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***Abstract*—In this contribution, a notion of the implicative normal form will be recalled and further used to establish a model of fuzzy rules that fits the best to the input set of data. This normal form has been introduced as an alternative to the Perfilieva's conjunctive normal form and it is a suitable model for a particular case of graded fuzzy rules. It will be shown that it has the universal approximation property for the class of extensional and functional fuzzy relations. Moreover, a suitable inference rule will be provided together with an analysis of its properties. It will be shown that all these results will pass also for the best data set fitting implicative normal form. Results in this field extend the theory of approximate reasoning as well as the theory of fuzzy functions.**

***Keywords*-Graded fuzzy rules; Normal forms; Approximate reasoning; Fuzzy functions; Fuzzy control; F-transform;**

I. Introduction

Within this contribution, we will take into account only the Implicative model [1], [2], [3] of fuzzy rules. They are not very popular in practical applications and the Mamdani-Assilian model [4], [5] is dominating. The causes of this fact are various [6] but the main one lies in the essential difference of suitability for an approximation of the relational dependency that can be represented using a fuzzy relation (see e.g. [2]). Informally speaking

- the Mamdani-Assilian model suites nearly for an arbitrary relation dependency;
- the Implicative model works well only for a functional dependency.

Strictly speaking, the extensionality is required from a relation that we approximate by the Mamdani-Assilian model; and additionally, the functionality is required in the case of the Implicative model. Provided that the requirements are fulfilled we can expect a well behaving model in the following sense: the Mamdani-Assilian model provides approximation of an ideal fuzzy relation from below and the Implicative model from the above and moreover, it can be done with an arbitrary precision.

First of all, we will present theoretical results relating to the generalized Implicative model of fuzzy rules (introduced in [7] and further studied in [8]) having the structure of the implicative normal form. This formalization can be interpreted in the context of the *graded fuzzy rules* (introduced in [9] as a generalization of the ordinary fuzzy rules). Graded fuzzy rules allow to involve an additional imprecision, uncertainty or vagueness related to each fuzzy rule in the form of particular degrees attached to the respective fragment of the normal form. We refer to [9], [10], [7] for more details on this topic.

In an analogy with [11], it will be shown that a specially chosen implicative normal forms can serve as the best approximating fuzzy relations for the extensional and functional fuzzy relation. This research is motivated mainly by Hájek's results in the field of fuzzy control [2] and Perfilieva's fuzzy transforms (F-transforms) over a residuated lattice [12]. F-transforms can be viewed as Perfilieva's normal forms (different from the implicative normal form) and they provide the best approximation of the extensional fuzzy relation. There are two F-transforms based on the operations of a residuated lattice. One can be identified with the Mamdani-Assilian model of fuzzy rules and the other one with the Implicative model [11]. But they do not exhibit well for the so called positive samples. They tend to be 1 nearly everywhere. To overcome this problem, we have focused on the functionality property (as in Hajek's work) and designed first, a new (implicative) normal form [7] and second, a new best approximation based on this normal form which we can interpret in terms of graded fuzzy rules and since it provides the best approximation, hence, we can state that these graded fuzzy rules are perfect.

The contribution is organized as follows: in the next section, we recall Hájek's approach to fuzzy rules; Section III will be devoted to the implicative normal form, its properties and consequences to fuzzy control. Newly in Section IV, we will construct the best approximation of the positive samples using an implicative normal form and moreover, we will provide an analysis of its behavior.

II. Basic Notions and Overview of the Hájek's Approach to Fuzzy Control

One of the original approaches for a fuzzy rule base construction generally works only for the so called *positive samples* w.r.t. some (binary) fuzzy relation F, i.e. $(c,d) \in X \times Y$ is called the positive sample w.r.t. F if $F(c,d) = 1$. The definition can be extended for sets in the following way: $C \subseteq X \times Y$ is a set of the positive samples w.r.t. F if each $(c,d) \in C$ is the positive sample w.r.t. F. Dually, we say that C is a set of the negative samples w.r.t. F if for each $(c,d) \in C$: $F(c,d) = 0$ (or $\neg F(c,d) = 1$, where $\neg x = x \to 0$ and $\to$ stands for some fuzzy implication that is usually interpreted as the residuum to a t-norm).

