C}_4(\bar{A})(m \boxplus i),$$

$$\mathfrak{C}_6(A)(i) = \mathfrak{C}_5(A)(i) \otimes \mathfrak{C}_4(A)(i),$$

where  $m = |\text{Dom}(A)|$  and  $m \boxplus i = \max(0, m - i)$ , are the c-measures with respect to  $\otimes$ . Again, a direct verification is rather complicated; therefore, we omit it here. Instead, we can use the representation theorem on page 21 and Theorem 3.10 on page 25. Take the following as a preview. Using these theorems, it is sufficient to find two appropriate mappings  $f, g : [0, 1] \rightarrow [0, 1]$  such that  $f$  is a  $\otimes$ -po-homomorphism, and  $g$  is a  $\oplus_d$ -po-homomorphism (see Definition 3.3). Set  $f(\alpha) = \alpha$  and  $g(\alpha) = 1 - \alpha$  for all  $\alpha \in [0, 1]$ . Moreover, consider the trivial homomorphism  $h(\alpha) = 1$  for all  $\alpha \in [0, 1]$  that were defined in the previous example. Trivially,  $f$  is a  $\otimes$ -po-homomorphism, and  $h$  is a  $\otimes$ -po-homomorphism as well as a  $\oplus_d$ -po-homomorphism. For  $g$ , we obtain  $g(\alpha \oplus \beta) = 1 - (\alpha \oplus \beta) = 1 - \max(1, \alpha + \beta) = g(\alpha) \otimes g(\beta)$ .<sup>10</sup> Moreover, if

<sup>9</sup>Recall that we suppose that  $\text{Dom}(A)$  is a non-empty and finite set and note that if  $\mathbf{L}$  is an rdr-lattice, then the complement of a fuzzy set  $A$  is generally defined by  $\bar{A}(x) = A(x) \rightarrow \perp$  (see Definition 3.8 in [9]).

<sup>10</sup>In fact, if  $\alpha + \beta \geq 1$ , then  $1 - (\alpha + \beta) = (1 - \alpha) + (1 - \beta) - 1 \leq 0$  and  $g(\alpha \oplus \beta) = 1 - 1 = 0 = (1 - \alpha) \otimes (1 - \beta) = g(\alpha) \otimes g(\beta)$ . If  $\alpha + \beta < 1$ , then  $1 - (\alpha + \beta) = (1 - \alpha) + (1 - \beta) - 1 > 0$  and  $g(\alpha \oplus \beta) = 1 - (\alpha + \beta) = (1 - \alpha) + (1 - \beta) - 1 = g(\alpha) \otimes g(\beta)$ .

$\alpha \leq \beta$ , then  $g(\alpha) \geq g(\beta)$ , and thus,  $g$  is a  $\oplus_d$ -po-homomorphism. Further, one can simply check (using Proposition 2.5) that

$$\mathfrak{C}_4(A)(i) = \bigotimes_{k=0}^i \mathfrak{C}_1(A)(k) = \bigotimes_{k=0}^i p_A^\wedge(k, y) = p_A^\otimes(i, y), \quad (9)$$

where  $y = \text{Dom}(A)$ . Finally, we can put

$$\begin{aligned} \mathfrak{C}_4(A)(i) &= \mathfrak{C}_{h,f}(A)(i) = p_{f(A)}^\otimes(i, y) = p_A^\otimes(i, y), \\ \mathfrak{C}_5(A)(i) &= \mathfrak{C}_{g,h}(A)(i) = p_{g(A)}^\otimes(m \boxminus i, y) = p_{\bar{A}}^\otimes(m \boxminus i, y) = \mathfrak{C}_4(\bar{A})(m \boxminus i), \\ \mathfrak{C}_6(A)(i) &= \mathfrak{C}_{g,f}(A)(i) = p_{\bar{A}}^\otimes(m \boxminus i, y) \otimes p_A^\otimes(i, y) = \mathfrak{C}_5(A)(i) \otimes \mathfrak{C}_4(A)(i), \end{aligned}$$

where  $f(A) = f \circ A$ ,  $g(A) = g \circ A$  (see Definition 3.5),  $m = |y| = |\text{Dom}(A)|$ , and, thus,  $\mathfrak{C}_k$  ( $k = 4, 5, 6$ ) is a c-measure with respect to  $\otimes$ . It should be noted that the choice of  $y$  (i.e.,  $y = \text{Dom}(A) = \text{Dom}(\bar{A})$ ) is substantial here, otherwise, we cannot write  $p_{\bar{A}}^\otimes(m \boxminus i, y) = \mathfrak{C}_4(\bar{A})(m \boxminus i)$ . One can simply check that the same is not true for the choice of  $y = \text{Supp}(A)$ .

Let  $A = \{0.5/a, 0.8/b, 0.1/c, 0.4/d, 0/e\}$ . Then,

$$\begin{aligned} \mathfrak{C}_4(A) &= \{1/0, 0.8/1, 0.3/2, 0/3, 0/4, 0/5, \dots\}, \\ \mathfrak{C}_5(A) &= \{0/0, 0/1, 0.5/2, 0.9/3, 1/4, 1/5, \dots\}, \\ \mathfrak{C}_6(A) &= \{0/0, 0/1, 0/2, 0/3, 0/4, 0/5, \dots\}, \end{aligned}$$

where  $\mathfrak{C}_5(A)$  is derived as follows. First, let us construct the complement of  $A$ , i.e.,

$$\bar{A} = \{0.5/a, 0.2/b, 0.9/c, 0.6/d, 1/e\}.$$

Furthermore, let us introduce

$$p_{\bar{A}}^\wedge(-, y) = \{1/0, 1/1, 0.9/2, 0.6/3, 0.5/4, 0.2/5, 0/6, \dots\},$$

and, by (9),

$$p_{\bar{A}}^\otimes(-, y) = \{1/0, 1/1, 0.9/2, 0.5/3, 0/4, 0/5, 0/6, \dots\}.$$

Finally, let us define

$$\mathfrak{C}_5(A) = p_{\bar{A}}^\otimes(5 \boxminus -, y) = \{0/0, 0/1, 0.5/2, 0.9/3, 1/4, 1/5, 1/6, \dots\}.$$

One can see that  $\mathfrak{C}_k$  ( $k = 4, 5, 6$ ) is an example of generalized FGCounts, FLCCounts and FECCounts provided by Wygralak in [22] (see also [23]). Note that the fuzzy set  $A$ , for which  $\mathfrak{C}_6(A) \equiv \emptyset$  holds, is called a singular fuzzy set in [8].

The following example presents measures that are c-measures with respect to  $\wedge$  as well as  $\otimes$ . Let us suppose that the structure of the membership degrees is the Goguen rlr-lattice  $\mathbf{L}_P$  on  $[0, 1]$  (see Ex. 2.1 in [9]); recall that  $\alpha \otimes \beta = \alpha \cdot \beta$  and  $\alpha \oplus \beta = \alpha + \beta - \alpha \cdot \beta$ , where  $\cdot$  and  $+$  are the operations of addition and multiplication of real numbers, respectively.

EXAMPLE 3.3 Let  $\mathfrak{C}_1$  be the c-measure from Ex. 3.1 and, for any  $A \in \mathfrak{F}\mathbf{fin}$  such that  $\text{Supp}(A) \neq \emptyset$ , let us define  $A^* : \text{Supp}(A) \rightarrow L$  as

$$A^*(x) = \begin{cases} 1, & \text{if } A(x) = 1; \\ 0, & \text{otherwise,} \end{cases}$$

and set  $A^* = \emptyset$ , if  $\text{Supp}(A) = \emptyset$ . One may see that  $A^*$  is a crisp set that expresses the *kernel* (denoted by  $\text{Ker}(A)$ ) of the fuzzy set  $A$ , i.e., the set  $\text{Ker}(A)$  contains all elements from  $\text{Dom}(A)$  that belong to  $A$  in the degree 1. Note that  $\text{Ker}(A) = \text{Ker}(A^*) = \text{Supp}(A^*)$ . Then,

$$\begin{aligned} \mathfrak{C}_7(A)(i) &= \mathfrak{C}_1(A^*)(i), \\ \mathfrak{C}_8(A)(i) &= \mathfrak{C}_1(\overline{A^*})(m \boxplus i), \\ \mathfrak{C}_9(A)(i) &= \mathfrak{C}_8(A)(i) \odot \mathfrak{C}_7(A)(i), \end{aligned}$$

where  $\overline{A^*}$  is the complement of  $A^*$  for  $\text{Dom}(A^*) \neq \emptyset$  and  $\overline{\emptyset} = \emptyset$ ,  $m = |\text{Dom}(A^*)|$ , and  $\boxplus$  is defined in Ex. 3.2, are c-measures with respect to  $\odot$ . Let us check this statement directly. It is easy to see that  $\mathfrak{C}_7$  can be directly expressed as follows:

$$\mathfrak{C}_7(A)(i) = \begin{cases} 1, & \text{if } |\text{Ker}(A)| \geq i, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\text{Ker}(A) = \text{Ker}(A^*)$ , then  $\mathfrak{C}_7(A) = \mathfrak{C}_7(A^*)$ . Let  $A, B \in \mathfrak{F}\mathbf{fin}$  such that  $\text{Supp}(A) \cap \text{Supp}(B) = \emptyset$  and  $i \in \mathbb{N}$ . Then,  $\mathfrak{C}_7(A \cup B)(i) = 1$  if and only if  $|\text{Ker}(A \cup B)| = |\text{Ker}(A) \cup \text{Ker}(B)| \geq i$  if and only if there exist  $k, l \in \mathbb{N}$  such that  $k+l = i$  and  $|\text{Ker}(A)| \geq k$  and  $|\text{Ker}(B)| \geq l$  if and only if  $\mathfrak{C}_7(A)(k) \odot \mathfrak{C}_7(B)(l) = 1$  for some  $k, l \in \mathbb{N}$  if and only if  $(\mathfrak{C}_7(A) + \mathfrak{C}_7(B))(i) = 1$ . Hence,  $\mathfrak{C}_7$  is additive. Since  $|\text{Ker}(A)| \leq |\text{Supp}(A)|$ , then, by the definition, we obtain  $\mathfrak{C}_7(A) = \perp$  for any  $i > |\text{Supp}(A)| \geq |\text{Ker}(A)|$ , and  $\mathfrak{C}_7$  satisfies the variability axiom. The consistency and singleton independency axioms immediately follow from the definition of  $\mathfrak{C}_7$ . Let  $\alpha, \beta \in L$ . Clearly, the preservation of the non-existence axiom is trivially satisfied. If  $\alpha \odot \beta = 1$ , then  $\alpha = \beta = 1$  and

$$1 = \mathfrak{C}_7(\{\alpha \odot \beta/x\})(1) = \mathfrak{C}_7(\{\alpha/x\})(1) \odot \mathfrak{C}_7(\{\beta/x\})(1) = 1 \odot 1 = 1.$$

If  $\alpha \odot \beta < 1$ , then  $\alpha < 1$  or  $\beta < 1$ , which implies that  $\mathfrak{C}_7(\{\alpha/x\})(1) = 0$  or  $\mathfrak{C}_7(\{\beta/x\})(1) = 0$ . Therefore, we obtain (supposing that  $\mathfrak{C}_7(\{\alpha/x\})(1) = 0$ )

$$0 = \mathfrak{C}_7(\{\alpha \odot \beta/x\})(1) = \mathfrak{C}_7(\{\alpha/x\})(1) \odot \mathfrak{C}_7(\{\beta/x\})(1) \leq 0 \odot 1 = 0.$$

Therefore,  $\mathfrak{C}_7$  satisfies the preservation of the existence axiom and is a c-measure with respect to  $\odot$ .

Further, one can express  $\mathfrak{C}_8$  as

$$\mathfrak{C}_8(A)(i) = \begin{cases} 1, & \text{if } |\text{Ker}(A)| \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

This immediately follows from the fact that  $|\text{Ker}(\overline{A^*})| = m - |\text{Ker}(A^*)|$ , where  $m = |\text{Supp}(A^*)| = |\text{Dom}(A^*)|$ . Hence,  $\mathfrak{C}_8(A)(i) = \mathfrak{C}_1(\overline{A^*})(m \boxplus i) = \top$  if and only if  $m \boxplus i \leq |\text{Ker}(\overline{A^*})| = m - |\text{Ker}(A^*)|$  if and only if  $i \geq |\text{Ker}(A^*)| = |\text{Ker}(A)|$ . Because the verification of the axioms can be performed by analogy, as in the case of  $\mathfrak{C}_7$ , we shall only check that the preservation of the non-existence axiom is satisfied. If  $\alpha \overline{\odot} \beta = 1$ , then it is easy to see that  $\alpha = 1$  or  $\beta = 1$  (for  $\oplus$ , we obtain  $\alpha = \beta = 1$ ), and thus,  $\mathfrak{C}_8(\{\alpha/x\})(0) = 0$  or  $\mathfrak{C}_8(\{\beta/x\})(0) = 0$ . Hence, we simply obtain (supposing  $\mathfrak{C}_8(\{\alpha/x\})(0) = 0$ )

$$0 = \mathfrak{C}_8(\{\alpha \overline{\odot} \beta/x\})(0) \leq \mathfrak{C}_8(\{\alpha/x\})(0) \odot \mathfrak{C}_8(\{\beta/x\})(0) \leq 0 \odot 1 = 0.$$

Note that this axiom would be false for an rdr-lattice with the operation  $\overline{\odot}$  for which  $\alpha \overline{\odot} \beta = \top$ , but  $\alpha, \beta < \top$  (e.g., the Łukasiewicz rdr-lattice).

Finally, we have

$$\mathfrak{C}_8(A)(i) = \begin{cases} 1, & \text{if } |\text{Ker}(A)| = i, \\ 0, & \text{otherwise.} \end{cases}$$

The verification is trivial; therefore, we completely omit it here.

Let  $A = \{0.5/a, 1/b, 1/c, 0.4/d, 0/e\}$ . Then,

$$\begin{aligned} \mathfrak{C}_7(A) &= \{1/0, 1/1, 1/2, 0/3, 0/4, \dots\}, \\ \mathfrak{C}_8(A) &= \{0/0, 0/1, 1/2, 1/3, 1/4, \dots\}, \\ \mathfrak{C}_9(A) &= \{0/0, 0/1, 1/2, 0/3, 0/4, \dots\}. \end{aligned}$$

A consequence of the additivity of the c-measure is the following statement.

**PROPOSITION 3.4** *Let  $\mathfrak{C}$  be a c-measure and  $A \in \mathfrak{Ffin}$  such that  $\text{Supp}(A) \subseteq \{x_1, \dots, x_m\}$ . Then,*

$$\mathfrak{C}(A)(i) = \bigvee_{\substack{i_1, \dots, i_m \in \mathbb{N} \\ i_1 + \dots + i_m = i}} \bigodot_{k=1}^m \mathfrak{C}(\{A(x_k)/x_k\})(i_k) \quad (10)$$

for each  $i \in \mathbb{N}$ .

**PROOF.** Let  $A \in \mathfrak{Ffin}$ . Due to Proposition 3.2, without loss of generality, we may suppose that  $\text{Dom}(A) = \{x_1, \dots, x_m\}$ . Using the additivity of  $\mathfrak{C}$  applied on the singletons, we obtain

$$\begin{aligned} \mathfrak{C}(A)(i) &= \mathfrak{C}(\{A(x_1)/x_1\} \cup \dots \cup \{A(x_m)/x_m\})(i) = \\ &= (\mathfrak{C}(\{A(x_1)/x_1\}) + \dots + \mathfrak{C}(\{A(x_m)/x_m\}))(i). \end{aligned}$$

Clearly, this formula may be reformulated to obtain (10).  $\square$



PROPOSITION 3.5 *Let  $\mathfrak{C}$  be a  $c$ -measure and  $A \in \mathfrak{F}\mathfrak{in}$  such that  $\text{Supp}(A) \subseteq \{x_1, \dots, x_m\}$ . Then,*

$$\mathfrak{C}(A)(i) = \bigvee_{\substack{i_1, \dots, i_m \in \{0,1\} \\ i_1 + \dots + i_m = i}} \bigodot_{k=1}^m \mathfrak{C}(\{A(x_k)/x_k\})(i_k) \quad (11)$$

for each  $0 \leq i \leq m$ .

