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Fuzzy Logic: A Powerful Tool for Modeling of Vagueness

Vilém Novák and Antonín Dvořák[1]

1 Introduction

The goal of this paper is twofold: *first*, to present fuzzy logic (more precisely, mathematical fuzzy logic) as a group of well established formal systems and *second*, to give arguments for its use as a tool for modeling of the vagueness phenomenon. We concentrate especially on the following two formal systems of fuzzy logic: earlier *fuzzy logic with evaluated syntax* and more recent *fuzzy type theory*. We provide an overview of these systems based mainly on works [40, 26, 28] and discuss various aspects of them from the point of view of vagueness modeling. In the end, Section 6 provides fuzzy logic models of several phenomena occurring in literature of vagueness, especially a previously unpublished model of *higher-order vagueness*.

Fuzzy logic has for a long time been a subject of controversy among philosophers, logicians, and fuzzy logicians. While the first raise many arguments against the ability of fuzzy logic to capture the vagueness phenomenon, the latter argue the opposite. One surprising fact is that philosophers quite often cite papers that are 30–40 years old (cf. [22, 45]) in their arguments, thus ignoring that fuzzy logic has made significant progress. However, the situation is improving, and the discussion is gradually becoming well substantiated and fruitful for both sides, see e.g., the recent book [47], and also the book [46] and analysis of its formal aspects in [17]. In the recent book [49], very interesting and inspiring manifestations of vagueness are discussed as well as the role of fuzzy logic in the modeling of it. One of the goals of this paper is to help filling this gap and to make an attempt at clarification of some misinterpretations occurring in the philosophical literature. We believe that contributions in the present volume will contribute to this clarification and will start a new round of discussions and fruitful research.

One of the objections against fuzzy logic is the allegation that it lacks the rule of modus ponens. This would indeed be a serious deficiency, because modus ponens is the most common and natural deduction rule (for Hilbert systems). It would be doubtful what such a logic, which cannot correctly express and use modus ponens, could serve for. However, already the first precisely introduced formal system of fuzzy logic published by J. Pavelka in [43], of course, includes a sound modus ponens rule. The first author of this paper has extended Pavelka's logic to first-order in [26] and proved a generalization of the famous Gödel completeness theorem. Since then, a lot of other works elaborating on formal fuzzy logic in detail have been published. As

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a consequence, many arguments against fuzzy logic turn out to be barren; they are either wrong or non-substantiated.

It is the fact that vagueness is a fairly mysterious phenomenon that plays a significant (if not crucial) role in human thinking. Thus, one may agree that a working mathematical theory of vagueness phenomenon is necessary. We argue that until now, this goal is most successfully accomplished by fuzzy logic, which attempts to grasp vagueness by introducing a well established and substantiated structure of truth degrees and by using the latter for modeling the way vagueness manifests itself in various situations.

Let us stress that we do not have the pretension to claim that fuzzy logic explains vagueness, because this would require that fuzzy logic would explain in detail all causes of vagueness and describe all situations and ways in which vagueness occurs; and also why it is unavoidable. Fuzzy logic is not here to clarify causes of vagueness. We argue that it is a reasonable mathematical model having enough power to deal with vagueness – nothing more, nothing less. It means that fuzzy logic just provides mathematical tools that allow us to model situations in which vague expressions occur. We can see fuzzy logic more as a logic of ordered structures, and its usefulness in modeling of vagueness depends on our acceptance/rejection of prominent role of truth degrees in describing vagueness.

It seems that the use of vague expressions is very natural for the human mind. For example, we often say "almost white dress", "very strong engine", "too unpleasant a situation", etc. There is naturally a lot of options how vague expressions can be modeled. Epistemicism, supervaluations, various forms of contextualism, etc. As argued in [49] and elsewhere, using some form of degrees offers distinct advantages over other theories. Also, we judge practically everything in our lives using various kinds of degrees that can be either specific (temperature, money) or abstract.[1] To characterize vagueness, certain degrees of intensity of the property of interest are introduced. The degrees are taken from a scale that should have some necessary properties to serve as a good means; except for being an ordered set, it in general should capture the continuity feature of vagueness and so, it should be either uncountable or dense.[2] Furthermore, to be able to represent various operations with the properties possessed by objects only partly, we must endow it with additional operations. The result is a specific algebra — the *algebra of truth values*. A lot of important mathematical results have been obtained in the theory of these algebras. For example, see the books [15, 5, 40, 39, 48] and many papers scattered throughout journals.

We also argue that the rise of vagueness is closely related to some *indiscernibility relation*. For example, a movie is a sequence of pictures. When projected at sufficient speed, we are unable to distinguish them one from another and the result is a vague phenomenon that we regard as a continuous movement. Similarly, the shape of a heap of stones is also vague, and when adding or removing one stone, its shape changes indiscernibly. This is the core of the *sorites paradox*. Another example is the so-called

[1]Abstract degrees can occur when we evaluate our success, health, beauty, etc. Then these degrees have no measuring units, as is the case of, e.g., temperature.

[2]However, in practical applications it often suffices to have some finite set of degrees at disposal. Nevetheless, we believe that from the theoretical point of view we need in principle at least dense infinite set of truth values.

ostensive definition, that is, learning by example: e.g., a mother shows her child a given plant and says "this is a tree". After several repetitions with different trees, the child captures this idea and can correctly point out various trees that may be significantly different from those originally shown. This means that he/she captured a certain indiscernibility relation enabling him/her to classify trees[3].

The following hypothesis has been extensively discussed in [27]:

Hypothesis 1 *Vagueness of natural phenomena, i.e. phenomena recordable by the human mind, usually using natural language, is a consequence of the indiscernibility phenomenon.*

There are at least two relevant ways to construct a mathematical model of the indiscernibility relation. One way is developed inside the alternative set theory [51] and the second one is based on introducing degrees. The second way leads to the concept of fuzzy equality (fuzzy equivalence), an idea contained in the works of many authors (e.g., [21, 23, 42]). It can be demonstrated that membership degrees in fuzzy sets can be quite naturally derived from fuzzy equality (cf. [27, 44]).

This paper is structured as follows: In Section 2, we characterize the vagueness phenomenon as one of two facets (together with uncertainty) of a more general phenomenon of indeterminacy. Hence, while the mathematical model of uncertainty is probability theory, fuzzy logic is a mathematical model of vagueness. In Section 3, the general principles of fuzzy logic are presented. Section 4 provides an overview of fuzzy logic in a narrow sense, and Section 5 deals with a fuzzy-logic model of the semantics of the so-called evaluative linguistic expressions that are essential bearers of vagueness in natural language. Finally, Section 6 focuses on other issues of vagueness analyzed from the point of view of fuzzy logic, starting with the sorites paradox and also including a model of higher-order vagueness. For brevity, we had to relax strict precision in some places to maintain clarity of our explanation.

2 Indeterminacy

We argue that vagueness is one facet of a more general phenomenon that should be called *indeterminacy*. Among these facets we can distinguish *uncertainty* and *vagueness*. Both of them characterize situations in which the amount and extent of our knowledge is crucial. There is, however, an essential distinction between both facets and it must be respected by any theory that claims to deal with indeterminacy (cf., e.g., [9, 20, 40, 31]).

Note that in reality, we often meet indeterminacy *with both its facets* present, i.e. *vague* phenomena are at the same time *uncertain*. Let us briefly summarize the main features of both facets.

2.1 Uncertainty

The *uncertainty* phenomenon emerges when there is a *lack of knowledge* about the *occurrence* of some *event*. The event is encountered when an experiment (process, test,

[3] One may argue that, in fact, some relevant properties are abstracted which are the same for all trees. However, this is not convincing because extension of such properties — a grouping of objects being trees — still does not have clear boundaries and remains vague. This argument can be better a core of explanation where does the considered indiscernibility come from.

etc.) is to proceed, the result of which is not known to us. It may also refer to a variety of potential outcomes, ways of solution, choices, etc. *Randomness* is a specific form of uncertainty arising in connection with time. There is no randomness (uncertainty) once the experiment is realized and the result is known to us. From this point of view, uncertainty is an epistemological concept. Note that it is connected with the question of whether a given event may be observed within some time period, or not. This becomes apparent in the typical example of tossing a dice. The phenomenon to occur is *the number of dots on the dice* and it occurs after the experiment (i.e. tossing the dice once) has been realized. Thus, we refer here to the future. However, the variety of potential events may raise an even more abstract uncertainty that is less dependent on time. We may, for example, analyze uncertainty in potentiality (that is, lack of knowledge) without a necessary reference to time, or with reference to the past (such as a posterior Bayesian probability). Note that pure uncertainty in an abstract way refrains from the character of events of concern, i.e., they can be either crisply or vaguely delineated.

