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# Arithmetics of Extensional Fuzzy Numbers – Part I: Introduction

Michal Holčapek and Martin Štěpnička

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**Centre of Excellence IT4Innovations, Division of the University of Ostrava**
**Institute for Research and Applications of Fuzzy Modeling**
**30. dubna 22, 701 03 Ostrava 1, Czech Republic**

tel.: +420-59-7091401 fax: +420-59-6120478
e-mail: michal.holcapekosu.cz

# Arithmetics of Extensional Fuzzy Numbers – Part I: Introduction

Michal Holčapek
Centre of Excellence IT4Innovations
Division of the University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 70103 Ostrava
Czech Republic
Email: Michal.Holcapek@osu.cz

Martin Štěpnička
Centre of Excellence IT4Innovations
Division of the University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 70103 Ostrava
Czech Republic
Email: Martin.Stepnicka@osu.cz

***Abstract*—Up to our best knowledge, distinct so far existing arithmetics of fuzzy numbers, usually stemming from the Zadeh's extensional principle, do not preserve some of the important properties of the standard arithmetics of classical (real) numbers. Obviously, although we cannot expect that a generalization of standard arithmetic will preserve precisely all its properties however, at least the most important ones should be preserved.**

**We present a novel framework of arithmetics of extensional fuzzy numbers that preserves more or less all the important (algebraic) properties of the arithmetic of real numbers and thus, seems to be an important seed for further investigations on this topic. The suggested approach arithmetics of extensional fuzzy numbers is demonstrated on many examples and besides the algebraic properties, it is also shown that it carries some desirable practical properties.**

## I. Introduction

Since fuzzy numbers were designed in order to generalize standard real numbers, it is unquestionable that arithmetics of fuzzy numbers should generalize the standard one as well. The importance of fuzzy numbers, i.e. fuzzy sets representing vaguely defined real numbers, has been published many times [1], [2]. Developing working arithmetics of fuzzy numbers is obviously of the same of importance. However, despite a tremendous job has been done in this field [1], [2], [3], [4], up to our best knowledge, so far existing arithmetics of fuzzy numbers do not carry even some very important algebraic properties that are naturally valid for the arithmetic of real numbers.

By these very important algebraic properties we basically mean the following three properties

$$a + (-a) = 0, \tag{1}$$

$$a \cdot a^{-1} = 1, \tag{2}$$

$$(a + b) \cdot c = ac + bc, \tag{3}$$

i.e. existence of an identity element for both summation and multiplication and the distributivity law, that are naturally valid for any $a, b, c \in \mathbb{R}$.

As we know, none of the properties (1-3) generally holds for $a, b, c \in \mathcal{F}(\mathbb{R})$ where $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy sets on real numbers. And neither restrictions to certain subsets of $\mathcal{F}(\mathbb{R})$ such as R-L fuzzy numbers [5] or L-U fuzzy numbers [6] helps sufficiently in order to obtain a kind of valid variant of the three properties recalled above.

In our opinion the main problem is that fuzzy sets, although defined on the universe of real numbers and restricted by distinct technical conditions (e.g. upper semi-continuity of R-L fuzzy numbers), are still fuzzy sets. That is, they are generalizations of classical sets, not generalizations of real numbers that could be considered as vague representations of real numbers. If we apply the Zadeh's extensional principle to classical (crisp) sets we would not get a working arithmetic of classical set and thus, we may hardly expect that its generalization to fuzzy sets would again gain these lost properties. Let us recall that the fact that fuzzy numbers are rather fuzzified version of crisp sets or more precisely intervals, was noted by several authors, see e.g. [7], [8], [9].

We present a novel approach to arithmetics of fuzzy numbers that gathers several basic principles that are highly desirable such as:

- we restrict our considerations to fuzzy numbers that are clearly conceptually created from real numbers because arithmetic operations work on numbers but not on crisp sets;
- we do not stick to the Zadeh's extensional principle as the only and basic principle from which our approach should stem;
- we try to capture an intuitive nature of arithmetics and lower the required computational costs.

The goal is to present a "fully working" (in a broad sense) scheme for arithmetics of fuzzy numbers with its motivation, conceptual, technical and algebraic points of view that may be inherited for basically all purposes were fuzzy numbers are naturally used.

## II. Conceptual pilots

### A. Basic notions

As foreshadowed above, we do not find all fuzzy sets usually called fuzzy numbers to be appropriate candidates being dealt as summands or multipliers in the generalized arithmetic operations. Once more, we recall and stress that arithmetic operations do not generally work on crisp sets [9] and thus, if fuzzy sets naturally extend crisp sets, one may hardly assume obtaining a working arithmetic on fuzzy sets. Therefore, if we want to develop a working arithmetic of fuzzy numbers, the fuzzy sets that are called fuzzy numbers have to be natural and mainly conceptual generalizations of real numbers, i.e., appropriate models of vaguely understood real numbers.

If a fuzzy set $x \in \mathcal{F}(\mathbb{R})$ is supposed to model the meaning e.g. of an amount "about five", naturally it should be constructed using the real number 5 and a tolerance of values close to 5. The closeness may be easily modelled within the fuzzy framework by the use of the so called similarity relation [10] (also fuzzy equivalence or fuzzy equality [11]) that generalizes the crisp equality.

