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Type $\langle 1,1 \rangle$ fuzzy quantifiers determined by fuzzy measures defined on residuated lattices. Part II: Permutation and isomorphism invariances

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Type $\langle 1, 1\rangle$ fuzzy quantifiers determined by fuzzy measures defined on residuated lattices. Part II: Permutation and isomorphism invariances

Abstract

This paper is a continuation of the first part of our study on type $\langle 1, 1\rangle$ fuzzy quantifiers determined by fuzzy measures, in which the basic notions and examples were introduced. Here, we study the properties of permutation invariance and isomorphism invariance of fuzzy quantifiers determined by pairs of functionals $(\mathcal{S}, \varphi)$ and their special cases, e.g., fuzzy quantifiers defined by fuzzy measures and cardinal fuzzy measure spaces. Both properties belong among the basic semantic properties of generalized quantifiers.

Keywords: fuzzy quantifier, fuzzy logic, permutation invariance, fuzzy measure

1. Introduction

This paper is a continuation of our work on generalized quantifiers of type $\langle 1, 1\rangle$ determined by fuzzy measures [2]. The first part contained basic definitions and examples. We provided introductory facts about generalized quantifiers and fuzzy quantifiers, and we summarized basic notions of fuzzy measures and integrals from [5]. Then we introduced a notion of residuated lattice operations (rl-operations) which serve as a general means of combining arguments of our fuzzy quantifiers and show several properties of these operations. The last part of that article contained several models of well-known natural language quantifiers of type $\langle 1, 1\rangle$, e.g., “many”, “almost all”, “few”, etc. using the theory we developed.

Now we can study various semantic properties of generalized quantifiers. These properties are essential from the point of view of adequacy of our models with respect to natural language semantics. If our model of natural language quantifier (e.g., “many”) would not possess semantic properties of permutation and isomorphism invariance, conservativity, extension, etc., it would hardly be possible to consider this model to be adequate.

Semantic properties of fuzzy quantifiers were studied by Glöckner [6] and elaborated by Holčapek in [8]. In this part of our paper, we concentrate on the following two (closely related) properties: permutation invariance (PI), and isomorphism invariance (ISOM). These properties hold if quantizes are invariant with respect to permutations (bijective mappings) on the universe of discourse (PI) and with respect to bijections among different universes of discourse (ISOM).

Practically all topic neutral natural language quantifiers possess these properties, which are considered to be necessary conditions for their logicality [14]. These quantifiers are sensitive only to cardinalities of subsets of the universe of discourse, and not to the individual nature of elements of these subsets. Examples of PI and ISOM quantifiers of type $\langle 1, 1\rangle$ are “some”, “all”, “almost all”. “many”, ”most”, “few”, etc. Example of quantifiers which are not PI or ISOM are “John’s” (in sentence “John’s apples are green”), “most of Mary’s” (in “Most of Mary’s computers are Apples”), etc. We can see that these quantifiers refer to particular individuals.

There is one important difference between properties of PI and ISOM. PI is a *local* property — it is defined with respect to a universe of discourse. On the other hand, ISOM is a *global* property - it should hold for bijections among *arbitrary* universes. ISOM is a stronger property: if a quantifier is ISOM, it is PI as well. Opposite implication is not valid in general. Quantifiers, which are PI but not ISOM are in some sense “unnatural” — they act differently for various universes. For example, quantifier defined as “if 23 is in the universe of discourse, many, otherwise some” returns the same value as quantifier “many” for universes which contain the number 23, and the same value as quantifier “some” for other universes. This is PI (because any universe either contains 23 or not), but it is certainly not ISOM - this property fails for bijections between universes containing 23 and those which do not contain them. This is the reason why ISOM is considered more important from the point of view of natural language semantics - it excludes these unnatural examples of quantifiers. However, it is simpler to start our investigation with the PI property, and then use notions and methods used for PI for the investigation of ISOM.

We will built our theory of PI and ISOM fuzzy quantifiers determined by fuzzy measure spaces as follows: First we will study the most general case of fuzzy quantifiers determined by a pair of functionals $(\mathcal{S}, \varphi)$ (see [2], Section 4). Then we will proceed to so-called fuzzy quantifiers definable by fuzzy measures (the functional $\mathcal{S}$ which assigns a fuzzy measure space to a universe M and quantifier arguments A, B is independent of the second argument B). The majority of important natural language quantifiers belong to this class. Finally we will also study so-called cardinal fuzzy quantifiers determined by functional independent of both A and B. We will show the relationship between properties PI and ISOM of our fuzzy quantifiers and properties of functional $\mathcal{S}$ (so-called pi-closedness and iso-closedness of $\mathcal{S}$). In simplified terms, we will be able to show that if $\mathcal{S}$ is pi-(iso-)closed, then the corresponding fuzzy quantifier is PI (ISOM). The opposite implication is not valid in general, but it can be shown that if a fuzzy quantifier Q is PI (ISOM), then there exists a fuzzy quantifier Q' which is based on pi-(iso-)closed functional $\mathcal{S}'$ and which coincides with the original fuzzy quantifier Q.

This paper is structured as follows. In Section 2 we provide necessary preliminaries on fuzzy measures and integrals. We concentrate on isomorphisms of fuzzy measure spaces and on cardinal fuzzy measure spaces, which are essential for the investigation of PI and ISOM properties. Examples show that there exist interesting relationships to number theoretical notions of asymptotical densities. In Section 3 we investigate the PI property first. Let us notice that we work with *global* quantifiers. It means that they should be defined on an arbitrary universe of discourse. In our case, we need to define a functional which assigns a fuzzy measure space to any universe. We show how the PI property is related to the corresponding properties (called *weak pi-closedness* and *pi-closedness*) of these functionals. Then we advance to the study of ISOM property and show that in this case, we need to take into account another functional which assigns the rl-operation to any universe. For ISOM it is necessary that this functional should assign the same (in some sense) operation to an arbitrary universe. Finally, Section 4 contains conclusions.

2. Preliminaries

In this paper, we suppose that the structure of truth values is a complete residuated lattice. For the definition, we refer to the first part of our study on fuzzy quantifiers of type $\langle 1, 1\rangle$ [2] (Section 2.1) or to [1, 13, 16]. A fuzzy set is a mapping $A : M \to L$, where M is a (possibly empty) universe of discourse and L is the support of a complete residuated lattice $\mathbf{L}$. For definitions, notation and properties of fuzzy sets, see Section 2.2 of [2].

2.1. Fuzzy measure spaces and their isomorphisms

In this subsection, we recall basic notions of fuzzy measure theory that have been proposed and investigated in [5] and that are used in this paper. We are primarily interested in isomorphisms between fuzzy measure spaces that are extremely important in the investigation of the permutation and isomorphism invariance of fuzzy quantifiers, which will be studied later.

Definition 2.1. Let A be a non-empty fuzzy set on M. A subset $\mathcal{F}$ of $\mathcal{F}(A)$ is an *algebra of fuzzy sets on* A if the following conditions are satisfied

(i) $1_\emptyset, A \in \mathcal{F}$,

(ii) if $X \in \mathcal{F}$, then $A \setminus X \in \mathcal{F}$,

(iii) if $X, Y \in \mathcal{F}$, then $X \cup Y \in \mathcal{F}$.

A pair $(A, \mathcal{F})$ is called a *fuzzy measurable space* (on A) if $\mathcal{F}$ is an algebra of fuzzy sets on A.

Definition 2.2. Let $(A, \mathcal{F})$ be a fuzzy measurable space. A mapping $\mu : \mathcal{F} \to L$ is called a *fuzzy measure* on $(A, \mathcal{F})$ if

(i) $\mu(1_\emptyset) = \bot$ and $\mu(A) = \top$,

(ii) if $B, C \in \mathcal{M}$ such that $B \subseteq C$, then $\mu(B) \leq \mu(C)$.

A triplet $(A, \mathcal{F}, \mu)$ is called a *fuzzy measure space* if $(A, \mathcal{F})$ is a fuzzy measurable space and μ is a fuzzy measure on $(A, \mathcal{F})$. We denote by $\mathbf{Fms}(M)$ the class of all fuzzy measure spaces defined on a non-empty universe M.

In our investigation of semantic properties of fuzzy quantifiers defined by fuzzy measures and integrals, we need to construct isomorphisms between fuzzy measurable and fuzzy measure spaces.

Definition 2.3. Let $(A, \mathcal{F})$ and $(B, \mathcal{G})$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{F} \to \mathcal{G}$ is an *isomorphism between* $(A, \mathcal{F})$ *and* $(B, \mathcal{G})$ if

(i) g is a bijective mapping with $g(1_\emptyset) = 1_\emptyset$,

(ii) $g(X \cup Y) = g(X) \cup g(Y)$ and $g(A \setminus X) = B \setminus g(X)$ hold for any $X, Y \in \mathcal{F}$,

(iii) there exists a bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ with $X(m) = g(X)(f(m))$ for any $X \in \mathcal{F}$ and $m \in \mathrm{Dom}(A)$.

Let $f : X \to Y$ be a mapping and $Z \subseteq X$. We denote by $f \upharpoonright Z$ the restriction of f to Z. The following theorem shows that each isomorphism of fuzzy measurable spaces is derived from a bijective mapping by the Zadeh's extension and its restriction to the algebra of fuzzy sets.

Theorem 2.1. *Let* $(A, \mathcal{F})$*,* $(B, \mathcal{G})$ *be fuzzy measurable spaces and* $g : \mathcal{F} \to \mathcal{G}$ *be a surjective mapping. Then,* g *is an isomorphism between* $(A, \mathcal{F})$ *and* $(B, \mathcal{G})$ *if and only if there exists a bijective mapping* $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ *such that* $g = f^{\to} \upharpoonright \mathcal{F}$.

Proof. See [5]. □

Definition 2.4. Let $(A, \mathcal{F}, \mu)$ and $(B, \mathcal{G}, \mu')$ be fuzzy measure spaces. We say that a mapping $g : \mathcal{F} \to \mathcal{G}$ is an *isomorphism between* $(A, \mathcal{F}, \mu)$ *and* $(B, \mathcal{G}, \mu')$ if

(i) g is an isomorphism between $(A, \mathcal{F})$ and $(B, \mathcal{G})$,

(ii) $\mu(X) = \mu'(g(X))$ for any $X \in \mathcal{F}$.

If g is an isomorphism between fuzzy measure spaces $\mathbf{A} = (A, \mathcal{F}, \mu)$ and $\mathbf{B} = (B, \mathcal{G}, \mu')$, then we will write $g(A, \mathcal{F}, \mu) = (B, \mathcal{G}, \mu')$ or shortly $g(\mathbf{A}) = \mathbf{B}$. If an isomorphism g between $\mathbf{A}$ and $\mathbf{B}$ is determined by a bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ (see Theorem 2.1), then we will write $f^{\to}(\mathbf{A}) = \mathbf{B}$.[1]

Definition 2.5. Let $\mathbf{A} = (A, \mathcal{F}, \mu)$ be a fuzzy measure space and f be a bijection from $M = \mathrm{Dom}(A)$ to a set M'. Then, a fuzzy measure space *generated by* $\mathbf{A}$ *and* f is the measure space $\mathbf{A}_f = (f^{\to}(A), \mathcal{F}_f, \mu_f)$, where $\mathrm{Dom}(f^{\to}(A)) = M'$, $\mathcal{F}_f = \{f^{\to}(X) \mid X \in \mathcal{F}\}$ and $\mu_f(X) = \mu((f^{-1})^{\to}(X))$ for all $X \in \mathcal{F}_f$.

[1] Note that $f^{\to}(\mathbf{A}) = \mathbf{B}$ is not a precise expression with respect to the previous expression by g, since $f^{\to} : \mathcal{F}(M) \to \mathcal{G}(M)$. Nevertheless, the precise expression $(f^{\to} \upharpoonright \mathcal{F})(\mathbf{A}) = \mathbf{B}$ can lead, in our opinion, to unclear formulations.

It is easy to see that $\mathbf{A}_f$ is a fuzzy measure space, and it holds that $f^{\rightarrow}(\mathbf{A}) = \mathbf{A}_f$. We will often refer to $\mathbf{A}_f$ by $f^{\rightarrow}(\mathbf{A})$. We say that a system $\mathcal{A}$ of fuzzy measure spaces from $\mathbf{Fms}(M)$ is *closed under isomorphisms in* $\mathbf{Fms}(M)$ if the following holds: if $\mathbf{A} \in \mathcal{A}$ and $\mathbf{B} \in \mathbf{Fms}(M)$ are isomorphic, then $\mathbf{B} \in \mathcal{A}$. In the following text, we will, for the sake of simplicity, omit the term "under isomorphisms" in "closed under isomorphisms" and say only "closed system of fuzzy measure spaces in $\mathbf{Fms}(M)$". Note that there are closed systems of fuzzy measure spaces containing non-isomorphic fuzzy measure spaces. If a system $\mathcal{A}$ of mutually isomorphic fuzzy measure spaces in $\mathbf{Fms}(M)$ is closed, then we say that $\mathcal{A}$ is a *closed system of mutually isomorphic fuzzy measure spaces in* $\mathbf{Fms}(M)$. Obviously, each closed system is a union of closed systems of mutually isomorphic fuzzy measure spaces.

In the following, let $\mathrm{Perm}(M)$ denote the set of all permutation on a set M.

Lemma 2.2. *A system $\mathcal{A}$ of fuzzy measure spaces in $\mathbf{Fms}(M)$ is closed if and only if $f^{\rightarrow}(\mathbf{A}) \in \mathcal{A}$ for any $\mathbf{A} \in \mathcal{A}$ and any permutation f on M.*

PROOF. See [5]. □

In [5], we defined the concept of a cardinal fuzzy measure space. The denotation "cardinal" means that fuzzy measures from these fuzzy measure spaces are invariant under the same cardinality of fuzzy sets. Let A be a non-empty fuzzy set. Then, we can say that two fuzzy sets $X, Y \in \mathcal{F}(A)$ have the same cardinality if there exists a permutation f on $\mathrm{Dom}(A)$ such that $f^{\rightarrow}(X) = Y$. A cardinal fuzzy measure space is defined as follows.

Definition 2.6. We say that $(A, \mathcal{F}, \mu)$ is a *cardinal* fuzzy measure space if

(i) if $X \in \mathcal{F}$, then $f^{\rightarrow}(X) \in \mathcal{F}$,

(ii) $\mu(X) = \mu(f^{\rightarrow}(X))$

hold for any $X \in \mathcal{F}$ and for any permutation f on $\mathrm{Dom}(A)$.

The following three examples present (finite as well as infinite) cardinal and non-cardinal fuzzy measure spaces. Notice that an isomorphism of fuzzy measurable spaces do not guarantee an isomorphism of fuzzy measure spaces. The invariance of a fuzzy measure under the isomorphism has to be verified, too (see Definition 2.4). All examples are constructed over a complete residuated lattice with the support $[0, 1]$.

Example 2.1. Let M be a finite set and $A \in \mathcal{F}(M)$ be a non-empty fuzzy set. Consider $\mathbf{A}^r = (A, \mathcal{F}(A), \mu^r_A)$, where $\mu^r_A : \mathcal{F}(A) \rightarrow [0, 1]$ is defined by

$$\mu^r_A(X) = \frac{\sum_{m \in M} X(m)}{\sum_{m \in M} A(m)} \tag{1}$$

(cf. Example 2.3 in [2]). One can simply check that $\mathbf{A}^r$ is a cardinal fuzzy measure space if A is a constant fuzzy set, i.e. $A(m) = c$ for any $m \in M$ (cf. Lemma 3.8. in [5]).

Example 2.2. Let M be an infinite universe, $\mathcal{F} \subseteq \mathcal{F}(M)$ be an algebra of fuzzy sets on M and

$$\mu(X) = \begin{cases} 1, & \text{if there exists an infinite subset } Y \text{ of } M \text{ such that } 1_Y \subseteq X; \\ 0, & \text{otherwise.} \end{cases}$$

One can simply check that $(1_M, \mathcal{F}, \mu)$ is a cardinal fuzzy measure space.

Example 2.3. Let N be the set of natural numbers and define μ in a much more interesting way than in Example 2.1 (cf. Example 3.5 in [3]). Let $F : (\mathcal{F}(N) \times N) \times N \to \mathcal{F}(N)$ be given by

$$F(X, n)(m) = \begin{cases} X(m), & \text{if } m \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Let A be a non-empty fuzzy set on N and $\mathcal{F}_A \subseteq \mathcal{F}(A)$ be an algebra of fuzzy sets on A. Put $n_A = \min(\mathrm{Supp}(A))$ and define a fuzzy measure $\mu_{A,n} : \mathcal{F}_A \to [0, 1]$ with respect to n for any $n \geq n_A$ by

$$\mu_{A,n}(X) = \frac{\sum_{m \in \mathrm{Supp}(X)} F(X, n)(m)}{\sum_{m \in \mathrm{Supp}(A)} F(A, n)(m)}. \tag{3}$$

For $n < n_A$, define $\mu_{A,n}(X) = 0$ for any $X \neq A$ and $\mu_{A,n}(A) = 1$. It is easy to see that $(A, \mathcal{F}_A, \mu_{A,n})$ is a fuzzy measure space for any $n \in N$.

Let $A = 1_N$ and $\mathcal{F}_{1_N} = \mathcal{F}(1_N)$. Then, $(1_N, \mathcal{F}_{1_N}, \mu_{1_N,n})$ is not cardinal for any $n \in N$. In fact, let $n \in N$ and consider a permutation f on N such that $f(m) = m$ for any $m < n$ and $f(n) > n$. If $n = 0$, then $f(0) > 0$ and

$$\mu_{1_N,0}(\{1/0\}) = 1 > 0 = \mu_{1_N,0}(\{1/f(0)\}).$$

If $n > 0$, put

$$X = \{1/0, \ldots, 1/n\}.$$

Clearly, $f^{\to}(X) = \{1/0, \ldots, 1/n-1, 1/f(n)\}$ and

$$\mu_{1_N,n}(X) = 1 > 1 - \frac{1}{n} = \mu_{1_N,n}(f^{\to}(X)),$$

whence $(1_N, \mathcal{F}_{1_N}, \mu_{1_N,n})$ is not cardinal. Let us define

$$\begin{aligned} \underline{\mu_A}(X) &= \liminf_{n\to\infty} \mu_{A,n}(X), \\ \overline{\mu_A}(X) &= \limsup_{n\to\infty} \mu_{A,n}(X). \end{aligned} \tag{4}$$

Obviously, if A is a non-empty fuzzy set with a finite support, then $\mu_{A,n}(X) = \mu_{A,m}(X)$ for any $m \geq n$, where $n = \max(\mathrm{Supp}(A))$. Hence, $\underline{\mu_A}$ and $\overline{\mu_A}$ are defined for any non-empty fuzzy set A.[2] Note that $\underline{\mu_A}$ and $\overline{\mu_A}$ (for $A = 1_N$

[2]One can see that $\underline{\mu_A} = \overline{\mu_A}$ for A with a finite support and both fuzzy measures generalize the fuzzy measure μ^r_A from Example 2.1 with finite M, $M \subseteq N$.

and $\mathcal{F}_{1_N} = \mathcal{F}(1_N)$) are examples of lower and upper weighted asymptotic densities, respectively, well known in the number theory (see [11] and the references therein). Again, neither $(1_N, \mathcal{F}, \underline{\mu_{1_N}})$ nor $(1_N, \mathcal{F}, \overline{\mu_{1_N}})$ are cardinal fuzzy measure spaces. In fact, it is well known that the set of even numbers has the measure ($\underline{\mu_{1_N}}$ or $\overline{\mu_{1_N}}$) equal to $\frac{1}{2}$ and the set of prime numbers equal to 0. Now, it is sufficient to consider a permutation on M such that the set of even numbers is transformed to the set of prime numbers.