The mathematical model (i.e. quantified characterization) of uncertainty is provided especially by *probability theory*. In everyday terminology, probability can be thought of as a numerical measure of the likelihood that a particular event will occur. There are also other mathematical theories addressing the mentioned abstract uncertainty, for example possibility theory, belief measures, and others.

2.2 Vagueness

The *vagueness phenomenon* arises when trying to *group* together objects that have a certain property φ. The result is an *actualized* grouping X of objects which is not necessarily a set because the property φ may be *vague*. That is, it may be impossible to characterize all the elements of the given grouping precisely and unambiguously — there can exist *borderline* objects for which it is unclear whether they have the property φ (and thus, whether they belong to X), or not. On the other hand, it is always possible to characterize at least some *typical objects* (prototypes), i.e. objects having typically the property in concern and also objects typically not having it. For example, everybody can point to a “blue jacket” or a “long bridge,” but it is impossible to show all blue jackets and all long bridges.[4]

Vagueness is opposite to exactness and we argue that it cannot be avoided in the human way of regarding the world. Any attempt to explain a description with extensive amount of details necessarily leads us to the use of vague concepts. The reason is that precise description contains an abundant number of details (cf. the *incompatibility principle* formulated by L. A. Zadeh in [54]). To understand such a description, we must group these details together — and this can hardly be done precisely. Natural language plays a non-substitutable role here. However, the problem lies deeper in the way people actually regard the phenomena around them. Unlike uncertainty where we always have to consider whether some phenomenon *occurs or not*, vagueness concerns the way the *phenomenon itself* is delineated, no matter whether it will occur or not.

A typical feature of vagueness is its continuity: a small difference between objects

[4]There are vague concepts which do not possess an existing prototype, e.g. *perfect tool*, *honest politician*, etc. These prototypes are nevertheless possible to be imagined in our mind.

cannot lead to abrupt change in the decision of whether either of them has, or does not have a vague property (cf. Black's "museum of applied logic" in [2]). The transition from having a (vague) property to not having it is smooth.

There is also discussion about the role of fuzzy logic with respect to probability and possibility theory. This can be, in our opinion, well explained on the basis of the concepts of actuality and potentiality. This idea was presented in [31].

3 General principles of fuzzy logic

3.1 Global characterization of fuzzy logic

Informally, we may say that mathematical fuzzy logic (MFL) is a special family of many-valued logics addressing the vagueness phenomenon and developing tools for its modeling via truth degrees taken from an ordered scale. These logics are expected to preserve as many properties of classical logic as possible. Furthermore, the following is required from MFL:

(a) It must be a well established sound formal system.

(b) It must be a system open to new connectives, non-commutativity of conjunction and disjunction, generalized quantifiers, and possibly to other specific phenomena.

(c) It should provide tools suitable for providing solutions to specific problems arising when dealing with vagueness, and for development of special techniques and concepts. Among such tools we can rank evaluative linguistic expressions, linguistic variables, fuzzy IF-THEN rules, fuzzy quantification, defuzzification, fuzzy equality, etc.

(d) Special inference schemata including sophisticated inference schemes of human reasoning (e.g., compositional rule of inference, reasoning based on natural language expressions, non-monotonic reasoning, abduction, etc.) should be expressible in it.

Let us stress that many of these requirements are already fulfilled by the available formal systems of MFL.

In our opinion, the fundamental classification of this research field is to *fuzzy logic in a narrow sense* (FLn) and to that *in a broader sense* (FLb). FLn is a mathematical fuzzy logic as such - mathematical study of formal systems of fuzzy logic. FLb is as an extension of FLn whose aim is to develop a formal theory of *the human way of reasoning that would include a mathematical model of the meaning of some expressions of natural language (evaluative linguistic expressions), the theory of generalized quantifiers and their use in human reasoning*[5]. One of the foreseen goals of FLb is to develop a formal logic that could be applied in human-like behaving robots[6]. The history of MFL has been nicely summarized by P. Hájek in [16]. The following is a (uncomplete) list of relevant books on fuzzy logic: [14, 15, 24, 5, 25, 40, 39].

[5] For a treatment of generalized quantifiers in FLb, see [33]. Model of commonsense reasoning is elaborated in [37].

[6] FLb is a logic with its own paradigm and should not be confused with the Zadeh's concept of "fuzzy logic in wide sense".

3.2 Truth degrees

A lot of controversy arose about degrees of truth. According to our analysis above, their use as a suitable means for the modeling of vagueness naturally emerge when we try to characterize an actualized grouping. If the latter is not delineated sharply then two truth degrees become insufficient and we must to resort to a richer scale of them.

Let us stress that they have little use alone. A frequent example[7] is "I love you in the degree 0.954867283", or even worse — "...in the degree $\sqrt{2}/2$". Of course, nobody will ever say such a strange sentence. However, it is quite natural to say "I love you *very much*". This sentence contains hidden degrees which, however, are not specified precisely but using an *evaluative linguistic expression* — see below.

We conclude that *specific truth values are assigned only in the model* and in practice we never deal with them directly without a wider context. It is important, however, to *compare them.* Consequently, in our opinion what is important is the course of truth values w.r.t. various kinds of models and not their specific values. In other words, in a typical situation where a universe is a subject of an ordered scale (e.g. real line) modeling e.g. heights of people, a fuzzy set being a mathematical model of the above discussed grouping of objects is a function $A : U \longrightarrow L$ (L is a set of truth degrees) whose most important characteristic is its *shape*, i.e. the way of A's course.

4 Fuzzy logic in narrow sense

In this part, we provide basic information about mathematical fuzzy logic and several approaches to it. Detailed information and discussion can be found in recent article [36].

4.1 Structure of truth values

The reasoning behind the structure of truth values in fuzzy logic has been initiated by J. A. Goguen in his famous paper [13]. Extensive analysis on what the structure of truth values should look like can also be found in the book [40], Section 4.2. Goguen gave a lot of arguments in favor of the assumption that there must be two conjunctions in fuzzy logic. The most essential argument is that the ordinary conjunction which joins formulas is not the same as the (strong) conjunction joining premises in modus ponens. The conjunction is seemingly unique when considering two truth values only. When introducing more of them, it splits into two (in general) different operations. For sound treatment of modus ponens, we need an interplay between strong conjunction and an operation which models material implication. It turned out that this interplay can be expressed by the adjunction property below.

From the algebraic point of view we arrive at the assumption[8] that truth degrees should form a *residuated lattice*[9]:

(1) $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$

[7] P. Hájek, personal discussion.

[8] It was proposed and substantiated already in seminal paper [13]

[9] It is possible to consider more general structures (noncommutative, nonintegral, etc). We are convinced that from the point of view of modeling of vagueness and its manifestations, residuated lattices are a suitable basis for appropriate logics.

which has the following properties:

(a) $\mathcal{L} = \langle L, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ is a lattice with the smallest element $\mathbf{0}$ and greatest element $\mathbf{1}$ (i.e., $a \leq b$ iff $a \wedge b = a$, as usual).

(b) The operation $\otimes$ is usually called *product* and it is associative ($(a \otimes b) \otimes c = a \otimes (b \otimes c)$), commutative ($a \otimes b = b \otimes a$) and $a \otimes \mathbf{1} = a$, $\qquad a, b, c \in L$.

(c) *Adjunction:* $a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$, $\qquad a, b, c \in L$ where $\rightarrow$ is called *residuation* (sometimes also *(fuzzy) implication*).

When using this structure in logical analysis, it turned out that the following property is also indispensable, because it guarantees soundness and completeness of resulting logics with respect to linearly ordered algebras:

(d) *Prelinearity:* $(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}$, $a, b \in L$.

A prelinear residuated lattice is called an *MTL-algebra*.

A more specific and quite often used structure is that of BL-algebra. This is MTL algebra fulfilling, moreover, also the *divisibility* $a \otimes (a \rightarrow b) = a \wedge b$. Even more specific is MV-algebra.

On the basis of this definition, we can introduce the following additional operations: *Negation* $\neg a = a \rightarrow \mathbf{0}$, *power* $a^n = a \otimes a \otimes \cdots \otimes a$ and *biresiduation* $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ and also *sum* $a \oplus b = \neg(\neg a \otimes \neg b)$. The algebra is linearly ordered if for all $a, b \in L$, $a \leq b$ or $b \leq a$.

It can be shown that a residuated lattice is a generalization of a boolean algebra where in general, however,

$$a \vee \neg a \neq \mathbf{1}$$

as well as

$$a \wedge \neg a \neq \mathbf{0}.$$

On the other hand, $a \otimes \neg a = \mathbf{0}$ and $a \oplus \neg a = \mathbf{1}$ for all $a \in L$. If we put $L = \{\mathbf{0}, \mathbf{1}\}$ then the residuated lattice reduces just to the algebra for classical logic where $\wedge = \otimes$ is conjunction, $\vee = \oplus$ is disjunction, $\rightarrow$ is implication and $\neg$ is negation.