*Definition 1:* Let $*$ be a left-continuous t-norm. Binary fuzzy relation on the set of real numbers $S : \mathbb{R}^2 \to [0,1]$ is called $*$*-similarity* if the following holds

$$S(x,x) = 1,$$
$$S(x,y) = S(y,x),$$
$$S(x,y) * S(y,z) \leq S(x,z),$$

for all $x, y, z \in \mathbb{R}$. Furthermore, we say that $S$ is *shift-invariant* if for all $x, y, h \in \mathbb{R}$ it holds

$$S(x,y) = S(x+h, y+h).$$

*Remark 1:* Neither the above definition of a similarity relation nor the latter definitions do have to be restricted on the universe of real numbers but with respect to the goal of our paper, we may freely afford this restriction and avoid considering further more general universes of discourse.

Since the similarity relation generalizes classical equality, it may be very useful in definitions of further notions that again generalize the classical ones. For example, extensionality is a standard notion from the set theory that may be easily generalized for fuzzy sets with help of the use of the similarity relation [12], [13].

*Definition 2:* [14] Let $S$ be a $*$-similarity relation on $\mathbb{R}$. A fuzzy set $a \in \mathcal{F}(\mathbb{R})$ is said *extensional* w.r.t. $S$ if

$$a(x) * S(x,y) \leq a(y),$$

holds for any $x, y \in \mathbb{R}$.

Extensional fuzzy sets compose an important class of fuzzy sets that represent collections of objects having the same property if they are close to each other. Although not each fuzzy set is extensional, for each fuzzy set we may define its extensional hull that is an extensional fuzzy superset of the given fuzzy set.

*Definition 3:* Let $S$ be a $*$-similarity relation on $\mathbb{R}$ and $a$ be a fuzzy set on $\mathbb{R}$. Fuzzy set $\widehat{a} \in \mathcal{F}(\mathbb{R})$ is an *extensional hull* of $a$ if it is its least fuzzy superset that is extensional w.r.t. $S$.

*Theorem 1:* [10] Let $S$ be a $*$-similarity relation on $\mathbb{R}$ and $a$ be a fuzzy set on $\mathbb{R}$. The extensional hull of $a$ is a fuzzy set $\widehat{a} \in \mathcal{F}(\mathbb{R})$ that is given as follows:

$$\widehat{a}(x) = \bigvee_{y \in \mathbb{R}} a(y) * S(x,y). \tag{4}$$

Since crisp set is a special case of a fuzzy set, we may easily determine an extensional hull of a crisp set as well. If we furthermore consider a crisp number $x \in \mathbb{R}$, represented by a singleton $\tilde{x} \in \mathcal{F}(\mathbb{R})$, i.e., by a set that contains the only element $x$, we may construct an extensional hull of $x$ with respect to a given similarity relation $S$. This notion is usually called *fuzzy point* and has been successfully used in many investigation e.g. by F. Klawonn [11], [10], [15].

Indeed, fuzzy point is conceptually the genuine representation of the vaguely defined real number since it is composed as its extensional hull. In other words, it expresses a real number with its neighborhood, i.e. with its class of (fuzzy) equivalence given by the similarity relation $S$. Thus, we precisely conceptually meet the original goal of modelling expression such as "about five" by dealing with crisp number 5 and a tolerance of close values (the influence of "about") that is given by the chosen similarity relation $S$. But because a fuzzy point could be generally constructed from an arbitrary point in an arbitrary space, not necessarily from a real number, for the purpose of developing arithmetics of fuzzy numbers we will rather restrict our focus to the universe of discourse $\mathbb{R}$ and re-name it as *extensional fuzzy number*.

*Definition 4:* [15] Let $S$ be a $*$-similarity relation on $\mathbb{R}$, $x \in \mathbb{R}$ and $\tilde{x} \in \mathcal{F}(\mathbb{R})$ be a singleton. *Extensional fuzzy number (fuzzy point)* $x_S \in \mathcal{F}(\mathbb{R})$ is the extensional hull of $\tilde{x}$ with respect to $S$.

For the sake of simplicity and clarity in our notation, we will not distinguish between $x \in \mathbb{R}$ and a singleton $\tilde{x}$ that attains normality at $x$ and freely talk e.g. about extensional hulls of a real number $x$.

*Lemma 1:* Extensional hull of $x$ w.r.t. a $*$-similarity relation $S$ on $\mathbb{R}$ may be expressed as follows

$$x_S(y) = S(x,y)$$

for any $y \in \mathbb{R}$.

PROOF: By the direct use of (4). □

Despite Lemma 1 is trivial, it is very important because it shows that dealing with extensional fuzzy numbers is as easy as dealing with chosen similarity relations evaluated in the given real numbers.

### B. Elementary but working example

Let us restrict our focus on fuzzy numbers that are extensional w.r.t. a single fixed similarity relation $S$. We will get a very simple but yet working example that will be described below.