Example 2.4. Here, we will propose an extension of the fuzzy measures $\mu_{A,n}$, $\underline{\mu_A}$ and $\overline{\mu_A}$ provided for fuzzy sets on N to fuzzy sets on an arbitrary countable universe M. Let M be an arbitrary countable set (finite or denumerable) and $h : \mathcal{F}(M) \to \mathcal{F}(N)$ be a mapping such that $h(A) \subseteq h(B)$, whenever $A \subseteq B$, $h(1_\emptyset) = 1_\emptyset$. Then, for any non-empty fuzzy set $A \in \mathcal{F}(M)$ such that $h(A) \neq 1_\emptyset$ and any algebra $\mathcal{F}_A \subseteq \mathcal{F}(A)$, we can define an h-fuzzy measure $\mu^h_{A,n} : \mathcal{F}_A \to [0, 1]$ with respect to $n \in N$ by

$$\mu^h_{A,n}(X) = \mu_{h(A),n}(h(X)), \tag{5}$$

where $\mu_{h(A),n}$ is the fuzzy measure defined in the previous example. The extension of the h-fuzzy measure for n going to the infinity may be done by

$$\underline{\mu^h_A}(X) = \liminf_{n\to\infty} \mu^h_{A,n}(X) = \underline{\mu_{h(A)}}(h(X)), \tag{6}$$

$$\overline{\mu^h_A}(X) = \limsup_{n\to\infty} \mu^h_{A,n}(X) = \overline{\mu_{h(A)}}(h(X)). \tag{7}$$

If $M = N$, then, putting $h = \mathrm{id}_N$, we obtain the definitions of $\mu_{A,n}$, $\underline{\mu_A}$ and $\overline{\mu_A}$ from the previous example. One way how to define h is to consider an injective mapping $f : M \to N$ and to put $h = f^{\to}$. Another way is, for example, to define h by

$$h^\star(X)(n) = \bigvee_{\substack{Y\subseteq M\\ |Y|=n}} \bigwedge_{m\in Y} X(m), \tag{8}$$

for any $n \in N$.

It should be noted that $h^\star(A)$ is a generalized cardinal number in the sense of Wygralak's cardinal theory for vaguely defined objects (see [17, 18]). One can simply check that $(A, \mathcal{F}_A, \mu^{h^*}_A)$, for $\mu^{h^*}_A \in \{\mu^{h^*}_{A,n}, \underline{\mu^{h^*}_A}, \overline{\mu^{h^*}_A}\}$, is a fuzzy measure space for any non-empty fuzzy set $A \in \mathcal{F}(M)$ and

$$\mu^{h^*}_A(X) = \mu^{h^*}_{f^{\to}(A)}(f^{\to}(X)) \tag{9}$$

for any $X \in \mathcal{F}_A$ (this fact immediately follows from the equality $h^*(A) = h^*(f^{\to}(A))$). Moreover, $(1_N, \mathcal{F}_{1_N}, \underline{\mu^{h^\star}_{1_N}})$ is a cardinal fuzzy measure space in contrast to $(1_N, \mathcal{F}_{1_N}, \underline{\mu_{1_N}})$ discussed in the previous example.[3]

[3]For example, if P and E denote the sets of prime and even numbers, respectively, then $h^\star(1_P) = h^\star(1_E) = 1_N$ and $\underline{\mu^{h^\star}_{1_N}}(1_P) = \underline{\mu^{h^\star}_{1_N}}(1_N) = \underline{\mu^{h^\star}_{1_N}}(1_E) = 1$, but $\underline{\mu_{1_N}}(1_P) = 0 < \underline{\mu_{1_N}}(1_E) = 0.5 < \underline{\mu_{1_N}}(1_N) = 1$.

A relation between closed systems of fuzzy measure spaces and cardinal fuzzy measure spaces is stated in the following lemma.

Lemma 2.3. *A set* $\{\mathbf{A}\}$ *forms a closed system of fuzzy measure spaces in* $\mathbf{Fms}(M)$ *if and only if* $\mathbf{A}$ *is a cardinal fuzzy measure space.*

PROOF. See [5]. □

One can see that the set of all permutations on a set endowed by the operation of composition forms a maximal permutation group. Then a natural generalization of the concept of cardinal fuzzy measure space can be done by considering an arbitrary permutation group instead of the maximal one.

Definition 2.7. Let $\mathbf{A} = (A, \mathcal{F}, \mu)$ be a fuzzy measure space and G be a group of permutations on $\mathrm{Dom}(A)$. We say that $\mathbf{A}$ is *closed under isomorphisms with respect to* G if $g^{\rightarrow}(\mathbf{A}) = \mathbf{A}$ for any $g \in G$.

The following examples show fuzzy measure spaces being closed under isomorphisms with respect to a permutation group.

Example 2.5. Let us consider the fuzzy measure space $\mathbf{A}^r = (A, \mathcal{F}(A), \mu^r_A)$ defined in Example 2.1 and $G = \{g \mid g \in \mathrm{Perm}(M) \ \&\ g^{\rightarrow}(A) = A\}$, where $M = \mathrm{Dom}(A)$. Then, G is a permutation group, $g^{\rightarrow}(\mathcal{F}(A)) = \mathcal{F}(A)$ for any $g \in G$ and

$$\mu^r_A(g^{\rightarrow}(X)) = \frac{\sum_{m\in M} g^{\rightarrow}(X)(m)}{\sum_{m\in M} A(m)} = \frac{\sum_{m\in M} X(g^{-1}(m))}{\sum_{m\in M} A(m)} = \mu^r_A(X)$$

holds for any $X \in \mathcal{F}(A)$ and $g \in G$. Hence, $\mathbf{A}^r$ is closed under isomorphisms with respect to G.

Example 2.6. Recall that $(1_N, \mathcal{F}(1_N), \overline{\mu_{1_N}})$ defined in Example 2.3 is not closed under isomorphisms with respect to the group of all permutations on N (i.e., it is not a cardinal fuzzy measure space). Now, let X be a crisp subset of N such that $\overline{\mu_{1_N}}(1_X) = 0$ (e.g., X is a finite subset of N or an infinite subset as the set of prime numbers) and G be a group of all permutations g on N with $g(n) = n$ for any $n \in N \setminus X$. It is easy to see that G is a permutation group and, trivially, $g^{\rightarrow}(\mathcal{F}(1_N)) = \mathcal{F}(1_N)$ for any $g \in G$. To show that $(1_N, \mathcal{F}(1_N), \overline{\mu_{1_N}})$ is closed under isomorphisms with respect to G we have to check that $\overline{\mu_{1_N}}(Y) = \overline{\mu_{1_N}}(g^{\rightarrow}(Y))$ holds for any $Y \in \mathcal{F}(1_N)$. But this immediately follows from the fact that $\overline{\mu_{1_N}}(1_X) = 0$ and $\overline{\mu_{1_N}}$ is null-additive, i.e., $\overline{\mu_{1_N}}(A \cup B) = \overline{\mu_{1_N}}(A)$ holds for any $A, B \in \mathcal{F}(1_N)$ with $\overline{\mu_{1_N}}(B) = 0$.[4] In fact,

[4] From the definition of $\overline{\mu_{1_N}}$, we have $\overline{\mu_{1_N}}(A \cup B) \leq \limsup_{n\to\infty}(\mu_{1_N,n}(A) + \mu_{1_N,n}(B)) \leq \limsup_{n\to\infty} \mu_{1_N,n}(A) + \limsup_{n\to\infty} \mu_{1_N,n}(B) = \overline{\mu_{1_N}}(A) + \overline{\mu_{1_N}}(B) = \overline{\mu_{1_N}}(A)$, where $\sup_i(a_i + b_i) \leq \sup_i a_i + \sup_i b_i$ is applied. Since $\overline{\mu_{1_N}}(A) \leq \overline{\mu_{1_N}}(A \cup B)$ follows from the monotony of $\overline{\mu_{1_N}}$, we obtain the desired equality.

let $Y \in \mathcal{F}(1_N)$ and put $Y_1 = Y \upharpoonleft X$and $Y_2 = Y \upharpoonleft (N \setminus X)$. Since $\overline{\mu_{1_N}}(1_X) = 0$ then, from the monotony of $\overline{\mu_{1_N}}$ and $Y_1 \subseteq 1_X$, we also have $\overline{\mu_{1_N}}(Y_1) = 0$. Moreover, for any $g \in G$, we have $g^{\rightarrow}(Y) = g^{\rightarrow}(Y_1 \cup Y_2) = g^{\rightarrow}(Y_1) \cup Y_2$ and $g^{\rightarrow}(Y_1) \subseteq 1_X$ which implies $\overline{\mu_{1_N}}(g^{\rightarrow}(Y_1)) = 0$. From the null-additivity of $\overline{\mu_{1_N}}$, we obtain $\overline{\mu_{1_N}}(Y) = \overline{\mu_{1_N}}(Y_1 \cup Y_2) = \overline{\mu_{1_N}}(Y_2)$ and similarly $\overline{\mu_{1_N}}(g^{\rightarrow}(Y)) = \overline{\mu_{1_N}}(g^{\rightarrow}(Y_1) \cup Y_2) = \overline{\mu_{1_N}}(Y_2)$. Hence, we have $\overline{\mu_{1_N}}(Y) = \overline{\mu_{1_N}}(g^{\rightarrow}(Y))$. Note that the same result cannot be obtained for an arbitrary fuzzy set X with $\overline{\mu_{1_N}}(X) = 0$ and also for the lower asymptotic weighted density $\underline{\mu_{1_N}}$, which is not null-additive.

Example 2.7. One can verify that each fuzzy measure space $(A, \mathcal{F}(A), \mu_A^{h^\star})$ for $h^\star$ defined by (8) in Example 2.4 is closed under isomorphisms with respect to $G = \{f \mid f \in \mathrm{Perm}(M)\ \&\ f^{\rightarrow}(A) = A\}$, where $M = \mathrm{Dom}(A)$.

The following theorem is a generalization of Lemma 3.9 in [5] and shows the construction of fuzzy measure spaces closed under isomorphisms with respect to a permutation group. Note that this construction will be later used for the characterization of fuzzy quantifiers that are, in some sense, invariant with respect to permutations from a permutation group.

Theorem 2.4. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space and G be a group of permutations on* $\mathrm{Dom}(A)$*. Then $(B, \mathcal{G}, \nu)$ defined as follows: $B = \bigcup_{g \in G} g^{\rightarrow}(A)$, $\mathcal{G}$ is the least algebra of fuzzy sets on B containing the set*

$$\mathcal{T} = \bigcup_{g \in G} g^{\rightarrow}(\mathcal{F}) \tag{10}$$

and

$$\nu(X) = \bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_{g,X}} \mu_g(Y),$$

where $\mathcal{F}_{g,X} = \{Y \mid Y \in g^{\rightarrow}(\mathcal{F})\ \&\ Y \subseteq X\}$, is a fuzzy measure space closed under isomorphisms with respect to G.

Proof. Let $\mathbf{A} = (A, \mathcal{F}, \mu)$ be a fuzzy measure space, G be a group of permutation on $\mathrm{Dom}(A)$ and denote $g^{\rightarrow}(\mathbf{A}) = \mathbf{A}_g = (A_g, \mathcal{F}_g, \mu_g)$ for any $g \in G$. One can simply show that $\mathbf{B} = (B, \mathcal{G}, \nu)$ defined by the presumptions of theorem is a fuzzy measure space. We will prove that $\mathbf{B}$ is also closed under isomorphisms with respect to G.

Let $f \in G$. Obviously, we have

$$f^{\rightarrow}(B)(m) = f^{\rightarrow}\Big(\bigcup_{g \in G} g^{\rightarrow}(A)\Big)(m) = \Big(\bigcup_{g \in G} g^{\rightarrow}(A)\Big)(f^{-1}(m)) =$$

$$\bigcup_{g \in G} g^{\rightarrow}(A)(f^{-1}(m)) = \bigcup_{g \in G} f^{\rightarrow} \circ g^{\rightarrow}(A)(m) = \bigcup_{g \in G} (f \circ g)^{\rightarrow}(A)(m) = B(m)$$

for any $m \in \mathrm{Dom}(A)$, where obviously $G = \{f \circ g \mid g \in G\}$ holds for any $f \in G$.[5] To prove $f^{\to}(\mathcal{G}) = \mathcal{G}$, it is sufficient to show $f^{\to}(\mathcal{T}) = \mathcal{T}$. In fact, if $X \in \mathcal{G}$ is derived by $\cup$ and $\setminus$ from $X_1, \dots, X_n \in \mathcal{T}$, then $f^{\to}(X)$ is derived by the same application of the mentioned operations as X from $f^{\to}(X_1), \dots, f^{\to}(X_n)$. Hence, if $f^{\to}(X_1), \dots, f^{\to}(X_n) \in \mathcal{T}$, then necessary $f^{\to}(X) \in \mathcal{G}$. If $X \in \mathcal{T}$, then, according to the definition of $\mathcal{T}$ and the fact that $f \in G$ implies $f^{-1} \in G$, we have $(f^{-1})^{\to}(X) \in \mathcal{T}$. Hence, $X = f^{\to}((f^{-1})^{\to}(X)) \in f^{\to}(\mathcal{T})$, and thus $\mathcal{T} \subseteq f^{\to}(\mathcal{T})$. If $X \in f^{\to}(\mathcal{T})$, then there exists $Y \in \mathcal{T}$ such that $f^{\to}(Y) = X$. Since $f \in G$, then $X = f^{\to}(Y) \in \mathcal{T}$, and thus $f^{\to}(\mathcal{T}) \subseteq \mathcal{T}$. Hence, we obtain $\mathcal{T} = f^{\to}(\mathcal{T})$, which implies $f^{\to}(\mathcal{G}) = \mathcal{G}$.

Obviously, $f^{\to}(\mathcal{F}_g) = \{f^{\to}(Y) \mid Y \in \mathcal{F}_g\} = \{(f \circ g)^{\to}(X) \mid X \in \mathcal{F}\} = \mathcal{F}_{f \circ g}$. If $f \in G$ and $X \in \mathcal{G}$, then $f^{\to}$ is a bijective mapping of $\mathcal{F}_{g,X}$ onto $\mathcal{F}_{f \circ g, f^{\to}(X)}$ and $\mu_g(Y) = \mu_{f \circ g}(f^{\to}(Y))$ for any $g \in G$ and $Y \in \mathcal{F}_{g,X}$. In fact, if $Y \in \mathcal{F}_{g,X}$, then $Y \subseteq X$ implies $f^{\to}(Y) \subseteq f^{\to}(X)$ and $f^{\to}(Y) \in \mathcal{F}_{f \circ g, f^{\to}(X)}$. Since f is a permutation, then $f^{\to}$ is necessary an injective mapping. If $Z \in \mathcal{F}_{f \circ g, f^{\to}(X)}$, then $Y = (f^{-1})^{\to}(Z) \in \mathcal{F}_{g,X}$ and $f^{\to}(Y) = Z$. Hence, $f^{\to}$ is surjective. Further, if $Y \in \mathcal{F}_{g,X}$, then $(g^{-1})^{\to}(Y) \in \mathcal{F}$ and

$$\mu_g(Y) = \mu((g^{-1})^{\to}(Y)) = \mu_{f \circ g}(f^{\to} \circ g^{\to} \circ (g^{-1})^{\to}(Y)) = \mu_{f \circ g}(f^{\to}(Y)).$$

Hence, we have

$$\nu(X) = \bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_{g,X}} \mu_g(Y) = \bigvee_{g \in G} \bigvee_{Z \in \mathcal{F}_{f \circ g, f^{\to}(X)}} \mu_{f \circ g}(Z) = \bigvee_{h \in G} \bigvee_{Z \in \mathcal{F}_{h, f^{\to}(X)}} \mu_h(Z) = \nu(f^{\to}(X))$$

and $\mathbf{B}$ is closed under isomorphisms with respect to G. □

Example 2.8. Let us consider $(1_N, \mathcal{F}(1_N), \overline{\mu_{1_N}})$ defined in Example 2.3 and discussed in Example 2.6. As it has been shown, this fuzzy measure space is not closed under isomorphisms with respect to any permutation group over N (different from the trivial group $(\{\mathrm{id}_N\}, \circ)$). Let $X \subseteq N$ and $G = \{g \mid g \in \mathrm{Perm}(N) \ \&\ g^{\to}(1_X) = 1_X\}$. According to the previous theorem, we can construct a fuzzy measure space $(B, \mathcal{G}, \nu)$ closed under isomorphisms with respect to G as follows:

(i) put $B = \bigcup_{g \in G} g^{\to}(1_N) = 1_N$;

(ii) put $\mathcal{G}$ the least algebra of fuzzy sets that contains the set

$$\mathcal{T} = \bigcup_{g \in G} g^{\to}(\mathcal{F}(1_N)),$$

since $g^{\to}(\mathcal{F}(1_N)) = \mathcal{F}(1_N)$ for any $g \in G$, then $\mathcal{G} = \mathcal{F}(1_N)$;

[5] Recall that $f^{\to}(A)(m) = A(f^{-1}(m))$ for any $m \in \mathrm{Dom}(f^{\to}(A))$ and $(f \circ g)^{\to} = f^{\to} \circ g^{\to}$.

(iii) define

$$\nu(A) = \bigvee_{g\in G} \bigvee_{Y\in\mathcal{F}_{g,A}} \overline{\mu_{g,1_N}}(Y) = \bigvee_{g\in G} \bigvee_{Y\in\mathcal{F}_A} \overline{\mu_{1_N}}(g^{\to}(Y)).$$

One can simply check that if $X = N$, then $\nu(A) = 1$, whenever there exists an infinite crisp set Y such that $1_Y \subseteq A$. For example, the fuzzy measure of the set P of prime numbers is under ν equal to 1 (in contrast to 0 for $\overline{\mu_{1_N}}$). In fact, it is sufficient to consider a permutation g on N such that $g^{\to}(1_P) = 1_{N\setminus P}$ and, from the null-additivity of $\overline{\mu_{1_N}}$, we obtain $\overline{\mu_{1_N}}(1_{N\setminus P}) = 1$. On the other hand, the fuzzy sets with finite supports have the measure under ν equal to 0.