In [2], Hájek used a set of the positive samples to create a fuzzy rule base using an indistinguishability (or similarity) relation. This procedure can be interpreted from the algebraic point of view it in the following way:

- **Requirements**:
  1) For the sequel, let X, Y be nonempty sets of objects and
  $$\mathscr{L} = \langle L, *, \to_*, \wedge, \vee, 0, 1\rangle,$$
  be a complete residuated lattice.
  2) Moreover, let $F \underset{\sim}{\subseteq} X \times Y$ (i.e. a fuzzy relation on $X \times Y$), $\approx_1 \underset{\sim}{\subseteq} X^2, \approx_2 \underset{\sim}{\subseteq} Y^2$ be fuzzy (similarity in the original text) relations and denote $\bar{x} = [x,y] \in X \times Y$, $\bar{x}' = [x', y'] \in X \times Y$;
  3) $C \subseteq X \times Y$ be a non-empty set of the positive samples w.r.t. F.
- **Models** of a collection of fuzzy rules are defined as
$$\mathrm{Mamd}(x,y) =_{df} \bigvee_{(c,d)\in C} ((c \approx_1 x) * (d \approx_2 y)),$$
$$\mathrm{Rules}(x,y) =_{df} \bigwedge_{(c,d)\in C} ((x \approx_1 c) \to_* (y \approx_2 d)).$$
- Then the following **properties** can be proved:
  1) If F is 1-extensional (for the definition see the following section) then $\mathrm{Mamd} \subseteq F = 1$, where $\subseteq$ is defined as $A \subseteq B \equiv_{df} \bigwedge_{x\in X}(A(x) \to_* B(x))$ for the unary fuzzy relations A, B and analogously for n-ary fuzzy relations.
  2) If F is 1-functional (for the definition see the following section) then $F \subseteq \mathrm{Rules} = 1$.
  3) If F is 1-extensional and 1-functional then
  $$\bigwedge_{x\in X} \bigvee_{(c,d)\in C} (x \approx_1 c) * (x \approx_1 c) \leq (\mathrm{Rules} \subseteq \mathrm{Mamd}), \tag{1}$$
  which consequently gives us the following estimations
  $$\bigwedge_{x\in X} \bigvee_{(c,d)\in C} (x \approx_1 c) * (x \approx_1 c) \leq \begin{array}{l}(F \approx \mathrm{Mamd}),\\ (F \approx \mathrm{Rules}),\end{array} \tag{2}$$
  where $F \approx S \equiv_{df} (F \subseteq S) * (S \subseteq F)$.

In the original source, Hájek investigated also properties of fuzzy control based on Mamd and $(\max, *)$ compositional rule of inference.

Let us keep the requirements 1) – 3) formulated in this section for the sequel.

III. The Implicative Normal Form

In this section, we are going to recall an implicative variant of the conjunctive normal form (implicative normal form for short) and to show its properties. Let us quote from [7]: "This normal form differs from the Perfilieva's conjunctive normal form and it has been introduced for a distinct purpose. While the Perfilieva's conjunctive normal form is suitable for extensional fuzzy relations and negative samples, the implicative normal form has been aimed for functional fuzzy relations and positive samples."

Definition 1 *[7] Let C be a non-empty subset of $X \times Y$.*
The conjunctive normal form – implicative variant *(*INF *for short) for a fuzzy relation F w.r.t. C is*
$$\mathrm{INF}^C_F(\bar{x}) =_{df} \bigwedge_{(c,d)\in C} [F(c,d) \to_* [(c \approx_1 x) \to_* (d \approx_2 y)]]. \tag{3}$$
For the sake of brevity and in order to simplify the distinction between the conjunctive normal forms, we will call the conjunctive normal form – implicative variant *as the* implicative normal form.