Consider a similarity relation $S$ on $\mathbb{R}$ and consider the set of all fuzzy numbers extensional w.r.t. $S$:

$$\mathcal{F}_S(\mathbb{R}) = \{x_S \mid x \in \mathbb{R}\}.$$

Let us define operations $+$ and $\cdot$ on $\mathcal{F}_S(\mathbb{R})$ as follows:

$$x_S + y_S = (x+y)_S,$$
$$x_S \cdot y_S = (xy)_S,$$

for any $x_S, y_S \in \mathcal{F}_S(\mathbb{R})$. Furthermore, both operations are obviously commutative. Further, let us define the identity elements for both operations as follows:

$$\mathbf{0} = 0_S = S(0, -),$$
$$\mathbf{1} = 1_S = S(1, -).$$

Indeed, one may easily check that

$$x_S + \mathbf{0} = x_S + 0_S = (x + 0)_S = x_S,$$
$$x_S \cdot \mathbf{1} = x_S \cdot 1_S = (x \cdot 1)_S = x_S$$

holds for arbitrary $x_S \in \mathcal{F}_S(\mathbb{R})$. Finally, let us define the inverse elements with respect to both operations as follows:

$$-(x_S) = (-x)_S, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}),$$
$$(x_S)^{-1} = (1/x)_S, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}) \setminus \{\mathbf{0}\}.$$

Again, one can easily check that

$$x_S + (-x_S) = \mathbf{0}, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}),$$
$$x_S \cdot (x_S)^{-1} = \mathbf{1}, \quad \forall\, x_S \in \mathcal{F}_S(\mathbb{R}) \setminus \{\mathbf{0}\}.$$

Please note, that the identity elements are not singletons which is the main difference to the usual arithmetics of fuzzy numbers based on the Zadeh's extensional principle. The calculus based on the extensional principle directly enforces that only singletons $\tilde{0}, \tilde{1}$ that attain normalities at 0 or 1, respectively, may serve as identity elements. However, this is precisely the problem of such arithmetics because it is not possible to determine inverse elements $-a, a^{-1}$ that would yield

$$a + (-a) = \tilde{0}, \quad \forall\, a \in \mathcal{F}(\mathbb{R}),$$
$$a \cdot (a)^{-1} = \tilde{1}, \quad \forall\, a \in \mathcal{F}(\mathbb{R}) \setminus \{\tilde{0}\}.$$

This is again very natural. If we deal with vaguely defined numbers, identity elements are elements of the set of vaguely defined numbers and thus, one may hardly expect these identity elements to be precise and not vague. So, we may conclude that in the case of extensional-based arithmetics of fuzzy numbers based on the extensional principle enforces that identity elements are singletons $\tilde{0}, \tilde{1}$ is "guilty" for the non-existence of inverse elements in this calculus.

Furthermore, one may easily check that associativity and both distributive laws:

$$(x_S + y_S) \cdot z_S = (x_S \cdot z_S) + (y_S \cdot z_S),$$
$$z_S \cdot (x_S + y_S) = (z_S \cdot x_S) + (z_S \cdot y_S)$$

hold. Therefore, the algebraic structure $(\mathcal{F}_S(\mathbb{R}), +, \cdot, \mathbf{0}, \mathbf{1})$ is a *field*, i.e. the same algebraic structure that is formed by $(\mathbb{R}, +, \cdot, 0, 1)$.

Obviously, there is a one-to-one correspondence between the structures for $\mathcal{F}_S(\mathbb{R})$ and for $\mathbb{R}$ and the example is too simplistic in order to capture the needs of the intended calculus of fuzzy numbers. However, this structure that naturally grown up from fuzzy numbers extensional with respect to a single fixed similarity relation serves as a motivation example that shows that:

- the arithmetic based on extensional fuzzy numbers is intuitive, easy to compute and very natural;
- the so far often met problem of the non-existence of inverse elements is only a secondary consequence of the crispness of identity elements;
- considering the arithmetical operations based on extensional fuzzy numbers enforces that the identity elements are also vaguely defined and thus, allows the existence of inverse elements;

which are unquestionably encouraging observations.

Let us finally point out that on one hand, computations according to the above suggested arithmetic of extensional fuzzy numbers are intuitive since they follow the idea of computing "about five" plus "about three" is something "about (five plus three)". This is in our opinion much closer to the human's style of computing with imprecise numbers than computing with $\alpha$-cuts or parameters of functions that express the imprecise numbers. On the other hand, in the suggested arithmetic the tolerance expressed by "about" is always the same, not only in both summands but also in the computed result and this may seem very unnatural in many cases. Let this serves as a motivation for the latter section.

*Remark 2:* In the above text we have mentioned that in the simple motivation example there is a problem of the only measure of tolerance which might be rather unnatural. However, this is meant in such a way that we deal with the only similarity relation that expresses the tolerance which does not necessarily mean that we tolerate the same imprecision or vagueness over the whole universe of reals. This would be truth only in case of a shit-invariant similarity $S$ but one may easily construct a similarity $S$ that is not shift-invariant. For example,

$$S(x, y) = e^{-\left(\frac{|x-y|}{|x|\cdot|y|}\right)}$$

is a $*$-similarity relation $\mathbb{R}$ where $*$ is the product t-norm but it is not a shift-invariant similarity. And such similarity allows to take into account the scale of compared values in order to obtain, e.g., $S(1, 2) \neq S(10001, 10002)$ which is very natural. Thus, even the simple example with a single similarity captures a sort of flexibility that is desirable.