2.2. $\odot$-fuzzy integral

In this subsection, we present several facts about the $\odot$-fuzzy integrals that will be used in the proofs of semantic properties of fuzzy quantifiers. Details and discussion can be found in [4, 5]. First, let us recall the definition of the $\odot$-fuzzy integral.

Definition 2.8. Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space, $Z : \mathrm{Dom}(A) \to L$ and X be an $\mathcal{F}$-measurable fuzzy set. The *$\odot$-fuzzy integral of Z on X* is given by

$$\int_X^{\odot} Z \, d\mu = \bigvee_{Y\in\mathcal{F}_X^-} \bigwedge_{m\in\mathrm{Supp}(Y)} (Z(m) \odot \mu(Y)), \tag{11}$$

where $\mathcal{F}_X^- = \{Y \mid Y \in \mathcal{F} \setminus \{1_\emptyset\} \;\&\; Y \subseteq X\}$. If $X = A$, then we write $\int_X^{\odot} Z \, d\mu$.

In our investigation of permutation and isomorphism invariance of fuzzy quantifiers, the following isomorphism theorem for $\odot$-fuzzy integrals plays a major role.

Theorem 2.5. *Let $g^{\to}$ be an isomorphism between $(A, \mathcal{F}, \mu)$ and $(B, \mathcal{G}, \nu)$, $Z : \mathrm{Dom}(A) \to L$ be a mapping and X be a $\mathcal{F}$-measurable fuzzy set. Then,*

$$\int_X^{\odot} Z \, d\mu = \int_{g^{\to}(X)}^{\odot} Z \circ g^{-1} \, d\nu. \tag{12}$$

Proof. See [5]. □

The following theorem characterizes fuzzy integrals that are invariant with respect to a permutation group, i.e., the values of fuzzy integrals remain unchanged, when the integrated functions are transformed by the permutations from a group of permutations. This characterization is substantial for the proofs of theorems on the permutation invariance of fuzzy quantifiers investigated in Section 3.2. Note that the proof of this theorem needs to suppose the distributivity of $\odot$ over arbitrary meets and joins in the complete residuated lattice. For instance, this presumption for $\odot = \wedge$ is ensured in the divisible residuated lattice and for $\odot = \otimes$ in the MV-algebras.

Theorem 2.6. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $Z : \mathrm{Dom}(A) \to L$*. If* G *is a group of permutations on* $\mathrm{Dom}(A)$ *such that*

$$\int^{\odot} Z \; d\mu = \int^{\odot} Z \circ g \; d\mu, \tag{13}$$

holds for any $g \in G$*, then there exists a fuzzy measure space* $(B, \mathcal{G}, \nu)$ *closed under isomorphisms with respect to* G *for which*

$$\int^{\odot} Z \; d\mu = \int^{\odot} Z \; d\nu. \tag{14}$$

Proof. Let $\mathbf{L}$ be a complete residuated lattice such that $\odot$ is distributive over $\bigwedge$ and $\bigvee$. Let $\mathbf{A} = (A, \mathcal{F}, \mu)$ be a fuzzy measure space, $Z : \mathrm{Dom}(A) \to L$ and G be a group of permutations on $\mathrm{Dom}(A)$ such that (13) is satisfied for any $g \in G$. We will prove that (14) is satisfied for the fuzzy measure space $(B, \mathcal{G}, \nu)$ defined in Theorem 2.4 that is closed under isomorphisms with respect to G.

Put $g^{\rightarrow}(\mathbf{A}) = (A_g, \mathcal{F}_g, \mu_g)$ for any $g \in G$. Using Theorem 2.5, the presumption (13) can be rewritten as

$$\int^{\odot} Z \; d\mu = \int_A^{\odot} Z \; d\mu = \int_A^{\odot} Z \circ g \; d\mu =$$

$$\int_{A_g}^{\odot} (Z \circ g) \circ g^{-1} \; d\mu_g = \int_{A_g}^{\odot} Z \; d\mu_g = \int^{\odot} Z \; d\mu_g.$$

for any $g \in G$. Recall that we write $\int^{\odot} Z \; d\mu = \int_A^{\odot} Z \; d\mu$, whenever A is the support of the fuzzy measure space $(A, \mathcal{F}, \mu)$. From the distributivity of $\odot$ in $\mathbf{L}$, we have

$$\int^{\odot} Z \; d\nu = \int_B^{\odot} Z \; d\nu = \bigvee_{X \in \mathcal{G}^-} \bigwedge_{m \in \mathrm{Supp}(X)} (Z(m) \odot \nu(X)) =$$

$$\bigvee_{X \in \mathcal{G}^-} \Big(\Big(\bigwedge_{m \in \mathrm{Supp}(X)} Z(m) \Big) \odot \Big(\bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_{g,X}} \mu_g(Y) \Big) \Big) =$$

$$\bigvee_{X \in \mathcal{G}^-} \bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_{g,X}} \bigwedge_{m \in \mathrm{Supp}(X)} (Z(m) \odot \mu_g(Y)) \leq$$

$$\bigvee_{X \in \mathcal{G}^-} \bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_{g,X} \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (Z(m) \odot \mu_g(Y)) \leq$$

$$\bigvee_{X \in \mathcal{G}^-} \bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_g^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (Z(m) \odot \mu_g(Y)) =$$

$$\bigvee_{g \in G} \bigvee_{Y \in \mathcal{F}_g^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (Z(m) \odot \mu_g(Y)) = \bigvee_{g \in G} \int_{A_g}^{\odot} Z \; d\mu_g =$$

$$\bigvee_{g \in G} \int_A^{\odot} Z \; d\mu = \int_A^{\odot} Z \; d\mu = \int^{\odot} Z \; d\mu.$$

One could note that a straightforward consequence of the definition of ν is the fact that $\mu(Z) \leq \nu(Z)$ for any $Z \in \mathcal{F}^-$. Hence and from the fact that $\mathcal{F} \subseteq \mathcal{G}$, we have

$$\int^{\odot} Z \, d\mu = \int_A^{\odot} Z \, d\mu = \bigvee_{Y \in \mathcal{F}^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (Z(m) \odot \mu(Y)) \leq$$

$$\bigvee_{Y' \in \mathcal{G}^-} \bigwedge_{m \in \mathrm{Supp}(Y')} (Z(m) \odot \nu(Y')) = \int_B^{\odot} Z \, d\nu = \int^{\odot} Z \, d\nu,$$

which concludes the proof. □

As we have mentioned above, the cardinal fuzzy measure spaces form a subfamily of the family of all fuzzy measure spaces closed under isomorphisms with respect to the permutation groups.

Corollary 2.7. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $Z : \mathrm{Dom}(A) \to L$*. If (13) holds for any permutation* g *on* $\mathrm{Dom}(A)$*, then there exists a cardinal fuzzy measure space* $(B, \mathcal{G}, \nu)$ *for which (14) is true.*

Proof. This is a straightforward consequence of the fact that $(B, \mathcal{G}, \nu)$ defined in Theorem 2.4 is a cardinal fuzzy measure space. □

2.3. Operations on fuzzy sets determined by rl-operations

Let us recall the basic notions and results on the so-called residuated lattice operations from the Part I. ([2]) that will be used in this paper.

Definition 2.9. Let L be a set of constants. The set $\mathbf{P}(L)$ of *binary (residuated lattice) polynomials* with constants from L is the smallest set satisfying the following conditions

(i) $c \in \mathbf{P}(L)$ for any $c \in L$,

(ii) $x, y \in \mathbf{P}(L)$,

(iii) if $\psi_1, \psi_2 \in \mathbf{P}(L)$, then $\psi_1 \square \psi_2 \in \mathbf{P}(L)$ for any $\square \in \{\wedge, \vee, \otimes, \rightarrow\}$.

Definition 2.10. Let $\mathbf{L}$ be a residuated lattice with the support L. A binary residuated lattice polynomial $\psi \in \mathbf{P}(L)$ defines a binary *residuated lattice operation* (shortly rl-operation) on L by the following rules $(a, b \in L)$

(i) if $\psi = c$, then $\psi(a, b) = c$,

(ii) if $\psi = x$ or $\psi = y$, then $\psi(a, b) = a$ or $\psi(a, b) = b$, respectively,

(iii) if $\psi = \psi_1 \square \psi_2$, then $\psi(a, b) = \psi_1(a, b) \square \psi_2(a, b)$ for any operation $\square \in \{\wedge, \vee, \otimes, \rightarrow\}$.

The set of all rl-operations in a residuated lattice $\mathbf{L}$ determined by polynomials of $\mathbf{P}(L)$ will be denoted by $\mathbf{Rlo}(\mathbf{L})$ or shortly $\mathbf{Rlo}$ if the residuated lattice $\mathbf{L}$ is unmistakably determined by the context. Note that we use the same symbol for a polynomial and a binary operation defined by this polynomial.

Definition 2.11. Let $\psi_1, \psi_2 \in \mathbf{Rlo}(\mathbf{L})$. We say that ψ_1 *is equal to* $\psi_2$and denote it by $\psi_1 = \psi_2$ if $\psi_1(a,b) = \psi_2(a,b)$ holds for any $a, b \in L$.

Definition 2.12. Let M be a non-empty universe and $\psi \in \mathbf{Rlo}(\mathbf{L})$. We say that an operation $\varphi_M : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ is *determined by* ψ if

$$\varphi_M(A, B)(m) = \psi(A(m), B(m)) \tag{15}$$

holds for any $A, B \in \mathcal{F}(M)$ and $m \in M$.

The following proposition will serve in our analysis of *permutation invariance* of fuzzy quantifiers.

Proposition 2.8. *For any operation φ_M and any permutation f on M, we have*

$$\varphi_M(A, B) = \varphi_M(f^{\to}(A), f^{\to}(B)) \circ f \tag{16}$$

for all $A, B \in \mathcal{F}(M)$.

Proof. See [5]. □

Definition 2.13. Let φ_M and $\varphi_{M'}$ be determined by ψ and ψ' from $\mathbf{Rlo}(\mathbf{L})$, respectively. We say that φ_M and $\varphi_{M'}$ are *equivalent* if $\psi = \psi'$ in $\mathbf{L}$.

The following proposition is important for the analysis of *isomorphism invariance* of fuzzy quantifiers.

Proposition 2.9. *For any equivalent operations φ_M, $\varphi_{M'}$ and a bijective mapping $f : M \to M'$, we have*

$$\varphi_M(A, B) = \varphi_{M'}(f^{\to}(A), f^{\to}(B)) \circ f \tag{17}$$

for all $A, B \in \mathcal{F}(M)$.

Proof. See [5]. □

3. Permutation and isomorphism invariance of fuzzy quantifiers

There are two properties, which characterize quantifiers that ignore the identity of individuals: permutation invariance and isomorphism invariance. Typical examples of permutation invariant as well as isomorphism invariant quantifiers are *every, some, no, not all, many, few, at least, at most, exactly, more than,* *less than.* There are also non-permutation invariant and non-isomorphism invariant quantifiers as *John, neither Bill nor Jack*, etc. (see also Section 1). These properties were in fact included directly in the definition of generalized quantifiers in famous papers by Mostowski [12] and Lindström [10]. The explicit definitions of the concepts of permutation invariance and isomorphism invariance were introduced by van Benthem in [15] (he called the property of isomorphism invariance *quantity* and he denoted the corresponding class of quantifiers by QUANT). For further information, we refer to [9, 14]. For fuzzy quantifiers these properties were introduced by Glöckner in [6] (see also [7, 8]).

3.1. Definition of fuzzy quantifiers of type $\langle 1, 1\rangle$ determined by fuzzy measures

In the first part of this paper [2] on fuzzy quantifiers of type $\langle 1, 1\rangle$, we proposed fuzzy quantifiers defined using $\odot$-fuzzy integrals. For the general definition of fuzzy quantifiers of type $\langle 1, 1\rangle$, see [8]. To exclude a confusion, let us denote by

$$\int^{\odot}_{(A,\mathcal{F},\mu)} Z \; d\mu \quad \text{or simply} \quad \int^{\odot}_{\mathbf{A}} Z \; d\mu$$

the $\odot$-fuzzy integral from Section 2.2 defined on a fuzzy measure space $\mathbf{A} = (A, \mathcal{F}, \mu)$.

Definition 3.1. Let M be a non-empty universe, $\mathcal{S}(M) : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathbf{Fms}(M)$ be a mapping assigning to any $A, B \in \mathcal{F}(M)$ a fuzzy measure space $\mathcal{S}(M)(A, B)$ and φ_M be an operation on $\mathcal{F}(M)$. A *fuzzy quantifier of type $\langle 1, 1\rangle$ limited to M determined by* $(\mathcal{S}(M), \varphi_M)$ is a mapping

$$Q_M : \mathcal{F}(M) \times \mathcal{F}(M) \to L$$

defined by

$$Q_M(A, B) = \int^{\odot}_{\mathcal{S}(M)(A,B)} \varphi_M(A, B) \; d\mu \tag{18}$$

for any $A, B \in \mathcal{F}(M)$.

Definition 3.2. An *unlimited fuzzy quantifier of type $\langle 1, 1\rangle$ determined by a pair of functionals* $(\mathcal{S}, \varphi)$ is an unlimited fuzzy quantifier Q of type $\langle 1, 1\rangle$ assigning a fuzzy quantifier Q_M determined by $(\mathcal{S}(M), \varphi_M)$ to each non-empty universe M.

In the following part, for simplicity, we will write only "Q is determined by $(\mathcal{S}, \varphi)$" instead of "Q is a fuzzy quantifier of type $\langle 1, 1\rangle$ determined by a pair of functionals $(\mathcal{S}, \varphi)$".

As we have discussed in [2], it is reasonable to introduce a class of fuzzy quantifiers definable by fuzzy measures. This type of fuzzy quantifiers ensures the same behavior (measurement) of fuzzy quantifiers if the first argument (scope) is fixed. The precise definition is as follows.

Definition 3.3. We say that an unlimited fuzzy quantifier Q is *definable by fuzzy measures* if there exists a pair of functionals $(\mathcal{S}, \varphi)$ such that Q is determined by $(\mathcal{S}, \varphi)$ and $\mathcal{S}(M)(A, -)$ is a constant mapping for any non-empty universe M and $A \in \mathcal{F}(M)$.

Although two fuzzy quantifiers Q_M and Q'_M can be determined by different mappings $\mathcal{S}(M)$ and $\mathcal{S}'(M)$ (or φ_M and φ'_M), these fuzzy quantifiers may give the same values for the same fuzzy sets, i.e., their behavior is coincident. The precise definition of the coincidence of fuzzy quantifiers is as follows.

Definition 3.4. We say that *a fuzzy quantifier Q coincides with* Q' and denote it by $Q \equiv Q'$ if, for any M and $A, B \in \mathcal{F}(M)$, it holds that $Q_M(A, B) = Q'_M(A, B)$.

Note that the empty set as the universe is also included in the previous definition.

Notation: In proofs of following theorems, we will often use $x, y, \ldots$ to denote pairs of fuzzy sets $(A, B), (C, D), \ldots$ This abbreviation will significantly simplify the expressions of formulas and make the following text more readable. For example, if $(A, B), (C, D) \in \mathcal{F}(M)^2$ and f is a permutation on M, then $f^{\rightarrow}(A, B) = (f^{\rightarrow}(A), f^{\rightarrow}(B)) = (C, D)$ will be written as $f^{\rightarrow}(x) = y$ for $x = (A, B)$ and $y = (C, D)$.

3.2. Permutation invariance

Definition 3.5. A fuzzy quantifier Q is *permutation invariant* if, for any universe M, bijective mapping $f : M \rightarrow M$ and $A, B \in \mathcal{F}(M)$, we have

$$Q_M(A, B) = Q_M(f^{\rightarrow}(A), f^{\rightarrow}(B)). \tag{19}$$

The class of all permutation invariant fuzzy quantifiers is denoted by PI.

Obviously, the behavior of fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$ is closely related to the form (behavior) of functionals $\mathcal{S}$ and φ. Here, we need the notion saying something about the property of "being permutation invariant" for the functional $\mathcal{S}$. Let us recall that $\mathrm{Perm}(M)$ is the set of all permutations on M.

Definition 3.6. A functional $\mathcal{S}$ is *weakly pi-closed* if, for any non-empty universe M and $A, B, C, D \in \mathcal{F}(M)$ such that $(g^{\rightarrow}(A), g^{\rightarrow}(B)) = (C, D)$ for a permutation g on M, there exists $f \in \mathrm{Perm}(M)$ for which $(f^{\rightarrow}(A), f^{\rightarrow}(B)) = (C, D)$ and

$$f^{\rightarrow}(\mathcal{S}(M)(A, B)) = \mathcal{S}(M)(C, D). \tag{20}$$

An equivalent definition is as follows.

Proposition 3.1. *A functional $\mathcal{S}$ is weakly pi-closed if and only if, for any non-empty universe M and $A, B \in \mathcal{F}(M)$, the equality*

$$\mathcal{S}(M)(f^{\rightarrow}(A), f^{\rightarrow}(B)) = f^{\rightarrow}(\mathcal{S}(M)(A, B)) \tag{21}$$

holds for at least one permutation f on M.

PROOF. Obvious. □

Theorem 3.2. *Let Q be determined by $(\mathcal{S},\varphi)$. If $\mathcal{S}$ is weakly pi-closed, then Q is permutation invariant.*

PROOF. The condition of being permutation invariant is trivially satisfied for $M=\emptyset$. Let us suppose that $M\neq\emptyset$ and $x,y\in\mathcal{F}(M)^2$ such that $g^{\rightarrow}(x)=y$ for a permutation g on M. Since $\mathcal{S}$ is weakly pi-closed, then there exists $f\in\mathrm{Perm}(M)$ for which $f^{\rightarrow}(x)=g^{\rightarrow}(x)=y$ and $f^{\rightarrow}(\mathcal{S}(M)(x))=\mathcal{S}(M)(f^{\rightarrow}(x))=\mathcal{S}(M)(y)$. Hence and from Theorem 2.5 and Proposition 2.8, we have

$$Q_M(x)=\int^{\odot}_{\mathcal{S}(M)(x)}\varphi_M(x)\;d\mu=\int^{\odot}_{f^{\rightarrow}(\mathcal{S}(M)(x))}\varphi_M(x)\circ f^{-1}\;d\mu'=$$
$$\int^{\odot}_{\mathcal{S}(M)(f^{\rightarrow}(x))}\varphi_M(f^{\rightarrow}(x))\;d\mu'=Q_M(f^{\rightarrow}(x))=Q_M(g^{\rightarrow}(x)),$$

where μ' denotes the fuzzy measure of the fuzzy measure space $f^{\rightarrow}(\mathcal{S}(M)(x))$ and $\varphi_M(x)\circ f^{-1}=\varphi_M(f^{\rightarrow}(x))$ is applied. Hence, Q is permutation invariant. □

In the following two examples, we restrict ourselves to fuzzy quantifiers that are defined for finite universes only. Note that fuzzy quantifiers defined in such a way are called *finite fuzzy quantifiers* (see Definition 6 in [8] and Section 4 of [2]).