An essential property which holds in all residuated lattices is the following:

$$a \otimes (a \rightarrow b) \leq b, \qquad a, b \in L. \tag{2}$$

This property is an easy consequence of the adjunction (property (c)) and it assures us that modus ponens is a sound inference rule.

We may add various conditions to residuated lattices to obtain stronger structures. These lead to various formal systems of fuzzy logic. Let us name, for example, *basic fuzzy logic* (BL), MTL-logic (MTL), IMTL-logic (IMTL), Łukasiewicz logic (Ł), Gödel logic (G), product logic Π, and others. There are also fairly complicated ŁΠ-algebras which have two (different) products and two implications and which determine ŁΠ fuzzy logic (ŁΠ). For the details see [6, 10, 14, 15, 40] and elsewhere.

Fundamental examples.
There are three fundamental examples of the structure of truth values, namely *standard Gödel, standard product, and standard Lukasiewicz MV-algebra*, which play essential roles in the theory of fuzzy logic. All of them have the support $L = [0, 1]$ and so, the operations $\vee, \wedge$ are minimum and maximum, respectively. Furthermore,

Gödel algebra	*product algebra*
$a \otimes b = a \wedge b$	$a \otimes b = a \cdot b$
$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{otherwise} \end{cases}$
$\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$	$\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$

In *standard Lukasiewicz MV-algebra*, we set

$$\begin{aligned} a \otimes b &= 0 \vee (a + b - 1), \\ a \to b &= 1 \wedge (1 - a + b), \\ \neg a &= 1 - a. \end{aligned}$$

For many properties of all these algebras see [5, 15, 40].

Subdirect representation.
One of the objections to fuzzy logic is the linearity of the ordering of the set of truth values. Though this is indeed true in the examples above, there is an infinite number of various examples of algebras of truth values that are either linearly or only partially ordered. Any of them can be used when modeling the vagueness phenomenon. The following general algebraic result is of a special importance.[10]

THEOREM 1 *Every MTL-algebra $\mathcal{L}$ is isomorphic to a subdirect product of* linearly ordered *MTL-algebras.*

The essential informal consequence of this theorem is that fuzzy logic may be confined only to linearly ordered structures of truth values! To verify a truth of some formula requires to verify it in an arbitrary linearly ordered structure of truth values. This result significantly reduces objections towards the usually assumed linearity of ordering of truth values since the required generality is not significantly harmed.

On the other hand, one cannot apply Theorem 1 arbitrarily. We must never forget that truth values are mainly technical means for modeling more complicated vague phenomena. When considering, e.g., the evaluative predications “John is tall” and “Mary is rich,” then given concrete *John* and *Mary*, these predications can be true in various degrees. There is no reason to reject their comparison, i.e., John can be tall in a degree, say 0.9, while Mary can be rich in a degree, say 0.6. These *truth values* can be compared but *not the meaning* of these clearly incomparable statements. We should

[10]Because logics BL. G. Ł, Π, etc. are special cases of MTL, it is straightforward to formulate and prove analogous theorems for them.

not mix the meaning of such statements. That would require a more sophisticated model (one possible is discussed below) than simple truth value assignment in a model that bears only partial information.

4.2 Fuzzy and classical logic

As all formal logical systems, fuzzy logic strictly distinguishes syntax from semantics. There are two essential approaches: fuzzy logic with *traditional syntax* and that with *evaluated syntax* ($\mathrm{Ev_L}$).

Fuzzy logic with traditional syntax

This approach is promoted by many mathematicians[11]. In fact, this is a class of various logical systems which are determined by the assumed structure of truth values. They generalize syntax of classical logic by adding a new connective of *strong conjunction* (**&**) (and, possibly, some other ones) and modifying special axioms. Each of these systems has a finite list of schemes of logical axioms and inference rule of *modus ponens*.[12] The fundamental concept of provability is classical, i.e., a formula A is *provable* in a theory T, $T \vdash A$, if there exists a formal proof of A from T.

As mentioned, all these formal systems of fuzzy logic differ from classical logic by considering two conjunction connectives, namely *minimum conjunction* $A \wedge B$ which is interpreted by minimum ($\wedge$), and *strong conjunction* $A \mathbin{\&} B$ which is interpreted by the product $\otimes$. Their use depends on a local character of the conjuncts, i.e., the resulting truth degree should be computed from truth degrees of the respective conjuncts with respect to their meaning, too. For example *slow and safe car* would better lead to strong conjunction while *big and beautiful house* to minimum conjunction. In general, we should use strong conjunction whenever we join formulas without a priori knowledge about their content since it is safer. However, the choice of proper conjunction should be a subject of further research.

Other connectives are the following: *implication* $A \Rightarrow B$ interpreted by $\rightarrow$, *negation* $\neg A := A \Rightarrow \bot$ interpreted by $\neg a$, *equivalence* $A \Leftrightarrow B$ interpreted by $\leftrightarrow$, *disjunction* $A \vee B$ interpreted by ($\vee$) and *strong disjunction* $A \nabla B := \neg(\neg A \mathbin{\&} \neg B)$.

As mentioned, the inference rules are *modus ponens* (from A and $A \Rightarrow B$ infer B) and also *generalization* (from A infer $(\forall x)A$).

Given a language J of predicate fuzzy logic we may naturally introduce the concept of a structure for J as follows:

$$(3) \quad \mathcal{M} = \langle \langle M, \mathcal{L} \rangle, \{F_f \mid f \in \mathrm{Func}_J\}, \{r_P \mid P \in \mathrm{Pred}_J\}, \{m_{\mathbf{u}} \mid \mathbf{u} \in \mathrm{OConst}_J\} \rangle,$$

in Tarski's sense where M is a set, $\mathcal{L}$ is a specific algebra of truth values, $F_f : M^n \longrightarrow M$ is an n-ary function assigned to each n-ary functional symbol f, $r_P : M^n \longrightarrow L$ is an n-ary fuzzy relation assigned to each n-ary predicate symbol and $m_{\mathbf{u}} \in M$ is an element representing object constant. Given an assignment p of elements from M to variables, we define a truth value $\mathcal{M}_p(A) \in L$ of each formula A of the given language in a standard way [15].

The following completeness theorem holds for many kinds of fuzzy logics (e.g., MTL, IMTL, BL, G and other ones). We will often use the term "fuzzy logic" without

[11] P. Hájek, F. Esteva, S. Gottwald, L. Godo, F. Montagna, D. Mundici, and others.

[12] There are studied also systems with infinite number of axioms or with other inference rules. We will not deal with them here.

closer specification of the concrete system if unnecessary. The details including full proofs can be found in the books [15, 5, 39, 40] and in the papers [8, 6, 10, 7, 18].

THEOREM 2 *Let T be a theory of fuzzy logic. Then $T \vdash A$ iff $\mathcal{M}(A) = 1$ for each model $\mathcal{M} \models T$.*

We can also introduce the concept of a *crisp formula* that is a formula A for which $\vdash A \vee \neg A$. Since the boolean lattice $\{\mathbf{0}, \mathbf{1}\}$ is also an MTL-algebra we may take *classical logic as a sort of limit "fuzzy" logic*, i.e. it is a fuzzy logic in which *all formulas are crisp.*

Fuzzy logic with evaluated syntax

This approach was initiated by Pavelka in [43] and extended to first-order in [26]. It is a more radical departure from classical logic since it allows evaluation of formulas also in syntax simply by assuming that axioms may not be fully true. Consequently, axioms may form a fuzzy set. But this means a departure from the traditional conception of syntax.

Let F_J be the set of all well-formed formulas of language J. The fundamental concept is that of an *evaluated formula* a/A where $A \in F_J$ is a formula and $a \in L$ is its syntactic evaluation. This means that on a syntactical level, we a priori assume that a certain formula may be true at least in the degree a, which can be generally different from $\mathbf{1}$. Hence, the designated truth values (values which express maximal truth) are replaced by the *maximality principle* saying that if the same formula is assigned more truth values then its final truth assignment is equal to the maximum (supremum) of all of them. Consequently, all truth values are equally important.

Syntactical derivation in $\mathrm{Ev}_{\text{Ł}}$ may end up with a formula that is true in *arbitrary degree* during the derivation and to progress the degree further (unlike traditional syntax where the derivation ends up only with a formula true in the degree $\mathbf{1}$). It can be demonstrated that these principles lead to transparent generalization of classical logic both in semantics as well as in syntax.

The truth values of this logic must form the standard Łukasiewicz MV-algebra. It can be proved that only this algebra (and its isomorphs) can ensure completeness — see below.