Generally for non-symmetric $\approx_1, \approx_2$, we can introduce also a variant of the above defined INF by juxtaposition of the variables and constants
$$\mathrm{INF}'^C_F(\bar{x}) =_{df} \bigwedge_{(c,d)\in C} [F(c,d) \to_* [(x \approx_1 c) \to_* (y \approx_2 d)]]. \tag{4}$$
Obviously, most of the results are valid for both implicative normal forms therefore, we will deal only with INF in the sequel and we will explain differences where it will be necessary.

A. Approximation Properties of the Implicative Normal Form

As noted at the beginning of this section, the implicative normal forms are suited to approximate functional fuzzy relations (the non-graded version is used in the field of fuzzy control see e.g. [13], [2], [14]). The following definition is in the graded form inspired by [15].

Definition 2 *A relation F is said to be a-functional w.r.t. $\approx_{1,2}$ if*
$$a = \bigwedge_{\bar{x},\bar{x}'\in X\times Y} [[(x \approx_1 x') * F(\bar{x}) * F(\bar{x}')] \to_* (y \approx_2 y')].$$
We will shortly denote the right side of the above equality by $\mathrm{Fun}_{\approx_{1,2}} F$.

Approximation properties of an implicative normal form is summarized in the following two theorems (analogous to the Hajek's results).

Theorem 3 *[7] If F is a-functional then*

$$a \leq \bigwedge_{\bar{x} \in X \times Y} (F(\bar{x}) \rightarrow_* \mathrm{INF}_F^C(\bar{x})).$$

It means that the degree of functionality $\mathrm{Fun}_{\approx_{1,2}} F$ is the lower estimation of the inclusion F and INF_F^C denoted as $F \subseteq \mathrm{INF}_F^C$, i.e. $\mathrm{Fun}_{\approx_{1,2}} F \leq F \subseteq \mathrm{INF}_F^C$.

Definition 4 *A relation F is said to be b-extensional w.r.t. $\approx_{1,2}$ if*

$$b = \bigwedge_{\bar{x}, \bar{x}' \in X \times Y} [[(x \approx_1 x') * (y \approx_2 y') * F(\bar{x})] \rightarrow_* F(\bar{x}')].$$

We will shortly denote the right side of the above equality by $\mathrm{Ext}_{\approx_{1,2}} F$.

Theorem 5 *[7] Let P and P' be given by*

$$P(x) =_{df} \bigvee_{(c,d) \in C} [(c \approx_1 x) * (c \approx_1 x)], \tag{5}$$

$$P'(x) =_{df} \bigvee_{(c,d) \in C} [(c \approx_1 x) * (x \approx_1 c)]. \tag{6}$$

If F is a-functional and b-extensional and moreover, C is a set of the positive samples w.r.t. F then

$$a * b * \bigwedge_{x \in X} P(x) \leq \bigwedge_{x \in X, y \in Y} (F(x,y) \leftrightarrow_* \mathrm{INF}_F^C(x,y)).$$

The result for $\mathrm{INF}_F'^C$ with symmetric $\approx_2$ is the same inequality with P' instead of P. The above inequality provides the lower estimation of the quality of approximation using the implicative normal form. The estimation is computed using the degrees of extensionality and functionality and moreover using the degree of lets say “the good covering property” that is represented by the expression $\bigwedge_{x \in X} P(x)$. Informally speaking, the higher are degrees of the requirements (extensionality, functionality and good covering) the better is the resulting approximation.