## III. Arithmetics of extensional fuzzy numbers

The previous section was closed by a motivating example of the fully working arithmetic of fuzzy numbers that were extensional w.r.t. a single similarity relation that formed an field. However, the drawback of dealing with the only similarity relation, i.e., the only tolerance or vagueness measure, is obvious as well as the natural way out of the drawback - employing a whole system of similarity relations into the suggested arithmetic.

For this purpose, we need to define the transitive closure of similarity relations.

*Definition 5:* [16] Let $*$ be a left-continuous t-norm and let $S$ be a reflexive and symmetric binary fuzzy relation on $\mathbb{R}$. Then the binary fuzzy relation on $\mathbb{R}$ denoted by $\mathrm{cl}(S)$ is the *transitive closure* of $S$ if it is its least superset that is a $*$-similarity relation, i.e.,

$$\mathrm{cl}(S) = \bigcap \{T \mid T \text{ is } *\text{-similarity and } S \subseteq T\}$$

where $\bigcap$ stands for the standard intersection of fuzzy sets.

Obviously, if $S$ is a $*$-similarity relation then $\mathrm{cl}(S) = S$. In the other case the transitive closure may be determined as follows.

*Lemma 2:* [16] Let $*$ be a left-continuous t-norm and let $S$ be a reflexive and symmetric binary fuzzy relation on $\mathbb{R}$. Then

$$\mathrm{cl}(S) = \bigcup_{n \in \mathbb{N}} S^{(n)}$$

where $\bigcup$ stands for the standard union of fuzzy sets and

$$S^{(1)}(x,y) = S(x,y),$$
$$S^{(n)}(x,y) = \bigvee_{z \in \mathbb{R}} \left( S^{(1)}(x,z) * S^{(n-1)}(z,y) \right), \quad n \geq 2$$

for $x, y \in \mathbb{R}$

Finally, we may consider the transitive closure of the union of two given similarity relations:

$$\mathrm{cl}(S \cup T)$$

as a binary operation on similarities.

*Lemma 3:* Operation of the transitive closure of union of $*$-similarity relations is associative.

PROOF: By direct employment of all involved definitions □

Please note, that the intersection of $*$-similarity relations is again a $*$-similarity relation and thus, for a given set of $*$-similarity relations $\mathcal{S}$ we may put:

$$S^* = \bigcap \mathcal{S}.$$

For the rest of the paper, without explicit mentioning we will consider only such systems of similarity relations $\mathcal{S}$ for which the following two conditions hold:

- $S^* \subseteq \mathcal{S}$,
- $\mathrm{cl}(S \cup T) \in \mathcal{S}$ for arbitrary $S, T \in \mathcal{S}$.

The next natural step is to extend the motivating example of a functional arithmetic of fuzzy numbers that are extensional w.r.t a similarity relation to arithmetics of fuzzy numbers that are extensional w.r.t. a similarity from a given system of similarity relations $\mathcal{S}$. Such extension would allow us to deal with distinct "measures of vagueness or tolerance" form the given system.

Let us denote the set of all fuzzy numbers that are extensional w.r.t. a similarity from a given system of similarity relations $\mathcal{S}$ by $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$:

$$\mathcal{F}_{\mathcal{S}}(\mathbb{R}) = \{x_S \mid x \in \mathbb{R} \text{ and } S \in \mathcal{S}\}$$

and define operations $+$ and $\cdot$ on $\mathcal{F}_{\mathcal{S}}(\mathbb{R})$ as follows:

$$x_S + y_T = (x+y)_{\mathrm{cl}(S \cup T)},$$
$$x_S \cdot y_T = (x \cdot y)_{\mathrm{cl}(S \cup T)}$$

for $S, T \in \mathcal{S}$. The identity element may be defined as follows:

$$\mathbf{0} = 0_{S^*} = S^*(0,-),$$
$$\mathbf{1} = 1_{S^*} = S^*(1,-)$$

one can check that indeed

$$x_S + \mathbf{0} = x_S + 0_{S^*} = (x+0)_{\mathrm{cl}(S \cup S^*)} = x_S,$$
$$x_S \cdot \mathbf{1} = x_S \cdot 1_{S^*} = (x \cdot 1)_{\mathrm{cl}(S \cup S^*)} = x_S$$

and moreover, with help of Lemma 3, the obtained arithmetic will be associative and distributive, i.e., the following distributive laws:

$$(x_R + y_S) \cdot z_T = (x_R \cdot z_T) + (y_S \cdot z_T),$$
$$z_T \cdot (x_R + y_S) = (z_T \cdot x_R) + (z_T \cdot y_S)$$

will hold for any $R, S, T \in \mathcal{S}$.

However, if we approach the inverse elements similarly to the elementary motivation example:

$$-(x_S) = (-x)_S, \quad \forall\, x_S \in \mathcal{F}_{\mathcal{S}}(\mathbb{R}),$$
$$(x_S)^{-1} = (1/x)_S, \quad \forall\, x_S \in \mathcal{F}_{\mathcal{S}}(\mathbb{R}) \smallsetminus \{0_S \mid S \in \mathcal{S}\}$$

we will encounter the following problems:

$$x_S + (-x_S) = 0_S \neq 0_{S^*} = \mathbf{0},$$
$$x_S \cdot (x_S)^{-1} = 1_S \neq 1_{S^*} = \mathbf{1},$$

for all $S \in \mathcal{S}$ such that $S \neq S^*$. Thus, the inverse elements do not work as the classical inverse element and do not give us the identity elements $\mathbf{0}$ and $\mathbf{1}$, respectively. However, at least the obtained elements $0_S, 1_S$ are constructed with help of $x = 0$ and $x = 1$, respectively, and the only difference is the measure of vagueness given by the similarity relation $S$.