Since the syntax deals with evaluated formulas, we must introduce evaluated inference rules. For example, *modus ponens* takes the form

$$\frac{a/A,\ c/A \Rightarrow B}{a \otimes c/B}$$

and similarly also generalization. This rule is *sound* in the sense that it is not possible to derive a conclusion B with the truth value smaller than the conjunction of truth values of the respective premises A and $A \Rightarrow B$ in any interpretation. This is assured by the property (2) of residuated lattices.

A *fuzzy theory* T is a fuzzy set of formulas. More specifically, it is determined by a triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, R \rangle$$

where LAx is a fuzzy set of logical axioms (axioms common to fuzzy logic with evaluated syntax), SAx is a fuzzy set of special axioms (axioms specific for the particular theory) and R a set of evaluated inference rules. This definition is motivated by the idea that axioms need not be fully true (convincing). A typical example are the assumptions leading to sorites paradox (the axiom $(\forall n)(H(n) \Rightarrow H(n+1))$ saying that "if n stones do not form a heap then $n+1$ do not form a heap either" can hardly be fully convincing).

We also introduce the concept of *evaluated proof* which means that each proof w_A of a formula A has its value $\mathrm{Val}(w_A) \in L$. In correspondence with the maximality principle we obtain the *provability degree* of a formula which generalizes the classical concept of provability:

$$T \vdash_a A \quad \text{iff} \quad a = \bigvee\{\mathrm{Val}(w_A) \mid w_A \text{ is a proof of } A \text{ in } T\}$$

where $\mathrm{Val}(w_A)$ is a *value* of evaluated proof, see [40], Definition 4.4, page 99. This means that we can have various proofs of the same formula which are "valuable" in various degrees. The final provability degree is the upper limit value of all possible proofs. Note that this is indeed a straightforward generalization of the classical provability where each formula in the proof can be taken as evaluated by the degree **1** (and so, in classical case it is sufficient to find one proof only).

The truth degree $T \models_a A$ is the infimum of all truth values $\mathcal{M}_p(A)$ of A in all models $\mathcal{M} \models T$ where $\mathcal{M}$ is a structure (3) such that

$$\mathcal{M}_p(A) \geq \mathrm{SAx}(A)$$

for all special axioms of T. This means that the truth of a formula is the lower limit of all possibilities — we are most pessimistic.

The following is a generalization of the Gödel completeness theorem ($J(T)$ is the language of a fuzzy theory T).

THEOREM 3 *For every fuzzy theory T and every formula $A \in F_{J(T)}$*

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A.$$

For more details including all proofs see [40].

4.3 Fuzzy type theory

It turns out that for full-fledged treatment of vagueness we cannot make do with the apparatus of predicate fuzzy logic, especially if we want to model the meaning of natural language expressions. We need to be able to express properties of subsets of objects, speak about functions, etc.[13] Thus, the need for higher-order fuzzy logic arises. Such logic is *fuzzy type theory* (FTT) which is a generalization of classical type theory which was initiated by B. Russel, A. Church and L. Henkin. For full treatment of FTT we refer particularly to [28]. Here, we provide only a sketchy idea. Let us also mention that there exists another version of higher-order fuzzy logic, called

[13]It is, however, possible to interpret higher-order fuzzy logic in the first-order one, but it would be quite inconvenient to apply it in this way.

fuzzy class theory (FCT) [3, 4], which starts from the axiomatization of the notion of fuzzy set, and then proceeds to formal definitions of operations on fuzzy sets, fuzzy relations, etc. Utilizing all this, it aims to become the general formal frame for fuzzy mathematics.

The structure of truth values of FTT is supposed to form one of the following: a complete IMTL$_\Delta$-algebra (see [10]), the standard Łukasiewicz$_\Delta$ MV-algebra, or a BL$_\Delta$-algebra.

The most important for applications in linguistics is the standard Łukasiewicz$_\Delta$ MV-algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \oplus, \Delta, \rightarrow, 0, 1\rangle$$

where

$$\begin{aligned} \wedge &= \text{minimum}, & \vee &= \text{maximum},\\ a \otimes b &= 0 \vee (a + b - 1), & a \rightarrow b &= 1 \wedge (1 - a + b),\\ \neg a &= a \rightarrow 0 = 1 - a, & a \oplus b &= 1 \wedge |a + b|,\\ \Delta(a) &= \begin{cases} 1 & \text{if } a = 1,\\ 0 & \text{otherwise.}\end{cases} \end{aligned}$$

The corresponding FTT is denoted by Ł-FTT.

Fuzzy equality.

The crucial concept in FTT is that of a *fuzzy equality*. This is a fuzzy relation $\doteq : M \times M \longrightarrow L$ which has the following properties:

(i) reflexivity $\quad [m \doteq m] = \mathbf{1}$,

(ii) symmetry $\quad [m \doteq m'] = [m' \doteq m]$,

(iii) $\otimes$-transitivity $\quad [m \doteq m'] \otimes [m' \doteq m''] \leq [m \doteq m'']$

for all $m, m', m'' \in M$ where $[m \doteq m']$ denotes a *truth value* of $m \doteq m'$.

A special case of fuzzy equality on the algebra of truth values is *biresiduation* (see definition on page 11). An example of a fuzzy equality on $M = \mathbb{R}$ with respect to standard Łukasiewicz$_\Delta$ MV-algebra is

$$[m \doteq n] = 1 - (1 \wedge |m - n|), \qquad m, n \in \mathbb{R}.$$

The reason for considering fuzzy equality in FTT is motivated by the fact that the basic connective in classical type theory is the equality. This turned out to be in nice correspondence with Hypothesis 1, as will be noted below[14].

[14] This fact also led to search of a special algebra of truth values where fuzzy equality is the basic operation. The first exposition of the new fuzzy type theory based on this algebra can be found in [34, 35].

Syntax of FTT

It is a generalization of the lambda-calculus and is constructed in a classical way. The main difference from classical type theory is in the definition of additional special connectives. Note that all essential syntactical elements of FTT are formulas (alternatively, they can be called lambda-terms which is quite common in classical type theory).

As usual, each formula A has a certain type. The basic types are o (truth values) and ϵ (elements). These can be then iterated to more complex types.

Formulas of type o (truth value) can be joined by the following connectives: $\equiv$ (equivalence), $\vee$ (disjunction), $\wedge$ (conjunction), $\&$ (strong conjunction), $\boldsymbol{\nabla}$ (strong disjunction), $\Rightarrow$ (implication). General ($\forall$) and existential ($\exists$) quantifiers are defined as special formulas. For the details about their definition and semantics — see [28].

In general, if $A_{\beta\alpha}$ is a formula then it represents a function assigning elements of type β to elements of type α. Hence, $A_{o\alpha}$ represents a fuzzy set of elements. It can also be understood as a first-order property of elements of the type α. Similarly, $A_{(o\alpha)\alpha}$ represents a fuzzy relation (between elements of type α).

There are 17 *logical axioms* in IMTL$_\Delta$-FTT (its structure of truth values is IMTL$_\Delta$-algebra). This number is influenced mainly by the structure of truth values and it can vary for other FTT's. It would be out of scope of this paper to present and discuss all of them, see [40].

Let us only mention *axioms of descriptions* $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha$, where $\alpha = o, \epsilon$ and $\iota_{\alpha(o\alpha)}$ is a description operator which assigns to a fuzzy set an element which belongs in the fuzzy set to degree 1 (i.e., interpretation of this operator is a *defuzzification operation*). $\mathbf{E}_{(o\alpha)\alpha}$ is a special constant which represents a fuzzy equality.

FTT has two inference rules and the classical concept of provability. The rules of modus ponens and generalization are derived rules.

A *theory* T is a set of formulas of type o. We may also consider a specific formula $\dagger$ of type o which represents the middlemost truth value for which $\vdash \neg\dagger \equiv \dagger$ holds. Note that such a truth value, in general, need not be present in the given algebra of truth values. In the standard Łukasiewicz MV-algebra is $\dagger$ interpreted by the truth value 0.5.

A formula $\boldsymbol{\Delta} A_o$ is *crisp* (in the sense introduced above), i.e., its interpretation is either 0 or 1. There are formulas which are not crisp.

Semantics of FTT

is defined using a generalization of the concept of *frame* which is a system

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{L} \rangle$$

where $=_\alpha$ is a special fuzzy equality in each set M_α and $\mathcal{L}$ is an algebra of truth values. Moreover, for any types $\alpha, \beta \in$ *Types*, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$.

To define an interpretation of a given formula we must start with an assignment p of elements from M_α to variables x_α, $\alpha \in$ *Types*. Then each formula $A_{\beta\alpha}$ is interpreted in $\mathcal{M}$ with respect to the assignment p by a function $\mathcal{M}_p(A_{\beta\alpha}) : M_\alpha \longrightarrow M_\beta$.