PROOF. Let  $A \in \mathfrak{F}\mathfrak{in}$ . Due to Proposition 3.2, without loss of generality, we may suppose that  $\text{Dom}(A) = \{x_1, \dots, x_m\}$ . Obviously, (11) is true for  $i \leq 1$ . Let  $1 < i \leq m$ . Define  $M_i$  as a multi-subset  $\{i_1, \dots, i_m\}$  of  $\mathbb{N}$  such that  $i_1 + \dots + i_m = i$  and a multi-set  $I_{M_i} = \{i_k \mid i_k \in M_i \text{ and } i_k \notin \{0, 1\}\}$ . To prove the proposition, it is sufficient to show that for each  $M_i$  there exists  $M'_i$  such that  $I_{M'_i} = \emptyset$  and

$$\bigodot_{i_k \in M_i} \mathfrak{C}(\{A(x_k)/x_k\})(i_k) \leq \bigodot_{i_k \in M'_i} \mathfrak{C}(\{A(x_k)/x_k\})(i_k). \quad (12)$$

If  $I_{M_i} = \emptyset$  for some set  $M_i$ , then define  $M'_i = M_i$ . Let  $M_i$  such that  $I_{M_i} \neq \emptyset$  and  $i_{k_0} \in I_{M_i}$ . Then, there exist at least  $i_{k_0} - 1$  elements of  $M_i$  that are equal to 0. In fact, let us suppose that  $s < i_{k_0} - 1$  is the maximal number of elements of  $M_i$  that are equal to 0. Then,  $m - s$  elements of  $M_i$  are greater than 0. Therefore, we obtain  $i_1 + \dots + i_m \geq i_{k_0} + ((m-1) - s) > i_{k_0} + ((m-1) - (i_{k_0} - 1)) = m \geq i$ , which is a contradiction with the presumption  $i_1 + \dots + i_m = i$ . Let us define  $r = i_{k_0} - 1$  and choose  $r$  elements  $i_{k_1}, \dots, i_{k_r}$  from  $M_i$  such that  $i_{k_1} = \dots = i_{k_r} = 0$ . Due to (C2) and (C3), we obtain

$$\mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) = \mathfrak{C}(\{\perp/x_{k_0}\})(2) = \mathfrak{C}(\{A(x_l)/x_l\})(2) \in \{\perp, \top\}$$

for all  $l \in \{k_1, \dots, k_r\}$ . Since  $\mathfrak{C}$  is a mapping to the set of  $\odot$ -convex fuzzy sets, we obtain

$$\begin{aligned} & \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathfrak{C}(\{A(x_l)/x_l\})(0) = \\ & \mathfrak{C}(\{A(x_l)/x_l\})(2) \odot \mathfrak{C}(\{A(x_l)/x_l\})(0) \leq \mathfrak{C}(\{A(x_l)/x_l\})(1) \end{aligned}$$

for each  $l \in \{k_1, \dots, k_r\}$ , and, using (C5) and  $\mathfrak{C}(\{\perp/x_{k_0}\})(0) = \top$  (recall that  $\mathfrak{C}(\emptyset)(0) = \top$  from (C3)), we obtain

$$\begin{aligned} & \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) = \mathfrak{C}(\{\perp/x_{k_0}\})(2) \odot \top = \\ & \mathfrak{C}(\{\perp/x_{k_0}\})(2) \odot \mathfrak{C}(\{\perp/x_{k_0}\})(0) \leq \\ & \mathfrak{C}(\{\perp/x_{k_0}\})(1) = \mathfrak{C}(\{\perp \odot A(x_{k_0})/x_{k_0}\})(1) = \\ & \mathfrak{C}(\{\perp/x_{k_0}\})(1) \odot \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(1) \leq \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(1). \end{aligned}$$

Since  $\mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \in \{\perp, \top\}$  and  $\perp$  and  $\top$  are the idempotent elements of  $\mathbf{L}$  with respect to  $\odot$ , using the previous inequalities, we obtain

$$\begin{aligned} & \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathfrak{C}(\{A(x_{k_1})/x_{k_1}\})(0) \odot \cdots \odot \mathfrak{C}(\{A(x_{k_r})/x_{k_r}\})(0) = \\ & \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot (\mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathfrak{C}(\{A(x_{k_1})/x_{k_1}\})(0)) \odot \cdots \odot \\ & \quad (\mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathfrak{C}(\{A(x_{k_r})/x_{k_r}\})(0)) \leq \\ & \mathfrak{C}(\{A(x_{k_0})/x_{k_0}\})(1) \odot \mathfrak{C}(\{A(x_{k_1})/x_{k_1}\})(1) \odot \cdots \odot \mathfrak{C}(\{A(x_{k_r})/x_{k_r}\})(1). \end{aligned}$$

By defining  $M'_i = \{i'_1, \dots, i'_m\}$  such that  $i'_{k_0} = i'_{k_1} = \cdots = i'_{k_r} = 1$  and  $i'_l = i_l$  for all  $l \in M_i \setminus \{k_0, k_1, \dots, k_r\}$ , we obtain  $i'_1 + \cdots + i'_m = i$ . Hence,  $I_{M'_i} = I_{M_i} \setminus \{i_{k_0}\} \subset I_{M_i}$ , and

$$\bigodot_{i_k \in M_i} \mathfrak{C}(\{A(x_k)/x_k\})(i_k) \leq \bigodot_{i'_k \in M'_i} \mathfrak{C}(\{A(x_k)/x_k\})(i'_k).$$

Repeating this procedure, we obtain, using a finite number of steps, the desired set  $M'_i$  for which  $I_{M'_i} = \emptyset$ .  $\square$

### 3.2. A representation of fuzzy c-measures

In [3], a representation of the cardinalities of finite fuzzy sets was provided using two monotonic mappings  $f, g : [0, 1] \rightarrow [0, 1]$ . To introduce an analogical representation for the c-measures, we need to establish a generalization of the monotonic mappings.

**DEFINITION 3.2** Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be rdr-lattices. A mapping  $h : L_1 \rightarrow L_2$  is said to be a  $\odot$ -homomorphism of  $\mathbf{L}_1$  to  $\mathbf{L}_2$  if  $h$  is a homomorphism of the reduct  $(L_1, \odot_1, \top_1)$  of  $\mathbf{L}_1$  to the reduct  $(L_2, \odot_2, \top_2)$  of  $\mathbf{L}_2$ , i.e.,  $h(\alpha \odot_1 \beta) = h(\alpha) \odot_2 h(\beta)$  and  $h(\top_1) = \top_2$ . A mapping  $h : L_1 \rightarrow L_2$  is said to be a  $\overline{\odot}_d$ -homomorphism if  $h$  is a homomorphism from the reduct  $(L_1, \overline{\odot}_1, \perp_1)$  of  $\mathbf{L}_1$  to the reduct  $(L_2, \odot_2, \top_2)$  of  $\mathbf{L}_2$ , i.e.,  $h(\alpha \overline{\odot}_1 \beta) = h(\alpha) \odot h(\beta)$  and  $h(\perp_1) = \top_2$ .

**REMARK 3.4** Obviously, each homomorphism between the rdr-lattices (or the residuated lattices, which are the reducts of the original rdr-lattices) is a  $\odot$ -homomorphism.

The following lemma gives a characterization of the c-measures using  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms.

**LEMMA 3.6** Let  $f$  be a  $\odot$ -homomorphism and  $g$  be a  $\overline{\odot}_d$ -homomorphism from  $\mathbf{L}$  to  $\mathbf{L}$  such that  $f(\perp) \in \{\perp, \top\}$  and  $g(\top) \in \{\perp, \top\}$ . Let  $\mathfrak{C}_{g,f} : \mathfrak{F}\text{in} \rightarrow \mathfrak{N}$  be a mapping defined by the induction

$$\begin{aligned} \mathfrak{C}_{g,f}(\{\alpha/x\})(0) &= g(\alpha), \quad \mathfrak{C}_{g,f}(\{\alpha/x\})(1) = f(\alpha), \quad \text{and} \\ \mathfrak{C}_{g,f}(\{\alpha/x\})(k) &= f(\perp), \quad k > 1 \end{aligned} \tag{13}$$

hold for any  $\alpha \in L$  and  $x \in \mathbf{Count}$ ,

$$\mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(\{\perp/\emptyset\}) \quad (14)$$

holds for any  $A \in \mathfrak{Ffin}$  with the empty support, and

$$\mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(\{A(x_1)/x_1\}) + \cdots + \mathfrak{C}_{g,f}(\{A(x_m)/x_m\}), \quad (15)$$

where  $\text{Supp}(A) = \{x_1, \dots, x_m\}$ , holds for any  $A \in \mathfrak{Ffin}$  with a non-empty support. Then,  $\mathfrak{C}_{g,f}$  is a  $c$ -measure of finite fuzzy sets with respect to  $\odot$ .

PROOF. Because the proof is a bit longer, we shall structure it to several claims for a better orientation.

First, we shall prove the correctness of the definition of  $\mathfrak{C}_{g,f}$ , which is precisely stated in the following claim.

CLAIM 1  $\mathfrak{C}_{g,f}(A)$  is a  $\odot$ -convex fuzzy set for each  $A \in \mathfrak{Ffin}$ .

Let  $x \in \mathbf{Count}$  and  $\{\alpha/x\}$  be a singleton. Since  $f(\perp) = f(\perp \odot \alpha) = f(\perp) \odot f(\alpha) \leq f(\alpha)$  holds for any  $\alpha \in L$ , then, by the definition, we have

$$\mathfrak{C}_{g,f}(\{\alpha/x\})(0) \odot \mathfrak{C}_{g,f}(\{\alpha/x\})(2) = g(\alpha) \odot f(\perp) \leq \mathfrak{C}_{g,f}(\{\alpha/x\})(k)$$

for  $0 \leq k \leq 2$  (for  $k = 1$ , consider  $\mathfrak{C}_{g,f}(\{\alpha/x\})(1) = f(\alpha) \geq f(\perp) \geq g(\alpha) \odot f(\perp)$ ). Moreover, the following inequality

$$\mathfrak{C}_{g,f}(\{\alpha/x\})(i) \odot \mathfrak{C}_{g,f}(\{\alpha/x\})(k) \leq \mathfrak{C}_{g,f}(\{\alpha/x\})(j)$$

is trivially fulfilled for all  $i, j, k \in \mathbb{N}$  such that  $0 < i \leq j \leq k$ , and  $\mathfrak{C}_{g,f}$  is a mapping that assigns to each singleton a  $\odot$ -convex fuzzy set. Obviously, the fuzzy set  $\mathfrak{C}_{g,f}(A)$  is also  $\odot$ -convex for any  $A \in \text{cls}(\emptyset)$  by the  $\odot$ -convexity of  $\{\perp/x\}$ . Recall that a finite sum of  $\odot$ -convex fuzzy sets is an  $\odot$ -convex fuzzy set (see Theorem 4.1 in [9]). Hence, if  $A \in \mathfrak{Ffin}$  is a fuzzy set and  $\text{Supp}(A) = \{x_1, \dots, x_m\}$ , then  $\mathfrak{C}_{g,f}(A)$ , defined by (15), has to be  $\odot$ -convex and the claim is proved.

A straightforward consequence of the definition of  $\mathfrak{C}_{g,f}$  is the following claim.

CLAIM 2 If  $A \in \mathfrak{Ffin}$  and  $B \in \text{cls}(A)$ , then  $\mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(B)$ .

Further, we shall prove the following claim.

CLAIM 3  $\mathfrak{C}_{g,f}$  satisfies the additivity axiom.

Let  $A, B \in \mathfrak{Ffin}$  be fuzzy sets such that  $\text{Supp}(A) \cap \text{Supp}(B) = \emptyset$ . If  $A, B \in \text{cls}(\emptyset)$ , then, by Claim 2 and the fact that  $A \cup B \in \text{cls}(\emptyset)$ , we obtain

$$\begin{aligned} \mathfrak{C}_{g,f}(A \cup B) &= \mathfrak{C}_{g,f}(\emptyset) = \mathfrak{C}_{g,f}(\{\perp/x\}) = \\ &= \mathfrak{C}_{g,f}(\{\perp/x\}) + \mathfrak{C}_{g,f}(\{\perp/x\}) = \mathfrak{C}_{g,f}(A) + \mathfrak{C}_{g,f}(B), \end{aligned}$$

where  $\mathfrak{C}_{g,f}(\{\perp/x\}) = \mathfrak{C}_{g,f}(\{\perp/x\}) + \mathfrak{C}_{g,f}(\{\perp/x\})$  can be simply verified. If  $A \notin \text{cls}(\emptyset)$  with  $\text{Supp}(A) = \{x_1, \dots, x_r\}$  and  $B \in \text{cls}(\emptyset)$ , then, using the associativity of  $+$ , we obtain

$$\begin{aligned}\mathfrak{C}_{g,f}(A \cup B) &= \mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(\{A(x_1)/x_1\}) + \dots + \mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) = \\ &\mathfrak{C}_{g,f}(\{A(x_1)/x_1\}) + \dots + (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\})) = \\ &(\mathfrak{C}_{g,f}(\{A(x_1)/x_1\}) + \dots + \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})) + \mathfrak{C}_{g,f}(\{\perp/x\}) = \\ &\mathfrak{C}_{g,f}(A) + \mathfrak{C}_{g,f}(B),\end{aligned}$$

where  $\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) = \mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\})$  may be verified as follows:

1) for  $i = 0$ , we have

$$\begin{aligned}(\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\}))(0) &= \\ \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(0) \odot \mathfrak{C}_{g,f}(\{\perp/x\})(0) &= \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(0) \odot g(\perp) = \\ \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(0) \odot \top &= \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(0); \end{aligned}$$

2) for  $i = 1$ , we have

$$\begin{aligned}\gamma &= (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\}))(1) = \\ (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(0) \odot \mathfrak{C}_{g,f}(\{\perp/x\})(1)) \vee & \\ (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(1) \odot \mathfrak{C}_{g,f}(\{\perp/x\})(0)) &= \\ (g(A(x_r)) \odot f(\perp)) \vee (f(A(x_r)) \odot g(\perp)) &= \\ (g(A(x_r)) \odot f(\perp)) \vee (f(A(x_r)) \odot \top); &\end{aligned}$$

if  $f(\perp) = \perp$ , then  $\gamma = f(A(x_r))$ , and if  $f(\perp) = \top$ , then

$$\top = f(\perp) = f(\perp \odot \alpha) = f(\perp) \odot f(\alpha) = \top \odot f(\alpha) = f(\alpha)$$

for all  $\alpha \in L$ , whence  $\gamma = f(A(x_r)) = \top$ ; thus,

$$(\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\}))(1) = f(A(x_r)) = \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(1);$$

3) for  $i \geq 2$ , we have

$$\begin{aligned}\gamma &= (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\}))(i) = \\ (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(0) \odot \mathfrak{C}_{g,f}(\{\perp/x\})(i)) \vee & \\ (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(1) \odot \mathfrak{C}_{g,f}(\{\perp/x\})(i-1)) \vee & \\ \vdots & \\ \vee (\mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(i) \odot \mathfrak{C}_{g,f}(\{\perp/x\})(0)) &= \\ (g(A(x_r)) \odot f(\perp)) \vee (f(A(x_r)) \odot f(\perp)) \vee \dots \vee & \\ (f(\perp) \odot g(\perp)) = & \\ (g(A(x_r)) \odot f(\perp)) \vee (f(A(x_r)) \odot f(\perp)) \vee \dots \vee & \\ (f(\perp) \odot \top), &\end{aligned}$$

i.e., the term  $f(\perp)$  is contained inside each bracket of the last formula; if  $f(\perp) = \perp$ , then  $\gamma = f(\perp) = \perp$ ; if  $f(\perp) = \top$ , then  $\gamma = f(\perp) = \top$ ; thus,

$$(\mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{\perp/x\}))(i) = f(\perp) = \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})(i);$$

If  $A, B \notin \text{cls}(\emptyset)$  with  $\text{Supp}(A) = \{x_1, \dots, x_r\}$  and  $\text{Supp}(B) = \{y_1, \dots, y_s\}$ , then, using the associativity of  $+$  and the definition of  $\mathfrak{C}_{g,f}$ , we obtain

$$\begin{aligned} \mathfrak{C}_{g,f}(A \cup B) &= \\ \mathfrak{C}_{g,f}(\{A(x_1)/x_1\}) + \dots + \mathfrak{C}_{g,f}(\{A(x_r)/x_r\}) + \mathfrak{C}_{g,f}(\{B(y_1)/y_1\}) + \dots \\ + \mathfrak{C}_{g,f}(\{B(y_s)/y_s\}) &= (\mathfrak{C}_{g,f}(\{A(x_1)/x_1\}) + \dots + \mathfrak{C}_{g,f}(\{A(x_r)/x_r\})) + \\ (\mathfrak{C}_{g,f}(\{B(y_1)/y_1\}) + \dots + \mathfrak{C}_{g,f}(\{B(y_s)/y_s\})) &= \mathfrak{C}_{g,f}(A) + \mathfrak{C}_{g,f}(B). \end{aligned}$$

Hence, the mapping  $\mathfrak{C}_{g,f}$  satisfies the additivity axiom.

To verify the variability of  $\mathfrak{C}_{g,f}$ , the following claim is sufficient.

CLAIM 4 *If  $A \in \mathfrak{F}\mathbf{fin}$ , then  $\mathfrak{C}_{g,f}(A)(i) = f(\perp)$  for all  $i > |\text{Supp}(A)|$ .*

By the definition, the claim is true for  $\emptyset$  and for all singletons  $\{\alpha/x\}$ . By Claim 2, the satisfaction of this claim may be extended to all equivalent fuzzy sets with  $\emptyset$  and singletons. Let  $A \in \mathfrak{F}\mathbf{fin}$  with  $\text{Supp}(A) = \{x_1, \dots, x_m\}$ , where  $m > 1$ . From the additivity of  $\mathfrak{C}_{g,f}$  applied to the singletons (cf. Proposition 3.4), we obtain

$$\mathfrak{C}_{g,f}(A)(i) = \bigvee_{\substack{i_1, \dots, i_m \in \mathbb{N} \\ i_1 + \dots + i_m = i}} \mathfrak{C}_{g,f}(\{A(x_1)/x_1\})(i_1) \odot \dots \odot \mathfrak{C}_{g,f}(\{A(x_m)/x_m\})(i_m)$$

for each  $i \in \mathbb{N}$ . If  $i > m$ , then, for an arbitrary combination  $i_1, \dots, i_m \in \mathbb{N}$  such that  $i_1 + \dots + i_m = i$ , there exists  $i_k \in \{i_1, \dots, i_m\}$  such that  $i_k > 1$ . Since  $\mathfrak{C}_{g,f}(A(x_k)/x_k)(i_k) = f(\perp)$ , then the term  $f(\perp)$  is contained in

$$\mathfrak{C}_{g,f}(\{A(x_1)/x_1\})(i_1) \odot \dots \odot \mathfrak{C}_{g,f}(\{A(x_m)/x_m\})(i_m)$$

for any combination  $i_1, \dots, i_m \in \mathbb{N}$  with a sum equal to  $i$ . If  $f(\perp) = \perp$ , then  $\mathfrak{C}_{g,f}(A)(i) = \perp = f(\perp)$ . If  $f(\perp) = \top$ , then  $f(\alpha) = \top$  (as we have shown above) and we can consider

$$\begin{aligned} \mathfrak{C}_{g,f}(A)(i) &\geq \mathfrak{C}_{g,f}(\{A(x_1)/x_1\})(1) \odot \mathfrak{C}_{g,f}(\{A(x_2)/x_2\})(1) \odot \dots \odot \\ \mathfrak{C}_{g,f}(\{A(x_{m-1})/x_{m-1}\})(1) &\odot \mathfrak{C}_{g,f}(\{A(x_m)/x_m\})(i - (m - 1)) = \\ f(A(x_1)) \odot f(A(x_2)) \odot \dots \odot f(A(x_{m-1})) &\odot f(\perp) = \\ \top \odot \top \odot \dots \odot \top \odot \top &= \top = f(\perp), \end{aligned}$$

where  $i > m$  implies  $i - (m - 1) \geq 2$  and thus,  $\mathfrak{C}_{g,f}(\{A(x_m)/x_m\})(i - (m - 1)) = f(\perp)$ . Therefore, we again obtain  $\mathfrak{C}_{g,f}(A)(i) = \top = f(\perp)$ , which concludes the proof of this claim.

To prove the consistency axiom, it is sufficient to verify the following claim.

CLAIM 5 *If  $A$  is crisp, then  $\mathfrak{C}_{g,f}(A)(i) = f(\top)$  for  $i = |\text{Supp}(A)|$ .*

Obviously,  $\mathfrak{C}_{g,f}(\emptyset)(0) = g(\perp) = \top = f(\top)$  (by the definition) and the same equality holds for all fuzzy sets that are equivalent to the empty fuzzy set (using Claim 2). If  $A \in \mathfrak{Ffin}$  is crisp with  $\text{Supp}(A) = \{x_1, \dots, x_m\}$ , by setting  $i_1 = \dots = i_m = 1$ , we obtain  $i_1 + \dots + i_m = m = |\text{Supp}(A)|$ , and

$$\begin{aligned} \mathfrak{C}_{g,f}(A)(m) &\geq \mathfrak{C}_{g,f}(\{\top/x_1\})(1) \odot \dots \odot \mathfrak{C}_{g,f}(\{\top/x_m\})(1) = \\ &f(\top) \odot \dots \odot f(\top) = f(\top), \end{aligned}$$

where  $f(\top) \odot f(\top) = f(\top)$ , which follows from the definition of  $f$ .