Example 3.1. Let $\mathbf{A}^r=(A,\mathcal{F}(A),\mu^r_A)$, $A\neq 1_\emptyset$, denote a fuzzy measure space from Example 2.1. For any finite universe M, let us define $\mathcal{S}^r(M)(A,B)=\mathbf{A}^r$ for any $A\neq 1_\emptyset$, and $\mathcal{S}^r(M)(1_\emptyset,B)=(1_M,\{1_\emptyset,1_M\},\mu)$. Then, it is easy to see that $\mathcal{S}^r(M)$ is weakly pi-closed. We have to check that, for any $A,B\in\mathcal{F}(M)$, the equality (21) holds for at least one $f\in\mathrm{Perm}(M)$. It is not difficult to see that a sufficient condition for (21) to be valid for any $f\in\mathrm{Perm}(M)$ is the following. For any $Y\in\mathcal{F}(f^{\rightarrow}(A))$, it should hold that

$$\mu^r_{f^{\rightarrow}(A)}(Y)=(\mu^r_A)_f(Y)$$

(for the definition of $(\mu^r_A)_f$ see Definition 2.5). Indeed,

$$\mu^r_{f^{\rightarrow}(A)}(Y)=\frac{\sum_{m\in M}Y(m)}{\sum_{m\in M}f^{\rightarrow}(A)(m)}=\frac{\sum_{m\in M}Y(m)}{\sum_{f(m)\in M}f^{\rightarrow}(A)(f(m))}=$$
$$\frac{\sum_{m\in M}Y(m)}{\sum_{m\in M}A(m)}=\frac{\sum_{f^{-1}(m)\in M}(f^{-1})^{\rightarrow}(Y)(f^{-1}(m))}{\sum_{m\in M}A(m)}=$$
$$\frac{\sum_{m\in M}(f^{-1})^{\rightarrow}(Y)(m)}{\sum_{m\in M}A(m)}=(\mu^r_A)((f^{-1})^{\rightarrow}(Y))=(\mu^r_A)_f(Y),$$

where the equalities $A(m)=f^{\rightarrow}(A)(f(m))$ as well as $(f^{-1})^{\rightarrow}(Y)(f^{-1}(m))=Y(m)$ are used. Hence, $\mathcal{S}^r(M)$ is weakly pi-closed, and the quantifier $\mathsf{many}_M(A,B)$ determined by $\mathcal{S}^r(M)$ and $\varphi_M(A,B)=A\rightarrow B$ from Example 5.3 in [2] is permutation invariant according to Theorem 3.2.

Example 3.2. Let $\mathbf{L}$ be a complete residuated lattice with $L \neq \{\bot, \top\}$ and P be a functional assigning to any non-empty finite M a fuzzy set $P_M \in \mathcal{F}(M)$ such that $1_\emptyset \neq P_M \neq 1_M$. Let $\mathcal{S}^r(M)(A, B)$ be as in Example 3.1 and define a functional $\mathcal{S}^{r,P}(M)(A, B) = S^r(M)(A \cap P(M), B)$ if $A \cap P(M) \neq 1_\emptyset$, and $\mathcal{S}^{r,P}(M)(A, B) = (M, \{1_\emptyset, 1_M\}, \mu)$ otherwise.[6] Then, it is easy to show that $\mathcal{S}^{r,P}(M)$ is not weakly pi-closed. Let for some $M \neq \emptyset$ consider $A = P_M$ and $C = g^{\rightarrow}(A) = g^{\rightarrow}(P) \neq P$ (it is possible because $P \neq 1_M$). Then, for any $f \in \mathrm{Perm}(M)$ such that $f^{\rightarrow}(A) = C$ and $B, D \in \mathcal{F}(M)$ such that $f^{\rightarrow}(B) = D$, the equation

$$\mathcal{S}^{r,P}(M)(C, D) = f^{\rightarrow}(\mathcal{S}^{r,P}(M)(A, B))$$

cannot hold, because

$$\mathcal{S}^{r,P}(M)(C, D) = (C \cap P, \mathcal{F}(C \cap P), \mu^r_{C\cap P}) \neq (C, (\mathcal{F}(P))_f, (\mu^r_P)_f) =$$
$$(f^{\rightarrow}(P), (\mathcal{F}(P))_f, (\mu^r_P)_f) = f^{\rightarrow}(\mathcal{S}^{r,P}(M)(P, B)) = f^{\rightarrow}(\mathcal{S}^{r,P}(M)(A, B)).$$

It is sufficient to realize that, given the presuppositions we stated, $C \cap P_M \subset C$. Hence, $\mathcal{S}^{r,P}(M)$ is not weakly pi-closed.

The functional $\mathcal{S}^{r,P}(M)$ is intended for modeling of the natural language quantifier "many P's" (as in a sentence "Many Peter's apples are green"). A model of this quantifier can be defined as a fuzzy quantifier determined by $(\mathcal{S}^{r,P}(M), \varphi^P_M)$, where $\varphi^P_M(A, B) = (A \cap P) \to B$. It is easy to show (using Definition 3.5 directly) that this fuzzy quantifier is not permutation invariant.

Unfortunately, the opposite implication to Theorem 3.2 is not valid in general. This can be caused by a choice of a residuated lattice, but primarily by a choice of fuzzy measure spaces (obtained by $\mathcal{S}$) as well as by a form of the operation φ as the following simple example demonstrates.

Example 3.3. Let $\mathbf{L}$ be a complete residuated lattice, $c \in L$ be a constant, $\psi(a, b) = c \vee (a \wedge b)$ be an rl-operation on L and φ_M be determined by ψ for any non-empty universe M. Further, consider $\mathbf{M} = (M, \mathcal{F}, \mu)$, where $M = [0, 1]$, $\mathcal{F} = \{\emptyset, 1_{[0,0.5[}, 1_{[0.5,1]}, 1_M\}$ and

$$\mu(1_{[0,0.5[}) = \mu(1_{[0.5,1]}) = c.$$

Finally, define $\mathcal{S}(M)(A, B) = \mathbf{M}$ for any $A, B \in \mathcal{F}(M)$. Since

$$\varphi_M(A, B)(m) \wedge \mu(1_{[0,0.5[}) = (c \vee (A(m) \wedge B(m))) \wedge c = c$$

for any $m \in [0, 0.5[$ and analogously $\varphi_M(A, B)(m) \wedge \mu(1_{[0.5,1]}) = c$ for any $m \in [0.5, 1]$, then, for any permutation f on M and $A, B \in \mathcal{F}(M)$, we have

[6] In this case, μ is the unique fuzzy measure on the trivial fuzzy measurable space $(M, \{1_\emptyset, 1_M\})$.

(putting $\odot = \wedge$)

$$Q_M(A,B) = \int^{\wedge}_{\mathcal{S}(M)(A,B)} \varphi_M(A,B)\ d\mu = c \vee \bigwedge_{m\in M} \psi(A(m),B(m)) =$$

$$c \vee \bigwedge_{m\in M} \psi(f^{\rightarrow}(A)(f(m)), f^{\rightarrow}(B)(f(m))) =$$

$$c \vee \bigwedge_{m\in M} \psi(f^{\rightarrow}(A)(m), f^{\rightarrow}(B)(m)) =$$

$$\int^{\wedge}_{\mathcal{S}(M)(f^{\rightarrow}(A),f^{\rightarrow}(B))} \varphi_M(f^{\rightarrow}(A), f^{\rightarrow}(B))\ d\mu = Q_M(f^{\rightarrow}(A), f^{\rightarrow}(B)).$$

Hence, Q is permutation invariant. Nevertheless, $\mathcal{S}$ is not weakly pi-closed. In fact, let us consider $a,b \in [0,1]$ with $|a-b| > 0.5$ and $g \in \mathrm{Perm}(M)$ defined by $g(x) = x$ for $x \in [0,1] \setminus \{a,b\}$, $g(a) = b$ and $g(b) = a$. Further, put

$$A(m) = \begin{cases} 1, & \text{if } m = a, \\ 0.5, & \text{if } m = b, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Clearly, for any $f \in \mathrm{Perm}(M)$ with $(f^{\rightarrow}(1_M), f^{\rightarrow}(A)) = (g^{\rightarrow}(1_M), g^{\rightarrow}(A))$, we obtain $\mathbf{M} = \mathcal{S}(M)(f^{\rightarrow}(1_M), f^{\rightarrow}(A)) \neq f^{\rightarrow}(\mathcal{S}(M)(1_M, A)) = f^{\rightarrow}(\mathbf{M})$ (because $f^{\rightarrow}(1_{[0,0.5[}) \notin \mathcal{F}$, whence $\mathcal{F} \neq \mathcal{F}_f$) and, hence, $\mathcal{S}$ is not weakly pi-closed. For $\odot = \otimes$, the same result can be obtained by putting $c = \perp$.

Perhaps it would be possible to find several classes of special conditions (on residuated lattices, fuzzy measure spaces and rl-operations) under which permutation invariant fuzzy quantifiers are determined by weakly pi-closed functionals. Nevertheless, in our opinion, this way from case to case seems to be not so conceptual here. Therefore, in the following part, we will show that the class of all permutation invariant fuzzy quantifiers determined by $(\mathcal{S},\varphi)$ can be described by fuzzy quantifiers based on weakly pi-closed functionals. More precisely, to each permutation invariant fuzzy quantifier determined by $(\mathcal{S},\varphi)$ we can construct a fuzzy quantifier determined by a weakly pi-closed functional and φ that coincides with the original one. Let us suppose the existence of choice function.[7]

Theorem 3.3. *Let Q be determined by $(\mathcal{S},\varphi)$. If Q is permutation invariant, then there exists a fuzzy quantifier Q' determined by $(\mathcal{S}',\varphi)$ such that $\mathcal{S}'$ is weakly pi-closed and Q coincides with Q'.*

Proof. Let $x,y \in \mathcal{F}(M)^2$. We will write $x \sim y$ if there exists $f \in \mathrm{Perm}(M)$ for which $f^{\rightarrow}(x) = y$. Obviously, $\sim$ is an equivalence on $\mathcal{F}(M)^2$. Now, the idea of the proof is to introduce equivalence classes on $\mathcal{F}(M)^2$ by $\sim$ and to represent each class by one of its element (a representative). These representatives and

[7]We do not consider the presumption of the axiom of choice and the existence of choice function as a limitation in our study of fuzzy quantifiers.

some chosen permutations on M are used for a construction of a weakly pi-closed functional over which the new fuzzy quantifier coincides with the original one.

In order to ensure $Q_M = Q'_M$ for $M = \emptyset$, we put $Q'_\emptyset(\emptyset, \emptyset) = Q_\emptyset(\emptyset, \emptyset)$. Let $M \neq \emptyset$, denote $\mathcal{Q}$ the quotient set of $\mathcal{F}(M)^2$ by $\sim$, $h : \mathcal{Q} \to \mathcal{F}(M)^2$ be a choice function (i.e., $h([x]) \in [x]$),

$$\mathcal{R} = \{(x, y) \mid x, y \in \mathcal{F}(M)^2 \text{ and } x \sim y\}$$

and $r : \mathcal{R} \to \mathrm{Perm}(M)$ be a mapping that satisfies

$$r(x, y) = \begin{cases} f, & \text{if } x \neq y \text{ and } y = f^{\to}(x), \\ \mathrm{id}_M, & \text{otherwise.} \end{cases}$$

Clearly, there exists a mapping r. Let us define $\mathcal{S}'(M) : \mathcal{F}(M)^2 \to \mathbf{Fms}(M)$ by

$$\mathcal{S}'(M)(x) = (r(x, h([x]))^{-1})^{\to}(\mathcal{S}(M)(h([x]))). \tag{23}$$

Putting $z = h([x])$ and $r(x, z) = f_x$, we may rewrite the previous formula as

$$\mathcal{S}'(M)(x) = (f_x^{-1})^{\to}(\mathcal{S}(M)(z)). \tag{24}$$

One can see that $\mathcal{S}'$ is derived from the fuzzy measure spaces of the representatives and the chosen permutations are used for their isomorphic images. We will show that $\mathcal{S}'$ is weakly pi-closed, i.e., for any $(x, y) \in \mathcal{R}$, there exists $f \in \mathrm{Perm}(M)$ such that

1. $f^{\to}(x) = y$,
2. $f^{\to}(\mathcal{S}'(M)(x)) = \mathcal{S}'(M)(f^{\to}(x))$.

Let $(x, y) \in \mathcal{R}$ (note that $[x] = [y]$). Put $z = h([x])$, $f_x = r(x, z)$ and $f_y = r(y, z)$. According to the definition of $\mathcal{S}'(M)$, we have

$$\begin{aligned} \mathcal{S}'(M)(x) &= (f_x^{-1})^{\to}(\mathcal{S}(M)(z)) \\ \mathcal{S}'(M)(y) &= (f_y^{-1})^{\to}(\mathcal{S}(M)(z)). \end{aligned} \tag{25}$$

Plugging $\mathcal{S}(M)(z) = f_x^{\to}(\mathcal{S}'(M)(x))$ to the second equality in (25), we obtain

$$\mathcal{S}'(M)(y) = (f_y^{-1})^{\to} \circ f_x^{\to}(\mathcal{S}'(M)(x)). \tag{26}$$

According to the choice of the permutations f_x and f_y, we have $f_x^{\to}(x) = z = f_y^{\to}(y)$. Then, however, we obtain

$$y = (f_y^{-1})^{\to} \circ f_x^{\to}(x) \tag{27}$$

Putting $f = f_y^{-1} \circ f_x$, we obtain

$$f^{\to} = (f_y^{-1} \circ f_x)^{\to} = (f_y^{-1})^{\to} \circ f_x^{\to}.$$

Plugging $f^{\rightarrow}$ to (26) and (27) and combining these expressions, we obtain

$$f^{\rightarrow}(\mathcal{S}'(M)(x)) = \mathcal{S}'(M)(f^{\rightarrow}(x)),$$

and thus $\mathcal{S}'$ is a weakly pi-closed functional.

According to Theorem 3.2, the fuzzy quantifier Q' determined by $(\mathcal{S}', \varphi)$ is permutation invariant. Let us show that $Q_M = Q'_M$. Let $x \in \mathcal{F}(M)^2$ and put $h([x]) = z$. From the permutation invariance of Q and Q', we have

$$Q_M(x) = Q_M(z) = Q'_M(z) = Q'_M(x),$$

where $Q_M(z) = Q'_M(z)$ follows from the fact that $r(z, z) = 1_M$, and thus $\mathcal{S}'(M)(z) = \mathcal{S}(M)(z)$. Hence and from the equality $Q_\emptyset(\emptyset, \emptyset) = Q'_\emptyset(\emptyset, \emptyset)$, we obtain the coincidence of Q and Q'. □

A straightforward consequence of Theorems 3.2 and 3.3 is the following corollary characterizing the permutation invariant fuzzy quantifiers determined by a pair of functionals $(\mathcal{S}, \varphi)$.

Corollary 3.4. *A fuzzy quantifier Q determined by $(\mathcal{S}, \varphi)$ is permutation invariant if and only if there exists a weakly pi-closed functional $\mathcal{S}'$ such that the fuzzy quantifier Q' determined by $(\mathcal{S}', \varphi)$ coincides with Q.*

The previous corollary shows a nice relation between permutation invariance and weakly pi-closed functionals $\mathcal{S}$ and practically says that all permutation invariant fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$ may be defined in such a way that $\mathcal{S}$ is a weakly pi-closed functional.

Our analysis of permutation invariance will continue for fuzzy quantifiers definable by fuzzy measures. Recall that a fuzzy quantifier determined by $(\mathcal{S}, \varphi)$ is definable by fuzzy measures if $\mathcal{S}(M)(A, -)$ is a constant mapping for any nonempty universe M and $A \in \mathcal{F}(M)$. Let us start with an analogous statement to that in Theorem 3.2.

Theorem 3.5. *Let Q determined by $(\mathcal{S}, \varphi)$ be definable by fuzzy measures. If $\mathcal{S}$ is weakly pi-closed, then Q is permutation invariant.*

Proof. Since the fuzzy quantifiers definable by fuzzy measures form a subfamily of fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$, this statement follows immediately from Theorem 3.2. □

One would expect, that the opposite implication is also a simple consequence of Theorem 3.3 and the fact that an arbitrary fuzzy quantifier definable by fuzzy measures is a special case of fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$. Unfortunately, the situation is not so straightforward as the following example demonstrates.

Example 3.4. Let us consider the fuzzy quantifier Q, the fuzzy set A and the permutation g on M defined in Example 3.3. Put $B = g^{\rightarrow}(A)$. As it could be seen, there is no permutation f on M that satisfies $f^{\rightarrow}(A) = B$ and

$f^{\rightarrow}(\mathbf{M}) = \mathbf{M}$. Put $x = (1_M, A)$ and $y = (1_M, B)$, i.e., $x \sim y$ according to the notation used in the proof of Theorem 3.3. Without loss of generality, let us suppose that the mapping h in the construction of the weakly pi-closed functional $\mathcal{S}'$ (see the proof of Theorem 3.3) is such that $h([x]) = x$ and denote $r(x, h([x])) = r(x, x) = 1_M$ and $r(y, h([y])) = r(y, x) = f_y$ (note that $f_y^{\rightarrow}(\mathbf{M}) \neq \mathbf{M}$ according to the presumption on the permutations making the interrelation between A and B). By the construction of $\mathcal{S}'$, we have

$$\mathcal{S}'(M)(x) = (1_M^{-1})^{\rightarrow}(\mathcal{S}(M)(x)) = 1_M^{\rightarrow}(\mathbf{M}) = \mathbf{M},$$
$$\mathcal{S}'(M)(y) = (f_y^{-1})^{\rightarrow}(\mathcal{S}(M)(x)) = (f_y^{-1})^{\rightarrow}(\mathbf{M}).$$

Hence, we obtain

$$\mathcal{S}'(M)((f_y^{-1})^{\rightarrow}(x)) = (f_y^{-1})^{\rightarrow}(\mathcal{S}'(M)(x)),$$

but $\mathcal{S}'(M)(y) \neq \mathbf{M}$, and thus Q' determined by $(\mathcal{S}', \varphi)$ is not definable by fuzzy measures (because the second argument should not play a role).

It is easy to see that the problem which arose in the previous example is closely related to the construction of a weakly pi-closed functional $\mathcal{S}'$ that does not guarantee the definability of fuzzy quantifiers by fuzzy measures. One can see that $\mathcal{S}'(M)(y) = \mathbf{M}$ can be ensured if the interrelations provided by permutations between fuzzy sets in the second argument (i.e., between A and B in the previous example) do not have an influence on the values of the functional $\mathcal{S}'$. It has motivated us to modify the concept of being weakly pi-closed functional with respect to the independence on the interrelations between fuzzy sets in the second argument.

Definition 3.7. A functional $\mathcal{S}$ is *weakly pi-closed in the first component* if, for any non-empty universe M and $A, B, C, D \in \mathcal{F}(M)$ such that $g^{\rightarrow}(A) = C$ for a permutation g on M, there exists a permutation f on M with $f^{\rightarrow}(A) = C$ and

$$\mathcal{S}(M)(C, D) = f^{\rightarrow}(\mathcal{S}(M)(A, B)). \tag{28}$$

An equivalent definition is as follows.