A frame $\mathcal{M}$ is a *general model* of a theory T if interpretation of all its formulas is defined for any assignment p and all axioms of T are true in the degree $\mathbf{1}$ in $\mathcal{M}$. A formula A_o is true in T, $T \models A_o$ if it is true in the degree $\mathbf{1}$ in all models of T.

The following theorem has been proved in [28].

THEOREM 4 *$T \vdash A_o$ iff $T \models A_o$ holds for every theory T and a formula A_o.*

The following special crisp formulas are quite useful:

$$\Upsilon_{oo} \equiv \lambda z_o \cdot \neg\boldsymbol{\Delta}(\neg z_o),$$
$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg\boldsymbol{\Delta}(z_o \vee \neg z_o).$$

The Υ_{oo} expresses a property of being a non-zero truth value and $\hat{\Upsilon}_{oo}$ a general one (i.e. between $\mathbf{0}$ and $\mathbf{1}$).

REMARK 5 *The $\boldsymbol{\Delta}$ corresponds to the D-operator of supervaluation theory (e.g., $A \vdash C$ implies $\vdash \boldsymbol{\Delta}A \Rightarrow C$ as well as $\neg C \vdash \neg\boldsymbol{\Delta}A$) and $\hat{\Upsilon}$ corresponds to I-operator* indefinitely.

We will need the following property:

LEMMA 6 $\vdash \hat{\Upsilon}(\hat{\Upsilon}x_o \wedge y_o) \equiv (\hat{\Upsilon}x_o \wedge \hat{\Upsilon}y_o)$.

Proof Let $\mathcal{M}$ be a model and p an assignment. If $p(x_o) \in (0,1)$ then $\mathcal{M}_p(\hat{\Upsilon}(\hat{\Upsilon}x_o \wedge y_o)) = \mathcal{M}_p(\hat{\Upsilon}y_o) = \mathcal{M}_p(\hat{\Upsilon}x_o) \wedge \mathcal{M}_p(\hat{\Upsilon}y_o)$. If $p(x_o) \in \{0,1\}$ then $\mathcal{M}_p(\hat{\Upsilon}x_o) = 0$ and so, $\mathcal{M}_p(\hat{\Upsilon}(\hat{\Upsilon}x_o \wedge y_o)) = \mathcal{M}_p(\hat{\Upsilon}x_o) \wedge \mathcal{M}_p(\hat{\Upsilon}y_o) = 0$. The lemma holds by the completeness theorem. □

4.4 Scheme of the most important fuzzy logics

We have already mentioned that there exist many formal systems of fuzzy logic. We argue that not all these systems indeed deserve to be called "fuzzy logic". According to us, the main criterion should be the ability of such a system to serve as a powerful formal grounds for characterization of the vagueness phenomenon, fulfilling the agenda of fuzzy logic and potential for the development of fuzzy logic in a broader sense as a logic of natural human reasoning. The fuzzy logic systems that, according to their properties and strength, seem to fit best this aim are depicted in Figure 1. One can find there also axioms that must be added to obtain another logical system. The scheme outlines also the shift from Ł to $\mathrm{Ev_Ł}$, and in a sense, also a similar shift from ŁΠ. The dotted arrows back express representability in the corresponding logic expanded by logical (truth) constants without evaluated syntax, provided that the concept of provability is added as a special definition. Existence of higher order fuzzy logic (fuzzy type theory) for each of the emphasized logics with traditional syntax (including, of course, classical one) is also indicated; such a logic is ready for extension to FLb.

5 Trichotomous evaluative linguistic expressions

Trichotomous evaluative linguistic expressions (or, simply, evaluative expressions) are natural language expressions serving for describing a position on an ordered scale, for example, *small, medium, big, about twenty five, roughly one hundred, very short, more or less deep, not very tall, roughly warm or medium hot, quite roughly strong,*

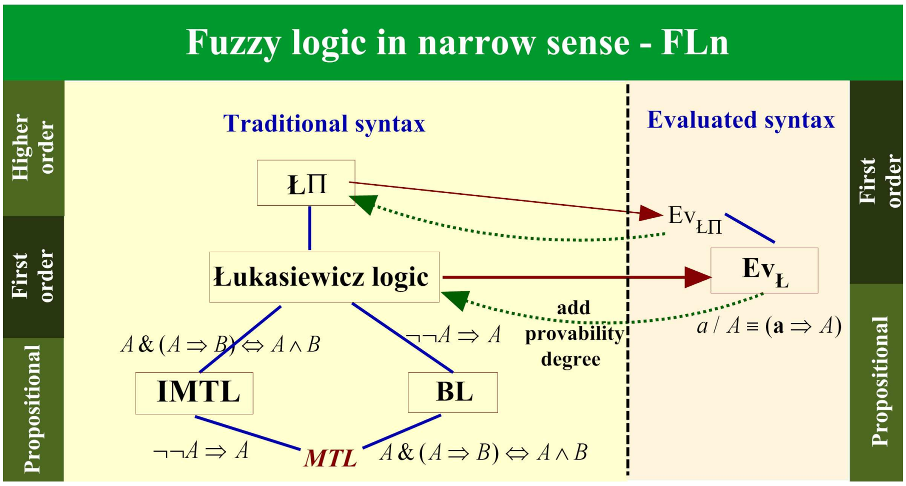

Figure 1. Scheme of the most important formal fuzzy logic systems that are relevant for modeling of the vagueness phenomenon.

roughly medium size, and many others[15]. They form a small but very important part of natural language and are present in its everyday use at practically any time. The reason is that people very often need to evaluate phenomena around them and make important decisions on the basis of them. Besides others, evaluative expressions are used when learning how to control (e.g., driving a car), and in many other activities. Motivation for their study in fuzzy logic has been provided by L. A. Zadeh in [55].

We argue that the meaning of evaluative linguistic expressions is the fundamental bearer of the vagueness phenomenon and, namely, that vagueness of their meaning is a consequence of the indiscernibility between objects. According to our discussion below Hypothesis 1, indiscernibility is in fuzzy logic modeled by fuzzy equality.

All the details about the formal theory of evaluative linguistic expressions can be found in [30, 32]. As usual, we distinguish *intension* (a property), and *extension* in a given *context* of use (i.e., a *possible world*; see [12])[16].

The mathematical representation of intension is a function defined on a set of contexts which assigns to each context a fuzzy set of elements. Intension leads to different truth values in various contexts but is invariant with respect to them. *Extension* is a class of elements (i.e., a fuzzy set) determined by the intension when setting a *specific context*. It depends on the particular context of use and changes whenever the context is changed. For example, the expression "tall" is a name of an intension being a property of some feature of objects, i.e. of their height. Its meaning can be, e.g., 30 cm when a beetle needs to climb a straw, 30 m for an electrical pylon, 200 m or more for a skyscraper, etc.

[15] Precise delineation of trichotomous evaluative linguistic expressions and their various subgroups can be found in [32].

[16] We follow the possible world semantics. In the theory of evaluative linguistic expressions, however, it is more convenient to replace the general term "possible world" by a more apt term "context".

Though the structure of evaluative expressions is more complicated, we will consider in this paper only the *pure evaluative expressions* that are expressions of the form ⟨linguistic hedge⟩⟨atomic evaluative expression⟩, where linguistic hedges are e.g. *very*, *more or less* and atomic evaluative expressions are *small*, *medium* and *big*. The essential characteristics of their meaning is the following:

(i) Extensions are classes of elements taken from nonempty, linearly ordered and bounded scale representing *context*. In this context, three distinguished limit points can be determined: *left bound*, *right bound*, and a *central point*. On Figure 2, these points are denoted by v_L, v_R and v_S, respectively.

(ii) Each of the above limit points is a starting point of some horizon running towards the next limit point in the sense of the ordering and vanishing beyond this point. These horizons on Figere 2 denoted by LH, RH and MH. Each horizon is determined by a reasoning analogous to that leading to the sorites paradox. Then extension of any evaluative expression is delineated by a specific horizon resulting from a shift of the above horizon. The modification corresponds to a linguistic hedge and is "small for big truth values" and "big for small ones".

(iii) Each scale is vaguely partitioned by the fundamental evaluative trichotomy consisting of a pair of antonyms, and a middle member (typically, "small, medium, big").

A formal logical theory of the meaning of evaluative linguistic expressions T^{Ev} in FTT is formed on the basis of the above characteristics. It is determined by 11 special axioms [32] and its language has means by which we can express formally the notions of context, horizon, and many others. The leading principle is based on Hypothesis 1, i.e. on a construction of a special fuzzy equality (the model of indiscernibility relation) characterizing the meaning of evaluative expressions.