The remaining axioms are immediate consequences of the definitions of  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms, and hence,  $\mathfrak{C}_{g,f}$  is a  $c$ -measure of the finite fuzzy sets.  $\square$

**THEOREM 3.7 (REPRESENTATION OF  $C$ -MEASURES)** *Let  $\mathfrak{C} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a mapping that satisfies the additivity axiom and  $\mathfrak{C}(A) = \mathfrak{C}(\emptyset)$  for any  $A \in \text{cls}(\emptyset)$ . Then, the following statements are equivalent:*

- (i)  $\mathfrak{C}$  is a  $c$ -measure of the finite fuzzy sets with respect to  $\odot$ ,
- (ii) there exist a  $\odot$ -homomorphism  $f : L \rightarrow L$  and a  $\overline{\odot}_d$ -homomorphism  $g : L \rightarrow L$  such that  $f(\perp) \in \{\perp, \top\}$ ,  $g(\top) \in \{\perp, \top\}$  and

$$\begin{aligned} \mathfrak{C}(\{\alpha/x\})(0) &= g(\alpha), \quad \mathfrak{C}(\{\alpha/x\})(1) = f(\alpha), \\ \mathfrak{C}(\{\alpha/x\})(k) &= f(\perp), \quad k > 1 \end{aligned}$$

hold for arbitrary  $\alpha \in L$  and  $x \in \mathbf{Count}$ .

**PROOF.** (i)  $\Rightarrow$  (ii) Let us suppose that  $\mathfrak{C}$  is a  $c$ -measure of the finite fuzzy sets and define two mappings  $f, g : L \rightarrow L$  as follows

$$f(\alpha) = \mathfrak{C}(\{\alpha/x\})(1) \quad \text{and} \quad g(\alpha) = \mathfrak{C}(\{\alpha/x\})(0),$$

where  $x \in \mathbf{Count}$  is an arbitrary set. According to (C5), we obtain

$$\begin{aligned} f(\alpha \odot \beta) &= \mathfrak{C}(\{\alpha \odot \beta/x\})(1) = \mathfrak{C}(\{\alpha/x\})(1) \odot \mathfrak{C}(\{\beta/x\})(1) = f(\alpha) \odot f(\beta), \\ g(\alpha \overline{\odot} \beta) &= \mathfrak{C}(\{\alpha \overline{\odot} \beta/x\})(0) = \mathfrak{C}(\{\alpha/x\})(0) \odot \mathfrak{C}(\{\beta/x\})(0) = g(\alpha) \odot g(\beta), \end{aligned}$$

and moreover,  $f(\top) = \mathfrak{C}(\{\top/x\})(1) = \top$  and  $g(\perp) = \mathfrak{C}(\{\perp/x\})(0) = \top$  hold due to (C3). Hence,  $f$  is a  $\odot$ -homomorphism and  $g$  is a  $\overline{\odot}_d$ -homomorphism of the relevant reducts. Since  $\mathfrak{C}(\{\perp/x\})$  (or  $\mathfrak{C}(\{\top/x\})$ ) is a crisp set, then, by (C3),  $\mathfrak{C}(\{\perp/x\})(1) = f(\perp) \in \{\perp, \top\}$  (or  $\mathfrak{C}(\{\top/x\})(0) = g(\top) \in \{\perp, \top\}$ ). Due to Corollary 3.3, we obtain  $\mathfrak{C}(\{\alpha/x\})(k) = f(\perp)$  for any  $k > 1$ .

(ii)  $\Rightarrow$  (i) Let  $\mathfrak{C}_{g,f}$  be the  $c$ -measure of the finite fuzzy sets defined by Lemma 3.6. For any singleton  $\{\alpha/x\} \in \mathfrak{Ffin}$ , we have  $\mathfrak{C}_{g,f}(\{\alpha/x\}) = \mathfrak{C}(\{\alpha/x\})$ . If  $A \in \mathfrak{Ffin}$  with  $\text{Supp}(A) = \emptyset$ , then  $\mathfrak{C}(A) = \mathfrak{C}(\{\perp/x\}) = \mathfrak{C}_{g,f}(\{\perp/x\}) = \mathfrak{C}_{g,f}(A)$ . If  $A \in \mathfrak{Ffin}$  with  $\text{Supp}(A) \neq \emptyset$ , then  $\mathfrak{C}(A) = \mathfrak{C}_{g,f}(A)$  follows from the additivity of  $\mathfrak{C}$ . Hence,  $\mathfrak{C}$  is a  $c$ -measure of the finite fuzzy sets.  $\square$

In the sequel we shall use  $\mathfrak{C}_{g,f}$  to denote a  $c$ -measure that is determined by a  $\odot$ -homomorphism  $f : L \rightarrow L$  and a  $\overline{\odot}_d$ -homomorphism  $g : L \rightarrow L$  for which  $f(\perp) \in \{\perp, \top\}$  and  $g(\top) \in \{\perp, \top\}$ . Moreover, we shall use  $\mathfrak{C}_f$  or  $\mathfrak{C}_g$  if  $g$  or  $f$  are the trivial  $\overline{\odot}_d$ -homomorphism or the  $\odot$ -homomorphism (i.e.,  $g(\alpha) = \top$  or  $f(\alpha) = \top$  for any  $\alpha \in L$ ), respectively.

### 3.3. Characterization of the $c$ -measures

In our further investigation of  $c$ -measures, some additional conditions on  $\odot$ -homomorphisms and  $\overline{\odot}_d$ -homomorphisms have to be imposed. More precisely, we shall consider the preservation of the partial ordering of  $L$  as it is defined below.

**DEFINITION 3.3** Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be rdr-lattices. We shall say that  $h : L_1 \rightarrow L_2$  is a  $\odot$ -po-homomorphism of  $\mathbf{L}_1$  to  $\mathbf{L}_2$  if  $h$  is a  $\odot$ -homomorphism, and  $h(\alpha) \leq h(\beta)$  for any  $\alpha, \beta \in L_1$  where  $\alpha \leq \beta$ . We shall say that  $h : L_1 \rightarrow L_2$  is a  $\overline{\odot}_d$ -po-homomorphism if  $h$  is a  $\overline{\odot}_d$ -homomorphism, and  $h(\alpha) \geq h(\beta)$  for any  $\alpha, \beta \in L_1$  where  $\alpha \leq \beta$ .

**EXAMPLE 3.5** It is easy to see that each  $\wedge$ -homomorphism is also a  $\wedge$ -po-homomorphism, and each  $\vee_d$ -homomorphism is also a  $\vee_d$ -po-homomorphism. In fact, if  $\alpha \leq \beta$  in  $L_1$  and  $h$  is, for example, a  $\vee_d$ -homomorphism, then  $h(\beta) = h(\alpha \vee \beta) = h(\alpha) \wedge h(\beta)$ , whence  $h(\beta) \leq h(\alpha)$ .

**EXAMPLE 3.6** Let  $\mathbf{L}$  be an rdr-lattice. Obviously, the trivial  $\odot$ -homomorphism ( $\overline{\odot}_d$ -homomorphism) is a  $\odot$ -po-homomorphism ( $\overline{\odot}_d$ -po-homomorphism). Consider  $h_n : L \rightarrow L$  defined by  $h_n(\alpha) = \alpha^n$  for some  $n \in \mathbb{N}$  where  $n > 1$ .<sup>11</sup> Then,  $h_n(\top) = \top^n = \top$  and, by the associativity of  $\otimes$ , we obtain that  $h_n(\alpha \otimes \beta) = (\alpha \otimes \beta)^n = \alpha^n \otimes \beta^n = h_n(\alpha) \otimes h_n(\beta)$ . Moreover, if  $\alpha \leq \beta$ , then  $h_n(\alpha) = \alpha^n \leq \beta^n = h_n(\beta)$  follows from the monotonicity of  $\otimes$ , whence  $h_n$  is a  $\otimes$ -po-homomorphism. Now, let  $\nu : L \rightarrow L$  be a one-to-one mapping of  $L$  onto  $L$  that reverses the partial ordering, i.e.,  $\alpha \leq \beta$  implies  $\nu(\beta) \leq \nu(\alpha)$ , and suppose that  $\nu(\alpha \oplus \beta) = \nu(\alpha) \otimes \nu(\beta)$ .<sup>12</sup> Consider  $h_{\nu,n} : L \rightarrow L$  defined by  $h_{\nu,n}(\alpha) = \nu(n\alpha)$  for any  $\alpha \in L$  and some  $n \in \mathbb{N}$  where  $n \geq 1$ .<sup>13</sup> Then,  $h_{\nu,n}(\perp) = \nu(n\perp) = \nu(\perp) = \top$ , where the last equality follows from the presumptions on  $\nu$ .<sup>14</sup> Furthermore,  $h_{\nu,n}(\alpha \oplus \beta) = \nu(n(\alpha \oplus \beta)) = \nu(n\alpha \oplus n\beta) = \nu(n\alpha) \otimes \nu(n\beta) = h_{\nu,n}(\alpha) \otimes h_{\nu,n}(\beta)$ . Finally,  $h_{\nu,n}(\alpha) \geq h_{\nu,n}(\beta)$  for any  $\alpha \leq \beta$ , which follows immediately from the definition of  $\nu$ , whence  $h_{\nu,n}$  is a  $\overline{\odot}_d$ -po-homomorphism.

**REMARK 3.7** Obviously, if  $h$  is a  $\odot$ -po-homomorphism, then  $\bigvee_{i \in I} h(\alpha_i) \leq h(\bigvee_{i \in I} \alpha_i)$ , whenever the supremum on both sides of the inequality exists, and

<sup>11</sup>Recall that  $\alpha^0 = \top$  and  $\alpha^n = \alpha \otimes \alpha^{n-1}$  for any  $n \geq 1$ .

<sup>12</sup>An example of rdr-lattice can be found in Remark 7.8 of the first part [9].

<sup>13</sup>Recall that  $0\alpha = \perp$  and  $\alpha^n = \alpha \oplus \alpha^{n-1}$  for any  $n \geq 1$ .

<sup>14</sup>It is easy to see that if  $\nu$  is a one-to-one mapping of  $L$  onto  $L$  and reverses the partial ordering, then  $\nu(\perp) = \top$ .

if  $h$  is a  $\overline{\odot}_d$ -po-homomorphism, then  $\bigvee_{i \in I} h(\alpha_i) \leq h(\bigwedge_{i \in I} \alpha_i)$ , whenever the infimum and the supremum exist. If the rdr-lattice is linearly ordered and  $I$  is a finite non-empty index set, then the previous inequalities may be changed to equalities.

DEFINITION 3.4 We shall say that a  $c$ -measure  $\mathfrak{C}$  *preserves*  $\subseteq$  if  $\mathfrak{C}(A) \subseteq \mathfrak{C}(B)$  whenever  $A \subseteq B$ , and the  $c$ -measure  $\mathfrak{C}$  *reverses*  $\subseteq$  if  $\mathfrak{C}(B) \subseteq \mathfrak{C}(A)$  whenever  $A \subseteq B$ .

THEOREM 3.8 Let  $\mathfrak{C}_{f,g}$  be a  $c$ -measure such that  $f$  is a  $\odot$ -po-homomorphism and  $g$  is a  $\overline{\odot}_d$ -po-homomorphism. Then,

- (i)  $\mathfrak{C}_{g,f}$  *preserves*  $\subseteq$  if and only if  $g$  is trivial, and
- (ii)  $\mathfrak{C}_{g,f}$  *reverses*  $\subseteq$  if and only if  $f$  is trivial.

PROOF. Here, we shall prove only (i), and (ii) may be proved using an analogous argument.

( $\Rightarrow$ ) If  $\mathfrak{C}_{g,f}$  preserves  $\subseteq$  and  $g$  is a non-trivial  $\overline{\odot}_d$ -po-homomorphism, then  $g(\alpha) < \top$  for some  $\alpha \in L$  with  $\perp < \alpha$ , whence  $\{\perp/x\} \subseteq \{\alpha/x\}$  for an arbitrary  $x \in \mathbf{Count}$ , and simultaneously, (by Theorem 3.7)

$$\mathfrak{C}_{g,f}(\{\perp/x\})(0) = g(\perp) > g(\alpha) = \mathfrak{C}_{g,f}(\{\alpha/x\})(0),$$

which contradicts the presumption on the preservation of  $\subseteq$ . Therefore,  $g$  has to be the trivial  $\overline{\odot}_d$ -po-homomorphism.

( $\Leftarrow$ ) Without lost of generality, let us suppose that  $A, B \in \mathfrak{Ffin}$  such that  $\text{Dom}(A) = \text{Dom}(B)$ . If  $g$  is the trivial  $\overline{\odot}_d$ -po-homomorphism and  $A \subseteq B$ , then

$$\mathfrak{C}_{g,f}(\{A(x)/x\})(0) = g(A(x)) = \top = g(B(x)) = \mathfrak{C}_{g,f}(\{B(x)/x\})(0)$$

and

$$\mathfrak{C}_{g,f}(\{A(x)/x\})(1) = f(A(x)) \leq f(B(x)) = \mathfrak{C}_{g,f}(\{B(x)/x\})(1)$$

hold for any  $x \in \text{Dom}(A)$ . The inequality  $\mathfrak{C}_{g,f}(A) \subseteq \mathfrak{C}_{g,f}(B)$  is a simple consequence of the monotonicity (in both arguments) of  $\odot$  and Proposition 3.5. Hence,  $\mathfrak{C}_{g,f}$  preserves  $\subseteq$ .  $\square$

In the first part of our contribution, we have introduced the fuzzy classes  $p_A^\odot$  and  $p_A^{\overline{\odot}}$  (for a summary of their properties, see Preliminaries) that determine a generalized cardinal with respect to an appropriate set  $y$ . The following statements show the relation between the  $c$ -measures and generalized cardinals  $p_A^\odot(-, y)$  and  $p_A^{\overline{\odot}}(-, y)$ .

THEOREM 3.9 Let  $\mathfrak{C}_{g,f}$  be a  $c$ -measure such that  $f$  is a  $\odot$ -po-homomorphism,  $g$  is a  $\overline{\odot}_d$ -po-homomorphism,  $A \in \mathfrak{Ffin}$  and  $y \in \mathfrak{Ffin}$  with  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$ . Then,

$$\mathfrak{C}_{g,f}(A)(i) \leq g(p_A^{\overline{\odot}}(m \boxplus i, y)) \odot f(p_A^\odot(i, y)), \quad (16)$$

where  $m = |y|$  and  $m \boxplus i = \max(0, m - i)$ , holds for any  $i \in \mathbb{N}$ .



PROOF. Let  $A \in \mathfrak{F}\mathfrak{in}$ ,  $y \in \mathfrak{F}\mathfrak{in}$  with  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$  and  $m = |y|$ . According to Proposition 3.5, we have

$$\mathfrak{C}(A)(i) = \bigvee_{\substack{i_1, \dots, i_m \in \{0,1\} \\ i_1 + \dots + i_m = i}} \bigodot_{k=1}^m \mathfrak{C}(\{A(x_k)/x_k\})(i_k) \quad (17)$$

for any  $0 \leq i \leq m$ . Let  $0 \leq i \leq m$  and  $T$  be the set of all mappings  $t : \{1, \dots, m\} \rightarrow \{0, 1\}$  for which  $\sum_{k=1}^m t(k) = i$ .<sup>15</sup> Define

$$g_t^A = \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \quad \text{and} \quad f_t^A = \bigodot_{l \in t^{-1}(1)} f(A(x_l)) \quad (18)$$

for any  $t \in T$ , where  $t^{-1}(\alpha) = \{k \mid k \in \{1, \dots, m\} \ \& \ t(k) = \alpha\}$  and  $\alpha \in \{0, 1\}$ . Due to Theorem 3.7, we have

$$\mathfrak{C}_{g,f}(\{A(x_k)/x_k\})(0) = g(A(x_k)) \quad \text{and} \quad \mathfrak{C}_{g,f}(\{A(x_l)/x_l\})(1) = f(A(x_l)),$$

whence (17) may be rewritten as

$$\mathfrak{C}_{g,f}(A)(i) = \bigvee_{t \in T} (g_t^A \odot f_t^A). \quad (19)$$

Since  $g$  is a  $\overline{\odot}_d$ -po-homomorphism and  $f$  is a  $\odot$ -po-homomorphism (see Remark 3.7), then

$$\begin{aligned} \mathfrak{C}_{g,f}(A)(i) &= \bigvee_{t \in T} (g_t^A \odot f_t^A) = \bigvee_{t \in T} \left( \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \odot \bigodot_{l \in t^{-1}(1)} f(A(x_l)) \right) = \\ &= \bigvee_{t \in T} \left( g \left( \bigodot_{k \in t^{-1}(0)} A(x_k) \right) \odot f \left( \bigodot_{l \in t^{-1}(1)} A(x_l) \right) \right) \leq \\ &= \left( \bigvee_{t \in T} g \left( \bigodot_{k \in t^{-1}(0)} A(x_k) \right) \right) \odot \left( \bigvee_{t \in T} f \left( \bigodot_{l \in t^{-1}(1)} A(x_l) \right) \right) \leq \\ &= g \left( \bigwedge_{t \in T} \bigodot_{k \in t^{-1}(0)} A(x_k) \right) \odot f \left( \bigvee_{t \in T} \bigodot_{l \in t^{-1}(1)} A(x_l) \right) = \\ &= g \left( \bigwedge_{\substack{z \subseteq y \\ |z|=m-i}} \bigodot_{x_k \in z} A(x_k) \right) \odot f \left( \bigvee_{\substack{z' \subseteq y \\ |z'|=i}} \bigodot_{x_l \in z'} A(x_l) \right) = \\ &= g(p_A^{\overline{\odot}}(m-i, y)) \odot f(p_A^{\odot}(i, y)) = g(p_A^{\overline{\odot}}(m \boxplus i, y)) \odot f(p_A^{\odot}(i, y)). \end{aligned}$$

Let  $i > m$ . Then,  $m \boxplus i = 0$ , whence  $p_A^{\overline{\odot}}(0, y) = \perp$  using (iii) of Proposition 2.8, and  $p_A^{\odot}(i, y) = \perp$  using (ii) of Proposition 2.4. Therefore, and using Claim 4 in the proof of Lemma 3.6, we simply obtain

$$\mathfrak{C}_{g,f}(A)(i) = f(\perp) = \top \odot f(\perp) = g(\perp) \odot f(\perp) = g(p_A^{\overline{\odot}}(m \boxplus i, y)) \odot f(p_A^{\odot}(i, y)),$$

<sup>15</sup>Note that  $T$  is a finite set.

where  $g(\perp) = \top$  follows from the definition of the  $\overline{\odot}_d$ -po-homomorphism.  $\square$

Before we demonstrate the characterization of the c-measures using generalized cardinals in linearly ordered lattices, we introduce one concept to modify a fuzzy set using a homomorphism of the reducts of rdr-lattices.