Example 6 *Let $\mathscr{L}$ be the standard Łukasiewicz algebra and*

$$F(\bar{x}) =_{df} y \leftrightarrow_* (x \cdot x)$$

on M^2, where $M = \{0.05 \cdot k |\ k = 0, 1, \dots, 20\}$. Then F is extensional w.r.t. $\approx_1, \approx_2$ defined as

$$x \approx_1 y =_{df} (x \leftrightarrow_* y) * (x \leftrightarrow_* y),$$
$$x \approx_2 y =_{df} x \leftrightarrow_* y.$$

In the terms of graded extensionality, it is

$$1\text{-extensional w.r.t. } \approx_1, \approx_2.$$

But if we change $\approx_1$ to $\leftrightarrow_*$ then we obtain that

$$F \text{ is } 0.75\text{-extensional w.r.t. } \approx_2, \approx_2.$$

And even worse, if we take the original $\approx_1$ and change $\approx_2$ to $\approx_1$ then we have that

$$F \text{ is } 0.19\text{-extensional w.r.t. } \approx_1, \approx_1.$$

Example 7 *In the setting of Example 6, we can compute the following degrees of functionality for F:*

F *is* 1*-functional w.r.t.* $\approx_1, \approx_2$;
F *is* 0.75*-functional w.r.t.* $\approx_2, \approx_2$;
F *is* 0.5*-functional w.r.t.* $\approx_1, \approx_1$;
F *is* 0.31*-functional w.r.t.* $\approx_2, \approx_1$.

If we change the background algebraic structure to the standard product algebra then the fist degree of functionality will remain the same and the rest will change to $0.25, 0.05, 0.0125$, *respectively.*

Example 8 *From Example 6 and 7, it follows that the precision of approximation of F using* INF *w.r.t. $\approx_1, \approx_2$ depends only on the suitable partition of $X \times Y$ such that $\bigwedge_{x \in X} P(x)$ is as high as possible (it also leads to an optimization problem) in the both cases of the background algebra. While e.g. in the case of $\approx_2, \approx_2$ and the Łukasiewicz standard algebra, we have that*

$$\underbrace{0.5}_{0.75 * 0.75} * \bigwedge_{x \in X} P(x) \leq \bigwedge_{x \in X, y \in Y} (F(x,y) \leftrightarrow_* \mathrm{INF}_F^C(x,y)).$$

From the sequence of examples, we see that the normal form based approximations and hence generally “fuzzy rule based approximations” are very sensitive to all choices of the input parameters such as the choice of the background algebra, binary fuzzy relations $\approx_1, \approx_2$ number of nodes and their distribution over the respective universe.

B. Consequences of the Implicative Normal Form Based Formalization to Fuzzy Control

It is important to investigate the outputs of the particular approximate inference (reasoning), i.e. the implicative normal form INF and the input fuzzy set A':

$$R_{\mathrm{INF}}: \quad \frac{A', \mathrm{INF}_F^C}{B'},$$

where

$$B'(y) =_{df} \bigvee_{x \in X} (A'(x) * \mathrm{INF}_F^C(x,y)).$$

The following theorem summarizes properties of R_{INF}.

Theorem 9 *[8] Let*

$$A_c(x) =_{df} (x \approx_1 c),$$
$$B_d(y) =_{df} F(c,d) \rightarrow_* (y \approx_2 d).$$

- $(A' \subseteq A_c) \leq (B' \subseteq B_d)$;
- $\bigvee_{x \in X} (A_c(x)) * (A_c \subseteq A') \leq (B_d \subseteq B')$;
- $\bigvee_{x \in X} (A_c(x)) * (A_c \approx A') \leq (B_d \approx B')$.

We see that R_{INF} provides an expected output without complicated requirements: only the expression $\bigvee_{x\in X}(A_c(x))$ influences the two above inequalities. This expression can be characterized as non-emptiness and we can read it as: "there exists $x \in A_c$".

The following properties relate to a position of the reconstructed implicative rule $A' \rightarrow_* B'$ and the ideal fuzzy relation F.