The T^{Ev} also has means to distinguish the meaning of evaluative expressions from the meaning of *evaluative predications* that are special linguistic expressions of the form

(4) X is ⟨linguistic hedge⟩⟨TE-adjective⟩

where X is a variable for some specific *feature* of objects (e.g., temperature, pressure, height, depth, etc.) and TE-adjective is an adjective of the kind "small, medium, big", for example "long, shallow, high", etc. The former are *canonical* TE-adjectives.

Intensions of evaluative expressions and predications are represented by specific formulas which are denoted by $(Sm\,\boldsymbol{\nu})$ (corresponds to "small"), $(Me\,\boldsymbol{\nu})$ (corresponds to "medium"), $(Bi\,\boldsymbol{\nu})$ (corresponds to "big") and where $\boldsymbol{\nu}$ is a formula assigned to a linguistic hedge. For the sake of brevity, we do not demonstrate the structure of them and so, we refer the reader to [32].

Possible extensions of evaluative expressions in a model are depicted in Fig. 2. In the picture, the context is determined by a triple $\langle v_L, v_S, v_R\rangle$ where v_L is a left bound, v_S central point and v_R the right bound. The LH, MH and RH are fuzzy sets interpreting the left, medium and right horizon, respectively. These fuzzy sets are determined by a special fuzzy equality $\approx_w$ constructed for each context w from one

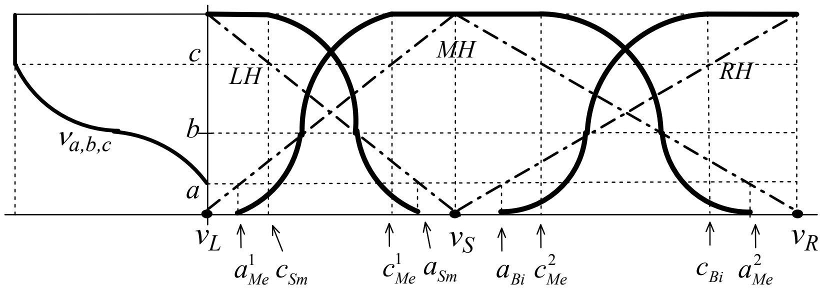

Figure 2. Scheme of the construction of extensions of evaluative expressions

universal fuzzy equality (cf. Hypothesis 1). The $\nu_{a,b,c}$ is a function specific for each linguistic hedge $\boldsymbol{\nu}$ which represents the corresponding horizon shift. The composition of each horizon and its shift provides extension of the evaluative predication (4) being a fuzzy set of elements.

It is possible to construct a model of the theory T^{Ev} of evaluative expressions. Hence, using the completeness theorem, we can prove the following:

THEOREM 7 *The theory of evaluative linguistic expressions is consistent.*

6 Some problems of vagueness from the point of view of fuzzy logic

6.1 Sorites paradox

The *sorites paradox* is usually considered as the crucial display of the vagueness phenomenon. It has been discussed in many papers and books both from a philosophical as well as from a mathematical point of view — see, e.g., [1, 2, 13, 19, 22, 40, 41, 53] and elsewhere.

Sorites paradox has several equivalent formulations and is essentially the same as the other well known paradox, the so-called falakros (bald men) paradox. Formally it can be presented as a sequence of deductions

$$\mathbb{FN}(0), \mathbb{FN}(0) \Rightarrow \mathbb{FN}(1), \mathbb{FN}(1), \ldots, \mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1), \mathbb{FN}(n+1), \ldots$$

where the predicate $\mathbb{FN}(n)$ means, e.g., "*n does not form a heap*", "*n is small*", "*n is feasible*", etc. Each $\mathbb{FN}(n+1)$ is obtained from $\mathbb{FN}(n)$ and $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ using modus ponens. The paradox arises when assuming $\mathbb{FN}(0)$ as well as the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ to be true.

We argue that the correct analysis of sorites should be based on the following assumptions:

(i) "0 does not form a heap" is valid.

(ii) "If n does not form a heap then $n+1$ does not form it" is *practically valid* but not completely valid.

(iii) "There is an n forming a heap" is valid.

We will show that in fuzzy logic these assumptions do not lead to contradiction.

The following theorem can be proved in Ev_{L}(see [19, 40]).

THEOREM 8 *Let T be a consistent fuzzy theory in which all Peano axioms are accepted in the degree 1 and $J(T)$ be the language of T. Let $1 \geq \varepsilon > 0$ and $\mathbb{FN} \notin J(T)$ be a new predicate. Then*

$$T^{+} = T \cup \left\{ {}^{1}\!\big/\mathbb{FN}(0), {}^{1-\varepsilon}\!\big/(\forall n)(\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)), {}^{1}\!\big/(\exists n)\neg\mathbb{FN}(n) \right\}$$

is a conservative extension of T.

In this theorem ${}^{1}\!\big/\mathbb{FN}(0)$ is an evaluated formula expressing that $\mathbb{FN}(0)$ is true — this is a formalization of the assumption (i). Similarly for (iii). The assumption (ii) requires the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ to be practically valid for values of ε close to 0. Setting values of ε provides various possible models of sorites.

Note that a specific heap, in fact, depends on a context. For example, a heap of stones of size 1 cm may have much smaller number of stones than a heap of stones of size 1 mm. But this is a matter of an individual model (!), the logic itself is not influenced. The context is in our solution accomplished by a specific value of ε.

We can also provide a solution of sorites in the fuzzy logic with traditional syntax, namely in BL (cf. [19]). This is achieved using a special predicate At interpreted as "almost true" which acts as a special unary connective. This solution takes a more specific form in connection with the concept of horizon as formalized in FTT discussed below.

6.2 Horizon

The concept of horizon plays a leading role in formation of the *alternative set theory* (AST) by P. Vopěnka [51, 52]. It can be informally characterized as follows.

(i) It is a threshold terminating *our view* of the world.

(ii) The world is not terminated by the horizon and continues beyond it.

(iii) The part of the world before horizon is delineated non-sharply, our ability to discern phenomena diminishes when getting closer to the horizon.

(iv) The horizon is not fixed and can be moved along in the world.

Note that the concept of horizon is used in AST to develop the theory of a *natural infinity* — an infinity that emerges in large sets that are finite but not located as a whole before the horizon.

We argue that the concept of horizon is analogous to sorites. Within the theory of evaluative expressions T^{Ev} outlined above, it is formalized using a special fuzzy equality $\approx_w$ introduced w.r.t. a context w. The sorites then can be obtained as follows.

Assume the theory of Peano arithmetics and let the context be $w_N = \langle 0, p, q\rangle$. Here, p represents a central point of the context w_N (see the discussion in Section 5 on p. 20). Furthermore, we put $\mathbb{FN}(n) = [0 \approx_{w_N} n]$ — the truth value of the statement "n is approximately equal to 0 in the context w" which can be informally interpreted as "n stones do not form a heap".

For the proofs of the following theorem — see [32].

THEOREM 9 *The following holds in arbitrary context w_N:*

(a) $\vdash \boldsymbol{\Delta}\mathbb{FN}(0)$,

(b) $\vdash (\forall n)((p \leq n) \Rightarrow \neg\mathbb{FN}(n))$,

(c) $\vdash (\exists m)(0 < m \,\&\, \hat{\Upsilon}(\mathbb{FN}(m)))$,

(d) $\vdash \neg(\exists n)(\boldsymbol{\Delta}\mathbb{FN}(n) \,\&\, \boldsymbol{\Delta}\neg\mathbb{FN}(n+1))$,

(e) $\vdash (\forall n)(\mathbb{FN}(n) \Rightarrow ((n \approx_{w_N} n+1) \Rightarrow \mathbb{FN}(n+1)))$.

The property (a) states classically that 0 surely does not form a heap; (b) states that each n behind central point p already forms a heap. This corresponds to our practical experience — in the given context, one can always show a heap. The property (c) states that there is a borderline number m which is neither surely not a heap nor surely the opposite. (d) states that there is no number n such that it is surely not a heap and $n+1$ is surely a heap. Finally, (e) states that the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ is *almost true* in the sense introduced by P. Hájek in [19] where the degree of being "almost true" is measured by the degree in which the fuzzy equality $(n \approx_{w_N} n+1)$ is true. This is again in correspondence with the experience since adding a stone to what is not yet a heap imperceptibly changes the shape of the former towards the heap.

Sorites in evaluative expressions.

The above theorem is a basis for analogous theorem concerning the meaning of evaluative expressions of the form "$\langle$linguistic hedge$\rangle$*small*" (for example, *small, very small, more or less small*, etc.). For simplicity, we have omitted the details; these can be found including full proofs in [32].