**DEFINITION 3.5** Let  $\mathbf{L}$  be an rdr-lattice,  $h : L \rightarrow L$  be a mapping and  $A \in \mathfrak{F}\mathbf{in}$ . We shall say that a fuzzy set  $B$  is made up of  $A$  through  $f$  if  $B = f \circ A$ , and we shall write  $B = f(A)$ .

It should be noted that one has to be very careful in dealing with fuzzy sets made up of other fuzzy sets through the mapping  $f : L \rightarrow L$ , because we can obtain an infinite (denumerable) fuzzy set, which may cause an inconsistency in our cardinal theory for finite fuzzy sets. In fact, if one assumes a mapping  $h : L \rightarrow L$  that assigns  $\top$  to  $\perp$  and assumes  $A$  is a finite fuzzy set with an infinite universe, then  $f(A)$  is not a finite fuzzy set!

**THEOREM 3.10** Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{g,f}$  be a c-measure such that  $f$  is a  $\odot$ -po-homomorphism,  $g$  a  $\overline{\odot}_d$ -po-homomorphism,  $A \in \mathfrak{F}\mathbf{in}$  and  $y \in \mathfrak{F}\mathbf{in}$  with  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$ . Then,

$$\mathfrak{C}_{g,f}(A)(i) = g(p_A^{\overline{\odot}}(m \boxplus i, y)) \odot f(p_A^{\odot}(i, y)), \quad (20)$$

$$\mathfrak{C}_{g,f}(A)(i) = p_{g(A)}^{\odot}(m \boxplus i, y) \odot (p_{f(A)}^{\odot}(i, y) \vee f(\perp)), \quad (21)$$

where  $m = |y|$  and  $m \boxplus i = \max(0, m - i)$ , holds for any  $i \in \mathbb{N}$ .

**PROOF.** Let  $\mathbf{L}$  be linearly ordered,  $A \in \mathfrak{F}\mathbf{in}$ ,  $y \in \mathfrak{F}\mathbf{in}$  such that  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$  and  $m = |y|$ . As we have shown in the proof of Theorem 3.9,  $\mathfrak{C}_{g,f}(A)(i) = g(p_A^{\overline{\odot}}(m \boxplus i, y)) \odot f(p_A^{\odot}(i, y))$  for any  $i > m$ . Let  $0 \leq i \leq m$ . To prove the first statement, it is sufficient to show that

$$\mathfrak{C}_{g,f}(A)(i) \geq g(p_A^{\overline{\odot}}(m \boxplus i, y)) \odot f(p_A^{\odot}(i, y)), \quad (22)$$

which may be rewritten as

$$\begin{aligned} & \bigvee_{t \in T} \left( \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \odot \bigodot_{l \in t^{-1}(1)} f(A(x_l)) \right) \geq \\ & \left( \bigvee_{t \in T} \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \right) \odot \left( \bigvee_{t \in T} \bigodot_{l \in t^{-1}(1)} f(A(x_l)) \right), \end{aligned} \quad (23)$$

where we use the notation from the proof of Theorem 3.9. Set

$$\alpha = \bigvee_{t \in T} \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \quad \text{and} \quad \beta = \bigvee_{t \in T} \bigodot_{l \in t^{-1}(1)} f(A(x_l)).$$

Since  $\mathbf{L}$  is linearly ordered and  $T$  is a finite set, then there exist  $t_1, t_2 \in T$  such that

$$g_{t_1}^A = \alpha \quad \text{and} \quad f_{t_2}^A = \beta, \quad (24)$$

where  $g_{t_1}^A$  and  $f_{t_2}^A$  are defined by (18). Thus, the inequality (23) is satisfied if the previous equalities in (24) hold for some  $t_1, t_2 \in T$  such that  $t_1 = t_2$ . Let  $P$  be the set of all pairs  $(t_1, t_2) \in T \times T$  for which (24) is true. Using the previous consideration, the set  $P$  is not-empty. Define a mapping  $n : P \rightarrow \mathbb{N}$  by

$$n(t_1, t_2) = |t_1^{-1}(0) \cap t_2^{-1}(1)|. \quad (25)$$

Obviously, if  $k \in t_1^{-1}(0) \cap t_2^{-1}(1)$ , then there exists  $l \in t_1^{-1}(1) \cap t_2^{-1}(0)$ . In fact, if  $t_1^{-1}(1) \cap t_2^{-1}(0) = \emptyset$ , then  $t_1^{-1}(1) = t_2^{-1}(1)$ , which implies  $t_1 = t_2$ , but this is a contradiction with  $t_1^{-1}(0) \cap t_2^{-1}(1) \neq \emptyset$ .<sup>16</sup> We shall prove (by contradiction) that there exists a pair  $(t_1, t_2) \in P$  such that  $n(t_1, t_2) = 0$ , i.e.,  $t_1 = t_2$ , which concludes the proof of the first statement.

Let us suppose that  $n(t_1, t_2) > 0$  for any  $(t_1, t_2) \in P$  and denote

$$n_0 = \min\{n(t_1, t_2) \mid (t_1, t_2) \in P\}.$$

Let  $(s_1, s_2) \in P$  be such that  $n(s_1, s_2) = n_0$ . Let  $k \in s_1^{-1}(0) \cap s_2^{-1}(1)$  and  $l \in s_1^{-1}(1) \cap s_2^{-1}(0)$ . Since  $\mathbf{L}$  is linearly ordered,  $A(x_l) \geq A(x_k)$  or  $A(x_k) > A(x_l)$ .

If  $A(x_l) \geq A(x_k)$ , then define  $r : \{1, \dots, m\} \rightarrow \{0, 1\}$  by

$$r(j) = \begin{cases} 0, & \text{if } j = k; \\ 1, & \text{if } j = l; \\ s_2(j), & \text{otherwise.} \end{cases}$$

Obviously,  $r \in T$ , since we only replace  $k$  in  $s_2^{-1}(1)$  by  $l$  from  $s_2^{-1}(0)$  to form  $r^{-1}(1)$ . Hence, we obtain  $r(k) = s_1(k) = 0$  and  $r(l) = s_1(l) = 1$ , which implies  $s_1^{-1}(0) \cap r^{-1}(1) = (s_1^{-1}(0) \cap s_2^{-1}(1)) \setminus \{k\}$ , i.e.,  $n(s_1, r) < n_0$ . Furthermore,

$$f_r^A = \bigcirc_{j \in r^{-1}(1)} f(A(x_j)) \geq \bigcirc_{j \in s_2^{-1}(1)} f(A(x_j)) = \beta,$$

since  $f(A(x_l)) \geq f(A(x_k))$ , which follows from the presumption  $A(x_l) \geq A(x_k)$  and the preservation of the partial ordering by  $f$ . Therefore, we obtain  $f_r^A = \beta$ , and thus,  $(s_1, r) \in P$ , which is a contradiction with the presumption on  $n_0$ .

If  $A(x_l) < A(x_k)$ , then define  $r : \{1, \dots, m\} \rightarrow \{0, 1\}$  by

$$r(j) = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j = l; \\ s_1(j), & \text{otherwise.} \end{cases}$$

---

<sup>16</sup>By a simple computation, one can prove that  $n(t_1, t_2) = |t_1^{-1}(1) \cap t_2^{-1}(0)|$ .

Analogously, one may verify that  $(r, s_2) \in P$  and  $n(r, s_2) < n_0$ , which is again a contradiction with the presumption on  $n_0$ . Therefore, we see that there exists  $(t_1, t_2) \in P$  such that  $n(t_1, t_2) = 0$ , i.e.,  $t_1 = t_2$ , and the first statement is proved.

To verify the second statement it is sufficient to prove that

$$g(p_A^{\bar{\odot}}(m \boxminus i, y)) = p_{g(A)}^{\bar{\odot}}(m \boxminus i, y) \quad \text{and} \quad f(p_A^{\bar{\odot}}(i, y)) = p_{f(A)}^{\bar{\odot}}(i, y) \vee f(\perp). \quad (26)$$

Let  $g$  be an arbitrary  $\bar{\odot}_d$ -po-homomorphism. If  $i \geq m$ , then  $m \boxminus i = 0$  and we trivially obtain (due to (iii) of Propositions 2.4 and 2.8)

$$g(p_A^{\bar{\odot}}(m \boxminus i, y)) = g(p_A^{\bar{\odot}}(0, y)) = g(\perp) = \top = p_{g(A)}^{\bar{\odot}}(0, y).$$

If  $0 \leq i < m$ , then, from the properties of  $g$ , we obtain

$$\begin{aligned} g(p_A^{\bar{\odot}}(m - i, y)) &= g\left(\bigwedge_{\substack{z \subseteq y \\ |z|=m-i}} \bar{\odot}_{x \in z} A(x)\right) = \\ &= \bigvee_{\substack{z \subseteq y \\ |z|=m-i}} \odot_{x \in z} g(A)(x) = p_{g(A)}^{\bar{\odot}}(m - i, y). \end{aligned}$$

Note that the infimum and the supremum are computed over a non-empty finite set, and  $g(\bigwedge_{i=1}^n \alpha_i) = \bigvee_{i=1}^n g(\alpha_i)$  holds in each linearly ordered rdr-lattice (see Remark 3.7). The equality  $g(\bar{\odot}_{i=1}^n \alpha_i) = \bar{\odot}_{i=1}^n g(\alpha_i)$  holds in each rdr-lattice, which follows from the presumption of  $g$ .

Let  $f$  be an arbitrary  $\odot$ -po-homomorphism. If  $f$  is trivial, then  $f(\alpha) = \top$  for any  $\alpha \in L$ , and

$$f(p_A^{\bar{\odot}}(i, y)) = \top = p_{f(A)}^{\bar{\odot}}(i, y) \vee \top = p_{f(A)}^{\bar{\odot}}(i, y) \vee f(\perp).$$

If  $f$  is non-trivial, then  $f(\perp) = \perp$ , and we have to prove that  $f(p_A^{\bar{\odot}}(i, y)) = p_{f(A)}^{\bar{\odot}}(i, y)$ . For  $i > m$ , we obtain (due to (iii) of Proposition 2.4)

$$f(p_A^{\bar{\odot}}(i, y)) = f(\perp) = \perp = p_{f(A)}^{\bar{\odot}}(i, y).$$

For  $i = 0$ , we obtain (due to (ii) of Proposition 2.4)

$$f(p_A^{\bar{\odot}}(0, y)) = f(\top) = \top = p_{f(A)}^{\bar{\odot}}(0, y).$$

If  $0 < i \leq m$ , then, from the properties of  $f$ , we obtain

$$f(p_A^{\bar{\odot}}(i, y)) = f\left(\bigvee_{\substack{z \subseteq y \\ |z|=i}} \odot_{x \in z} A(x)\right) = \bigvee_{\substack{z \subseteq y \\ |z|=i}} \odot_{x \in z} f(A)(x) = p_{f(A)}^{\bar{\odot}}(i, y),$$

where again  $f(\bigvee_{i=1}^n \alpha_i) = \bigvee_{i=1}^n f(\alpha_i)$  holds in each linearly ordered rdr-lattice (see Remark 3.7), and the equality  $f(\odot_{i=1}^n \alpha_i) = \odot_{i=1}^n f(\alpha_i)$  holds in each rdr-lattice, which follows from the presumption of  $f$ . Hence, (26) is verified, and the proof is finished.  $\square$

One may be surprised that  $y$  in the definition of  $\mathfrak{C}_{g,f}$  can move between the support and the universe of discourse of a fuzzy set. Recall that the value of  $p_A^\circ(i, y)$  (and  $p_A^\ominus(i, y)$ ) is computed as the supremum (infimum) over all the subsets of  $y$  where  $|y| = i$ . By a simple computation, or by using Proposition 7.3 in [9], one can prove that  $p_A^\circ(i, y) = p_A^\circ(i, y')$  for any  $y, y' \in \mathfrak{Fin}$  such that  $\text{Supp}(A) \subseteq y, y' \subseteq \text{Dom}(A)$ . For  $\ominus$ , the same equality is not true, but  $p_A^\ominus(m \boxminus i, y) = p_A^\ominus(m' \boxminus i, y')$ , where  $m = |y|$  and  $m' = |y'|$ , as the following example demonstrates.

EXAMPLE 3.8 Let us suppose that  $\mathbf{L}$  is the Lukasiewicz rdr-lattice and consider  $A = \{0.5/a, 0.8/b, 0.1/c, 0.4/d, 0/e\}$ . Set  $y = \text{Dom}(A)$ ,  $y' = \text{Supp}(A)$  and  $m = |y| = 5$ ,  $m' = |y'| = 4$ . Then, we obtain the following:

for  $i = 0$ ,

$$p_A^\ominus(5, y) = \bigoplus_{x \in y} A(x) = 1 = \bigoplus_{x \in y'} A(x) = p_A^\ominus(4, y');$$

for  $i = 1$ ,

$$p_A^\ominus(4, y) = \bigoplus_{x \in \{a,c,d,e\}} A(x) = 1 = \bigoplus_{x \in \{a,c,d\}} A(x) = p_A^\ominus(3, y');$$

for  $i = 2$ ,

$$p_A^\ominus(3, y) = \bigoplus_{x \in \{c,d,e\}} A(x) = 0.4 = \bigoplus_{x \in \{c,d\}} A(x) = p_A^\ominus(2, y').$$

Analogously, we obtain the equalities for the remaining values of  $i$ . One could notice that the subset of  $y$  used for the computation of  $p_A^\oplus(-, y)$  always contains the element  $e \in y$  for which  $A(e) = 0$  (i.e.,  $e \notin y'$ ), and thus, this element has no effect on the values of  $p_A^\oplus(-, y)$ .

Recall that  $\mathfrak{C}_g$  ( $\mathfrak{C}_f$ ) represents a  $c$ -measure, where  $f$  ( $g$ ) is a trivial homomorphism of the reducts (i.e.,  $f(\alpha) = g(\alpha) = \top$  for all  $\alpha \in L$ ).

COROLLARY 3.11 Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{g,f}$  be a  $c$ -measure such that  $f$  is a  $\odot$ -po-homomorphism and  $g$  is a  $\ominus_d$ -po-homomorphism. Then,

$$\mathfrak{C}_{g,f}(A)(i) = \mathfrak{C}_g(A)(i) \odot \mathfrak{C}_f(A)(i) \tag{27}$$

holds for any  $A \in \mathfrak{Fin}$  and  $i \in \mathbb{N}$ .

PROOF. Due to Theorem 3.10, we have (we put  $y = \text{Supp}(A)$  and  $m = |y|$ )

$$\mathfrak{C}_g(A)(i) = g(p_A^\ominus(m \boxminus i, y)) \text{ and } \mathfrak{C}_f(A)(i) = f(p_A^\circ(i, y))$$

whenever  $f$  or  $g$  are trivial. Substituting  $\mathfrak{C}_g(A)(i)$  and  $\mathfrak{C}_f(A)(i)$  into (20), we obtain (27).  $\square$

COROLLARY 3.12 *Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{g,f}$  be a c-measure with respect to  $\wedge$  such that  $f$  is a  $\wedge$ -homomorphism,  $g$  is a  $\vee$ -homomorphism,  $A \in \mathfrak{F}\mathbf{in}$  and  $y \in \mathfrak{F}\mathbf{in}$  with  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$ . Then,*

$$\mathfrak{C}_{g,f}(A)(i) = g(p_A^\wedge(i+1, y)) \wedge f(p_A^\wedge(i, y)) \quad (28)$$

*holds for any  $i \in \mathbb{N}$ .*

PROOF. Let  $\mathbf{L}$  be linearly ordered,  $A \in \mathfrak{F}\mathbf{in}$  and  $y \in \mathfrak{F}\mathbf{in}$  with  $\text{Supp}(A) \subseteq y \subseteq \text{Dom}(A)$  and  $|y| = m$ . By Theorem 3.10, it is sufficient to prove that

$$p_A^\wedge(i+1, y) = p_A^\vee(m \boxplus i, y). \quad (29)$$

If  $i \geq m$ , the equality immediately follows from (ii) of Proposition 2.4 and (iii) of Proposition 2.8. Let  $0 \leq i < m$ . Since  $\mathbf{L}$  is linearly ordered, we may consider the sequence  $x_1, \dots, x_m$  of the elements from  $y$  such that  $A(x_j) \geq A(x_{j+1})$  holds for any  $j = 1 \dots, m-1$ . Using the definition of  $p_A^\wedge(i, y)$  and  $p_A^\vee(i, y)$ , one may simply check that

$$p_A^\wedge(i+1, y) = A(x_{i+1}) = A(x_{m-i}) = p_A^\vee(m-i, y) = p_A^\vee(m \boxplus i, y),$$

and the proof is finished.  $\square$

COROLLARY 3.13 *Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{g,f}$  be a c-measure with respect to  $\wedge$  such that  $f$  is a  $\wedge$ -homomorphism and  $g$  is a  $\vee$ -homomorphism. Then,*

$$\mathfrak{C}_{g,f}(A)(i) = g(\mathfrak{C}_{\text{id}}(A)(i+1)) \wedge f(\mathfrak{C}_{\text{id}}(A)(i)), \quad (30)$$

*where  $\text{id}$  denotes the identity homomorphism, holds for any  $A \in \mathfrak{F}\mathbf{in}$  and  $i \in \mathbb{N}$ .*

PROOF. It immediately follows from the fact that  $\mathfrak{C}_{\text{id}}(A)(i) = p_A^\wedge(i, \text{Supp}(A))$ .  $\square$

REMARK 3.9 It is easy to see that  $\mathfrak{C}_{\text{id}} = \mathfrak{C}_1$ , where  $\mathfrak{C}_1$  is defined in Ex. 3.1.

In the following example, we shall introduce three c-measures based on Theorem 3.10.

EXAMPLE 3.10 Let us suppose that the product (Goguen) algebra is given (i.e.,  $\otimes = \cdot$ ). Define the strict negation  $\nu : [0, 1] \rightarrow [0, 1]$  by  $\nu(\alpha) = 1 - \alpha^2$  and the addition  $\oplus : [0, 1]^2 \rightarrow [0, 1]$  by

$$\alpha \oplus \beta = \nu^{-1}(\nu(\alpha) \otimes \nu(\beta)) = \sqrt{\alpha^2 + \beta^2 - \alpha^2\beta^2}.$$

It is well-known that  $\oplus$  is a continuous t-conorm (see [17]), and thus, we can define (see Ex. 2.1 in [9]) an rdr-lattice  $\mathbf{L}$  and interpret the membership degrees of the fuzzy sets in  $\mathbf{L}$ . Note that  $\nu^{-1}(\alpha) = \sqrt{1 - \alpha}$  and whence  $\nu(\alpha \oplus \beta) = \nu(\alpha) \otimes \nu(\beta)$ .