Proposition 3.6. *A functional $\mathcal{S}$ is weakly pi-closed in the first component if and only if, for any non-empty universe M and $A, B, D \in \mathcal{F}(M)$, the equality*

$$\mathcal{S}(M)(f^{\rightarrow}(A), D) = f^{\rightarrow}(\mathcal{S}(M)(A, B))$$

holds for at least one permutation f on M.

Proof. Obvious. □

Before we give an analogous statement to that in Theorem 3.3, we will prove two lemmas.

Lemma 3.7. *Let Q be a permutation invariant fuzzy quantifier definable by fuzzy measures. Then, for any non-empty universe M, permutations f and g on M and $A, B \in \mathcal{F}(M)$ such that $f^{\rightarrow} \circ g^{\rightarrow}(A) = g^{\rightarrow}(A)$, we have*

$$\int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow}(B)) \, d\mu_g = \int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), f^{\rightarrow} \circ g^{\rightarrow}(B)) \, d\mu_g, \tag{29}$$

where $\mathbf{A}_g = g^{\rightarrow}(\mathcal{S}(M)(A, B))$.

Proof. First, let us suppose that $g = 1_M$. We have to show that □

$$\int_{\mathbf{A}_{1_M}}^{\odot} \varphi_M(A, B) \, d\mu_{1_M} = \int_{\mathbf{A}_{1_M}}^{\odot} \varphi_M(A, f^{\rightarrow}(B)) \, d\mu_{1_M} \tag{30}$$

for any permutation f on M such that $f^{\rightarrow}(A) = A$. Since Q is permutation invariant fuzzy quantifier definable by fuzzy measures, then $\mathcal{S}(M)(A, B) = \mathcal{S}(M)(A, f^{\rightarrow}(B)) = \mathbf{A}_{1_M}$ and

$$\int_{\mathbf{A}_{1_M}}^{\odot} \varphi_M(A, B) \, d\mu_{1_M} = Q_M(A, B) =$$
$$Q_M(A, f^{\rightarrow}(B)) = \int_{\mathbf{A}_{1_M}}^{\odot} \varphi_M(A, f^{\rightarrow}(B)) \, d\mu_{1_M}.$$

Now, let us suppose an arbitrary permutation g on M. Due to Theorem 2.5, Proposition 2.8 and the previous equality for $g = 1_M$, we have

$$\begin{aligned} \int_{\mathbf{A}_{1_M}}^{\odot} \varphi_M(A, B) \, d\mu_{1_M} &= \int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow}(B)) \, d\mu_g = \\ \int_{\mathbf{A}_{1_M}}^{\odot} \varphi_M(A, h^{\rightarrow}(B)) \, d\mu_{1_M} &= \int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow} \circ h^{\rightarrow}(B)) \, d\mu_g \end{aligned} \tag{31}$$

for any permutation h on M with $h^{\rightarrow}(A) = A$. Let f be a permutation on M with $f^{\rightarrow} \circ g^{\rightarrow}(A) = g^{\rightarrow}(A)$ and put $h = g^{-1} \circ f \circ g$. Then,

$$h^{\rightarrow}(A) = (g^{-1} \circ f \circ g)^{\rightarrow}(A) = (g^{-1})^{\rightarrow} \circ f^{\rightarrow} \circ g^{\rightarrow}(A) = (g^{-1})^{\rightarrow} \circ g^{\rightarrow}(A) = A.$$

Moreover, we have

$$g^{\rightarrow} \circ h^{\rightarrow}(B) = g^{\rightarrow} \circ ((g^{-1})^{\rightarrow} \circ f^{\rightarrow} \circ g^{\rightarrow})(B) = f^{\rightarrow} \circ g^{\rightarrow}(B)$$

and thus $(g^{\rightarrow}(A), g^{\rightarrow} \circ h^{\rightarrow}(B)) = (g^{\rightarrow}(A), f^{\rightarrow} \circ g^{\rightarrow}(B))$. Using the equalities of fuzzy integrals in (31) that hold for $h^{\rightarrow}(A) = A$, we obtain

$$\int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), f^{\rightarrow} \circ g^{\rightarrow}(B)) \, d\mu_g = \int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow} \circ h^{\rightarrow}(B)) \, d\mu_g =$$
$$\int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow}(B)) \, d\mu_g,$$

which concludes the proof. □

Lemma 3.8. *Let Q be a permutation invariant fuzzy quantifier definable by fuzzy measures. Then, for any non-empty universe M, permutations f and g on M and $A, B \in \mathcal{F}(M)$ such that $f^{\to}(A) = g^{\to}(A)$, we have*

$$\int_{\mathbf{A}_g}^{\odot} \varphi_M(f^{\to}(A), f^{\to}(B))\, d\mu_g = \int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\to}(A), g^{\to}(B))\, d\mu_g, \qquad (32)$$

where $\mathbf{A}_g = g^{\to}(\mathcal{S}(M)(A, B))$.

Proof. Put $h = f \circ g^{-1}$. Clearly, h is a permutation on M and $h^{\to}(g^{\to}(A)) = f^{\to} \circ (g^{-1})^{\to}(g^{\to}(A)) = f^{\to}(A) = g^{\to}(A)$. Analogously, we obtain $h^{\to}(g^{\to}(B)) = f^{\to}(B)$ (note that $f^{\to}(B) \neq g^{\to}(B)$ in general). According to Lemma 3.7, we have

$$\int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\to}(A), g^{\to}(B))\, d\mu_g = \int_{\mathbf{A}_g}^{\odot} \varphi_M(g^{\to}(A), h^{\to}(g^{\to}(B)))\, d\mu_g =$$
$$\int_{\mathbf{A}_g}^{\odot} \varphi_M(f^{\to}(A), f^{\to}(B))\, d\mu_g,$$

where we put $g^{\to}(A) = f^{\to}(A)$ and $h^{\to}(g^{\to}(B)) = f^{\to}(B)$. □

Now, we can prove the main theorem for the permutation invariant fuzzy quantifiers definable by fuzzy measures.

Theorem 3.9. *Let Q be a fuzzy quantifier determined by $(\mathcal{S}, \varphi)$ and be definable by fuzzy measures. If Q is permutation invariant, then there exists a weakly pi-closed in the first component functional $\mathcal{S}'$ such that Q' determined by $(\mathcal{S}', \varphi)$ is definable by fuzzy measures and Q' coincides with Q.*

Proof. In order to ensure $Q_M = Q'_M$ for $M = \emptyset$, we put $Q'_{\emptyset}(\emptyset, \emptyset) = Q_{\emptyset}(\emptyset, \emptyset)$. Let $M \neq \emptyset$ and $A, B \in \mathcal{F}(M)$. We will write $A \overset{*}{\sim} B$ if there exists a permutation f on M such that $f^{\to}(A) = B$. Clearly, $\overset{*}{\sim}$ is an equivalence on $\mathcal{F}(M)$ (cf. $\sim$ from the proof of Theorem 3.3). Let $\mathcal{Q}$ denote the quotient set of $\mathcal{F}(M)$ by $\overset{*}{\sim}$, $h : \mathcal{Q} \to \mathcal{F}(M)$ be a choice function (i.e., $h^{\to}([A]) \in [A]$),

$$\mathcal{R} = \{(A, B) \mid (A, B) \in \mathcal{F}(M)^2 \text{ and } A \overset{*}{\sim} B\}.$$

Let $r : \mathcal{R} \to \mathrm{Perm}(M)$ be a mapping that satisfies

$$r(A, B) = \begin{cases} f, & \text{if } A \neq B \text{ and } B = f^{\to}(A), \\ 1_M, & \text{otherwise.} \end{cases}$$

One can see that there exists a mapping r. Further, let us define a functional $\mathcal{S}'(M) : \mathcal{F}(M)^2 \to \mathbf{Fms}(M)$ by

$$\mathcal{S}'(M)(A, B) = (r(A, h([A]))^{-1})^{\to}(\mathcal{S}(M)(h([A]), 1_{\emptyset})). \qquad (33)$$

One could note that our choice of the empty fuzzy set in the definition does not have any influence on the definition of the functional $\mathcal{S}'(M)$, in other words, replacing $1_\emptyset$ by any fuzzy set on M, we obtain the same definition, because Q is definable by fuzzy measures. For simplicity, put $h([A]) = Z$ and $r(A, Z) = f_A$. Then, the formula in (33) can be rewritten as

$$\mathcal{S}'(M)(A, B) = (f_A^{-1})^{\rightarrow}(\mathcal{S}(M)(A, 1_\emptyset)). \tag{34}$$

Now, let Q' be a fuzzy quantifier determined by $(\mathcal{S}', \varphi)$. A simple consequence of the definition of $\mathcal{S}'$ is the fact that Q' is definable by fuzzy measures (i.e., $\mathcal{S}'(M)(A, B) = \mathcal{S}'(M)(A, C)$ for any $M \neq \emptyset$ and $A, B, C \in \mathcal{F}(M)$). Let $A, B, C, D \in \mathcal{F}(M)$ and $g \in \mathrm{Perm}(M)$ be such that $g^{\rightarrow}(A) = C$ (i.e., $A \overset{*}{\sim} C$). Put $Z = h([A]) = h([C])$ and $f_A = r(A, Z)$ and $f_C = r(C, Z)$. Then,

$$\mathcal{S}'(M)(A, B) = (f_A^{-1})^{\rightarrow}(\mathcal{S}(M)(Z, 1_\emptyset)),$$
$$\mathcal{S}'(M)(C, D) = (f_C^{-1})^{\rightarrow}(\mathcal{S}(M)(Z, 1_\emptyset)).$$

Hence, we obtain

$$\mathcal{S}'(M)(C, D) = (f_C^{-1} \circ f_A)^{\rightarrow}(\mathcal{S}(M)(A, B)).$$

Putting $f = f_C^{-1} \circ f_A$, we obtain

$$f^{\rightarrow}(A) = (f_C^{-1} \circ f_A)^{-1}(A) = (f_C^{-1})^{\rightarrow} \circ f^{\rightarrow}(A) = (f_C^{-1})^{\rightarrow}(Z) = C$$

and $\mathcal{S}'(M)(C, D) = f^{\rightarrow}(\mathcal{S}'(M)(A, B))$. Hence, $\mathcal{S}'$ is weakly pi-closed in the first component.

Now, we will show that Q' is permutation invariant. Let $A, B \in \mathcal{F}(M)$ and put $h([A]) = Z$, $f_A = r(A, Z)$ and, for simplicity, $\mathbf{A} = \mathcal{S}(M)(Z, \emptyset)$ (we will use also the denotation $\mathbf{A}_f = f^{\rightarrow}(\mathbf{A})$). Let $f \in \mathrm{Perm}(M)$. We have to prove that $Q'_M(A, B) = Q'_M(f^{\rightarrow}(A), f^{\rightarrow}(B))$. Since $\mathcal{S}'$ is weakly pi-closed in the first component, there exists $g \in \mathrm{Perm}(M)$ such that $f^{\rightarrow}(A) = g^{\rightarrow}(A)$ and

$$\mathcal{S}'(M)(f^{\rightarrow}(A), f^{\rightarrow}(B)) = \mathcal{S}'(M)(g^{\rightarrow}(A), g^{\rightarrow}(B)) =$$
$$g^{\rightarrow}(\mathcal{S}'(M)(A, B)) = g^{\rightarrow}((f_A^{-1})^{\rightarrow}(\mathbf{A})) = \mathbf{A}_{g \circ f_A^{-1}},$$

where the first equality follows from the fact that $\mathcal{S}'(M)(A, -)$ is a constant mapping for any $A \in \mathcal{F}(M)$. Obviously, we can also write

$$g^{\rightarrow}(A) = g^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(Z) = f^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(Z) = f^{\rightarrow}(A).$$

Since Q is definable by fuzzy measures and it is permutation invariant, then we have (due to Lemma 3.8)

$$\int_{\mathbf{A}_{g \circ f_A^{-1}}}^{\odot} \varphi_M(f^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(Z), f^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(f_A^{\rightarrow}(B)))\, d\mu_{g \circ f_A^{-1}} =$$
$$\int_{\mathbf{A}_{g \circ f_A^{-1}}}^{\odot} \varphi_M(g^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(Z), g^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(f_A^{\rightarrow}(B)))\, d\mu_{g \circ f_A^{-1}},$$

where we consider $\mathcal{S}'(M)(Z, f_A^{\rightarrow}(B)) = \mathcal{S}(M)(Z, f_A^{\rightarrow}(B)) = \mathbf{A}$. This equality may be rewritten as follows

$$\int^{\odot}_{g^{\rightarrow}(\mathcal{S}'(M)(A,B))} \varphi_M(f^{\rightarrow}(A), f^{\rightarrow}(B))\ d\mu_g = \int^{\odot}_{g^{\rightarrow}(\mathcal{S}'(M)(A,B))} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow}(B))\ d\mu_g,$$

where, by the definition of $\mathcal{S}'$, we have

$$\mathbf{A}_{g\circ f_A^{-1}} = g^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(\mathbf{A}) = g^{\rightarrow} \circ (f_A^{-1})^{\rightarrow}(\mathcal{S}(M)(Z, 1_\emptyset)) = g^{\rightarrow}(\mathcal{S}'(M)(A, B)).$$

Using this equality and $\mathcal{S}'(M)(f^{\rightarrow}(A), f^{\rightarrow}(B)) = g^{\rightarrow}(\mathcal{S}'(M)(A, B))$, we obtain

$$Q'_M(A, B) = \int^{\odot}_{\mathcal{S}'(M)(A,B)} \varphi_M(A, B)\ d\mu = \int^{\odot}_{g^{\rightarrow}(\mathcal{S}'(M)(A,B))} \varphi_M(g^{\rightarrow}(A), g^{\rightarrow}(B))\ d\mu_g = \int^{\odot}_{g^{\rightarrow}(\mathcal{S}'(M)(A,B))} \varphi_M(f^{\rightarrow}(A), f^{\rightarrow}(B))\ d\mu_g = \int^{\odot}_{\mathcal{S}'(M)(f^{\rightarrow}(A),f^{\rightarrow}(B))} \varphi_M(f^{\rightarrow}(A), f^{\rightarrow}(B))\ d\mu_g = Q'_M(f^{\rightarrow}(A), f^{\rightarrow}(B)),$$

where the first equality is a consequence of Theorem 2.5 and Proposition 2.8. Hence, Q' is permutation invariant.

Finally, we will prove that $Q_M(A, B) = Q'_M(A, B)$ for any $A, B \in \mathcal{F}(M)$. Let $A, B \in \mathcal{F}(M)$ be arbitrary fuzzy sets. Put $h([A]) = Z$ and $f_A = r(A, Z)$. Since Q is permutation invariant, then

$$Q'_M(A, B) = Q'_M(f_A^{\rightarrow}(A), f_A^{\rightarrow}(B)) = Q'_M(Z, f_A^{\rightarrow}(B)) = Q_M(Z, f_A^{\rightarrow}(B)) = Q_M((f_A^{-1})^{\rightarrow}(Z), (f_A^{-1})^{\rightarrow}(f_A^{\rightarrow}(B))) = Q_M(A, B),$$

where the equality $Q'_M(Z, f_A^{\rightarrow}(B)) = Q_M(Z, f_A^{\rightarrow}(B))$ follows from the fact that $\mathcal{S}'(M)(Z, f_A^{\rightarrow}(B)) = \mathcal{S}(M)(Z, f_A^{\rightarrow}(B))$. Hence and from the equality $Q_\emptyset(\emptyset, \emptyset) = Q'_\emptyset(\emptyset, \emptyset)$, both fuzzy quantifiers coincide and the proof is finished. □

It should be noted that the weak pi-closeness in the first component of the functional $\mathcal{S}$ does not imply the permutation invariance of fuzzy quantifiers definable by fuzzy measures, in general. This is also the reason why the proof of the previous theorem is more complicated than the proof of Theorem 3.2. An example of a fuzzy quantifier determined by a functional weakly pi-closed in the first component, that is not permutation invariant, is given in the following example.

Example 3.5. Let $\mathbf{L}$ be a complete residuated lattice with the support $[0, 1]$ and N be the set of natural numbers with zero. Let us define $\mathcal{S}$ for N as follows

$$\mathcal{S}(N)(A, B) = \begin{cases} (1_N, \mathcal{F}(1_N), \overline{\mu_{1_N}}), & \text{if } A = 1_N, \\ (1_N, \{1_\emptyset, 1_N\}, \mu), & \text{otherwise,} \end{cases}$$

where $\overline{\mu_{1_N}}$ is defined in Example 2.3 and μ is the trivial fuzzy measure.

One may simply check that $\mathcal{S}(N)(A,-)$ is a constant mapping and, for any $A,B,C \in \mathcal{F}(N)$, the equality $\mathcal{S}(N)(f^{\rightarrow}(A),C) = f^{\rightarrow}(\mathcal{S}(N)(A,B))$ holds for at least one permutation f on N.[8]

Let Q be a fuzzy quantifier determined by $(\mathcal{S},\varphi)$ with $\mathcal{S}(N)(-,-)$ defined above and $\varphi = \cap$. Since $\mathcal{S}(N)(A,-)$ is a constant mapping and the condition to be weakly pi-closed in the first component is satisfied by $\mathcal{S}$ for N, we may suppose that the same holds for other universes, and thus Q is definable by fuzzy measures and, moreover, $\mathcal{S}$ is weakly pi-closed in the first component. Let E and P denote the sets of even and prime numbers, respectively. Clearly, there exists f on N such that $f^{\rightarrow}(1_E) = 1_P$. Then,

$$Q_M(1_N,1_E) = \int^{\odot}_{\mathcal{S}(M)(1_N,1_E)} \varphi(1_N,1_E)\ d\overline{\mu_{1_N}} = \int^{\odot}_{\mathcal{S}(M)(1_N,1_E)} 1_E\ d\overline{\mu_{1_N}} =$$

$$\mu_{1_N}(1_E) = 0.5 \neq 0 = \mu_{1_N}(1_P) =$$

$$\int^{\odot}_{\mathcal{S}(M)(1_N,1_E)} 1_P\ d\overline{\mu_{1_N}} = \int^{\odot}_{\mathcal{S}(M)(1_N,1_E)} \varphi(1_N,1_P)\ d\overline{\mu_{1_N}} = Q_M(1_N,1_P),$$

where we use

$$\int^{\odot}_{\mathcal{S}(M)(1_N,1_X)} 1_X\ d\overline{\mu_{1_N}} = \bigvee_{Y\in\mathcal{F}^{-}_{1_N}} \bigwedge_{m\in\mathrm{Supp}(Y)} (1_X(m) \odot \overline{\mu_{1_N}}(1_X)) = \overline{\mu_{1_N}}(1_X),$$

which holds for any $X \subseteq N$ and $\overline{\mu_{1_N}}(1_E) = 0.5$ and $\overline{\mu_{1_N}}(1_P) = 0$ (see Example 2.3). Thus, although Q is definable by fuzzy measures and $\mathcal{S}$ is weakly pi-closed in the first component, Q is not permutation invariant.