THEOREM 10 *Let $\boldsymbol{\nu}$ be a linguistic hedge. The following holds in arbitrary context w:*

(a) $\vdash \boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})(0)$,

(b) $\vdash (\exists p)(\boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})(p))$,

(c) $\vdash \neg(\exists n)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})(n) \,\&\, \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})(n+1))$,

(d) $\vdash (\forall n)((Sm\,\boldsymbol{\nu})(n) \Rightarrow At((Sm\,\boldsymbol{\nu})(n+1))$
where At is "measured" by the implication $(Sm\,\boldsymbol{\nu})\,n \Rightarrow (Sm\,\boldsymbol{\nu})(n+1)$.

This means that 0 is definitely small, there is a number that definitely is not small, there is no definitely small number n such that $n+1$ definitely is not small, and if n is small, then it is almost true that $n+1$ is also small. An analogous theorem can be proved also about *big* and *medium*.

THEOREM 11 *The following holds in arbitrary context w:*

$$T^{Ev} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})(x)\,\boldsymbol{\&}(x < y \Rightarrow \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})(y))),$$
$$T^{Ev} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(Bi\,\boldsymbol{\nu})(x)\,\boldsymbol{\&}(y < x \Rightarrow \boldsymbol{\Delta}\neg(Bi\,\boldsymbol{\nu})(y))).$$

According to this theorem, there is no "definitely ⟨*linguistic hedge*⟩ *small x*" followed by a "definitely not ⟨*linguistic hedge*⟩ *small y*" which we can paraphrase as there is no last "definitely ⟨*linguistic hedge*⟩ *small x*". Similarly, there is no first "definitely ⟨*linguistic hedge*⟩ *big x*" (x, y need not represent natural numbers). It is important to realize that both theorems above hold *in arbitrary context* (possible world) and so, they are quite general.

6.3 Tall and Taller

In [22], pp. 96–97 and elsewhere one can encounter a somewhat unclear discussion about "tall and taller" aiming at the critique of fuzzy logic. One of situations Keefe is mentioning is the following. Suppose that Tim is tall in the degree of 0.4 and Tek is tall in the degree of 0.5. Then, if we use degree-theoretic/fuzzy approach, and negation is interpreted using involutive operation, then statements "If Tim is tall then Tek is tall" and "If Tim is tall then Tek is not tall" should have identical truth value, which is, according to Keefe, counterintuitive. Another issue mentioned by Keefe is that the truth values of statements "Tek is tall and Tek is not tall" and "Tek is tall and Tek is tall" is identical, which is counterintuitive, too. Let us propose the following:

We will interpret *Tall* by evaluative expression "big" and *Short* by evaluative expression "small" (in the formal theory T^{Ev} outlined above) and *Taller* as a binary crisp relation

$$y \textit{ Taller } x := \boldsymbol{\Delta}(x < y)$$

(it is not a problem to interpret it also as a fuzzy relation).

Using the properties of T^{Ev}, we can easily prove the following.

THEOREM 12

(5) $\vdash \textit{Tall}\ (x)\,\boldsymbol{\&}(y \textit{ Taller } x) \Rightarrow \textit{Tall}\ (y).$

The interpretation is straightforward: if x is tall and y is taller than x then y is also tall — the truth degree of y being tall is greater than or equal to the truth degree of x being tall.

Let us return to Tim and Tek from the beginning of this subsection. We claim that there is a substantial difference between these statements. The first statement "If Tim is tall then Tek is tall" is an instance of provable formula (5). Therefore, all instances are true in degree 1.[17] The second statement "If Tim is tall then Tek is not tall" is an instance of non-provable (even false) formula $\mathit{Tall}\,(x)\,\mathbf{\&}(y\ \mathit{Taller}\ x) \Rightarrow \neg\,\mathit{Tall}\,(y)$. Some of its instances, however, can be true too, namely when the truth of its premises is low, as is the case here. We see no problem in it. We can guarantee that instances of (5) are always true. For the other formula, we can guarantee nothing.

From the theorem, if we define *not* as negation, i.e., $\mathit{not}\ \mathit{Tall}\,(x) \equiv \neg\,\mathit{Tall}\,(x)$, we can derive, for example,

$$\nvdash \mathit{Short}(x)\,\mathbf{\&}(y\ \mathit{Taller}\ x) \Rightarrow \mathit{Tall}\,(y),$$
$$\vdash \mathit{Tall}\,(x)\,\mathbf{\&}\,\mathit{not}\ \mathit{Tall}\,(x) \equiv \bot.$$

Hence, it is *not provable* that, if x is short and y is taller than x then y is tall. At the same time, the strong conjunction of x being tall and also not tall is false, as well as the strong conjunction of being tall and being tall. One can see that there is no controversy in this analysis.

6.4 Higher-order vagueness

Higher-order vagueness seems to be, in a sense, an artificial problem that arises when deliberating about consequences of the definition of a vague property that should have definitely positive, negative, and borderline cases. The main idea consists of thinking about the fact that vagueness has no "end", i.e. that there are no sharp boundaries in any respect. According to [22], there is neither a sharp boundary between positive and negative cases, nor between borderline cases and other sharp boundaries. It means, besides others, that if a vague property $A(x)$ of elements is given, then there are values of x for which "$A(x)$ is borderline" is itself borderline. In other words, the property "to be borderline" is also vague. Then we can iterate, i.e. there are borderline borderline cases as well as borderline cases of "definitely $A(x)$", "definitely $\neg A(x)$", etc. We can also say that higher-order vagueness stems from vagueness of the property "to be borderline", i.e., that there are elements "more or less borderline". Thus, elements having a given property A that are "typically borderline" should be distinguishable from elements that are borderline only "a little", i.e. close to prototypical examples of A or not A. We propose a fuzzy logic model of higher-order vagueness in Ł-FTT.

How can we recognize a borderline element in the fuzzy logic model? The answer follows from the answer to the question, what does it actually mean that a given property is vague? In fuzzy logic, this should be done in accordance with the assigned truth degrees.

For simplicity, we will consider only first-order properties of type $A_{o\alpha}$[18]. Let us now define the property "to be vague candiddate" formally as follows:

$$\mathit{VgC} := \lambda u_{o\alpha}\,(\forall t_o)(\exists y_\alpha)\mathbf{\Delta}(u_{o\alpha}y_\alpha \equiv t_o).$$

The formula $\mathit{VgC}\,A_{o\alpha}$ simply says that a formula $A_{o\alpha}$ is a candidate for vague formula if its interpretation is a surjective function to the set of all truth values. It goes without saying that this condition is meant as a necessary condition for a predicate

[17] By claiming that instance of some formula is true we mean that truth degree of an interpretation of this instance is 1.

[18] Of course, we can consider also possible worlds (contexts) in our theory.

to be considered vague. To obtain necessary and sufficient conditions, we need to consider also the continuity of vague predicates, some form of monotonicity, fulfillment of so-called principle of tolerance[19] etc. Anyway, we doubt that the effort to obtain necessary and sufficient conditions can be fruitful, and that these conditions satisfying everybody can be found. Therefore, we consider here only one basic condition, namely that of surjectivity. This definition can be justified by the very principle of fuzzy logic — to model vagueness via assignment of truth values from a scale (basically infinite) to elements so that $\mathcal{M}_p(A_{o\alpha}(x_\alpha)) = a \in L$ expresses a truth value of the statement "an element $m_\alpha = p(x_\alpha)$ has the property A". The truth values $a \neq \mathbf{0}, \mathbf{1}$ characterize borderline cases. Since there should be no truth value gaps, a vague property must principally attain any truth value from the scale (algebra) L. Note that, e.g. in Ł-FTT, this assumption enables us to model the Black's "museum of applied logic" (cf. [2]) — cutting arbitrary small piece of a chair corresponds to lowering the truth value $\mathcal{M}_p(\text{Chair}(x))$ by arbitrarily small ε.

Because in Ł-FTT, the equivalence $y_o \equiv z_o$ is interpreted by the continuous operation of biresiduation $\leftrightarrow$, $\vdash VgC\, \lambda t \cdot (y_o \equiv t)^2$ (twice application of the strong conjunction $\mathbf{\&}$) holds true.[20]

If the degree of $\mathcal{M}_p(A_{o\alpha}(x_\alpha))$ is close to 0 or 1 then the element $p(x_\alpha)$ is more definite, i.e., less borderline. The closer it is to the neutral value $\dagger$[21] the more it is borderline. Therefore, we will introduce below a formula $Brd_{o\alpha(o\alpha)}$ expressing that an element x_α of type α is a *borderline* case of a property $A_{o\alpha}$. We will do it, moreover, iteratively for arbitrary order.

First, we define recursively the following special formulas:

(6) $C_0 := \lambda z_o\, z_o,$

(7) $C_1 := \lambda z_o\, (C_0\, z_o \equiv \dagger)^2,$

... .