Let us define  $f, g : [0, 1] \rightarrow [0, 1]$  by  $f(\alpha) = \alpha^2$  and  $g(\alpha) = \nu(\alpha) = 1 - \alpha^2$  for any  $\alpha \in [0, 1]$ . As we have shown in Ex. 3.6,  $f$  and  $g$  are  $\otimes$ -po- and  $\oplus_d$ -po-homomorphisms, respectively. In fact, it is sufficient to define  $f = h_2$  and  $g = h_{\nu,1}$ , where  $h_2$  and  $h_{\nu,1}$  are from Ex. 3.6. Using Theorem 3.10, the c-measure determined by  $g$  and  $f$  has the following form by (20):

$$\mathfrak{C}_{g,f}(A)(i) = g(p_A^\oplus(m \boxminus i, y)) \otimes f(p_A^\otimes(i, y)) = (1 - p_A^\oplus(m \boxminus i, y)^2) \cdot p_A^\otimes(i, y)^2,$$

where we put  $y = \text{Supp}(A)$  and  $m = |y|$ . Let

$$A = \{0.5/x_1, 0.2/x_2, 0.9/x_3, 1/x_4, 0.8/x_5, 0.9/x_6, 1/x_7, 0.6/x_8\}$$

be a finite fuzzy set. After a simple computation,<sup>17</sup> we obtain (rounded to three digits)

$$\begin{aligned} p_A^\otimes(-, y) &= \{1/0, 1/1, 1/2, 0.9/3, 0.81/4, \\ &\quad 0.648/5, 0.389/6, 0.194/7, 0.039/8, 0/9, \dots\} \text{ and} \\ p_A^\oplus(-, y) &= \{0/0, 0.2/1, 0.529/2, 0.737/3, 0.913/4, \\ &\quad 0.984/5, 0.997/6, 1/7, 1/8, 1/9, \dots\}. \end{aligned}$$

Then, using

$$\mathfrak{C}_{g,f}(A) = (1 - p_A^\oplus(8 \boxminus i, y)^2) \cdot p_A^\otimes(i, y)^2,$$

we obtain

$$\begin{aligned} \mathfrak{C}_{g,f}(A) &= \{0/0, 0.2/1, 0.006/2, 0.026/3, 0.109/4, \\ &\quad 0.193/5, 0.109/6, 0.036/7, 0.002/8, 0/9, \dots\}. \end{aligned}$$

The same result can be derived by (21) where first the fuzzy set  $A$  is modified by  $g$  and  $f$ , and then, the c-measure of  $A$  is computed. Note that  $\mathfrak{C}_{g,f}$  is not an example of a generalized FECount, provided by Wygralak in [22] and [23], for  $f \neq \text{id}$ ; this holds for  $f = \text{id}$ .

REMARK 3.11 Assuming a linearly ordered rdr-lattice, a c-measure  $\mathfrak{C}_f$  determined by a  $\odot$ -po-homomorphism  $f$  may be expressed in the following form (using Proposition 2.5 for  $\odot = \otimes$ )

$$\mathfrak{C}_f(A)(i) = f(p_A^\odot(i, y)) = f(p_A^\wedge(0, y)) \odot \cdots \odot f(p_A^\wedge(i, y)), \quad (31)$$

where  $y = \text{Supp}(A)$ . For  $\odot = \wedge$ , we obtain a simple form  $\mathfrak{C}_f(A)(i) = f(p_A^\wedge(i, y))$ . Therefore, one can recognize that  $\mathfrak{C}_f$  results in a generalization of FGCounts, which is provided by Wygralak in [22, 23]. Note that an analogous extension of the original FGCount, where  $f$  is called a *pattern*, has been commented on in subsection 4.1.4 of [23]; nevertheless, a further development of

<sup>17</sup>One can use the fact that  $p_A^\otimes(i, y) = \otimes_{k=0}^i p_A^\wedge(k, y)$  and  $p_A^\oplus(i, y) = \oplus_{k=0}^i p_A^\vee(k, y)$  (for details, see Proposition 7.2 and 7.11 in [9]).

this stream has not yet been achieved. For a c-measure  $\mathfrak{C}_g$  determined by a  $\overline{\odot}_d$ -po-homomorphism  $g$ , one can simply derive

$$\mathfrak{C}_g(A)(i) = g(p_A^{\overline{\odot}}(m \boxminus i, y)) = g(p_A^{\wedge}(i+1, y)) \odot \cdots \odot g(p_A^{\wedge}(m, y)), \quad (32)$$

where  $y = \text{Supp}(A)$ ,  $m = |y|$  and  $p_A^{\wedge}(i+1, y) = p_A^{\vee}(m \boxminus i, y)$  is applied. For  $\odot = \wedge$ , we obtain  $\mathfrak{C}_g(A)(i) = g(p_A^{\wedge}(i+1, y))$ , and one can recognize a generalized FLCCount whenever  $g$  defines a negation on  $[0, 1]$  (see Subsection 1.2 in [23] or Subsection 11.1 in [12]). For a c-measure  $\mathfrak{C}_{g,f}$  determined by a  $\overline{\odot}_d$ -po-homomorphism  $g$  and  $\odot$ -po-homomorphism  $f$ , we obtain

$$\begin{aligned} \mathfrak{C}_{g,f}(A)(i) = & g(p_A^{\overline{\odot}}(m \boxminus i, y)) \odot f(p_A^{\odot}(i, y)) = \\ & f(p_A^{\wedge}(0, y)) \odot \cdots \odot f(p_A^{\wedge}(i, y)) \odot g(p_A^{\wedge}(i+1, y)) \odot \cdots \odot g(p_A^{\wedge}(m, y)), \end{aligned} \quad (33)$$

where  $y = \text{Supp}(A)$  and  $m = |y|$ , and one may see a further generalization of FECCounts introduced by Wygralak in [22] whenever  $g$  is a negation on  $[0, 1]$ . Let us stress that  $\mathfrak{C}_{g,f}$  need not be defined on  $[0, 1]$  and its form is derived from the axiomatic system, which is a difference from Wygralak's approach to defining fuzzy cardinalities of the forms (31)-(33).

#### 3.4. Valuation property for c-measures

One well-known property of the cardinality of sets is the *valuation property*, which states that

$$|x \cap y| + |x \cup y| = |x| + |y| \quad (34)$$

holds for arbitrary sets  $x$  and  $y$ . In [23], Wygralak proved that the valuation property is satisfied by generalized cardinals if and only if we restrict ourselves to the infimum and supremum (see Theorem 4.18 on page 88 in [23]). Here, moreover, we have to add the presumption on the linearity of the rdr-lattices as the following example demonstrates.

**EXAMPLE 3.12** Let  $L = (L, \wedge, \vee, \rightarrow, \ominus, \perp, \top)$  be the rdr-lattice with the support  $L = \{\perp, \alpha, \beta, \top\}$ ,  $\alpha \wedge \beta = \perp$  and  $\alpha \vee \beta = \top$ . The operations  $\rightarrow$  and  $\ominus$  are defined by the adjunction and the dual adjunction, respectively.<sup>18</sup> One may see that  $\alpha$  and  $\beta$  are incomparable elements with respect to  $\wedge$  naturally defined on  $L$ . Let us define  $f, g : L \rightarrow L$  as follows

$$\begin{aligned} f(x) &= x, \text{ for each } x \in L, \text{ and} \\ g(x) &= \begin{cases} x, & \text{if } x \in \{\alpha, \beta\}; \\ \perp, & \text{if } x = \top; \\ \top, & \text{if } x = \perp. \end{cases} \end{aligned}$$

One may simply verify that  $f$  is a  $\wedge$ -homomorphism and  $g$  a  $\vee_d$ -homomorphism (e.g.,  $g(\alpha \vee \beta) = g(\top) = \perp = g(\alpha) \wedge g(\beta)$ ). Let  $\mathfrak{C}_{f,g}$  be a c-measure determined

<sup>18</sup>For example, we obtain  $\alpha \ominus \beta = \beta \ominus \alpha = \top$ .



by  $f$  and  $g$  and  $A = \{\alpha/x\}$  and  $B = \{\beta/x\}$  be singletons, where  $x \in \mathbf{Count}$ . Since  $\alpha \wedge \beta = \perp$  and  $\alpha \vee \beta = \top$ , then we obtain

$$\begin{aligned} ((\mathfrak{C}_{g,f}(A \cap B) + \mathfrak{C}_{g,f}(A \cup B))(1) &= (\mathfrak{C}_{g,f}(\{\alpha \wedge \beta/x\}) + \mathfrak{C}_{g,f}(\{\alpha \vee \beta/x\}))(1) = \\ (\mathfrak{C}_{g,f}(\{\perp/x\})(0) \wedge \mathfrak{C}_{g,f}(\{\top/x\})(1)) \vee &(\mathfrak{C}_{g,f}(\{\perp/x\})(1) \wedge \mathfrak{C}_{g,f}(\{\top/x\})(0)) = \\ (g(\perp) \wedge f(\top)) \vee (g(\top) \wedge f(\perp)) &= (\top \wedge \top) \vee (\perp \wedge \perp) = \top. \end{aligned}$$

Conversely, we obtain

$$\begin{aligned} (\mathfrak{C}_{g,f}(A) + \mathfrak{C}_{g,f}(B))(1) &= (\mathfrak{C}_{g,f}(\{\alpha/x\}) + \mathfrak{C}_{g,f}(\{\beta/x\}))(1) = \\ (\mathfrak{C}_{g,f}(\{\alpha/x\})(0) \wedge \mathfrak{C}_{g,f}(\{\beta/x\})(1)) \vee &(\mathfrak{C}_{g,f}(\{\alpha/x\})(1) \wedge \mathfrak{C}_{g,f}(\{\beta/x\})(0)) = \\ (g(\alpha) \wedge f(\beta)) \vee (f(\alpha) \wedge g(\beta)) &= (\alpha \wedge \beta) \vee (\alpha \wedge \beta) = \perp \vee \perp = \perp. \end{aligned}$$

Therefore, the valuation property is not satisfied.

Before we state an analogous statement to Theorem 4.18 that is presented in [23], let us derive useful inequalities provided in the following lemma.

LEMMA 3.14 *Let  $\mathfrak{C}_{g,f}$  be a c-measure with respect to  $\wedge$  and  $A, B \in \mathfrak{Ffin}$ . Then,*

$$h(\alpha \wedge \beta) \wedge h'(\alpha \vee \beta) \geq h(\alpha) \wedge h'(\beta) \quad (35)$$

*holds for any  $\alpha, \beta \in L$  and  $(h, h') \in \{(g, g), (f, f), (g, f)\}$ .*

PROOF. Since  $\mathfrak{C}_{g,f}$  is a c-measure with respect to  $\wedge$ , then, using Ex. 3.5,  $f$  preserves and  $g$  reverses the partial ordering of the rdr-lattice. If  $(h, h') = (g, g)$ , then  $g(\alpha \wedge \beta) \geq g(\alpha \vee \beta)$ , which implies

$$g(\alpha \wedge \beta) \wedge g(\alpha \vee \beta) = g(\alpha \vee \beta) = g(\alpha) \wedge g(\beta),$$

since  $g$  is a  $\vee_d$ -homomorphism. Analogously, one can prove (35) for  $(h, h') = (f, f)$ . Let  $(h, h') = (g, f)$ . Then, we obtain

$$g(\alpha \wedge \beta) \wedge f(\alpha \vee \beta) \geq (g(\alpha) \vee g(\beta)) \wedge (f(\alpha) \vee f(\beta)) \geq g(\alpha) \wedge f(\beta),$$

and the proof is finished.  $\square$

Now, we may provide a weaker version of the valuation property that holds for rdr-lattices, where the distributivity of  $\wedge$  over  $\vee$  is supposed. Because we deal with finite index sets, the distributivity is ensured by the presumption that the rdr-lattice satisfies the prelinearity axiom (see Theorem 2.37 in [1]).

THEOREM 3.15 *Let  $\mathbf{L}$  satisfy the prelinearity axiom and  $\mathfrak{C}$  be a c-measure with respect to  $\wedge$ . Then,*

$$\mathfrak{C}(A \cap B) + \mathfrak{C}(A \cup B) \geq \mathfrak{C}(A) + \mathfrak{C}(B) \quad (36)$$

*holds for arbitrary  $A, B \in \mathfrak{Ffin}$ .*

PROOF. Let  $\mathfrak{C} = \mathfrak{C}_{g,f}$  be a c-measure determined by a  $\wedge$ -homomorphism  $f$  and a  $\vee_d$ -homomorphism  $g$  and  $A, B \in \mathfrak{F}\text{in}$ . Without loss of generality, let us suppose that  $\text{Dom}(A) = \text{Dom}(B) = \{x_1, \dots, x_m\}$ .

First, let us note that we trivially have

$$g(\alpha \vee \beta) \wedge f(\alpha \wedge \beta) \leq g(\alpha \wedge \beta) \wedge f(\alpha \vee \beta), \quad (37)$$

since  $g$  reverses and  $f$  preserves the ordering, and as a simple consequence of (35), we obtain

$$g(\alpha \wedge \beta) \wedge f(\alpha \vee \beta) \geq (g(\alpha) \wedge f(\beta)) \vee (g(\beta) \wedge f(\alpha)). \quad (38)$$

Let  $i, k, l \in \{0, \dots, m\}$  and  $k + l = i$ . Denote  $T_k$  and  $T_l$  the sets of all mappings  $t, s : \{1, \dots, m\} \rightarrow \{0, 1\}$  for which  $\sum_{u=1}^m t(u) = k$  and  $\sum_{v=1}^m s(v) = l$ , respectively. Similarly to (18), we define

$$g_t^C = \bigwedge_{u \in t^{-1}(0)} g(C(x_u)) \text{ and } f_t^C = \bigwedge_{v \in t^{-1}(1)} f(C(x_v)) \quad (39)$$

for any  $t \in T_k$ , where  $t^{-1}(\alpha) = \{k \mid k \in \{1, \dots, m\} \text{ \& } t(k) = \alpha\}$  and  $\alpha \in \{0, 1\}$ , and, by analogy, we define  $g_s^C$  and  $f_s^C$  for any  $s \in T_l$ . Using the same arguments as for (19), we obtain

$$\mathfrak{C}_{g,f}(C)(k) = \bigvee_{t \in T_k} (g_t^C \wedge f_t^C) \quad \text{and} \quad \mathfrak{C}_{g,f}(C)(l) = \bigvee_{s \in T_l} (g_s^C \wedge f_s^C).$$

Now, if  $t \in T_k$  and  $s \in T_l$ , then we define  $\xi(t, s) = \min(t, s)$  and  $\eta(t, s) = \max(t, s)$ , i.e.,  $\xi(t, s)(u) = \min(t(u), s(u))$  and  $\eta(t, s)(u) = \max(t(u), s(u))$  for any  $u = 1, \dots, m$ . For the sake of simplicity, we shall use  $t_* = \xi(t, s)$  and  $s_* = \eta(t, s)$  in the following part. Further, denote  $k_* = \sum_{u=1}^m t_*(u)$  and  $l_* = \sum_{v=1}^m s_*(v)$ . It is easy to see that  $k_* \neq k$  and  $l_* \neq l$  in general, but

$$k_* + l_* = k + l = i,$$

and thus  $t_* \in T_{k_*}$  and  $s_* \in T_{l_*}$ . As a simple consequence of the definition of  $t_*$  and  $s_*$ , we obtain

$$\begin{aligned} t_*^{-1}(0) \cap s_*^{-1}(0) &= t^{-1}(0) \cap s^{-1}(0); \\ t_*^{-1}(0) \cap s_*^{-1}(1) &= (t^{-1}(0) \cap s^{-1}(1)) \cup (t^{-1}(1) \cap s^{-1}(0)); \\ t_*^{-1}(1) \cap s_*^{-1}(0) &= \emptyset; \\ t_*^{-1}(1) \cap s_*^{-1}(1) &= t^{-1}(1) \cap s^{-1}(1), \end{aligned} \quad (40)$$

where the third equality follows from  $t_*(u) \leq s_*(u)$  that holds for any  $u = 1, \dots, m$ . Using these equalities and (37), one can simply derive<sup>19</sup>

$$g_t^{A \wedge B} \wedge f_t^{A \wedge B} \wedge g_s^{A \cup B} \wedge f_s^{A \cup B} \leq g_{t_*}^{A \wedge B} \wedge f_{t_*}^{A \wedge B} \wedge g_{s_*}^{A \cup B} \wedge f_{s_*}^{A \cup B}.$$

<sup>19</sup>Use the equality  $\bigwedge_{u \in t^{-1}(0)} g(A(x_u) \wedge B(x_u)) \wedge \dots \wedge \bigwedge_{u \in s^{-1}(1)} f(A(x_u) \vee B(x_u)) = \bigwedge_{u \in t^{-1}(0) \cap s^{-1}(0)} g(A(x_u) \wedge B(x_u)) \wedge g(A(x_u) \vee B(x_u)) \wedge \dots \wedge \bigwedge_{u \in t^{-1}(1) \cap s^{-1}(1)} f(A(x_u) \wedge B(x_u)) \wedge f(A(x_u) \vee B(x_u))$ , and the analogous equality for  $t_*$  and  $s_*$ .