Though we have proved two interesting statements for fuzzy quantifiers definable by fuzzy measures stating sufficient conditions for being permutation invariant and being determined by a weakly pi-closed in the first component functional $\mathcal{S}$, we cannot provide an analogous equivalence to that in Corollary 3.4, in general.

In the rest part of this subsection, we restrict ourselves to complete residuated lattices in which $\odot$ is distributive over $\bigwedge$ and $\bigvee$. Under this presumption, we will show that a type of such equivalence for fuzzy quantifiers definable by fuzzy measures may be proved.

The main idea is based on the possibility to construct a "bigger" fuzzy measure spaces that contain the original ones and, moreover, their isomorphic images with respect to a permutation group (see Theorem 2.4). Roughly speaking, this trick allows us to replace a fuzzy measure space $\mathbf{A} = (A,\mathcal{F},\mu)$ for which there exists a permutation f from a suitable permutation group G, where

[8]Note that if $A \neq 1_N$, then $\mathcal{S}(N)(f^{\rightarrow}(A),C) = f^{\rightarrow}(\mathcal{S}(N)(A,B))$ holds even for any permutation f on N, because $f^{\rightarrow}(1_N,\{1_\emptyset,1_N\},\mu) = (1_N,\{1_\emptyset,1_N\},\mu)$ (see Definition 2.5).

$\varphi_M(x) \circ g^{-1} = \varphi_M(x)$ holds for some x and any $g \in G$, such that $f^{\rightarrow}(\mathbf{A}) \neq \mathbf{A}$, by a "bigger" fuzzy measure space $\mathbf{B}$ that is closed under isomorphisms with respect to G. Furthermore, due to the equality $\varphi_M(x) \circ g^{-1} = \varphi_M(x)$ that is satisfied for any $g \in G$ we obtain by Theorem 2.6

$$\int_{\mathbf{A}}^{\odot} \varphi_M(x)\ d\mu = \int_{\mathbf{B}}^{\odot} \varphi_M(x)\ d\nu.$$

Since the new fuzzy measure space $\mathbf{B}$ is closed under isomorphisms with respect to a permutation group G, we can require a stronger property of the functional $\mathcal{S}$ to characterize the permutation invariance. This property, in essence, imitates the permutation invariance for fuzzy quantifiers.

Definition 3.8. We say that a functional $\mathcal{S}$ is *pi-closed* if, for any non-empty universe M, any fuzzy sets $A, B \in \mathcal{F}(M)$ and any permutation $f \in \mathrm{Perm}(M)$, we have

$$f^{\rightarrow}(\mathcal{S}(M)(A,B)) = \mathcal{S}(M)(f^{\rightarrow}(A), f^{\rightarrow}(B)). \tag{35}$$

An equivalent definition is as follows.

Proposition 3.10. *A functional $\mathcal{S}$ is pi-closed if and only if, for any non-empty universe M and $A, B \in \mathcal{F}(M)$, the equality*

$$\mathcal{S}(M)(f^{\rightarrow}(A), f^{\rightarrow}(B)) = f^{\rightarrow}(\mathcal{S}(M)(A,B))$$

holds for any permutation f on M.

PROOF. Obvious. □

A simple but very useful example of a pi-closed functional is a functional $\mathcal{S}$ defined by $\mathcal{S}(M)(-,-) = (M, \mathcal{F}, \mu)$ for any non-empty universe M, provided that

$$\mu(Y) = \mu(f^{\rightarrow}(Y))$$

holds for any $f \in \mathrm{Perm}(M)$ and $Y \in \mathcal{F}$.

Example 3.6. We have seen that, for the functional $S^r(M)$ from Example 3.1, the equation (35) holds for any finite universe M and any permutation $f \in \mathrm{Perm}(M)$. Hence, $\mathcal{S}^r$ is pi-closed.

Proposition 3.11. *If $\mathcal{S}$ is pi-closed, then the set*

$$\mathcal{A}_M = \{\mathcal{S}(M)(A,B) \mid A, B \in \mathcal{F}(M)\} \tag{36}$$

forms a closed under isomorphisms system of fuzzy measure spaces for any non-empty universe M.

Proof. If $\mathbf{A} = (A, \mathcal{F}, \mu)$ and $\mathbf{A} \in \mathcal{A}_M$, then there exists $x \in \mathcal{F}(M)^2$ such that $\mathcal{S}(M)(x) = \mathbf{A}$. Since $\mathcal{S}$ is pi-closed, then

$$f^{\rightarrow}(\mathbf{A}) = f^{\rightarrow}(\mathcal{S}(M)(x)) = \mathcal{S}(M)(f^{\rightarrow}(x)) = \mathbf{A}'$$

for any permutation f on M. From the definition of $\mathcal{A}_M$, we obtain that $f^{\rightarrow}(\mathbf{A}) \in \mathcal{A}_M$ for any permutation f on M and $\mathbf{A} \in \mathcal{A}_M$. According to Lemma 2.2, the set $\mathcal{A}_M$ is a closed system of fuzzy measure spaces. □

Remark 3.7. One could note that the opposite implication in not true and the pi-closed functionals define only a special class of closed under isomorphisms systems of fuzzy measure spaces.

Clearly, each pi-closed functional is also weakly pi-closed. Hence, the following theorem is a straightforward consequence of Theorem 3.2.

Theorem 3.12. *Let Q be determined by $(\mathcal{S}, \varphi)$. If $\mathcal{S}$ is pi-closed, then Q is permutation invariant.*

Note that we cannot provide an analogous statement to that in Theorem 3.3, because the functional $\mathcal{S}'$ constructed in the proof of that theorem is not pi-closed in general (it is only weakly pi-closed). Nevertheless, the proposition can be proved under the declared presumption on the distributivity of $\odot$ over arbitrary meets and joins as follows.

Theorem 3.13. *Let $\mathbf{L}$ be a complete residuated lattice such that $\odot$ is distributive over $\bigwedge$ and $\bigvee$. If Q determined by $(\mathcal{S}, \varphi)$ is permutation invariant, then there exists a fuzzy quantifier Q' determined by $(\mathcal{S}', \varphi)$ such that $\mathcal{S}'$ is pi-closed and Q coincides with Q'.*

Proof. The aim of the proof is to construct a pi-closed functional $\mathcal{S}'$ based on fuzzy measure spaces that are closed under isomorphisms over groups G_x of permutations having the property $f^{\rightarrow}(x) = x$, and then to show that the fuzzy quantifier Q' determined by the functionals $\mathcal{S}'$ and φ coincides with Q.

To ensure $Q_M = Q'_M$ for $M = \emptyset$, it is sufficient to put $Q'_\emptyset(\emptyset, \emptyset) = Q_\emptyset(\emptyset, \emptyset)$. Now, let us suppose that $M \neq \emptyset$. Let $x, y \in \mathcal{F}(M)^2$. We will write $x \sim y$ if there exists a permutation f on M such that $f^{\rightarrow}(x) = y$. Further, denote by H_{xy} the set of all permutations f on M such that $f^{\rightarrow}(x) = y$. One can see that H_{xy} forms a permutation group on M if $x = y$.[9] Denote this permutation group by G_x, i.e., $G_x = H_{xx}$. Due to Proposition 2.8, we have $\varphi_M(x) \circ f = \varphi_M((f^{-1})^{\rightarrow}(x)) = \varphi_M(x)$ for any permutation f from G_x. Then,

$$\int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu = \int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x) \circ f\ d\mu, \tag{37}$$

[9] Obviously, this is not true for arbitrary $x, y \in \mathcal{F}(M)^2$.

holds for any $f \in G_x$, where we put $\mathcal{S}(M)(x) = (A, \mathcal{F}, \mu)$. According to Theorem 2.6, there exists a fuzzy measure space $\mathbf{B}_x = (B_x, \mathcal{G}_x, \nu_x)$ closed under isomorphisms with respect to G_x, i.e., $f^{\rightarrow}(\mathbf{B}_x) = \mathbf{B}_x = \mathbf{B}_{f^{\rightarrow}(x)}$ for any $f \in G_x$, such that

$$\int_{\mathcal{S}(M)(x)}^{\odot} \varphi_M(x)\ d\mu = \int_{\mathbf{B}_x}^{\odot} \varphi_M(x)\ d\nu_x. \tag{38}$$

Let $x, z \in \mathcal{F}(M)^2$ such that $x \sim z$ and $f, g, t \in H_{xz}$. Then, $g^{\rightarrow} \circ (t^{-1})^{\rightarrow}(\mathbf{B}_z) = f^{\rightarrow} \circ (t^{-1})^{\rightarrow}(\mathbf{B}_z)$. In fact, we have

$$g^{\rightarrow} \circ (t^{-1})^{\rightarrow}(\mathbf{B}_z) = \mathbf{B}_z = f^{\rightarrow} \circ (t^{-1})^{\rightarrow}(\mathbf{B}_z), \tag{39}$$

since $g \circ t^{-1}, f \circ t^{-1} \in G_z$.

Now, let us denote $\mathcal{Q} = \mathcal{F}(M)^2 \backslash \sim$, consider a choice function $h : \mathcal{Q} \to \mathcal{F}(M)^2$, i.e., $h([x]) \in [x]$ for any $[x] \in \mathcal{Q}$, and define

$$\mathcal{R} = \{(x, y) \mid x, y \in \mathcal{F}(M)^2 \text{ and } x \sim y\}.$$

Further, let us consider a mapping $r : \mathcal{R} \to \mathrm{Perm}(M)$ that satisfies

$$r(x, y) = \begin{cases} f, & \text{if } x \neq y \text{ and } f^{\rightarrow}(x) = y, \\ 1_M, & \text{otherwise.} \end{cases}$$

Note that $r(x, y) \in H_{xy}$. Denote $r(x, h([x])) = f_x$ and define

$$\mathcal{S}'(M)(x) = (f_x^{-1})^{\rightarrow}(\mathbf{B}_{h([x])}). \tag{40}$$

Clearly, $\mathcal{S}'(M)(h([x])) = \mathbf{B}_{h([x])}$. Let us show that $\mathcal{S}'$ defined by (40) is a pi-closed functional.

Let $x, y \in \mathcal{F}(M)^2$ such that $x \sim y$ and put $h([x]) = h([y]) = z$. According to the definition of $\mathcal{S}'$, we have

$$\begin{aligned} \mathcal{S}'(M)(x) &= (f_x^{-1})^{\rightarrow}(\mathbf{B}_z), \\ \mathcal{S}'(M)(y) &= (f_y^{-1})^{\rightarrow}(\mathbf{B}_z), \end{aligned} \tag{41}$$

where $f_x = r(x, z)$ and $f_y = r(y, z)$. Let f be an arbitrary permutation for which $f^{\rightarrow}(x) = y$, i.e., $f \in H_{xy}$. Clearly, we have $f_y \circ f \in H_{xz}$. Since also $f_x \in H_{xz}$, then, by (39), we obtain

$$\mathbf{B}_z = f_x^{\rightarrow} \circ (f_x^{-1})^{\rightarrow}(\mathbf{B}_z) = (f_y \circ f)^{\rightarrow} \circ (f_x^{-1})^{\rightarrow}(\mathbf{B}_z) = f_y^{\rightarrow} \circ f^{\rightarrow} \circ (f_x^{-1})^{\rightarrow}(\mathbf{B}_z).$$

Hence, we have

$$(f_y^{-1})^{\rightarrow}(\mathbf{B}_z) = f^{\rightarrow} \circ (f_x^{-1})^{\rightarrow}(\mathbf{B}_z),$$

which can be rewritten as

$$\mathcal{S}'(M)(f^{\rightarrow}(x)) = (f_y^{-1})^{\rightarrow}(\mathbf{B}_z) = f^{\rightarrow} \circ (f_x^{-1})^{\rightarrow}(\mathbf{B}_z) = f^{\rightarrow}(\mathcal{S}'(M)(x)).$$

Thus, $\mathcal{S}'$ is a pi-closed functional.

Let us denote by Q' the fuzzy quantifier determined by $(\mathcal{S}', \varphi)$. Due to Theorem 3.12, we have $Q' \in \mathrm{PI}$. To show that $Q_M = Q'_M$, let $x \in \mathcal{F}(M)^2$ and, firstly, let us suppose that $h([x]) = x$. Then, $\mathcal{S}'(M)(x) = \mathbf{B}_x$ and, according to (38), we have

$$Q_M(x) = \int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu =$$
$$\int^{\odot}_{\mathbf{B}_x} \varphi_M(x)\ d\nu_x = \int^{\odot}_{\mathcal{S}'(M)(x)} \varphi_M(x)\ d\nu_x = Q'_M(x).$$

Now, let $x \in \mathcal{F}(M)^2$ be arbitrary. Since $Q, Q' \in \mathrm{PI}$ and $f^{\rightarrow}_x(x) = h([x])$, then, by the previous equality, we have

$$Q_M(x) = Q_M(f^{\rightarrow}_x(x)) = Q_M(h([x])) = Q'_M(h([x])) = Q'_M(f^{\rightarrow}_x(x)) = Q'_M(x).$$

From this and from the equality $Q_\emptyset = Q'_\emptyset$, we obtain the coincidence of Q and Q'. □

Corollary 3.14. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$. *A fuzzy quantifier* Q *determined by* $(\mathcal{S}, \varphi)$ *is permutation invariant if and only if there exists a pi-closed functional* $\mathcal{S}'$ *such that the fuzzy quantifier* Q' *determined by* $(\mathcal{S}', \varphi)$ *coincides with* Q.

Now, we will show that an analogous proposition presented above in the corollary holds also for fuzzy quantifiers definable by fuzzy measures. The following theorem is a straightforward consequence of Theorem 3.12.

Theorem 3.15. *Let* Q *determined by* $(\mathcal{S}, \varphi)$ *be definable by fuzzy measures. If* $\mathcal{S}$ *is pi-closed, then* Q *is permutation invariant.*

A weaker opposite implication is as follows (cf. Theorem 3.9).

Theorem 3.16. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$ *and* Q *be determined by* $(\mathcal{S}, \varphi)$. *If* Q *is definable by fuzzy measures and permutation invariant, then there exists a fuzzy quantifier* Q' *determined by* $(\mathcal{S}', \varphi)$ *such that* Q' *is definable by fuzzy measures,* $\mathcal{S}'$ *is pi-closed and* Q' *coincides with* Q.

Proof. The proof is more or less analogous to that of Theorem 3.13, only not one, but two equivalences will be constructed here. The first equivalence ensures the definability of the fuzzy quantifier by fuzzy measures, i.e., $Q_M(A, -)$ is a constant mapping for any M and $A \in \mathcal{F}(M)$, and the second one makes the fuzzy quantifier invariant with respect to permutations. Again, to ensure $Q_M = Q'_M$ for $M = \emptyset$, it is sufficient to put $Q'_\emptyset(\emptyset, \emptyset) = Q_\emptyset(\emptyset, \emptyset)$.

Let $M \neq \emptyset$ and $x, y \in \mathcal{F}(M)^2$. We will write $x \stackrel{*}{=} y$ if the first components of x and y are equal, i.e., if $x = (A, B)$ and $y = (C, D)$, then $A = C$. One could note that $\stackrel{*}{=}$ is an equivalence on $\mathcal{F}(M)$, and if $x = y$, then $x \stackrel{*}{=} y$.

Let H^*_{xy} denote the set of all permutations f on M for which $f^{\rightarrow}(x) \stackrel{*}{=} y$ and G^*_x the group of all permutations f on M satisfying $f^{\rightarrow}(x) \stackrel{*}{=} x$. One could simply prove that $G^*_x = H^*_{xx}$.[10]

Let Q determined by $(\mathcal{S}, \varphi)$ be definable by fuzzy measures, i.e., $\mathcal{S}(M)(A, -)$ is a constant mapping for any $A \in \mathcal{F}(M)$. This condition can be equivalently expressed as $\mathcal{S}(M)(x) = \mathcal{S}(M)(y)$, whenever $x \stackrel{*}{=} y$.

Let $x \in \mathcal{F}(M)$, $f \in G^*_x$ and put $\mathcal{S}(M)(x) = (A, \mathcal{F}, \mu)$. Then,

$$\int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu = \int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x) \circ f\ d\mu. \tag{42}$$

In fact, using Theorem 2.5 and the facts that $\mathcal{S}(M)((f^{-1})^{\rightarrow}(x)) = \mathcal{S}(M)(x)$, $Q_M(x) = Q_M((f^{-1})^{\rightarrow}(x))$ and $\varphi_M((f^{-1})^{\rightarrow}(x)) = \varphi_M(x) \circ f$, we have

$$\int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu = Q_M(x) = Q_M((f^{-1})^{\rightarrow}(x)) =$$

$$\int^{\odot}_{\mathcal{S}(M)((f^{-1})^{\rightarrow}(x))} \varphi_M((f^{-1})^{\rightarrow}(x))\ d\mu = \int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x) \circ f\ d\mu.$$

Since $\odot$ is distributive over $\bigwedge$ and $\bigvee$, then, supposing (42) in Theorem 2.6 to each $x \in \mathcal{F}(M)^2$, there exists a fuzzy measure space $\mathbf{B}_x = (B_x, \mathcal{G}_x, \nu_x)$ closed under isomorphisms with respect to G^*_x for which

$$\int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu = \int^{\odot}_{\mathbf{B}_x} \varphi_M(x)\ d\nu_x. \tag{43}$$

Let $\mathcal{P} = \mathcal{F}(M) \backslash \stackrel{*}{=}$, i.e., $y \in [x]$ if $x \stackrel{*}{=} y$. We will write $[x] \stackrel{*}{\sim} [y]$ if there exists a permutation f on M such that $f^{\rightarrow}(x) \stackrel{*}{=} y$. It is easy to see that the relation $\stackrel{*}{\sim}$ is an equivalence on $\mathcal{P}$ and let $\mathcal{Q} = \mathcal{P} \backslash \stackrel{*}{\sim}$, i.e., $[y] \in [[x]]$ if $[x] \stackrel{*}{\sim} [y]$.