In other words, $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, \cdot, \mathbf{0}, \mathbf{1})$ is not a field but it is an algebraic structure that is very close to a field. The question, what kind of algebraic structure it is, is partially answered in the next section. One can see that considering a set of identity elements instead of a single identity for each operations is the most natural way out of this problem.

*Example 1:* Consider the following system of embedded shift-invariant $*$-similarities where $*$ is the Łukasiewicz t-norm [17], [18]:

$$\mathcal{S} = \{S_p \mid p \in [a,b] \text{ and } S_p(x,y) = (1 - p|x-y|) \vee 0\}.$$

Since $S_p \cup S_{p'} = S_p$ for $p \leq p'$ and since all elements of $\mathcal{S}$ are transitive, it also holds that

$$\mathrm{cl}(S_p \cup S_{p'}) = S_p, \text{ for } p \leq p'.$$

Therefore,

$$x_{S_p} + y_{S_{p'}} = (x+y)_{S_{p''}}, \qquad p'' = \min\{p, p'\},$$
$$x_{S_p} \cdot y_{S_{p'}} = (x \cdot y)_{S_{p''}}, \qquad p'' = \min\{p, p'\}.$$

Finally $S^* = \bigcap \mathcal{S} = S_b$ and thus

$$\mathbf{0} = 0_{S_b} = S_b(0,-),$$
$$\mathbf{1} = 1_{S_b} = S_b(1,-).$$

where $S_b(x,y) = (1 - b|x-y|) \vee 0$.

Example 1 describes the arithmetic of triangular shaped fuzzy numbers that are very common and favorite. It should be stressed that w.r.t. often used arithmetics of such fuzzy numbers that stem from the Zadeh's extensional principle, i.e., that are $\alpha$-cut based arithmetics, there are at least three unquestionable advantages:

- the obtained triangular fuzzy numbers do not get inevitably wide;
- the triangular shape is preserved even in the case of the multiplication or the division, which is not only intuitive but also highly desirable (the fuzzy number is still a vaguely defined number with some tolerance of the vagueness);
- the computational efforts are very low since computing the central point is computationally as cheap as in the case of the real numbers and the only possibly expensive step - the computation of the transitive closure of two involved similarities - is in this case as cheap as the comparison of $p$ and $p'$.

One may argue, that the last argument in favor of the suggested approach is not that significant since the standard approach based on the Zadeh's extensional principle is very efficient if it is not applied in the $\alpha$-cut based scheme but with the use of parameters of the triangular fuzzy numbers. This is true but only in the case of addition and this particular shape of fuzzy sets i.e. this particular $\mathcal{S}$. However, let us consider a different example that is much more complicated in the case of the standard Zadeh's extensional principle based approach (especially for the multiplication), yet still as computationally cheap as the one given in Example 1, if we use the suggested approach based on extensional fuzzy numbers.

*Example 2:* Consider the following system of embedded shift-invariant $*$-similarities where $*$ is the product t-norm [17], [18]:

$$\mathcal{S} = \{S_p \mid p \in [a,b] \text{ and } S_p(x,y) = e^{-p|x-y|}\}.$$

Again, since $S_p \cup S_{p'} = S_p$ for $p \le p'$ and since all elements of $\mathcal{S}$ are transitive, it also holds that

$$\mathrm{cl}(S_p \cup S_{p'}) = S_p, \text{ for } p \le p'$$

and thus, as in the previous example:

$$\begin{aligned} x_{S_p} + y_{S_{p'}} &= (x+y)_{S_{p''}}, & p'' &= \min\{p,p'\},\\ x_{S_p} \cdot y_{S_{p'}} &= (x \cdot y)_{S_{p''}}, & p'' &= \min\{p,p'\}. \end{aligned}$$

Finally $S^* = \bigcap \mathcal{S} = S_b$ and thus

$$\begin{aligned} \mathbf{0} &= 0_{S_b} = S_b(0,-),\\ \mathbf{1} &= 1_{S_b} = S_b(1,-). \end{aligned}$$

where $S_b(x,y) = e^{-b|x-y|}$.

Example 2 describes the arithmetic that deals with the Gaussian-shaped[1] fuzzy sets. Again, as in the case of the arithmetic described in Example 1, the computation is even for the multiplication (and hence division too) computationally as cheap as in the case of real numbers and the only possibly expensive step - the computation of the transitive closure of two involved similarities - is in this case as cheap as the comparison of $p$ and $p'$. Again, it can be stated that the resulting fuzzy sets do not get inevitably wide.

*Remark 3:* Note that since all $S_p \in \mathcal{S}$ from Examples 1 and 2 are shift-invariant, it holds that $S_p(1,2) = S_{p'}(10001, 10002)$. However, the flexibility is not lost at all, the flexibility is provided by the variety of similarity relations that are at disposal in the system $\mathcal{S}$ and thus, one can easily model a similarity of small values by an $S_p$ for some high $p$ and a similarity of high values by an $S_{p'}$ for some small $p'$.