Theorem 10 *[8] Assume the notational convention as in Theorem 9 and moreover*

$$F_h(y) =_{df} \bigvee_{x\in X} F(x,y);$$

- *If F is a-functional then $a \le F \subseteq (A' \rightarrow_* B')$;*
- *If F is a-functional and b-extensional then*

$$a * b * \bigwedge_{x\in X} P(x) \le (A' \rightarrow_* B') \subseteq (A' \rightarrow_* F_h).$$

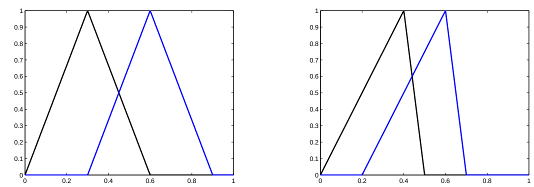
The first inequality shows that the degree of functionality of F estimates the degree of inclusion of F in $A' \rightarrow_* B'$. And the second inequality provides an estimation of the reverse inclusion, i.e. the degrees of extensionality, functionality and the good partition estimates inclusion of $A' \rightarrow_* B'$ in $A' \rightarrow_* F_h$. An open question remains whether it is possible to prove $(A' \rightarrow_* B') \subseteq F$.

Due to the space limitation, we will not go into the details of a relationship between an implicative normal forms and the graded fuzzy rules. We will only mention that a set of (non-empty and disjoint, see [16]) fuzzy sets e.g. $\{A_1,\ldots,A_n\}$ can be expressed by means of a similarity relation defined as $x \approx x' =_{df} \bigwedge_{i=1}^{n} A_i(x) \leftrightarrow_* A_i(x')$ using nodes $c_1,\ldots,c_n$ satisfying $x \approx c_i = A_i(x)$. Having this on mind, we can safely express a neighborhood of some node either using $\approx$ or directly by fuzzy set and their relation is clear. Therefore, we will freely come from the one to another in the following text.

Example 11 *As an illustration, let us consider Łukasiewicz standard algebra and two graded fuzzy rules*

$$^{0.9}\preceq/(\textit{If } x \in A_1 \textit{ then } y \in B_1) \quad \text{AND}$$
$$^{0.8}\preceq/(\textit{If } x \in A_2 \textit{ then } y \in B_2),$$

where A_i, B_i are (non-symmetric) triangular fuzzy numbers depicted on Figure 1. The resulting fuzzy relation is visualized on Figure 2. The output of the inference R_{INF} with the input fuzzy set A' (blue line on Figure 3(a), i.e. shifted A_1) is a fuzzy set B' drawn on Figure 3(b). On these figures, we demonstrate the last inequality in the above theorem: $(A_1 \approx A') \doteq 0.6667$ and $\bigvee_{x\in X}(A_1(x)) = 1$, hence, $0.6667 \le (0.9 \rightarrow_* B_1) \approx B'$. In the case of the input fuzzy set A' identical with A_1 (or A_2), we obtain exactly $0.9 \rightarrow_* B_1$ ($0.8 \rightarrow_* B_2$) as the output B'.



(a) Fuzzy sets on X, A_1 – black line; A_2 – blue line.

(b) Fuzzy sets on Y, B_1 – black line; B_2 – blue line

Figure 1. Fuzzy sets in the graded fuzzy rules from Example 11

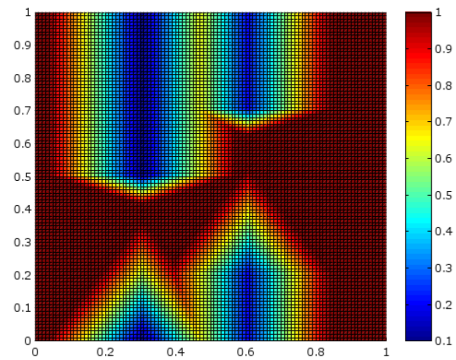


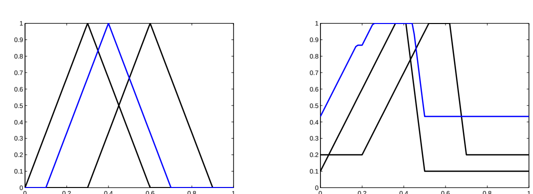
Figure 2. Graded fuzzy rules from Example 11