Further, let us consider two choice functions

$$q : \mathcal{Q} \to \mathcal{P},$$
$$p : \mathcal{P} \to \mathcal{F}(M)^2,$$

i.e., $q([[x]]) \in [[x]]$ for any $[[x]] \in \mathcal{Q}$ and $p([x]) \in [x]$ for any $[x] \in \mathcal{P}$, and put $h = p \circ q$. Define

$$\mathcal{R}^* = \{([x], [y]) \mid x, y \in \mathcal{F}(M) \text{ and } [x] \stackrel{*}{\sim} [y]\},$$
$$\mathcal{R} = \{(x, y) \mid x, y \in \mathcal{F}(M) \text{ and } [x] \stackrel{*}{\sim} [y]\},$$

$r^* : \mathcal{R}^* \to \mathrm{Perm}(M)$ in such a way that

$$r^*([x],[y]) = \begin{cases} f, & \text{if } [x] \neq [y] \text{ and } f^{\rightarrow}(x) \stackrel{*}{=} y, \\ 1_M, & \text{otherwise,} \end{cases}$$

[10] In comparison with the constructions of H_{xy} and G_x in the proof of Theorem 3.13, the second components of x and y are ignored here.

and $r : \mathcal{R} \to \mathrm{Perm}(M)$ by $r(x,y) = r^*([x],[y])$. One could simply verify that the definition of r^* does not depend on the choice of x and y (note that the first components of x and z with $z \in [x]$ are the same) and clearly r^* (and hence r) can be defined in different ways. Moreover, $r(x,y) \in H^*_{xy}$, because $r(x,y)^{\to}(x) \stackrel{*}{=} y$. Denote $r(x, h([[x]])) = f_x$ and define

$$\mathcal{S}'(M)(x) = (f_x^{-1})^{\to}(\mathbf{B}_{h([[x]])}), \tag{44}$$

where f_x^{-1} is the inverse mapping to f_x. Clearly, $\mathcal{S}'(M)(h([[x]])) = \mathbf{B}_{h([[x]])}$, because $r(h([[x]]), h([[x]])) = 1_M$.

Firstly, we will show that $\mathcal{S}'$ defined by (44) is a pi-closed functional, and thus the fuzzy quantifier determined by $(\mathcal{S}', \varphi)$ is permutation invariant. Let $x \in \mathcal{F}(M)^2$ and $f \in \mathrm{Perm}(M)$. Clearly, $[[x]] = [[f^{\to}(x)]]$. Put $f^{\to}(x) = y$ and $h([[x]]) = h([[y]]) = z$. According to the definition of $\mathcal{S}'$, we have

$$\begin{aligned} \mathcal{S}'(M)(x) &= (f_x^{-1})^{\to}(\mathbf{B}_z), \\ \mathcal{S}'(M)(y) &= (f_y^{-1})^{\to}(\mathbf{B}_z), \end{aligned} \tag{45}$$

where $f_x = r(x,z)$ and $f_y = r(y,z)$. Since $f_y \circ f \in H^*_{xz}$ and $f_x \in H^*_{xz}$, then, using (39) in the proof of Theorem 3.13 that also holds for H^*_{xz},[11] we obtain

$$\mathbf{B}_z = f_x^{\to} \circ (f_x^{-1})^{\to}(\mathbf{B}_z) = (f_y \circ f)^{\to} \circ (f_x^{-1})^{\to}(\mathbf{B}_z) = f_y^{\to} \circ f^{\to} \circ (f_x^{-1})^{\to}(\mathbf{B}_z).$$

Hence, we have

$$(f_y^{-1})^{\to}(\mathbf{B}_z) = f^{\to} \circ (f_x^{-1})^{\to}(\mathbf{B}_z),$$

which implies

$$\mathcal{S}'(M)(f^{\to}(x)) = (f_y^{-1})^{\to}(\mathbf{B}_z) = f^{\to} \circ (f_x^{-1})^{\to}(\mathbf{B}_z) = f^{\to}(\mathcal{S}'(M)(x)),$$

and $\mathcal{S}'$ is a pi-closed functional. Further, we will show that $\mathcal{S}'(M)(x) = \mathcal{S}'(M)(y)$, whenever $x \stackrel{*}{=} y$, and thus the fuzzy quantifier determined by $(\mathcal{S}', \varphi)$ is definable by fuzzy measures. If $x, y \in \mathcal{F}(M)^2$ and $x \stackrel{*}{=} y$, then $[x] = [y]$ and also $[[x]] = [[y]]$. Hence, we obtain $f_x = r(x,z) = r^*([x],[z]) = r^*([y],[z]) = r(y,z) = f_y$, which implies

$$\mathcal{S}'(M)(x) = (f_x^{-1})^{\to}(\mathbf{B}_z) = (f_y^{-1})^{\to}(\mathbf{B}_z) = \mathcal{S}'(M)(y).$$

Finally, let Q' be a fuzzy quantifier determined $(\mathcal{S}', \varphi)$ that is, as we have proved, permutation invariant and definable by fuzzy measures. We will show that $Q'_M = Q_M$. Let $x \in \mathcal{F}(M)^2$ and, firstly, let us suppose that $h([[x]]) = x$.

[11] It is sufficient to consider $f_x \circ f_x^{-1}, f_y \circ f \circ f_x^{-1} \in G^*_z$ and to apply $g^{\to}(\mathbf{B}_z) = \mathbf{B}_z$ which holds for any $g \in G^*_z$.

Since $\mathcal{S}'(M)(x) = \mathbf{B}_x$, then, according to (43), we have

$$Q_M(x) = \int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu =$$
$$\int^{\odot}_{\mathbf{B}_x} \varphi_M(x)\ d\nu_x = \int^{\odot}_{\mathcal{S}'(M)(x)} \varphi_M(x)\ d\nu_x = Q'_M(x).$$

Now, let $x \in \mathcal{F}(M)^2$ be arbitrary. Since $Q, Q' \in \mathrm{PI}$ and $f_x^{\rightarrow}(x) = h([[x]])$, then

$$Q_M(x) = Q_M(f_x^{\rightarrow}(x)) = Q_M(h([[x]])) = Q'_M(h([[x]])) = Q'_M(f_x^{\rightarrow}(x)) = Q'_M(x).$$

Hence and from the equality $Q_\emptyset = Q'_\emptyset$, we obtain the coincidence of Q and Q'. □

Now, we can present the main goal of this part stating a relation between permutation invariance and pi-closed functionals for fuzzy quantifiers definable by fuzzy measures.

Corollary 3.17. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*. A fuzzy quantifier* Q *definable by fuzzy measures is permutation invariant if and only if there exists a pi-closed functional* $\mathcal{S}$ *such that the fuzzy quantifier* Q' *determined by* $(\mathcal{S}, \varphi)$ *is definable by fuzzy measures and coincides with* Q*.*

In practice, it seems to be profitable to use fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$ with $\mathcal{S}(M)$ as a constant mapping, i.e., $\mathcal{S}(M)(A, B) = \mathbf{M}$ for any non-empty universe M and $A, B \in \mathcal{F}(M)$. If such a fuzzy quantifier is permutation invariant, then there is a question, whether $\mathbf{M}$ is a cardinal fuzzy measure space.

Definition 3.9. We say that a fuzzy quantifier Q determined by $(\mathcal{S}, \varphi)$ is *locally cardinal* if $\mathcal{S}(M)(-,-) = \mathbf{M}$ is a constant mapping for any non-empty universe M, and $\mathcal{S}$ is pi-closed.

A straightforward consequence of the definition of locally cardinal fuzzy quantifier is the following statement showing a close relation between locally cardinal fuzzy quantifiers and cardinal fuzzy measure spaces.

Lemma 3.18. *A fuzzy quantifier* Q *determined by* $(\mathcal{S}, \varphi)$ *is locally cardinal if and only if* $\mathcal{S}(M)(-,-)$ *is a cardinal fuzzy measure space for any non-empty universe* M*.*

Note that the term "locally" in the previous definition of a cardinal fuzzy quantifier is motivated by the fact that "to be pi-closed" is only a local property, which is limited to a universe. It means that we can have a fuzzy quantifier defined by cardinal fuzzy measure spaces, but the same cardinality of universes does not imply the isomorphism of the fuzzy measure spaces over these universes.[12]

[12] Consider $|M| = |M'|$, but $f^{\rightarrow}(\mathcal{S}(M)(-,-)) \neq \mathcal{S}(M')(-,-)$ for a bijection $f : M \rightarrow M'$. Later, we will see that the locally cardinal quantifiers are not isomorphism invariant in general.

Since the locally cardinal fuzzy quantifiers are determined by pi-closed functionals $\mathcal{S}$, they are permutation invariant. The opposite implication is also true under the presumption of the distributivity of $\odot$ over arbitrary meets and joins. Note that the fuzzy quantifier (limited to $M = [0, 1]$) from Example 3.3 is permutation invariant and is definable by fuzzy measures (i.e., $\mathcal{S}(M)(-, -)$ is a constant mapping), but $\mathcal{S}(M)(-, -)$ is not a cardinal fuzzy measure space.

Theorem 3.19. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$ *and* Q *be a fuzzy quantifier determined by* $(\mathcal{S}, \varphi)$ *with* $\mathcal{S}(M)(-, -) = \mathbf{M}$ *for any non-empty universe* M*. If* Q *is permutation invariant, then there exists a locally cardinal fuzzy quantifier coinciding with* Q*.*

Proof. Put $Q'_\emptyset = Q_\emptyset$. Further, according to the presumption on $\mathcal{S}$, the fact that Q is permutation invariant and Proposition 2.8, we have

$$Q_M(x) = \int^{\odot}_{\mathcal{S}(M)(x)} \varphi_M(x)\ d\mu = \int^{\odot}_{\mathbf{M}} \varphi_M(x)\ d\mu =$$

$$\int^{\odot}_{\mathbf{M}} \varphi_M(f^{\rightarrow}(x))\ d\mu = \int^{\odot}_{\mathbf{M}} \varphi_M(x) \circ f^{-1}\ d\mu = Q_M(f^{\rightarrow}(x))$$

for any permutation f on M. Due to Corollary 2.7, there is a cardinal fuzzy measure space $\mathbf{M}'$ for each $M \neq \emptyset$ such that

$$\int^{\odot}_{\mathbf{M}} \varphi_M(x)\ d\mu = \int^{\odot}_{\mathbf{M}'} \varphi_M(x)\ d\nu. \tag{46}$$

Now, it is sufficient to define $\mathcal{S}'(M)(-, -) = \mathbf{M}'$ and to consider Q' determined by $(\mathcal{S}', \varphi)$. The coincidence of Q and Q' immediately follows from $Q_\emptyset = Q'_\emptyset$ and the equality of integrals in (46). □

Corollary 3.20. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*. A fuzzy quantifier* Q *determined by* $(\mathcal{S}, \varphi)$ *with* $\mathcal{S}(M)(-, -) = \mathbf{M}$ *for any non-empty universe* M *is permutation invariant if and only if there exists a locally cardinal fuzzy quantifier* Q' *coinciding with* Q*.*

Example 3.8. We already showed that the quantifier $\mathsf{many}_M(A, B)$ from Example 5.3 in [2] is permutation invariant (see Example 3.1). Analogously, this can be shown also for other quantifiers from [2], namely, every, some, at least half and few.

3.3. Isomorphism invariance

Let us investigate the property that extends the permutation invariance from one universe (a local property) to the class of all bijective universes (a global property). In our investigation, we will deal with the concept of classes and mappings and relations on classes. Because many questions and problems are similar to the case of the permutation invariance, we will not go into details unless there is a substantial difference with respect to the case of permutation invariance.

Definition 3.10. We say that a fuzzy quantifier Q is *isomorphism invariant* if, for any universe M, bijective mapping $f : M \to M'$ and $A, B \in \mathcal{F}(M)$, we have

$$Q_M(A, B) = Q_{M'}(f^{\to}(A), f^{\to}(B)). \tag{47}$$

The class of all isomorphism invariant fuzzy quantifiers is denoted by ISOM.

It is well-known that the isomorphism invariance is a natural extension of permutation invariance, which is fixed to one universe, to the class of universes having the same cardinality. To study the relations between ISOM and the fuzzy quantifiers determined by fuzzy measures, we need to extend the concept of the weakly pi-closed functional used in the case of PI.

Let us denote by $\mathrm{Bij}(M, M')$ the set of all bijections from M to M'. The following concept naturally extends the property of being weakly pi-closed (cf. Definition 3.6).

Definition 3.11. We say that a functional $\mathcal{S}$ is *weakly iso-closed* if, for any non-empty universes M, M', fuzzy sets $A, B \in \mathcal{F}(M)$ and $C, D \in \mathcal{F}(M')$ such that $(g^{\to}(A), g^{\to}(B)) = (C, D)$ for a bijection g of M onto M', there exists $f \in \mathrm{Bij}(M, M')$ for which $(f^{\to}(A), f^{\to}(B)) = (C, D)$ and

$$f^{\to}(\mathcal{S}(M)(A, B)) = \mathcal{S}(M')(C, D). \tag{48}$$

An equivalent definition is as follows.

Proposition 3.21. *A functional $\mathcal{S}$ is weakly iso-closed if and only if, for any non-empty bijective universes M, M', $A, B \in \mathcal{F}(M)$, the equality*

$$\mathcal{S}(M')(f^{\to}(A), f^{\to}(B)) = f^{\to}(\mathcal{S}(M)(A, B)) \tag{49}$$

holds for at least one bijection f of M onto M'.

Proof. Obvious. □

Recall that φ_M and $\varphi_{M'}$ are the equivalent operations if they are determined by one rl-operation ψ (see Definition 2.13).

Definition 3.12. We say that a functional φ is *weakly global*, if, for any non-empty isomorphic universes M and M', the operations φ_M and $\varphi_{M'}$ are the equivalent operations. If the operations φ_M and $\varphi_{M'}$ are equivalent for any M and M', we say that φ is *global*.

Note that the denotation "global" expresses the fact that there exists a one rl-operation ψ on L such that each φ_M is determined by ψ, and thus the definitions of operations φ_M are independent from their universes M. The weakness of the global property is more or less technical and may be used in many cases, when we need not suppose the existence of one rl-operation for all universes. The following theorem is a global version of Theorem 3.2.

Theorem 3.22. *Let Q be determined by $(\mathcal{S},\varphi)$. If $\mathcal{S}$ is weakly iso-closed and φ is weakly global, then Q is isomorphism invariant.*

Proof. The condition of being isomorphism invariant is trivially satisfied for $M=\emptyset$. Let $M\neq\emptyset\neq M'$, $x\in\mathcal{F}(M)^2$, $y\in\mathcal{F}(M')^2$ and $g^{\rightarrow}(x)=y$ for some $g\in \mathrm{Bij}(M,M')$. Since $\mathcal{S}$ is weakly iso-closed, then there exists $f\in\mathrm{Bij}(M,M')$ with $f^{\rightarrow}(x)=y$ and

$$f^{\rightarrow}(\mathcal{S}(M)(x))=\mathcal{S}(M')(f^{\rightarrow}(x)).$$

Since φ is weakly global, then, according to Theorem 2.5 and Proposition 2.9, we have

$$Q_M(x)=\int^{\odot}_{\mathcal{S}(M)(x)}\varphi_M(x)\;d\mu=\int^{\odot}_{f^{\rightarrow}(\mathcal{S}(M)(x))}\varphi_M(x)\circ f^{-1}\;d\mu'=$$
$$\int^{\odot}_{\mathcal{S}(M')(f^{\rightarrow}(x))}\varphi_{M'}(f^{\rightarrow}(x))\;d\mu'=Q_{M'}(f^{\rightarrow}(x)),$$

where $\varphi_M(x)\circ f^{-1}=\varphi_{M'}(f^{\rightarrow}(x))$ is applied. Since $f^{\rightarrow}(x)=g^{\rightarrow}(x)$, then $Q_M(x)=Q_{M'}(f^{\rightarrow}(x))=Q_{M'}(g^{\rightarrow}(x))$. Hence, Q is isomorphism invariant. □

Example 3.9. Let us define a functional $\mathcal{S}^{\star}$ for any finite universe M and $A,B\in\mathcal{F}(M)$ as follows:

$$\mathcal{S}^{\star}(M)(A,B)=\begin{cases}(1_M,\{1_{\emptyset},1_M\},\mu), & \text{if } 15\in M;\\ (1_M,\mathcal{F}(1_M),\mu^r_{1_M}), & \text{otherwise,}\end{cases}$$

where $(1_M,\mathcal{F}(1_M),\mu^r_{1_M})$ is defined in Example 2.1. Now, let us consider a finite fuzzy quantifier $Q^{\star}$ determined by $(\mathcal{S}^{\star},\varphi)$, where φ is an arbitrary functional (e.g., $\varphi=\cap$). It is obvious that $\mathcal{S}^{\star}$ is (weakly) pi-closed and that the fuzzy quantifier $Q^{\star}$ determined by $(\mathcal{S}^{\star},\varphi)$ is PI (and locally cardinal). However, $\mathcal{S}^{\star}$ is not weakly iso-closed. In fact, it is sufficient to consider universes $M=\{14,15,16\}$ and $M'=\{c,d,e\}$ for which evidently the condition (49) cannot be satisfied. Moreover, using these universes one may simply demonstrate that $Q^{\star}$ is not ISOM.

Theorem 3.23. *Let Q be determined by $(\mathcal{S},\varphi)$. If Q is isomorphism invariant, then there exists a fuzzy quantifier Q' determined by $(\mathcal{S}',\varphi')$ such that $\mathcal{S}'$ is weakly iso-closed, φ' is weakly global, and Q' coincides with Q.*

Proof. Put $Q'_{\emptyset}=Q_{\emptyset}$. Further, let Card denote the class of all cardinal numbers different from 0 expressed by the least ordinal numbers that have the same cardinality. It means that a cardinal number α is a well ordered transitive set. The cardinal numbers (as sets) will serve as representatives in the classes of all universes having the same cardinality used in our construction.