## IV. ALGEBRAIC FRAMEWORK

### A. *MI-algebras*

In the above sections, we have introduced a novel approach to arithmetics of extensional fuzzy numbers that seems to be nearly working from an algebraic point of view. However, one big drawback seems to be present, that is that for $S \ne S^*$ the following holds:

$$x_S + (-x_S) = 0_S \ne \mathbf{0}.$$

Let us focus on the properties of such $0_S$. We may see that

$$\begin{gathered} 0_S = -0_S,\\ x_S + x_S = 0_S \text{ then } x_S = -x_S \text{ and thus } x_S = 0_S \end{gathered}$$

which are exactly the properties carried by $\mathbf{0}$. In other words, $0_S$ is a $\mathbf{0}$-like element. The same can be shown about $1_S$ w.r.t. the multiplication operation.

Therefore, we naturally introduce an algebraic structure that contains a set of identity elements. The introduced structure will be called MI-monoid where MI stands for *Many Identities*.

*Definition 6:* A triplet $(G, \circ, E)$ is said to be an *MI-monoid* if $E$ is a non-empty subset of $G$ and $\circ$ is a binary operation on $G$ such that for any $x, y, z \in G$ and $a, b \in E$ the following conditions are satisfied:

(M1) $x \circ (y \circ z) = (x \circ y) \circ z$,
(M2) $\exists e \in E, \forall x \in G : x \circ e = e \circ x = x$,
(M3) $a \circ b \in E$,
(M4) $x \circ x = a$ then $x \in E$,
(M5) $x \circ a = a \circ x$.

Elements from $E$ are called *pseudoidentities* and the element $e \in E$ satisfying (M2) is called *(strong) identity element*.

*Remark 4:* Let us note that it can be shown that there exists just one element $e \in E$ such that (M2) is fulfilled.

[1] We are fully aware of the fact that (due to their unlimited support) Gaussian-shaped fuzzy sets on the do not fit into most of the schemes of fuzzy numbers. However, we do not see any theoretical nor conceptual reason for that exclusion and we are convinced that this restriction is only of a technical nature and stems from the $\alpha$-cut based arithmetics. On the contrary, in our opinion such fuzzy sets perfectly fit into the scheme that considers computing with vaguely defined numbers.

One can check that $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}})$ where

$$E^0_{\mathcal{S}} = \{0_S \mid S \in \mathcal{S}\}$$

is an MI-monoid with the strong identity element $\mathbf{0}$ and similarly also $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), \cdot, E^1_{\mathcal{S}})$ where

$$E^1_{\mathcal{S}} = \{1_S \mid S \in \mathcal{S}\}$$

is an MI-monoid with the strong identity element $\mathbf{1}$.

*Definition 7:* An *MI-monoid* $(G, \circ, E)$ is said to be *good* if for any $x \in G$ and $a, b \in E$ the following holds:

(MG) $x \circ a = b$ implies $x \in E$.

It can be shown that both above mentioned examples of MI-monoids of extensional fuzzy numbers, i.e, $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}})$ and $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), \cdot, E^1_{\mathcal{S}})$ are good MI-monoids. However, is there a non-trivial example of an MI-monoid that is not good? The answer is positive and we provide readers with two very natural examples of such MI-monoids.

*Example 3:* Let $G$ be the set of all real intervals

$$G = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\},$$

$E^0$ denotes the set of all intervals containing 0

$$E^0 = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b \text{ and } 0 \in [a, b]\}$$

and let the operation $+$ on $G$ is defined as follows:

$$[a, b] + [c, d] = [a + c, b + d].$$

Then the strong identity element is $[0, 0]$ and $(G, +, E^0)$ is an abelian[2] MI-monoid that is not good because, e.g.,

$$[-5, 5] + [1, 2] = [-4, 7]$$

but $[1, 2] \notin E$.

Similarly, let $G$ be defined as above, $E^1$ denotes the set of all intervals containing 1 and let the operation $\cdot$ is defined as follows

$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)].$$

Then the strong identity element is $[1, 1]$ and $(G, \cdot, E^1)$ is an abelian MI-monoid that is not good because, e.g.,

$$[0.2, 2] \cdot [2, 4] = [0.4, 8]$$

but $[2, 4] \notin E$.

Of course, both examples of MI-monoids that are not good, can be enhanced by a re-definition of the sets of identity elements in such a way that they contain only intervals symmetrically distributed around 0 or 1, respectively. For further details related to algebraic properties of MI-monoids, MI-submonoids and their homomorphisms, we refer to the consecutive article [19].

MI-monoid is just a first step that allows us to deal with more identity-like elements but the next natural step is to introduce inverse elements and thus, a structure that naturally generalizes a group.

[2]We follow the standard terminology and commutative algebras are said to be abelian.

*Definition 8:* An abelian MI-monoid $(G, \circ, E)$ is said to be an *MI-pregroup* if for any $x \in G$ there exists $x^{-1} \in G$ that satisfies

(G1) $x \circ x^{-1} \in E$,
(G2) $(x^{-1})^{-1} = x$.
(G3) $e^{-1} = e$.
(G4) $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.

The element $x^{-1}$ is called an *inversion (inverse element)* of $x$.