Then, we have

$$\begin{aligned}
(\mathfrak{C}(A \cap B) + \mathfrak{C}(A \cup B))(i) &= \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} (\mathfrak{C}(A \cap B)(k) \wedge \mathfrak{C}(A \cup B)(l)) = \\
&= \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} \left( \bigvee_{t \in T_k} (g_t^{A \cap B} \wedge f_t^{A \cap B}) \right) \wedge \left( \bigvee_{s \in T_l} (g_s^{A \cup B} \wedge f_s^{A \cup B}) \right) = \\
&= \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} \bigvee_{t \in T_k} \bigvee_{s \in T_l} \left( g_t^{A \cap B} \wedge f_t^{A \cap B} \wedge g_s^{A \cup B} \wedge f_s^{A \cup B} \right) = \\
&= \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} \bigvee_{t \in T_k} \bigvee_{s \in T_l} \left( g_{t_*}^{A \cap B} \wedge f_{t_*}^{A \cap B} \wedge g_{s_*}^{A \cup B} \wedge f_{s_*}^{A \cup B} \right) = \\
&= \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} \bigvee_{t \in T_k} \bigvee_{s \in T_l} \bigwedge_{u \in t_*^{-1}(0) \cap s_*^{-1}(0)} (g(A(x_u) \wedge B(x_u)) \wedge g(A(x_u) \vee B(x_u))) \wedge \\
&\quad \bigwedge_{u \in t_*^{-1}(0) \cap s_*^{-1}(1)} (g(A(x_u) \wedge B(x_u)) \wedge f(A(x_u) \vee B(x_u))) \wedge \\
&\quad \bigwedge_{u \in t_*^{-1}(1) \cap s_*^{-1}(1)} (f(A(x_u) \wedge B(x_u)) \wedge f(A(x_u) \vee B(x_u))) = \alpha,
\end{aligned}$$

where the third equality is a consequence of the prelinearity and the fourth equality follows from the fact that  $k_* + l_* = i$ . Obviously,  $\bigwedge_{i \in I} (\alpha_i \vee \beta_i) \geq \bigwedge_{i \in I_1} \alpha_i \wedge \bigwedge_{i \in I_2} \beta_i$  holds for  $I = I_1 \cup I_2$ .<sup>20</sup> Using this inequality and the fact that  $t_*^{-1}(0) \cap s_*^{-1}(1) = (t^{-1}(0) \cap s^{-1}(1)) \cup (t^{-1}(1) \cap s^{-1}(0))$ , we obtain

$$\begin{aligned}
&\bigwedge_{u \in t_*^{-1}(0) \cap s_*^{-1}(1)} ((g(A(x_u)) \wedge f(B(x_u))) \vee (f(A(x_u)) \wedge g(B(x_u)))) \geq \\
&\bigwedge_{u \in t^{-1}(0) \cap s^{-1}(1)} (g(A(x_u)) \wedge f(B(x_u))) \wedge \bigwedge_{u \in t^{-1}(1) \cap s^{-1}(0)} (f(A(x_u)) \wedge g(B(x_u))).
\end{aligned}$$

Applying Lemma 3.14, (38) and the previous inequality, we obtain

$$\begin{aligned}
\alpha &\geq \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} \bigvee_{t \in T_k} \bigvee_{s \in T_l} \bigwedge_{u \in t_*^{-1}(0) \cap s_*^{-1}(0)} (g(A(x_u)) \wedge g(B(x_u))) \wedge \\
&\quad \bigwedge_{u \in t_*^{-1}(0) \cap s_*^{-1}(1)} ((g(A(x_u)) \wedge f(B(x_u))) \vee (f(A(x_u)) \wedge g(B(x_u)))) \wedge \\
&\quad \bigwedge_{u \in t_*^{-1}(1) \cap s_*^{-1}(1)} f(A(x_u)) \wedge f(B(x_u)) \geq
\end{aligned}$$

<sup>20</sup>In fact,  $\bigwedge_{i \in I_1} (\alpha_i \vee \beta_i) \geq \bigwedge_{i \in I_1} \alpha_i$  and  $\bigwedge_{i \in I_2} (\alpha_i \vee \beta_i) \geq \bigwedge_{i \in I_2} \beta_i$ . Then  $\bigwedge_{i \in I} (\alpha_i \vee \beta_i) = \bigwedge_{i \in I_1} (\alpha_i \vee \beta_i) \wedge \bigwedge_{i \in I_2} (\alpha_i \vee \beta_i) \geq \bigwedge_{i \in I_1} \alpha_i \wedge \bigwedge_{i \in I_2} \beta_i$ .

$$\begin{aligned}
& \bigvee_{\substack{k,l \in \mathbb{N} \\ k+l=i}} \bigvee_{t \in T_k} \bigvee_{s \in T_l} \bigwedge_{u \in t^{-1}(0) \cap s^{-1}(0)} (g(A(x_u)) \wedge g(B(x_u))) \wedge \\
& \bigwedge_{u \in t^{-1}(0) \cap s^{-1}(1)} (g(A(x_u)) \wedge f(B(x_u))) \wedge \bigwedge_{u \in t^{-1}(1) \cap s^{-1}(0)} (f(A(x_u)) \wedge g(B(x_u))) \wedge \\
& \bigwedge_{u \in t^{-1}(1) \cap s^{-1}(1)} (f(A(x_u)) \wedge f(B(x_u))) = \bigvee_{\substack{k,l \in \mathbb{N} \\ k+l=i}} \bigvee_{t \in T_k} \bigvee_{s \in T_l} (g_t^A \wedge f_t^A \wedge g_s^B \wedge f_s^B) = \\
& \bigvee_{\substack{k,l \in \mathbb{N} \\ k+l=i}} \left( \bigvee_{t \in T_k} (g_t^A \wedge f_t^A) \right) \wedge \left( \bigvee_{s \in T_l} (g_s^B \wedge f_s^B) \right) = \\
& \bigvee_{\substack{k,l \in \mathbb{N} \\ k+l=i}} (\mathfrak{C}_{g,f}(A)(k) \wedge \mathfrak{C}_{g,f}(B)(l)) = (\mathfrak{C}_{g,f}(A) + \mathfrak{C}_{g,f}(B))(i),
\end{aligned}$$

where the last but one equality follows from the prelinearity.  $\square$

The following statement shows that the valuation property is satisfied under the presumption of the linearity (cf., Theorem 3.7 in [3]).

**THEOREM 3.16** *Let  $\mathbf{L}$  be linearly ordered and  $\mathfrak{C}$  be a  $c$ -measure with respect to  $\wedge$ . Then,  $\mathfrak{C}$  satisfies the valuation property.*

**PROOF.** Let  $A, B \in \mathfrak{F}\text{fin}$ , and, without loss of generality, suppose that  $y = \text{Dom}(A) = \text{Dom}(B) = \text{Supp}(A \cup B)$ . Define  $x_A = \{x \in y \mid A(x) < B(x)\}$  and  $x_B = y \setminus x_A$ . It is easy to see that  $A(x) \geq B(x)$  for any  $x \in x_B$ . Hence, we obtain  $A(x) = A(x) \wedge B(x)$  and  $B(x) = A(x) \vee B(x)$  for any  $x \in x_A$ , and  $A(x) = A(x) \vee B(x)$  and  $B(x) = A(x) \wedge B(x)$  for any  $x \in x_B$ . According to the additivity axiom, we obtain

$$\begin{aligned}
\mathfrak{C}(A) + \mathfrak{C}(B) &= \sum_{x \in x_A} \mathfrak{C}(\{A(x)/x\}) + \sum_{x \in x_B} \mathfrak{C}(\{A(x)/x\}) + \\
& \sum_{x \in x_A} \mathfrak{C}(\{B(x)/x\}) + \sum_{x \in x_B} \mathfrak{C}(\{B(x)/x\}) = \sum_{x \in x_A} \mathfrak{C}(\{A(x)/x\}) + \\
& \sum_{x \in x_B} \mathfrak{C}(\{B(x)/x\}) + \sum_{x \in x_A} \mathfrak{C}(\{B(x)/x\}) + \sum_{x \in x_B} \mathfrak{C}(\{A(x)/x\}) = \\
& \sum_{x \in y} \mathfrak{C}(\{A(x) \wedge B(x)/x\}) + \sum_{x \in y} \mathfrak{C}(\{A(x) \vee B(x)/x\}) = \mathfrak{C}(A \cap B) + \mathfrak{C}(A \cup B),
\end{aligned}$$

and the proof is finished.  $\square$

**REMARK 3.13** In contrast to Theorem 4.18 in [23] stating that the valuation property is satisfied only for the fuzzy set operations defined by the infimum and the supremum, here, there is a problem in proving that the satisfaction of the valuation property implies the choice of the infimum and the supremum. One

may see that the culprits of this failure are considered  $\bar{\odot}_d$ -homomorphisms and  $\odot$ -homomorphisms. In fact, we obtain, for a generalized valuation property,<sup>21</sup>

$$\begin{aligned} & (\mathfrak{C}_{g,f}(\{\alpha \odot \alpha/x\}) + \mathfrak{C}_{g,f}(\{\alpha \bar{\odot} \alpha/x\}))(0) = \\ & g(\alpha \odot \alpha) \odot g(\alpha \bar{\odot} \alpha) = g(\alpha \odot \alpha) \odot g(\alpha) \odot g(\alpha) = \\ & g(\alpha) \odot g(\alpha) = (\mathfrak{C}_{g,f}(\{\alpha/x\}) + \mathfrak{C}_{g,f}(\{\alpha/x\}))(0); \end{aligned}$$

however, how can we prove that

$$g(\alpha \odot \alpha) \odot g(\alpha) \odot g(\alpha) = g(\alpha) \odot g(\alpha) \quad (41)$$

implies  $\odot = \wedge$ ? For example, (41) is true for  $\odot = \wedge$  and for  $g(\alpha \odot \alpha) = \top$  as well. A much more complicated problem arises for  $i = 1$ . Therefore, we leave the full characterization of the generalized valuation property for c-measures as an open problem.

#### 4. A relationship between graded equipollence and fuzzy c-measures

Recall that a fuzzy set  $B$  is made up of  $A$  through  $f : L \rightarrow L$  if  $B = f \circ A$  and we write  $B = f(A)$ . It is easy to see that  $A$  and  $f(A)$  have the same domain, but the fact that  $h \in \text{Bij}(A, B)$  does not, in general, imply that  $h \in \text{Bij}(f(A), f(B))$ .

Let us start our investigation with the theorem that demonstrates a relationship between the degrees in which there exist one-to-one correspondences between the fuzzy sets made up of the other ones through  $f$  and  $g$  and the similarity of the generalized cardinals obtained by a c-measure  $\mathfrak{C}_{g,f}$ . Practically, this theorem defines a lower bound derived by one-to-one correspondences between fuzzy sets to estimate the degree of similarity of the generalized cardinals that express the number of elements contained in these fuzzy sets.

**THEOREM 4.1** *Let  $\mathfrak{C}_{g,f}$  be a c-measure. Then,*

$$[g(A) \sim_h^\odot g(B)] \odot [f(A) \sim_h^\odot f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)] \quad (42)$$

*holds for any  $A, B \in \mathfrak{Ffin}$  such that  $|\text{Dom}(A)| = |\text{Dom}(B)| = m$  and  $h \in \text{Perm}(A, B)$ .*

**PROOF.** Let  $A, B \in \mathfrak{Ffin}$  such that  $|\text{Dom}(A)| = |\text{Dom}(B)| = m$  and  $h \in \text{Perm}(A, B)$ . If  $m = 0$ , then  $A = B = \emptyset$ , whence  $\mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(B)$ , and (42) is trivially satisfied.

Let  $m > 0$ . Set  $z = \text{Dom}(A) = \{x_1, \dots, x_m\}$  and  $h(z) = \text{Dom}(B) = \{y_1, \dots, y_m\}$ , where we suppose that  $y_k = h(x_k)$ . Let  $0 \leq i \leq m$  be arbitrary

<sup>21</sup>Consider  $(A \odot B)(x) = A(x) \odot B(x)$  and  $(A \bar{\odot} B)(x) = A(x) \bar{\odot} B(x)$  as a generalization of  $\cap$  and  $\cup$  and put  $A = B = \{\alpha/x\}$ .

and  $T$  be the set of all mappings  $t : \{1, \dots, m\} \rightarrow \{0, 1\}$  for which  $\sum_{k=1}^m t(k) = i$ . Furthermore, define

$$g_t^A = \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \text{ and } f_t^A = \bigodot_{l \in t^{-1}(1)} f(A(x_l)), \quad (43)$$

where  $t^{-1}(\alpha) = \{k \mid k \in \{1, \dots, m\} \ \& \ t(k) = \alpha\}$  and  $\alpha \in \{0, 1\}$ . Using the same arguments as for (19) (the proof of Theorem 3.9), we obtain

$$\mathfrak{C}_{g,f}(A)(i) = \bigvee_{t \in T} (g_t^A \odot f_t^A). \quad (44)$$

Analogous to (43), let us define  $g_t^B$  and  $f_t^B$ . Recall that  $h(x_k) = y_k$  for any  $k \in t^{-1}(0) \cup t^{-1}(1) = \{1, \dots, m\}$ . Then,

$$\begin{aligned} \mathfrak{C}_{g,f}(A)(i) \leftrightarrow \mathfrak{C}_{g,f}(B)(i) &= \left( \bigvee_{t \in T} (g_t^A \odot f_t^A) \right) \leftrightarrow \left( \bigvee_{t \in T} (g_t^B \odot f_t^B) \right) \geq \\ \bigwedge_{t \in T} ((g_t^A \odot f_t^A) \leftrightarrow (g_t^B \odot f_t^B)) &\geq \bigwedge_{t \in T} ((g_t^A \leftrightarrow g_t^B) \odot (f_t^A \leftrightarrow f_t^B)) \geq \\ \bigwedge_{t \in T} \left( \left( \bigodot_{k \in t^{-1}(0)} g(A(x_k)) \right) \leftrightarrow \left( \bigodot_{k \in t^{-1}(0)} g(B(y_k)) \right) \right) \odot \\ \left( \left( \bigodot_{l \in t^{-1}(1)} f(A(x_l)) \right) \leftrightarrow \left( \bigodot_{l \in t^{-1}(1)} f(B(y_l)) \right) \right) &\geq \\ \bigwedge_{t \in T} \left( \bigodot_{k \in t^{-1}(0)} (g(A(x_k)) \leftrightarrow g(B(y_k))) \right) \odot \\ \left( \bigodot_{l \in t^{-1}(1)} (f(A(x_l)) \leftrightarrow f(B(y_l))) \right) &\geq \\ \left( \bigodot_{k=1}^m (g(A(x_k)) \leftrightarrow g(B(y_k))) \right) \odot \left( \bigodot_{k=1}^m (f(A(x_k)) \leftrightarrow f(B(y_k))) \right) &= \\ \left( \bigodot_{x \in z} (g(A)(x) \leftrightarrow g(B)(h(x))) \right) \odot \left( \bigodot_{x \in z} (f(A)(x) \leftrightarrow f(B)(h(x))) \right) &= \\ [g(A) \sim_h^\circ g(B)] \odot [f(A) \sim_h^\circ f(B)]. \end{aligned}$$

If  $i > m$ , then  $\mathfrak{C}_{g,f}(A)(i) \leftrightarrow \mathfrak{C}_{g,f}(B) = f(\perp) \leftrightarrow f(\perp) = \top$ . Hence, and from the previous inequality for  $1 \leq i \leq m$ , we obtain the desired statement.  $\square$

A straightforward consequence of Proposition 3.2 is the following assertion that defines the lower bound for the similarity of fuzzy sets with different cardinalities of their universes.

**COROLLARY 4.2** *Let  $\mathfrak{C}_{g,f}$  be a  $c$ -measure and  $A, B \in \mathfrak{F}\text{fin}$ . Then,*

$$[g(C) \sim_h^\circ g(D)] \odot [f(C) \sim_h^\circ f(D)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)] \quad (45)$$

holds for any  $C \in \text{cls}(A)$  and  $D \in \text{cls}(B)$  such that  $|\text{Dom}(C)| = |\text{Dom}(D)| = m$  and  $h \in \text{Perm}(C, D)$ .

REMARK 4.1 Using the same arguments presented in the proof of Theorem 4.1, one can verify that the presumption on the same cardinality of domains of used fuzzy sets may be omitted in the previous statements if the fuzzy sets made up through  $f$  are defined in a more restrictive way by  $f^*(A)(x) = f(A(x))$  for any  $x \in \text{Supp}(A)$  and  $f^*(A)(x) = \perp$  for any  $x \in \text{Dom}(A) \setminus \text{Supp}(A)$ . In this case, it holds

$$[g^*(A) \sim_h^\circ g^*(B)] \odot [f^*(A) \sim_h^\circ f^*(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)]$$

for any  $A, B \in \mathfrak{F}\text{fin}$  and  $h \in \text{Bij}(A, B)$ . Although, this statement seems to be more satisfactory than the statement presented in Theorem 4.1, the restrictive definition of the fuzzy sets made up through  $f$  has some disadvantages upon further investigation.

A consequence of the previous theorem is the following corollary, where c-measures that are only determined by non-trivial homomorphisms are considered. Recall that  $\mathfrak{C}_f$  ( $\mathfrak{C}_g$ ) demonstrates that  $g$  ( $f$ ) is the trivial homomorphism of the reducts, i.e.,  $g(\alpha) = \top$  ( $f(\alpha) = \top$ ) and, hence,  $g_t^A = \top$  ( $f_t^A = \top$ ) in (19).

COROLLARY 4.3 Let  $\mathfrak{C}_{g,f}$  be a c-measure. Then,

- (i)  $[f(A) \sim^\circ f(B)] \leq [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)]$  and
- (ii)  $[g(A) \sim^\circ g(B)] \leq [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)]$

hold for any  $A, B \in \mathfrak{F}\text{fin}$  such that  $|\text{Dom}(A)| = |\text{Dom}(B)| = m$ .

PROOF. We shall prove (i), the statement (ii) may be verified by the same arguments.

Let  $g$  be the trivial  $\overline{\odot}_d$ -homomorphism, i.e.,  $g(\alpha) = \top$  for any  $\alpha \in L$ , then  $[g(A) \sim_h^\circ g(B)] = \top$  for any  $h \in \text{Perm}(A, B)$  and, using Theorem 4.1, the following inequality

$$[f(A) \sim_h^\circ f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)] \quad (46)$$

is true for any  $h \in \text{Perm}(A, B)$ . By (4) in Theorem 2.1, we obtain

$$[f(A) \sim^\circ f(B)] = \bigvee_{h \in \text{Perm}(A, B)} [f(A) \sim_h^\circ f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)],$$

and the proof is finished.  $\square$

REMARK 4.2 One may omit the presumption on the cardinality of domains in the previous corollary by replacing  $f(A)$  ( $g(A)$ ) with  $f^*$  ( $g^*(A)$ ).

DEFINITION 4.1 Let  $\mathbf{L}$  be an rdr-lattice and  $h : L \rightarrow L$  be a mapping. We shall say that  $h$  is  $(k, \odot)$ -compatible with  $\leftrightarrow$  if

$$\underbrace{(\alpha \leftrightarrow \beta) \odot \cdots \odot (\alpha \leftrightarrow \beta)}_{k\text{-times}} \leq h(\alpha) \leftrightarrow h(\beta) \quad (47)$$

for any  $\alpha, \beta \in L$ .

Obviously,  $h$  is  $(k, \wedge)$ -compatible with  $\leftrightarrow$ , if  $(\alpha \leftrightarrow \beta) \leq h(\alpha) \leftrightarrow h(\beta)$ . For example,  $h(\alpha) = (-\alpha)^k$  is  $(k, \otimes)$ -compatible with  $\leftrightarrow$ .