Let $\mathcal{S}'(\alpha)$ be the mapping defined in the same way as in the proof of Theorem 3.3 for any $\alpha\in$ Card. More precisely, we consider $\mathcal{S}'$ to be defined only for the universes that belong to CARD, $\mathcal{S}'$ for further universes will be defined later. Recall that, for any $x \in \mathcal{F}(\alpha)^2$, and a permutation g on α, there exists a permutation f on α such that $f^{\rightarrow}(x) = g^{\rightarrow}(x)$ and

$$f^{\rightarrow}(\mathcal{S}'(\alpha)(x)) = \mathcal{S}'(\alpha)(f^{\rightarrow}(x)). \tag{50}$$

Moreover, if Q'_α (limited to α) is defined by $\mathcal{S}'(\alpha)(-,-)$ and φ_α, we have $Q_\alpha = Q'_\alpha$ (see the proof of Theorem 3.3).[13]

Let $\mathrm{V} = \bigcup_{M \neq \emptyset} \mathcal{F}(M)$ and $\mathbf{R} \subseteq \mathrm{V} \times \mathrm{CARD}$ be a (class) relation naturally defined by $(M, \alpha) \in R$ if M and α have the same cardinality (i.e., there exists a bijective mapping of M onto α) and consider a mapping $\mathbf{F} : \mathbf{R} \to \mathrm{Bij}$, where Bij denotes the class of all bijective mappings such that

$$\mathbf{F}(M, \alpha) = \begin{cases} f, & \text{if } M \neq \alpha \text{ and } f \text{ is a bijection of } M \text{ onto } \alpha, \\ 1_M, & \text{otherwise.} \end{cases} \tag{51}$$

Define

$$\mathcal{S}'(M)(x) = (\mathbf{F}(M,\alpha)^{-1})^{\rightarrow}(\mathcal{S}'(\alpha)(\mathbf{F}(M,\alpha)^{\rightarrow}(x)). \tag{52}$$

Putting $f_M = \mathbf{F}(M, \alpha)$, we may rewrite (52) by

$$\mathcal{S}'(M)(x) = (f_M^{-1})^{\rightarrow}(\mathcal{S}'(\alpha)(f_M^{\rightarrow}(x))). \tag{53}$$

Let us show that $\mathcal{S}'$ defined by (53) for any non-empty universe is a weakly iso-closed functional. Let $f \in \mathrm{Bij}(M, M')$ and $\alpha = |M|$. Put $f_M = \mathbf{F}(M, \alpha)$ and $f_{M'} = \mathbf{F}(M', \alpha)$. If $x \in \mathcal{F}(M)^2$, $y \in \mathcal{F}(M')^2$ and $f^{\rightarrow}(x) = y$ for some $f \in \mathrm{Bij}(M, M')$, then there exists a permutation g on α such that $g^{\rightarrow}(f_M^{\rightarrow}(x)) = f_{M'}^{\rightarrow}(y)$ (note that $f_M^{\rightarrow}(x), f_{M'}^{\rightarrow}(x) \in \mathcal{F}(\alpha)$). In fact, denote $\alpha_x = f_M^{\rightarrow}(x)$ and $\alpha_y = f_{M'}^{\rightarrow}(y)$ and let $f \in \mathrm{Bij}(M, M')$. Then,

$$(f_{M'})^{\rightarrow} \circ f^{\rightarrow} \circ (f_M^{-1})^{\rightarrow}(\alpha_x) = \alpha_y.$$

Hence, it is sufficient to put $g = f_{M'} \circ f \circ f_M^{-1}$. From the construction of $\mathcal{S}'(\alpha)$, there exists $h \in \mathrm{Perm}(\alpha)$ such that $g^{\rightarrow}(f_M^{\rightarrow}(x)) = h^{\rightarrow}(f_M^{\rightarrow}(x)) = f_{M'}^{\rightarrow}(y)$ and

$$h^{\rightarrow}(\mathcal{S}'(\alpha)(f_M^{\rightarrow}(x))) = \mathcal{S}'(\alpha)(h^{\rightarrow}(f_M^{\rightarrow}(x))) = \mathcal{S}'(\alpha)(f_{M'}^{\rightarrow}(y)). \tag{54}$$

According to the definition of $\mathcal{S}'$, we have

$$\begin{aligned} \mathcal{S}'(\alpha)(f_M^{\rightarrow}(x)) &= f_M^{\rightarrow}(\mathcal{S}'(M)(x)), \\ \mathcal{S}'(\alpha)(f_{M'}^{\rightarrow}(y)) &= f_{M'}^{\rightarrow}(\mathcal{S}'(M')(y). \end{aligned} \tag{55}$$

Plugging (54) into the second formula in (55), we obtain

$$h^{\rightarrow}(\mathcal{S}'(\alpha)(f_M^{\rightarrow}(x))) = f_{M'}^{\rightarrow}(\mathcal{S}'(M')(y)).$$

[13]Note that in this part, we cannot define Q' for an arbitrary universe M, nevertheless, if we restrict ourselves to the sets from CARD and define Q' only over CARD, then Q' coincides with Q restricted to CARD.

After a simple manipulation with the previous formula one can derive

$$(f_M^{-1})^{\rightarrow}(\mathcal{S}'(\alpha)(f_M^{\rightarrow}(x))) = (f_M^{-1})^{\rightarrow} \circ (h^{-1})^{\rightarrow} \circ f_{M'}^{\rightarrow}(\mathcal{S}'(M')(y)),$$

and, using (53), we obtain

$$\mathcal{S}'(M)(x) = (f_M^{-1} \circ h^{-1} \circ f_{M'})^{\rightarrow}(\mathcal{S}'(M')(y)).$$

Putting $t = f_{M'}^{-1} \circ h \circ f_M$ (one can simply check that $t^{\rightarrow}(x) = y$), the previous formula can be rewritten as

$$t^{\rightarrow}(\mathcal{S}'(M)(x)) = \mathcal{S}'(M')(t^{\rightarrow}(x)). \tag{56}$$

Hence, the functional $\mathcal{S}'$ is weakly iso-closed.

If we define φ' in such way that φ'_M for any non-empty universe M is equivalent with φ_α, whenever $|M| = \alpha$, and thus the functional φ' is weakly global, we obtain, by Theorem 3.22, that Q' determined by $(\mathcal{S}', \varphi')$ is isomorphism invariant.

Finally, we will prove that Q' coincides with Q, i.e., $Q_M = Q'_M$ for any $M \neq \emptyset$. Let $x \in \mathcal{F}(M)^2$. Put $f_M = \mathbf{F}(M, \alpha)$. Since Q is isomorphism invariant and $Q_\alpha = Q'_\alpha$ for any $\alpha \in \text{CARD}$, we have

$$Q_M(x) = Q_\alpha(f^{\rightarrow}(x)) = Q'_\alpha(f^{\rightarrow}(x)) = Q'_M(x),$$

and the proof is finished. □

Corollary 3.24. *A fuzzy quantifier Q determined by $(\mathcal{S}, \varphi)$ is isomorphism invariant if and only if there exist a weakly iso-closed functional $\mathcal{S}'$ and a weakly global functional φ' such that the fuzzy quantifier Q' determined by $(\mathcal{S}', \varphi')$ coincides with Q.*

Obviously, the statement in Theorem 3.22 (that states: if $\mathcal{S}$ is weakly iso-closed and φ is weakly global, then Q determined by them is ISOM) is also true for fuzzy quantifiers definable by fuzzy measures. To show an opposite implication analogous to that in Theorem 3.9, we should introduce a concept of functional weakly iso-closed in the first component. Since the statement as well as its proof can be done analogously to that in Theorem 3.23 and again a relation between isomorphism invariant fuzzy quantifiers and weakly iso-closed in the first component functionals $\mathcal{S}$ and weakly global functionals φ cannot be proved, we will omit it here.

In the rest of this section, we will suppose the distributivity of $\odot$ over $\bigwedge$ and $\bigvee$. The following concept naturally extends the property of being pi-closed (cf. Definition 3.8) and, in essence, imitates the isomorphism invariance for fuzzy quantifiers.

Definition 3.13. We say that a functional $\mathcal{S}$ is *iso-closed* if, for any non-empty universes M, M', fuzzy sets $A, B \in \mathcal{F}(M)$ and bijection $f \in \text{Bij}(M, M')$, there is

$$f^{\rightarrow}(\mathcal{S}(M)(A, B)) = \mathcal{S}(M')(f^{\rightarrow}(A), f^{\rightarrow}(B)). \tag{57}$$

An equivalent definition is as follows.

Proposition 3.25. *A functional $\mathcal{S}$ is iso-closed if and only if, for any non-empty bijective universes M, M', $A, B \in \mathcal{F}(M)$, the equality*

$$\mathcal{S}(M')(f^{\rightarrow}(A), f^{\rightarrow}(B)) = f^{\rightarrow}(\mathcal{S}(M)(A, B))$$

holds for any bijection $f \in \mathrm{Bij}(M, M')$.

Proof. Obvious. □

Proposition 3.26. *Let $\mathcal{S}$ be iso-closed and $f : M \to M'$ be a bijective mapping. Then,*

(i) $\mathcal{A}_M = \{\mathcal{S}(M)(A, B) \mid A, B \in \mathcal{F}(M)\}$ forms a closed system of fuzzy measure spaces for any non-empty universe M,

(ii) there exists a bijective mapping $g : \mathcal{A}_M \to \mathcal{A}_{M'}$ such that $g \circ h^{\rightarrow} = h'^{\rightarrow} \circ g$, where $h' = f \circ h \circ f^{-1}$, holds for any permutation h on M.

Proof. The statement (i) can be proved analogously as in Lemma 3.11. Let h be a permutation on M. Since f and f^{-1} are bijective mappings, h' is a permutation on M'. Put $g(\mathbf{A}) = f^{\rightarrow}(\mathbf{A})$ for any $\mathbf{A} \in \mathcal{A}_M$. Then,

$$g \circ h^{\rightarrow}(\mathbf{A}) = f^{\rightarrow} \circ h^{\rightarrow}(\mathbf{A}) = (f \circ h)^{\rightarrow}(\mathbf{A}) =$$
$$(h' \circ f)^{\rightarrow}(\mathbf{A}) = h'^{\rightarrow} \circ f^{\rightarrow}(\mathbf{A}) = h'^{\rightarrow} \circ g(\mathbf{A})$$

for any $\mathbf{A} \in \mathcal{A}_M$. Hence, $g \circ h^{\rightarrow} = h'^{\rightarrow} \circ g$ and (ii) is proved. □

Because each iso-closed functional $\mathcal{S}$ is also weakly iso-closed, the following theorem is a straightforward consequence of Theorem 3.22.

Theorem 3.27. *Let Q be determined by $(\mathcal{S}, \varphi)$. If $\mathcal{S}$ is iso-closed and φ is weakly global, then Q is isomorphism invariant.*

A weaker opposite implication under the presumption on the distributivity of $\odot$ is as follows.

Theorem 3.28. *Let $\mathbf{L}$ be a complete residuated lattice such that $\odot$ is distributive over $\bigwedge$ and $\bigvee$. If Q determined by $(\mathcal{S}, \varphi)$ is permutation invariant, then there exists a fuzzy quantifier Q' determined by $(\mathcal{S}', \varphi')$ such that $\mathcal{S}'$ is iso-closed, φ' is weakly global, and Q' coincides with Q.*

Proof. Because the proof can be done analogously to that of Theorem 3.23, we will prove it briefly. Let $\mathcal{S}'(\alpha)$ be defined for any $\alpha \in$ Card by the pi-closed functional S' introduced in (40) of the proof of Theorem 3.13. For any $x \in \mathcal{F}(\alpha)^2$ and $f \in \mathrm{Perm}(\alpha)$, the equality $f^{\rightarrow}(\mathcal{S}'(\alpha)(x)) = \mathcal{S}'(\alpha)(f^{\rightarrow}(x))$ holds. Let us define $\mathcal{S}'(M)(x)$ by the formula (53).

Let $M \neq \emptyset$, $x \in \mathcal{F}(M)$ and $f \in \mathrm{Bij}(M, M')$. Put $f_M = \mathbf{F}(M, \alpha)$, $f_{M'} = \mathbf{F}(M', \alpha)$ and $h = f_{M'} \circ f \circ f_M^{-1}$. Clearly, $h^{\rightarrow}(f_M^{\rightarrow}(x)) = f_{M'}^{\rightarrow}(f^{\rightarrow}(x))$. Since

$$h^{\rightarrow}(\mathcal{S}'(\alpha)(f_M^{\rightarrow}(x)) = \mathcal{S}'(\alpha)(f_{M'}^{\rightarrow}(f^{\rightarrow}(x))), \tag{58}$$

then, according to (55), this equality may be rewritten as

$$h^{\rightarrow} \circ f_M^{\rightarrow}(\mathcal{S}'(M)(x)) = f_{M'}^{\rightarrow}(\mathcal{S}'(M')(f^{\rightarrow}(x))),$$

which implies

$$\mathcal{S}'(M')(f^{\rightarrow}(x)) = (f_{M'}^{-1})^{\rightarrow} \circ h^{\rightarrow} \circ f_M^{\rightarrow}(\mathcal{S}'(M)(x)) = f^{\rightarrow}(\mathcal{S}'(M)(x)).$$

Hence, $\mathcal{S}'$ is an iso-closed functional. The rest of the proof precisely follows the verification of the coincidence of Q' determined by $(\mathcal{S}', \varphi')$ and Q given in the proof of Theorem 3.23. □

A straightforward consequence of Theorems 3.27 and 3.28 is the following statement stating that the isomorphism invariant fuzzy quantifiers (determined by a pair of functionals) are closely related to the iso-closed and a weakly global functionals $\mathcal{S}$ and φ, respectively.

Corollary 3.29. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*. A fuzzy quantifier* Q *determined by* $(\mathcal{S}, \varphi)$ *is isomorphism invariant if and only if there exist an iso-closed functional* $\mathcal{S}'$ *and a weakly global functional* φ' *such that the fuzzy quantifier* Q' *determined by* $(\mathcal{S}', \varphi')$ *coincides with* Q*.*

In the last part of this section, we will study the isomorphism invariance for the fuzzy quantifiers definable by fuzzy measures and for globally cardinal fuzzy quantifiers (the locally cardinal fuzzy quantifiers have been introduced in Definition 3.9). The following theorem is a straightforward consequence of Theorem 3.27.

Theorem 3.30. *Let* Q *determined by* $(\mathcal{S}, \varphi)$ *be definable by fuzzy measures. If* $\mathcal{S}$ *is iso-closed and* φ *is weakly global, then* Q *is isomorphism invariant.*

A weaker opposite statement is as follows. Again, this statement needs the presumption on the distributivity of $\odot$.

Theorem 3.31. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$ *and* Q *be determined by* $(\mathcal{S}, \varphi)$*. If* Q *is definable by fuzzy measures and isomorphism invariant, then there exists a fuzzy quantifier* Q' *determined by* $(\mathcal{S}', \varphi')$ *such that* Q' *is definable by fuzzy measures,* $\mathcal{S}'$ *is iso-closed,* φ' *is weakly global, and* Q' *coincides with* Q*.*

Proof. Let us define $\mathcal{S}'(\alpha)$ for any $\alpha \in \mathrm{Card}$ using (44) in the proof of Theorem 3.16, define $\mathcal{S}'(M)(x)$ by (53) and φ' is derived from φ (using φ_α for $\alpha \in \mathrm{Card}$) to be weakly global. It is easy to see that to complete the proof it is sufficient to show that Q' determined by $(\mathcal{S}', \varphi')$ is definable by fuzzy measures, i.e., $\mathcal{S}'(M)(A, -)$ is a constant function for any $A \in \mathcal{F}(M)$. From the definition of $\mathcal{S}'(\alpha)$, we know that $\mathcal{S}'(\alpha)(A, -)$ is a constant function for any $A \in \mathcal{F}(\alpha)$. Let $M \neq \emptyset$, $A, B, C \in \mathcal{F}(M)$ be arbitrary fuzzy sets and put $f_M = \mathbf{F}(M, \alpha)$. Then,

$$\mathcal{S}'(M)(A, B) = (f_M^{-1})^{\rightarrow}(\mathcal{S}'(\alpha)(f^{\rightarrow}(A), f^{\rightarrow}(B))) =$$
$$(f_M^{-1})^{\rightarrow}(\mathcal{S}'(\alpha)(f^{\rightarrow}(A), f^{\rightarrow}(C))) = \mathcal{S}'(M)(A, C),$$

and, hence, $\mathcal{S}'(M)(A, -)$ is a constant mapping. □

A straightforward consequence of Theorems 3.30 and 3.31 is the following statement stating that, under the presumption on the distributivity of $\odot$, the isomorphism invariant fuzzy quantifiers definable by fuzzy measures are closely related to the iso-closed and weakly global functionals $\mathcal{S}$ and φ, respectively.

Corollary 3.32. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*. A fuzzy quantifier* Q *definable by fuzzy measures is isomorphism invariant if and only if there exists an iso-closed functional* $\mathcal{S}$ *and weakly global* φ *such that the fuzzy quantifier* Q' *determined by* $(\mathcal{S}, \varphi)$ *is definable by fuzzy measures and coincides with* Q*.*

One could note that a locally cardinal fuzzy quantifier need not be isomorphism invariant (cf. Example 3.9). Therefore, we establish a global variant of cardinal fuzzy quantifiers as follows.

Definition 3.14. We say that a fuzzy quantifier Q determined by $(\mathcal{S}, \varphi)$ is *globally cardinal* if $\mathcal{S}(M)(-, -) = \mathbf{M}$ is a constant mapping, $\mathcal{S}$ is iso-closed, and φ is weakly global.

Clearly, globally cardinal fuzzy quantifiers are defined by cardinal fuzzy measure spaces. Moreover, they are isomorphism invariant. The opposite statement is also true supposing the distributivity of $\odot$ over arbitrary meets and joins. (cf. Theorem 3.19).

Theorem 3.33. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$ *and* Q *be a fuzzy quantifier determined by* $(\mathcal{S}, \varphi)$ *with* $\mathcal{S}(M)(-, -) = \mathbf{M}$ *for any non-empty universe* M*. If* Q *is isomorphism invariant, then there exists a globally cardinal fuzzy quantifier that coincides with* Q*.*

Proof. Let $\mathcal{S}(M)(-, -)$ be a constant mapping for any non-empty universe M. Define $\mathcal{S}'$ in the same way as in the proof of Theorem 3.31. Since $\mathcal{S}'(\alpha)(-, -)$ is a constant mapping, then $\mathcal{S}'(M)(-, -) = (f_M^{-1})^{\rightarrow}(\mathcal{S}'(\alpha)(-, -))$ is again a constant mapping, and Q' determined by $(\mathcal{S}', \varphi')$ is a globally cardinal fuzzy quantifier. □

Corollary 3.34. *Let* $\mathbf{L}$ *be a complete residuated lattice such that* $\odot$ *is distributive over* $\bigwedge$ *and* $\bigvee$*. A fuzzy quantifier* Q *determined by* $(\mathcal{S}, \varphi)$ *with* $\mathcal{S}(M)(-,-) = \mathbf{M}$ *for any non-empty universe* M *is isomorphism invariant if and only if there exists a globally cardinal fuzzy quantifier coinciding with* Q*.*

Example 3.10. It is not difficult to see that the functional $\mathcal{S}^r(M)$ from Example 3.1 is iso-closed. Hence, the quantifier $\mathsf{many}_M(A, B)$ from Example 5.3 in [2] is isomorphism invariant. Analogously, we can show also the isomorphism invariance of other quantifiers from [2], namely, every, some, at least half, and few.

Example 3.11. For finite M, let $\mathcal{S}^r(M)$ be the same as in Example 3.1. Let the finite fuzzy quantifier Q' be determined by $(\mathcal{S}^r, \varphi')$, where $\varphi'_M(A, B) = A \to B$ if $a \in M$, and $\varphi'_M(A, B) = (A \otimes A) \to B$ otherwise. Then, Q is PI and $\mathcal{S}^r$ is iso-closed. Whether φ'_M is weakly global (and, consequently, whether Q' is ISOM) depends on the underlying residuated lattice. For example, φ'_M is not weakly global for the Łukasiewicz algebra, but it is weakly global for the Gödel algebra, because the operation $\otimes$ is idempotent for the Gödel algebra.

4. Conclusion

In this part of our contribution on fuzzy quantifiers determined by fuzzy measures, we investigated two closely related fundamental semantic properties: permutation invariance (PI) and isomorphism invariance (ISOM). We showed how these properties are related to properties of functionals $\mathcal{S}$ and φ from the definition of fuzzy quantifiers based on $\odot$-fuzzy integrals (see Definition 3.1). It turned out that results on PI and ISOM require some non-trivial methods to solve them. We are convinced that they can be interesting from the mathematical point of view, too. In the following part, we will investigate another two important semantic properties [14]: extension (EXT) and conservativity (CONS).

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