(8) $C_n := \lambda z_o\, (C_{n-1} z_o \equiv \dagger)^2.$

LEMMA 13 *If* $\vdash VgC\, u_{o\alpha}$ *then* $\vdash VgC(C_n u_{o\alpha})$ *for all* n.

Proof The statement is trivial for $n = 0$. Let the property hold for some n, $\mathcal{M}$ be a model and p an assignment to variables such that $p(t_o) = a \in L$. Then $\mathcal{M}_p((C_n z_o \equiv \dagger)^2 \equiv t_o) = 1$ if $\mathcal{M}_p(C_n z_o \equiv \dagger)^2 = a$ and it holds if we find some $b \in L$ such that $\mathcal{M}_p(C_n z_o \equiv \dagger) = b$ and $b \otimes b = a$. Such b, however, exists by the assumption and properties of Łukasiewicz algebra. Therefore, $\mathcal{M}_p((\exists z_o)\mathbf{\Delta}((C_n z_o \equiv \dagger)^2 \equiv t_o)) = 1$ for arbitrary p which leads to $\mathcal{M}_p((\forall t_o)(\exists z_o)\mathbf{\Delta}((C_n z_o \equiv \dagger)^2 \equiv t_o)) = 1$. We conclude that $\vdash VgC(C_{n+1} u_{o\alpha})$. □

LEMMA 14

(a) $\vdash \hat{\Upsilon}(z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.

[19]It says that two observationally indistinguishable (with respect to predicate P) objects a and b should possess equal truth values $P(a)$ and $P(b)$.

[20]It means that for a fixed y_0, the unary predicate $\lambda t \cdot (y_0 \equiv t)\, \mathbf{\&} (y_0 \equiv t)$ is a surjective function to the set of truth values.

[21]Recall that in the standard semantics, interpretation of $\dagger$ is the truth value 0.5.

(b) $\vdash \hat{\Upsilon} z_o \wedge \hat{\Upsilon}(z_o \equiv \dagger)^2 \equiv \hat{\Upsilon}(z_o \equiv \dagger)^2$.

(c) $\vdash (z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.

(d) $\vdash \hat{\Upsilon} z_o \wedge (z_o \equiv \dagger)^2 \equiv (z_o \equiv \dagger)^2$.

Proof We will use the completeness theorem. Let $\mathcal{M}$ be a model and p an assignment of elements of $\mathcal{M}$ to variables.

(a) If $\mathcal{M}_p(\hat{\Upsilon}(z_o \equiv \dagger)^2) = 0$ then the formula in (a) is true in the degree 1 in $\mathcal{M}_p$. Let $\mathcal{M}_p(\hat{\Upsilon}(z_o \equiv \dagger)^2) = 1$. Then $\mathcal{M}_p(z_o) \notin \{0, 0.5, 1\}$ and so $\mathcal{M}_p(\hat{\Upsilon} z_o) = 1$. Consequently, (a) is provable.

(b) is a consequence of (a).

(c) If $\mathcal{M}_p(\hat{\Upsilon} z_o) = 1$ then the truth value of (c) in $\mathcal{M}_p$ is also 1. Let $\mathcal{M}_p(\hat{\Upsilon} z_o) = 0$. Then $\mathcal{M}_p(z_o) \in \{0, 1\}$ and so, $\mathcal{M}_p((z_o \equiv \dagger)^2) = 0$ and so, the truth value of (c) in $\mathcal{M}_p$ is 1. Consequently, (c) is provable.

(d) is a consequence of (c). □

Now, we define the predicate "to be borderline" on the level n as follows:

(9) $\quad Brd^{(1)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2$

(10) $\quad Brd^{(n)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(Brd^{(n-1)}\, u_{o\alpha} x_\alpha) \wedge (Brd^{(n-1)}\, u_{o\alpha} x_\alpha \equiv \dagger)^2$.

This definition is based on the following idea: all elements x for which Ax is true in a degree different from 0, 1 are borderline. The closer this degree is to 0.5 the more indefinite this element is; if the truth degree is 0.5 then it is "typically borderline", i.e. neither close to definitely Ax, nor close to definitely not Ax. Similar reasoning can be done on any level n.

LEMMA 15 *The following holds true:*

(11) $\quad \vdash Brd^{(n)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_{n-1}(u_{o\alpha} x_\alpha)) \wedge C_n(u_{o\alpha} x_\alpha)$

for all $n \geq 1$.

Proof After rewriting, we obtain for $n = 1$

$$\vdash Brd^{(1)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2,$$

i.e. $\vdash Brd^{(1)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge C_1(u_{o\alpha} x_\alpha)$. Furthermore, for $n = 2$ we have

$$\begin{aligned}(12) \quad \vdash Brd^{(2)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(\hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2) \wedge \\ ((\hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_1(u_{o\alpha} x_\alpha) \equiv \dagger)^2) \equiv \dagger)^2.\end{aligned}$$

Then, using Lemma 14 and the properties of FTT we obtain

$$\vdash Brd^{(2)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_1(u_{o\alpha} x_\alpha)) \wedge (C_1(u_{o\alpha} x_\alpha) \equiv \dagger)^2,$$

i.e. $\vdash Brd^{(2)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_1(u_{o\alpha} x_\alpha)) \wedge C_2(u_{o\alpha} x_\alpha)$. Analogously we proceed up to an arbitrary n. □

THEOREM 16

(a) If $\vdash VgC\, u_{o\alpha}$ *then* $\vdash VgC(Brd^{(n)}\, u_{o\alpha})$ *for all* $n \geq 1$*.*

(b) To every formula $A_{o\alpha}$ *(vague property) and every* $n \geq 1$ *the following is provable:*

(i) $\vdash (\exists x_\alpha)\, Brd^{(n)}\, A_{o\alpha} x_\alpha$,

(ii) $\vdash (\exists x_\alpha)\hat{\Upsilon}(Brd^{(n)}\, A_{o\alpha} x_\alpha)$.

Proof (a) is a consequence of Lemmas 13 and 15.
(b) Both statements follow from (a).

□

This theorem states that the property "to be borderline" is vague candidate. Moreover, the case (b) states that there is an element that is "borderline borderline ... borderline", and also an element that is "definitely borderline ... borderline".

Another possibility to model higher-order vagueness in fuzzy logic is based on the use of the sophisticated theory of fuzzy IF-THEN rules in FLb (see [29, 38]), i.e. the rules of the form

$$\mathsf{IF}\ X \text{ is } \mathcal{A}\ \mathsf{THEN}\ Y \text{ is } \mathcal{B}$$

where $\mathcal{A}, \mathcal{B}$ are trichotomous evaluative linguistic expressions of natural language, or they can be arbitrary vague properties. Using these rules we can characterize the vagueness of a given predicate of arbitrary order. Then using the so-called *perception-based logical deduction* [29] we can show that there is an element $p(x_\alpha)$ which is borderline in an arbitrary order.

7 Conclusions

Mathematical fuzzy logic is a group of well established and powerful formal logical tools. We argue that it provides a reasonable and well working mathematical model of the vagueness phenomenon. The main idea consists in introducing a specific algebra of truth values by which the truth value of the statement "an element x has the property A" is expressed. Note that all available analyzes of vagueness exhibit a clear classification of elements to *typical* ones, i.e. those having a property A, those not having it, and *borderline* ones. This immediately gives rise to three truth values. More truth values just provide a more subtle classification. We are convinced that any attempt to avoid degrees is hopeless and can lead only to impoverishing the theory. Though the degree-theoretical approach need not be omnipotent, it certainly can be very helpful.

The degrees are hidden also in the *supervaluation* theory (see, for example, [50]). Its main idea is that vague phenomena can be made precise in a variety of different ways. The truth that an element has a vague property is its supervaluation, which is a function of the tentative ordinary (classical) truth valuations of this proposition. For each way of making it precise, we get a new tentative classical valuation indicating whether the proposition, as thus made precise, is true or false. If every way of making the proposition precise makes it (classically) true, all of its tentative valuations will be true. If every precise version of the proposition is false, all of the tentative classical valuations are false. Otherwise, we get a mixture of tentative valuations. The result of supervaluation of the vague proposition is true if all the tentative valuations are true, and it is false if they are all false; otherwise it is undefined. We argue that this situation can be embedded into fuzzy logic and thus, in fuzzy set theory. One attempt has already been done in [11], where the authors suggest the introduction of some measure on a class K of models. More detailed research still needs to be done.

We have also outlined one possible way to model higher-order vagueness in fuzzy logic. Though the theory is not yet finished, the results seem to be promising.

The great power of fuzzy logic lies also in its ability to model the meaning of vague expressions of natural language. Until now, the theory was confined to a small though important class of evaluative linguistic expressions, conditional clauses, and partly also to the class of linguistic quantifiers [33]. We argue that the results are in accordance with the intuitive meaning and have a wide potential for various kinds of applications.

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