THEOREM 4.4 Let  $\mathfrak{C}_{g,f}$  be a c-measure such that  $f$  is a  $\odot$ -homomorphism being  $(k, \odot)$ -compatible with  $\leftrightarrow$  and  $g$  is a  $\overline{\odot}_d$ -homomorphisms being  $(l, \odot)$ -compatible with  $\leftrightarrow$ ,  $A, B \in \mathfrak{F}\mathbf{in}$  such that  $|\text{Dom}(A)| = |\text{Dom}(B)| = m$  and  $h \in \text{Perm}(A, B)$ . Then,

$$[A \sim_h^\odot B]^{k+l} \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)], \quad (48)$$

where  $\alpha^{k+l} = \alpha$  for  $\odot = \wedge$ . Specifically, if  $\mathfrak{C}_{g,f}$  is a c-measure with respect to  $\wedge$ , then

$$[A \sim^\wedge B] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)]. \quad (49)$$

PROOF. Let  $A, B \in \mathfrak{F}\mathbf{in}$  and  $h \in \text{Perm}(A, B)$ . By Theorem 4.1, we obtain

$$[g(A) \sim_h^\odot g(B)] \odot [f(A) \sim_h^\odot f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)]. \quad (50)$$

Since  $g$  is  $(l, \odot)$ -compatible with  $\leftrightarrow$  and  $\text{Dom}(A) = \text{Dom}(g(A)) = \text{Dom}(h)$ , then

$$\begin{aligned} [g(A) \sim_h^\odot g(B)] &= \bigodot_{x \in \text{Dom}(g(A))} (g(A)(x) \leftrightarrow g(B)(h(x))) \geq \\ \bigodot_{x \in \text{Dom}(A)} (A(x) \leftrightarrow B(h(x)))^l &= \left( \bigodot_{x \in \text{Dom}(A)} (A(x) \leftrightarrow B(h(x))) \right)^l = [A \sim_h^\odot B]^l. \end{aligned}$$

Analogously, one may prove that  $[A \sim_h^\odot B]^k \leq [f(A) \sim_h^\odot f(B)]$ . Therefore, we obtain

$$\begin{aligned} [A \sim_h^\odot B]^{k+l} &= [A \sim_h^\odot B]^l \odot [A \sim_h^\odot B]^k \leq \\ [g(A) \sim_h^\odot g(B)] \odot [f(A) \sim_h^\odot f(B)] &\leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)]. \end{aligned}$$

The second inequality is a straightforward consequence of the idempotence of  $\wedge$ .  $\square$

In [9], we showed that a class of all equipollent fuzzy sets, where two fuzzy sets are equipollent if their fuzzy substantial segments with respect to  $\odot$  (or  $\overline{\odot}$ ) are equipollent in the degree  $\top$ , can be represented by a non-increasing (non-decreasing) generalized cardinal. The following three theorems provide a representation using generalized cardinals for a given c-measure. Of course, there



is a problem in proving such a representation in full generality; nevertheless, the restrictions still give a wide variety of c-measures for which a corresponding equipollence can be introduced (cf., Theorems 4.3, 4.51 and 4.78 in [23]). Recall that a homomorphism  $h$  is said to be a *monomorphism* if  $h$  is a one-to-one mapping.

**THEOREM 4.5** *Let  $\mathbf{L}$  be linearly ordered and, for  $\odot = \otimes$ , satisfy the conditional cancellation law for  $\otimes$ . Let  $\mathfrak{C}_f$  be a c-measure such that  $f$  is a  $\odot$ -pomonorphism and  $A, B \in \mathfrak{F}\text{fin}$ . If  $s$  and  $t$  are substantial segments of  $A$  and  $B$  with respect to  $\odot$ , respectively, then  $[A^s \sim^\odot B^t] = \top$  if and only if  $\mathfrak{C}_f(A) = \mathfrak{C}_f(B)$ .*

**PROOF.** Set  $y = \text{Supp}(A)$  and  $y' = \text{Supp}(B)$ .

( $\Rightarrow$ ) Let us suppose that  $[A^s \sim^\odot B^t] = \top$ . Using Theorem 2.6, we obtain  $p_A^\odot(-, y) = p_B^\odot(-, y')$ . By Theorem 3.10, we have  $\mathfrak{C}_f(A)(i) = f(p_A^\odot(i, y)) = f(p_B^\odot(i, y')) = \mathfrak{C}_f(B)(i)$  for any  $i \in \mathbb{N}$ . Hence, we obtain  $\mathfrak{C}_f(A) = \mathfrak{C}_f(B)$ .

( $\Leftarrow$ ) Let us suppose that  $\mathfrak{C}_f(A) = \mathfrak{C}_f(B)$ , i.e.,  $f(p_A^\odot(i, y)) = f(p_B^\odot(i, y'))$  for any  $i \in \mathbb{N}$ . Since  $f$  is a one-to-one mapping, then  $p_A^\odot(i, y) = p_B^\odot(i, y')$ , and, by Theorem 2.6, we obtain  $[A^s \sim^\odot B^t] = \top$ .  $\square$

**THEOREM 4.6** *Let  $\mathbf{L}$  be linearly ordered and, for  $\odot = \oplus$ , satisfy the conditional cancellation law for  $\oplus$ . Let  $\mathfrak{C}_g$  be a c-measure such that  $g$  is a  $\overline{\odot}_d$ -pomonorphism and  $A, B \in \mathfrak{F}\text{fin}$ . If  $s$  and  $t$  are substantial segments of  $A$  and  $B$  with respect to  $\overline{\odot}$ , respectively, then  $[A^s \sim^{\overline{\odot}} B^t] = \top$  and  $|\text{Supp}(A)| = |\text{Supp}(B)|$  if and only if  $\mathfrak{C}_g(A) = \mathfrak{C}_g(B)$ .*

**PROOF.** Set  $y = \text{Supp}(A)$  and  $y' = \text{Supp}(B)$ .

( $\Rightarrow$ ) If  $[A^s \sim^{\overline{\odot}} B^t] = \top$  and  $|y| = |y'| = m$ , where  $s$  and  $t$  are substantial segments of  $A$  and  $B$  with respect to  $\overline{\odot}$ , respectively, then, by Theorem 2.10, we obtain  $p_A^{\overline{\odot}}(-, y) = p_B^{\overline{\odot}}(-, y')$ , and, from Theorem 3.10, we obtain  $\mathfrak{C}_g(A)(i) = g(p_A^{\overline{\odot}}(m \boxplus i, y)) = g(p_B^{\overline{\odot}}(m \boxplus i, y')) = \mathfrak{C}_g(B)(i)$  for any  $i \in \mathbb{N}$ .

( $\Leftarrow$ ) If  $\mathfrak{C}_g(A) = \mathfrak{C}_g(B)$ , then from Theorem 3.10 and the presumption on  $g$ , we obtain  $p_A^{\overline{\odot}}(m \boxplus i, y) = p_B^{\overline{\odot}}(m' \boxplus i, y')$ , where  $m = |y| = |\text{Supp}(A)|$  and  $m' = |y'| = |\text{Supp}(B)|$ . Let us show that  $m = m'$ . If  $m > m'$ , then  $p_A^{\overline{\odot}}(m \boxplus m', y) = p_B^{\overline{\odot}}(0, y') = \perp$ , which implies that  $p_A^{\overline{\odot}}(1, y) = \bigwedge_{x \in y} A(x) = \perp$  according to (iv) and (vi) of Proposition 2.8. However, this is a contradiction with the assumption about  $y = \text{Supp}(A)$ . Analogously, we obtain a contradiction for  $m < m'$ , and thus,  $m = m'$ . Hence, from (ii) of Proposition 2.8, we obtain  $p_A^{\overline{\odot}}(-, y) = p_B^{\overline{\odot}}(-, y')$ , and  $[A^s \sim^{\overline{\odot}} B^t] = \top$  is a consequence of Theorem 2.10.  $\square$

To state an analogous equivalence to the equivalencies provided in the previous two theorems for a c-measure determined by a non-trivial homomorphism between the reducts of the rdr-lattices, we have to assume that the rdr-lattices also have no zero divisor. Recall that an rdr-lattice  $\mathbf{L}$  has no zero divisor if  $\alpha \otimes \beta = \perp$  implies  $\alpha = \perp$  or  $\beta = \perp$ . Note that the same condition was assumed in Wygralak's work (see Theorem 4.78 in [23]), because the strict t-norms used in the work has no zero divisor.

**THEOREM 4.7** *Let  $\mathbf{L}$  be linearly ordered and, for  $\odot = \otimes$ , have no zero divisor and satisfy the conditional cancellation law for  $\otimes$ . Let  $\mathfrak{C}_{g,f}$  be a c-measure such that  $g$  is a  $\overline{\odot}_d$ -po-monomorphism,  $f$  is a  $\odot$ -po-monomorphism and  $A, B \in \mathfrak{F}\text{fin}$ . Then,  $[A \sim^{\odot} B] = \top$  if and only if  $\mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(B)$ .*

**PROOF.** ( $\Rightarrow$ ) This immediately follows from Corollary 4.2 and the fact that  $g$  and  $f$  are monomorphisms of the reducts (the existence  $h$  for which  $[C \sim_h^{\odot} D] = \top$ , where  $C \in \text{cls}(A)$  and  $D \in \text{cls}(B)$  are appropriate fuzzy sets, is self-evident); for an alternative verification, one can also use Theorem 4.8, which will be stated later.

( $\Leftarrow$ ) The proof of this portion is motivated by the proof of Theorem 4.78 in [23]. Let the presumptions of the theorem be satisfied. Let  $\mathfrak{C}_{g,f}(A) = \mathfrak{C}_{g,f}(B)$  and set  $y = \text{Supp}(A)$  and  $y' = \text{Supp}(B)$ . Before we start with the proof of this implication, we will establish several useful claims.

**CLAIM 1** *If  $\alpha \oplus \beta = \top$  then  $\alpha = \top$  or  $\beta = \top$ .*

We know that  $\mathbf{L}$  has no zero divisor and  $g$  is a monomorphism of reducts. Let us suppose that  $\alpha \oplus \beta = \top$  and  $\alpha \neq \top$  and  $\beta \neq \top$ . Since  $g(\alpha) \neq \perp$  and  $g(\beta) \neq \perp$  ( $g$  is a monomorphism), then  $\perp = g(\top) = g(\alpha \oplus \beta) = g(\alpha) \otimes g(\beta) > \perp$ , which is a contradiction.

**CLAIM 2**  *$\mathbf{L}$  satisfies the conditional cancellation law for  $\oplus$ .*

Since  $g$  is a monomorphism, then  $\alpha \oplus \beta = \alpha \oplus \gamma < \top$ , which implies  $g(\alpha) \otimes g(\beta) = g(\alpha) \otimes g(\gamma) > \perp$ . Since  $\mathbf{L}$  satisfies the conditional cancellation for  $\otimes$ , we obtain  $g(\beta) = g(\gamma)$ , and hence,  $\beta = \gamma$ .

**CLAIM 3**  $|y| = |y'|$ .

Set  $m = |y|$  and  $m' = |y'|$ . Let us suppose that  $m < m'$ . Then, we obtain (from Theorem 3.10 and  $|y| < m'$ )

$$\begin{aligned} \mathfrak{C}_{g,f}(A)(m') &= \perp = g(p_B^{\overline{\odot}}(0, y')) \odot f(p_B^{\odot}(m', y')) = \\ &f(p_B^{\odot}(m', y')) = \mathfrak{C}_{g,f}(B)(m'). \end{aligned}$$

If  $\odot = \wedge$ , then  $p_B^{\wedge}(m', y') = \bigwedge_{x \in y'} B(x) > \perp$ . Since  $\mathbf{L}$  has no zero divisor, then  $p_B^{\otimes}(m', y') = \bigotimes_{x \in y'} B(x) > \perp$  as well. Therefore, we obtain  $\mathfrak{C}_{g,f}(B)(m') = f(p_B^{\otimes}(m', y')) > \perp$ , which is a contradiction. Analogously, we obtain a contradiction for  $m > m'$ , and thus,  $m = m'$ .

Now let us suppose that  $\mathfrak{C}_{g,f}$  is a c-measure with respect to  $\otimes$ . We shall prove that  $p_A^{\wedge}(-, y) = p_B^{\wedge}(-, y')$ . Hence, by Theorem 2.7, we simply obtain  $[A \sim^{\wedge} B] = \top$ , which trivially implies  $[A \sim^{\otimes} B] = \top$ . By Claim 3, we can set  $m = |y| = |y'|$ . According to the definition of  $p_A^{\otimes}$ , the desired equality is trivially true for  $m = 0$  and  $m = 1$ . Let  $m \geq 2$  and set

$$I = \{i \mid i \in \mathbb{N} \ \& \ p_A^{\wedge}(i, y) \neq p_B^{\wedge}(i, y')\}. \quad (51)$$

If  $I = \emptyset$ , then clearly the desired equality is satisfied. Let us suppose that  $I \neq \emptyset$ . If  $I = \{i\}$ , then, from Remark 3.11, we obtain

$$\begin{aligned}\mathfrak{C}_{g,f}(A)(i) &= f(p_A^\wedge(0, y)) \otimes \cdots \otimes f(p_A^\wedge(i, y)) \otimes \\ g(p_A^\wedge(i+1, y)) \otimes \cdots \otimes g(p_A^\wedge(m, y)) &= f(p_B^\wedge(0, y')) \otimes \cdots \otimes f(p_B^\wedge(i, y')) \otimes \\ g(p_B^\wedge(i+1, y')) \otimes \cdots \otimes g(p_B^\wedge(m, y')) &= \mathfrak{C}_{g,f}(B)(i).\end{aligned}$$

Since  $y = \text{Supp}(A)$  and  $f$  and  $g$  are monomorphisms, we obtain

$$f(p_A^\wedge(k, y)) \wedge g(p_A^\wedge(l, y)) > \perp$$

for any  $k = 0, \dots, i$  and  $l = i+1, \dots, m$ , and hence,  $\mathfrak{C}_{g,f}(A)(i) > \perp$  ( $\mathbf{L}$  has no zero divisor). Since  $p_A^\wedge(k, y) = p_B^\wedge(k, y')$  for all  $k \neq i$ , then, by the conditional cancellation law for  $\otimes$ , we obtain  $f(p_A^\wedge(i, y)) = f(p_B^\wedge(i, y'))$ , which implies  $p_A^\wedge(i, y) = p_B^\wedge(i, y')$  and is a contradiction.

Let us suppose that  $I = \{i_1, \dots, i_k\}$ , where  $k \geq 2$  and  $i_1 < \dots < i_k$ . Consider the following two equalities:

$$\begin{aligned}\mathfrak{C}_{g,f}(A)(i_{k-1}) &= \mathfrak{C}_{g,f}(B)(i_{k-1}) \text{ and} \\ \mathfrak{C}_{g,f}(A)(i_k) &= \mathfrak{C}_{g,f}(B)(i_k).\end{aligned}$$

By the same arguments used above, one may check that  $\mathfrak{C}_{g,f}(A)(i_{k-1}) > \perp$  and  $\mathfrak{C}_{g,f}(A)(i_k) > \perp$ . Using Remark 3.11 and applying the conditional cancellation law for  $\otimes$ , we can derive the following equalities from the previous ones:

$$\begin{aligned}\alpha \otimes g(p_A^\wedge(i_k, y)) &= \beta \otimes g(p_B^\wedge(i_k, y')) \text{ and} & (52) \\ \alpha \otimes f(p_A^\wedge(i_k, y)) &= \beta \otimes f(p_B^\wedge(i_k, y')), & (53)\end{aligned}$$

where  $\alpha = f(p_A^\wedge(i_1, y)) \otimes \cdots \otimes f(p_A^\wedge(i_{k-1}, y))$  and  $\beta = f(p_B^\wedge(i_1, y')) \otimes \cdots \otimes f(p_B^\wedge(i_{k-1}, y'))$ . Then, we can analyze four cases to provide a contradiction with the presumption that  $I \neq \emptyset$ :

- Case 1: if  $\alpha \leq \beta$  and  $p_A^\wedge(i_k, y) > p_B^\wedge(i_k, y')$ , then  $g(p_A^\wedge(i_k, y)) < g(p_B^\wedge(i_k, y'))$  and  $\alpha \otimes g(p_A^\wedge(i_k, y)) < \beta \otimes g(p_B^\wedge(i_k, y'))$ , which is a contradiction with (52);<sup>22</sup>
- Case 2: if  $\alpha \leq \beta$  and  $p_A^\wedge(i_k, y) < p_B^\wedge(i_k, y')$ , then  $f(p_A^\wedge(i_k, y)) < f(p_B^\wedge(i_k, y'))$  and  $\alpha \otimes f(p_A^\wedge(i_k, y)) < \beta \otimes f(p_B^\wedge(i_k, y'))$ , which is a contradiction with (53);
- Case 3: if  $\alpha > \beta$  and  $p_A^\wedge(i_k, y) < p_B^\wedge(i_k, y')$ , then  $g(p_A^\wedge(i_k, y)) > g(p_B^\wedge(i_k, y'))$  and  $\alpha \otimes g(p_A^\wedge(i_k, y)) > \beta \otimes g(p_B^\wedge(i_k, y'))$ , which is a contradiction with (52);

---

<sup>22</sup>Note that if  $\alpha, \beta, \gamma, \delta \in L \setminus \{\perp\}$  such that  $\alpha < \beta$  and  $\gamma < \delta$  and  $\mathbf{L}$  has no zero divisor and satisfies the conditional cancellation for  $\otimes$ , then  $\alpha \otimes \gamma < \beta \otimes \delta$ . In fact, it holds that  $\alpha \otimes \gamma < \beta \otimes \gamma$  and  $\beta \otimes \gamma < \beta \otimes \delta$ ; otherwise, it is sufficient to use the conditional cancellation for  $\otimes$  for a contradiction. Therefore, we obtain the desired inequality.

Case 4: if  $\alpha > \beta$  and  $p_A^\wedge(i_k, y) > p_B^\wedge(i_k, y')$ , then  $f(p_A^\wedge(i_k, y)) > f(p_B^\wedge(i_k, y'))$  and  $\alpha \otimes f(p_A^\wedge(i_k, y)) > \beta \otimes f(p_B^\wedge(i_k, y'))$ , which is a contradiction with (53).

Thus,  $p_A^\wedge(i_k, y) = p_B^\wedge(i_k, y')$ , which implies that  $I$  has to be empty, and the proof of case  $\otimes$  is finished.