Since we still aim at a structure appropriate for arithmetics of extensional fuzzy numbers, we may assume commutativity as an automatic property but it is clear that the commutativity is not necessary, see [19] for a more general approach to such algebraic structures.

We may define a set of *symmetric* elements of $G$, denoted by $S_G$, as follows

$$S_G = \{x \in G \mid x = x^{-1}\}.$$

*Lemma 4:* If $x \in G$ is symmetric then $x \in E$, i.e. $S_G \subseteq E$.

PROOF: Let $x \in S_G$ and let $b \in E$ be such that $x \circ x^{-1} = b$. Then $x \circ x^{-1} = x \circ x = b$ which is due to (M4) an element of $E$. □

Now, we may finally approach the definition of a group-like structure.

*Definition 9:* Let $(G, \circ, E)$ be an abelian MI-pregroup and $S_G$ be the set of symmetric elements. Then $G$ is said to be an *MI-group* if for any $x, y \in G$ and for any $a \in S_G$ the following *cancellation property* holds

(G5) $x \circ a = y \circ a$ implies $x = y$.

*Example 4:* The structure $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, E^0_{\mathcal{S}})$ with inverse elements defined as $x_S^{-1} = (-x)_S$ is a good abelian MI-group.

The structure $(\mathcal{F}_{\mathcal{S}}(\mathbb{R}) \setminus E^0_{\mathcal{S}}, \cdot, E^1_{\mathcal{S}})$ with inverse elements defined as $x_S^{-1} = (1/x)_S$ is a good abelian MI-group.

In both examples, all pseudoidentity elements are symmetric, that is $E = S_G$.

*Example 5:* Let us consider examples of structures with real intervals from Example 3. For the additive MI-monoid we may define inverse elements as $[a, b]^{-1} = [-b, -a]$ and we obtain an abelian MI-group that is not good.

For the multiplicative structure we first omit $E^0$ elements from $G$ in order to obtain reduced the structure $(G \setminus E^0, \cdot, E^1)$. Then we may define inverse elements as $[a, b]^{-1} = [1/b, 1/a]$ and we obtain an abelian MI-group that is not good.

Please, note, that in these examples, not all pseudoidentities are symmetric, that is $S_G \subset E$. Indeed, check e.g. $[-3, 2] \in E^0$ and its inversion in the additive structure $[-2, 3] \in E^0$ but $[-3, 2] \neq [-2, 3]$.

Again, for further details related algebraic properties of MI-groups, MI-subgroups and their homomorphisms, we refer to the consecutive article [19] and here we proceed forward towards a field structure.

*Definition 10:* An algebra $(G, +, \cdot, E^0, E^1)$ is said to be an *MI-field*

(F1) $(G, +, E^0)$ is an abelian MI-group
(F2) $(G \setminus E^0, \cdot, E^1 \setminus E^0)$ is an abelian MI-group

(F3) for any $x, y, z \in G$ the following

$$z \cdot (x + y) = zx + zy \quad \text{(distributive law)}$$

holds.

*Example 6:* For any system of similarity relations $\mathcal{S}$ the structure

$$(\mathcal{F}_{\mathcal{S}}(\mathbb{R}), +, \cdot, E^0_{\mathcal{S}}, E^1_{\mathcal{S}})$$

where operations $+, \cdot$ are defined as above, is an MI-field.

*B. Some reflections on computation over fuzzy quantities*

It has been a well-known fact [20] that if we consider a set of standard (R-L, L-U or most usual definitions) fuzzy numbers $\mathcal{F}(\mathbb{R})$ and operations $+, \cdot$ defined with help of the Zadeh's extensional principle, we get the following two monoids

$$(\mathcal{F}(\mathbb{R}), +, \tilde{0}) \text{ and } (\mathcal{F}(\mathbb{R}), \cdot, \tilde{1}).$$

But due to the singleton identity elements $\tilde{0}$ and $\tilde{1}$ we are not able to determine inverse elements and thus, we cannot proceed further towards group structures or even to a field structure. This shows the importance of introducing the MI-monoid, i.e., a structure that generalizes a monoid, although it is not a problem to get a monoid with the previous approaches based on the extensional principle.

Our work introduced in this paper and followed in [19] is mainly motivated by the lack of a field-like structure for the so far existing arithmetics of fuzzy numbers. Up to our best knowledge most of the articles focusing on arithmetics of fuzzy numbers were rather application, technically or computationally oriented than investigating algebraic properties. However, there are interesting and very valuable exceptions. In our opinion, we may highlight one of such exceptions that deserves a special focus among all other authors who also devoted their research to algebraic backgrounds of arithmetics of fuzzy numbers. It is the work of M. Mareš that was published in many articles in late 80s and early 90s and that is mostly collected in the book *"Computation over fuzzy quantities"* [20].

Obviously, although our approach and the above recalled approach to computation over fuzzy quantities stem from completely different roots and thus, lead to different results, there is a non-trivial overlap of both theories.

In [20] we may find out that the author also finds the non-existence of inverse elements as one of two crucial problems. The other one is - not surprisingly - the fact that the distributive law does not hold. And as in our case, also M. Mareš states:

"it would be absurd to expect that arithmetic manipulation with vague fuzzy numbers will result in a crisp number, either $\mathbf{0}$ or $\mathbf{1}$".