Now let us suppose that  $\mathfrak{C}_{g,f}$  is a c-measure with respect to  $\wedge$ . Again, it is sufficient to show that  $p_A^\wedge(-, y) = p_B^\wedge(-, y')$  for  $m \geq 2$ . Using Corollary 3.12 (see also Remark 3.11), we obtain

$$\begin{aligned}\mathfrak{C}_{g,f}(A)(0) &= g(p_A^\wedge(1, y)), \\ \mathfrak{C}_{g,f}(A)(m) &= f(p_A^\wedge(m, y)),\end{aligned}$$

and thus,  $p_A^\wedge(1, y) = p_B^\wedge(1, y')$  and  $p_A^\wedge(m, y) = p_B^\wedge(m, y')$ . Moreover, we trivially have  $p_A^\wedge(0, y) = p_B^\wedge(0, y')$ . Let  $I$  be the set of indexes defined by (51). If  $I = \emptyset$ , then the desired equality is satisfied. Let us suppose that  $I \neq \emptyset$ . If  $I = \{i\}$ , then  $1 < i < m$ , and we can write the two following equalities:

$$\begin{aligned}f(p_A^\wedge(i-1, y)) \wedge g(p_A^\wedge(i, y)) &= f(p_B^\wedge(i-1, y')) \wedge g(p_B^\wedge(i, y')) \text{ and} \\ f(p_A^\wedge(i, y)) \wedge g(p_A^\wedge(i+1, y)) &= f(p_B^\wedge(i, y')) \wedge g(p_B^\wedge(i+1, y')),\end{aligned}$$

where  $p_A^\wedge(i, y) \neq p_B^\wedge(i, y')$ ,  $p_A^\wedge(i-1, y) = p_B^\wedge(i-1, y')$  and  $p_A^\wedge(i+1, y) = p_B^\wedge(i+1, y')$ . Hence, we simply obtain

$$\begin{aligned}f(p_A^\wedge(i-1, y)) &= f(p_B^\wedge(i-1, y')) \leq g(p_A^\wedge(i, y)) \wedge g(p_B^\wedge(i, y')) \text{ and} \\ g(p_A^\wedge(i+1, y)) &= g(p_B^\wedge(i+1, y')) \leq f(p_A^\wedge(i, y)) \wedge f(p_B^\wedge(i, y')).\end{aligned}$$

If  $p_A^\wedge(i, y) < p_B^\wedge(i, y')$ , then  $g(p_A^\wedge(i, y)) \leq g(p_A^\wedge(i+1, y))$ , and one can simply derive

$$f(p_B^\wedge(i-1, y')) < g(p_A^\wedge(i, y)) \leq g(p_A^\wedge(i+1, y)) < f(p_B^\wedge(i, y')).$$

However, this is a contradiction with  $f(p_B^\wedge(i, y')) \leq f(p_B^\wedge(i-1, y'))$ . Analogously, we obtain a contradiction for  $p_A^\wedge(i, y) > p_B^\wedge(i, y')$  and  $I \neq \{i\}$ .

Let us assume that  $I = \{i_1, \dots, i_k\}$ , where  $k \geq 2$  and  $1 < i_1 < \dots < i_k < m$ . Consider the following two equalities:

$$\begin{aligned}\mathfrak{C}_{g,f}(A)(i_1 - 1) &= \mathfrak{C}_{g,f}(B)(i_1 - 1) \text{ and} \\ \mathfrak{C}_{g,f}(A)(i_k) &= \mathfrak{C}_{g,f}(B)(i_k).\end{aligned}$$

Both equalities can be rewritten as

$$\begin{aligned}f(p_A^\wedge(i_1 - 1, y)) \wedge g(p_A^\wedge(i_1, y)) &= f(p_B^\wedge(i_1 - 1, y')) \wedge g(p_B^\wedge(i_1, y')) \text{ and} \\ f(p_A^\wedge(i_k, y)) \wedge g(p_A^\wedge(i_k + 1, y)) &= f(p_B^\wedge(i_k, y')) \wedge g(p_B^\wedge(i_k + 1, y')).\end{aligned}$$

Since  $i_1$  is the least and  $i_k$  is the largest element of  $I$ , we obtain  $p_A^\wedge(i_1 - 1, y) = p_B^\wedge(i_1 - 1, y')$  and  $p_A^\wedge(i_k + 1, y) = p_B^\wedge(i_k + 1, y')$ . Then,

$$\begin{aligned}f(p_A^\wedge(i_1 - 1, y)) &= f(p_B^\wedge(i_1 - 1, y')) \leq g(p_A^\wedge(i_1, y)) \wedge g(p_B^\wedge(i_1, y')) \text{ and} \\ g(p_A^\wedge(i_k + 1, y)) &= g(p_B^\wedge(i_k + 1, y')) \leq f(p_A^\wedge(i_k, y)) \wedge f(p_B^\wedge(i_k, y')).\end{aligned}$$

If  $p_A^\wedge(i_1, y) < p_B^\wedge(i_1, y')$ , then  $g(p_B^\wedge(i_1, y')) < g(p_A^\wedge(i_1, y))$ . Moreover,  $p_A^\wedge(i_1, y) \geq p_A^\wedge(i_k + 1, y)$  by definition, and this implies  $g(p_A^\wedge(i_1, y)) \leq g(p_A^\wedge(i_k + 1, y))$ . Therefore, one can simply derive

$$f(p_B^\wedge(i_1 - 1, y')) < g(p_A^\wedge(i_1, y)) \leq g(p_A^\wedge(i_k + 1, y)) \leq f(p_B^\wedge(i_k, y')),$$

which is a contradiction with  $f(p_B^\wedge(i_k, y')) \leq f(p_B^\wedge(i_1 - 1, y'))$ . Analogously, one can construct a contradiction for  $p_A^\wedge(i_1, y) > p_B^\wedge(i_1, y')$ . Hence, the presumption  $I \neq \emptyset$  is false, and the proof of case  $\wedge$  is finished.  $\square$

REMARK 4.3 The previous theorems show a way to introduce, under some pre-suppositions concerning the rdr-lattices and homomorphisms  $g$  and  $f$  of reducts, a non-graded equipollence on the class of all finite fuzzy sets to a given type of a c-measure, which guarantees the relationship between a functional approach to the cardinality of finite fuzzy sets (using fuzzy substantial segments) and the approach based on generalized cardinals obtained by a c-measure. For example, let us consider a c-measure  $\mathfrak{C}_f$  with respect to  $\odot$ . One can state that finite fuzzy sets  $A$  and  $B$  are *equipollent* (denoted by  $A \sim B$ ) if  $[A^s \sim^\odot B^t] = \top$ , where  $A^s, B^t$  are fuzzy substantial segments with respect to  $\odot$ . In this case, it holds that  $A \sim B$  if and only if  $\mathfrak{C}_f(A) = \mathfrak{C}_f(B)$ . An interesting question is if the provided equipollence is save for the constructions with fuzzy sets proposed in [9], e.g.,

$$\text{if } A \sim B \text{ and } C \sim D, \text{ then } A \times C \sim B \times D.$$

To verify this implication one has to check that  $[A^s \sim^\odot B^t] = [C^u \sim^\odot D^v] = \top$  does imply  $[(A \times C)^x \sim^\wedge (B \times D)^y] = \top$ . Because the constructions of the equipollences seem to be equivalent to those proposed by Wygralak in [22, 23], we believe that the answer to the question is positive. Nevertheless, the proper verification is left for future research.

Until now all relationships between the graded equipollence and the similarity of finite fuzzy set c-measures have been in the form of an inequality. The following theorem shows that an equality can arise only in very special cases.

THEOREM 4.8 *Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{g,f}$  be a c-measure such that  $f$  is a  $\odot$ -po-homomorphism and  $g$  is a  $\overline{\odot}_A$ -po-homomorphism. Then,*

$$[g(A) \sim^\odot g(B)] \odot [f(A) \sim^\odot f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)] \quad (54)$$

for any  $A, B \in \mathfrak{Ffin}$  such that  $|\text{Dom}(A)| = |\text{Dom}(B)| = m$ . Specifically, if  $\mathfrak{C}_g$  and  $\mathfrak{C}_f$  are c-measures with respect to  $\odot = \wedge$ , then

$$(i) [g(A) \sim^\wedge g(B)] = [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)] \quad \text{and}$$

$$(ii) [f(A) \sim^\wedge f(B)] = [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)]$$

hold for any  $A, B \in \mathfrak{Ffin}$  such that  $|\text{Dom}(A)| = |\text{Dom}(B)| = m$ .

PROOF. From Corollary 3.11, we obtain  $\mathfrak{C}_{g,f}(A)(i) = \mathfrak{C}_g(A)(i) \odot \mathfrak{C}_f(A)(i)$  and similarly for  $\mathfrak{C}_{g,f}(B)(i)$ . Using Corollary 4.3, we obtain

$$\begin{aligned} [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)] &\geq \bigwedge_{i \in \mathbb{N}} (\mathfrak{C}_{g,f}(A)(i) \leftrightarrow \mathfrak{C}_{g,f}(B)(i)) = \\ &\bigwedge_{i \in \mathbb{N}} ((\mathfrak{C}_g(A)(i) \odot \mathfrak{C}_f(A)(i)) \leftrightarrow (\mathfrak{C}_g(B)(i) \odot \mathfrak{C}_f(B)(i))) \geq \\ &\bigwedge_{i \in \mathbb{N}} ((\mathfrak{C}_g(A)(i) \leftrightarrow \mathfrak{C}_g(B)(i)) \odot (\mathfrak{C}_f(A)(i) \leftrightarrow \mathfrak{C}_f(B)(i))) \geq \\ &\left( \bigwedge_{i \in \mathbb{N}} (\mathfrak{C}_g(A)(i) \leftrightarrow \mathfrak{C}_g(B)(i)) \right) \odot \left( \bigwedge_{i \in \mathbb{N}} \mathfrak{C}_f(A)(i) \leftrightarrow \mathfrak{C}_f(B)(i) \right) = \\ &[\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)] \odot [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)] \geq [g(A) \sim^\circ g(B)] \odot [f(A) \sim^\circ f(B)], \end{aligned}$$

and the first statement is proved.

Furthermore, we shall only prove (i). The second statement may be verified by analogy. Let  $\odot = \wedge$  and  $f$  be trivial. Set  $y = \text{Dom}(A)$  and  $y' = \text{Dom}(B)$ . From (21), we have

$$\begin{aligned} \mathfrak{C}_g(A)(i) &= p_{g(A)}^\wedge(i, y) \text{ and} \\ \mathfrak{C}_g(B)(i) &= p_{g(B)}^\wedge(i, y'). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} [\mathfrak{C}_g(A) \approx C_g(B)] &= \bigwedge_{i \in \mathbb{N}} (\mathfrak{C}_g(A)(i) \leftrightarrow \mathfrak{C}_g(B)(i)) = \\ &\bigwedge_{i \in \mathbb{N}} (p_{g(A)}^\wedge(i, y) \leftrightarrow p_{g(B)}^\wedge(i, y')) = [p_{g(A)}^\wedge(-, y) \approx p_{g(B)}^\wedge(-, y')]. \end{aligned}$$

Since  $\text{Dom}(g(A)) = y$  and  $\text{Dom}(g(B)) = y'$  with  $|y| = |y'|$ , then, as a result of Theorem 2.7, we obtain

$$[\mathfrak{C}_g(A) \approx C_g(B)] = [p_{g(A)}^\wedge(-, y) \approx p_{g(B)}^\wedge(-, y')] = [g(A) \sim^\wedge g(B)],$$

and the statement is proved.  $\square$

## 5. Conclusion

This paper is the second part of our contribution to the work on the graded approach to cardinality of finite fuzzy sets. Here, we proposed an axiomatic system for fuzzy cardinality (c-)measures by assigning to each finite fuzzy set a generalized cardinal expressing how many elements this fuzzy set has. We proved that each fuzzy c-measure may be represented by appropriate homomorphisms of reducts of an rdr-lattice. Under the presumptions on the linearity of the rdr-lattices and the ‘‘preservation’’ of ordering by the homomorphisms, we

proved that each fuzzy  $c$ -measures can be expressed as the product of a non-decreasing and a non-increasing fuzzy  $c$ -measure and this is analogous to the case of generalized FECounts proposed by Wygralak in [22, 23]. Because a non-increasing fuzzy  $c$ -measure may be interpreted as a generalized FGCount while a non-decreasing fuzzy  $c$ -measure may be interpreted as a generalized FLCount, we obtain

$$\text{FECCount} = \text{FGCount} \odot \text{FLCount}.$$

Furthermore, we discussed the valuation property that holds assuming the linearity of the rdr-lattice.

In the second part of this paper, we were interested in the question of whether there is a relation between fuzzy  $c$ -measures and the graded equipollence defined in [9]. Although, both approaches to cardinality of finite fuzzy sets use different tools, we proved several interesting interrelations between them. We proved that fuzzy  $c$ -measures usually give “less sensitive” results when comparing the power of the fuzzy sets with using the graded equipollence. This result means that equipollent fuzzy sets in a low degree can have very similar cardinality provided by a fuzzy  $c$ -measure. A full graded correspondence between both approaches was proved only for the case  $\odot = \wedge$  (see Corollary 4.8). Conversely, we proved that, for fuzzy  $c$ -measures describing a generalized FGCount, FLCount and FECCount, we can establish a corresponding non-graded equipollence. This crisp relation is defined by the graded equipollence, which is, in some cases, applied to the fuzzy substantial segments of fuzzy sets.

In our opinion, it is obviously naive to expect a rich fuzzy cardinality theory with a full correspondence between a functional approach based on a graded equipollence and an approach that uses generalized cardinals. Nevertheless, this imperfection should not stop the research and development in this field, because many questions about fuzzy cardinalities, which have been pointed out by Wygralak in his seminal papers and that have appeared in this contribution, are not still answered. In particular, one may recognize the absence of a definition stating when one fuzzy set has less cardinality than another fuzzy set. Additionally, the absence of a verification whether the generalized cardinals derived by fuzzy  $c$ -measures respect the constructions with fuzzy sets, which enable the introduction of adequate generalized cardinal arithmetic (see Remark 4.3) or the absence of fully working applications that use the results of the proposed theories. Among other questions, one can see a task that would fully explain why finite fuzzy sets are defined as fuzzy sets with finite supports, introduce the cardinal theory for countable fuzzy sets (i.e., fuzzy sets with a countable supports) by analogous tools as for finite fuzzy sets, or investigate some of the concepts, e.g., finiteness, singularity or transitivity of fuzzy sets,<sup>23</sup> in a graded conception. Finally, it should be noted that Novák in [14] and Běhounek and

---

<sup>23</sup>Singular fuzzy sets have been introduced by Dyczkowski and Wygralak in [8]. To the best of our knowledge, we have not seen a definition of a transitive or a well-order fuzzy set as a fuzzy counterpart to the very important concepts in set theory, which is a transitive and a well-ordered set.

Cintula in [2] recently suggested two general frameworks for fuzzy sets, which are, namely, fuzzy type theory and fuzzy class theory, respectively. Both theories provide appropriate tools for a possible construction of a fuzzy cardinal theory, which could be an interesting subject of further research.

## References

- [1] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic Publisher, New York, 2002.
- [2] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.
- [3] J. Casasnovas and J. Torrens. An axiomatic approach to fuzzy cardinalities of finite fuzzy sets. *Fuzzy Sets and Systems*, 133:193–209, 2003.
- [4] M. Delgado, D. Sánchez, M.J. Martín-Bautista, and M.A. Vila. A probabilistic definition of a nonconvex fuzzy cardinality. *Fuzzy Sets and Systems*, 126:177–190, 2002.
- [5] Glad Deschrijver and Pavol Král'. On the cardinalities of interval-valued fuzzy sets. *Fuzzy Sets and Systems*, 158(15):1728–1750, 2007.
- [6] D. Dubois. A new definition of the fuzzy cardinality of finite fuzzy sets preserving the classical additivity property. *BUSEFAL*, 8:65–67, 1981.
- [7] D. Dubois and H. Prade. Fuzzy cardinality and the modeling of imprecise quantification. *Fuzzy Sets and Systems*, 16:199–230, 1985.
- [8] K. Dyczkowski and M. Wygralak. On triangular norm-based generalized cardinals and singular fuzzy sets. *Fuzzy Sets and Systems*, 133(2):211–226, 2003.
- [9] M. Holčapek. A graded approach to cardinal theory of finite fuzzy sets, part I: Graded equipollence. *Submitted to Inf. Science*.
- [10] M. Holčapek. An axiomatic approach to fuzzy measures like set cardinality for finite fuzzy sets. Hüllermeier, E. (ed.) et al., Information processing and management of uncertainty in knowledge-based systems. Theory and methods. IPMU 2010, Dortmund, Germany. Proceedings. Part I. Berlin: Springer. Communications in Computer and Information Science 80, 505–514., 2010.
- [11] M. Holčapek. Graded equipollence and fuzzy c-measures of finite fuzzy sets. In *Proc. of 2011 IEEE International Conference on Fuzzy Systems*, pages 2375–2382, DnE Taiwan, 2011.
- [12] E.P. Klement, R. Mesiar, and E. Pap. *Triangular norms*, volume 8 of *Trends in Logic*. Kluwer Academic Publisher, Dordrecht, 2000.



- [13] P. Lubczonok. *Aspects of Fuzzy Spaces with Special Reference to Cardinality, Dimension and Order Homomorphisms*. PhD thesis, Rhodes University, 1991.
- [14] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149:235–273, 2005.
- [15] D. Ralescu. Cardinality, quantifiers and the aggregation of fuzzy criteria. *Fuzzy Sets and Systems*, 69:355–365, 1995.
- [16] A.P. Šostak. Fuzzy cardinals and cardinality of fuzzy sets. *Algebra and Discrete Mathematics*, pages 137–144, 1989. Latvian State University, Riga (in Russian).
- [17] Siegfried Weber. A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms. *Fuzzy Sets Syst.*, 11:115–134, 1983.
- [18] M. Wygalak. Fuzzy cardinals based on the generalized equality of fuzzy subsets. *Fuzzy Sets and Systems*, 18:143–158, 1986.
- [19] M. Wygalak. Generalized cardinal numbers and operations on them. *Fuzzy Sets and Systems*, 53(1):49–85, 1993.
- [20] M. Wygalak. Questions of cardinality of finite fuzzy sets. *Fuzzy Sets and Systems*, 102(2):185–210, 1999.
- [21] M. Wygalak. An axiomatic approach to scalar cardinalities of fuzzy sets. *Fuzzy Sets and Systems*, 110(2):175–179, 2000.
- [22] M. Wygalak. Fuzzy sets with triangular norms and their cardinality theory. *Fuzzy Sets and Systems*, 124(1):1–24, 2001.
- [23] M. Wygalak. *Cardinalities of Fuzzy Sets*. Kluwer Academic Publisher, Berlin, 2003. isbn: 3-5400-0337-1.
- [24] L.A. Zadeh. A computational approach to fuzzy quantifiers in natural languages. *Comp. Math. with Applications*, 9:149–184, 1983.
- [25] L.A. Zadeh. A theory of approximate reasoning. In J.E. Hayes, D. Michie, and L.I. Mikulich, editors, *Machine intelligence*, volume 9, pages 149–194. John Wiley and Sons, 1983.