The way how the above mentioned idea is employed in [20] is little bit different than in our approach. The author first defines so called *$y$-symmetric fuzzy quantity* $a \in \mathcal{F}(\mathbb{R})$ as a fuzzy set for which the following holds:

$$a(y + x) = a(y - x), \quad x \in \mathbb{R} \tag{5}$$

and denotes the set of $y$-symmetric fuzzy quantities as $\mathbb{S}_y$. Set of symmetric fuzzy quantities is a union of all the $y$-symmetric ones over the whole real domain. Obviously, all the extensional fuzzy sets involved in our approach are symmetric as well, which comes directly from the definition of the similarity relation. The important point in the approach provided in [20] is that the so called *additive equivalence* is defined. Two fuzzy quantities $a, b \in \mathcal{F}(\mathbb{R})$ are said to be *additively equivalent*[3] (denoted by $a \sim_+ b$) iff there exist 0-symmetric quantities $s_1, s_2 \in \mathbb{S}_0$ such that

$$a + s_1 \sim_+ b + s_2. \tag{6}$$

In other words, although the approach based on the Zadeh's extensional principle (which is still kept in [20]) does enforce crisp identity elements, the author defines a sort of tolerance of given equalities up to some 0-symmetric fuzzy quantities. And in our approach, these 0-symmetric fuzzy quantities would be elements of $S_G$ and consequently, due to Lemma 4, they would be elements of $E^0_{\mathcal{S}}$, i.e., pseudoidentities.

The crucial difference consists in the fact that 0-symmetric elements from (6) may be arbitrarily wide. While in our approach, that does not involve the extensional principle and suggests a completely different calculus, the width (vagueness) of psudoidentity elements is given by the width (vagueness) of the original elements.

Note, that as our approach uses the similarity relation as a crucial conceptual object, the same may be said about the approach by M. Mareš. The difference lies in the fact that we use it for the conceptual construction of extensional fuzzy numbers, i.e., to a construction of objects on which we develop the arithmetic. In the case of [20], the author used the similarity relation as a generalization of the classical equality in order to weaken the algebraic properties. Particularly, it can be shown that $\sim_+$ is a $*$-similarity relation for $*$ being the Gödel t-norm. Thus, $a = b$ implies $a \sim_+ b$ and the monoid structure axioms are also valid in their weakened form

$$a + b \sim_+ b + a,$$
$$a + (b + c) \sim_+ (a + b) + c,$$
$$a + \mathbf{0} \sim_+ a$$

but furthermore, in this weakened form, also the other desirable properties hold:

$$a + s \sim_+ a, \quad s \in \mathbb{S}_0,$$
$$a + (-a) \sim_+ \mathbf{0}$$

and thus, a sort of group-like structure is obtained.

Although the whole calculus is still based on the extensional principle and thus, it is different to the one we suggested, the overlap of crucial ideas is obvious. No matter that our approach provides some unquestionable advantages such as: computational simplicity; limitation of undesirable widening of fuzzy numbers; preserving the pre-given shape of fuzzy numbers; a field-like (MI-field) structure, i.e., the distributivity

[3] Multiplicative equivalence is defined analogously.

law and the intuitive way of calculating with extensional fuzzy numbers, we are very grateful for the existence of the work by M. Mareš that was mainly gathered in [20] and then followed e.g. in [21] that motivated us.

This paper is a tribute dedicated to the memory of Milan Mareš – our colleague and teacher.

## V. Conclusion

We have recalled standard approaches to arithmetics of fuzzy numbers stemming from the Zadeh's extensional principle and we have recalled the drawbacks of such approaches. We have listed the crucial points for such drawback - the fact that fuzzy numbers do not conceptually stem from numbers but from sets and the fact that the Zadeh's extensional principle enforces that only the singletons $\tilde{0}, \tilde{1}$ may serve as the identity elements.

With these observations and motivated by the importance of arithmetics of vaguely defined numbers, we have introduced a novel approach that tries to mimic the arithmetic of real numbers from the very roots. It means that we have started from the conceptually clear definition of an extensional fuzzy number and introduced arithmetic operations on such objects that obeys the human's intuition. Further, we have shown that with this approach we get an algebraic structure that is isomorphic to the field of real numbers.

Motivated by this and by the natural requirement to generalize the structure for more similarity relations that model tolerance measures, we came up with a generalization of such arithmetic for whole (even uncountable) systems of similarity relations. We have demonstrated very promising algebraic properties that are very similar to the highly desirable properties of groups and fields with the only difference, the non-uniqueness of identity elements in such structures. Moreover, we have shown that if the system of similarities is appropriately chosen, e.g. as in case of the embedded systems of shift-invariant similarities, the suggested arithmetic provides us with unquestionable practical advantages including the computational efficiency.

Thus, we have introduced novel algebras: MI-monoids, MI-groups and finally MI-fields, that capture these property due to a set of (pseudo)identity elements. We have shown, that are approaches as well as other approaches to generalization of arithmetic operations on more complicated objects (extensional fuzzy numbers, real interval) fall into these newly define algebras that are very close to the standard algebras serving as algebraic backgrounds for the arithmetic of reals.

The detailed investigation of the properties of these algebraic structures is provided in the consecutive article [19].

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