IV. The Best Approximation Using the Implicative Normal Form

In the following, we will combine two implicative normal forms to receive a desired approximation property. Let us consider the following expression:

$$\begin{aligned}
&\text{INF}^{C}_{\text{INF}^{D}_{F}}(x,y) = \\
&\quad \bigwedge_{(c,d)\in C} \left[\text{INF}^{D}_{F}(c,d) \rightarrow_* [(c \approx_1 x) \rightarrow_* (d \approx_2 y)]\right] \\
&= \bigwedge_{(c,d)\in C} \Big[\Big[\bigwedge_{(p,q)\in D} (F(p,q) \rightarrow_* [(p \approx_1 c) \rightarrow_* (q \approx_2 d)])\Big] \\
&\qquad \rightarrow_* [(c \approx_1 x) \rightarrow_* (d \approx_2 y)]\Big]. \qquad (7)
\end{aligned}$$

Observe that $\text{INF}^{C}_{\text{INF}^{D}_{F}}$ follows the idea used in the F-transform technique where an aggregated value is used



(a) A_1, A_2 – black line; A' – blue line.

(b) $0.9 \rightarrow_* B_1, 0.8 \rightarrow_* B_2$ – black line; B' – blue line

Figure 3. Inference over the graded fuzzy rules from Example 11

inside of the respective normal form instead of the functional value of the approximated fuzzy relation. Hence, we fix some nodes $\{(c_i,d_i) \mid (c,d) \in C\}$ that are used to create the input $\{(c \approx_1 x) \mid (c,d) \in C\}$ and output $\{(d \approx_2 y) \mid (c,d) \in C\}$ fuzzy sets and we aggregate the functional values $\{F(p,q) \mid (p,q) \in D\}$ in every neighborhood of these nodes using the implicative normal form as follows:

$$\mathrm{INF}_F^D(c,d) = \bigwedge_{(p,q)\in D} (F(p,q) \to_* [(p \approx_1 c) \to_* (q \approx_2 d)]). \quad (8)$$

Let us investigate properties of $\mathrm{INF}^C_{\mathrm{INF}_F^D}$.

Theorem 12 *Let C,D be non-empty subsets of $X \times Y$.*

1) *For an arbitrary $x \in D$:*
$$F(x) \le \mathrm{INF}^C_{\mathrm{INF}_F^D}(x); \quad (9)$$
2) *INF_F^D is the greatest element of the following set*
$$S = \{g \mathrel{\tilde{\subseteq}} X \times Y \mid F(x) \le \mathrm{INF}^C_g(x) \text{ for any } x \in D\}$$
3) *Let moreover D be a set of the positive samples w.r.t. F, $\approx_2$ be symmetric and transitive w.r.t. $*$, F be a-extensional and b-functional. Then*
$$a * b * \bigwedge_{x\in D} P(x) \le \bigwedge_{x \in D} (\mathrm{INF}^C_{\mathrm{INF}_F^D}(x) \leftrightarrow_* F(x)) \quad (10)$$

Proof: Due to the space limitation, we will provide only sketches of the proofs.

1) $F(x) \le \mathrm{INF}^C_{\mathrm{INF}_F^D}(x)$ follows from the following property that is valid on an arbitrary residuated lattice $f * a * (f * a \to_* b) \le b$ which is equivalent to $f \le (f * a \to_* b) \to_* (a \to_* b)$.
2) From $F(x) \le \mathrm{INF}^C_g(x)$, it follows that $g(x) \le \mathrm{INF}^D_F(x), \forall x \in D$ for an arbitrary $g \in S$.
3) The inequality (10) follows from a comparison of $\mathrm{INF}^C_{\mathrm{INF}_F^D}$ and $\mathrm{INF}^C_{\mathrm{DNF}_F^D}$, where
$$\mathrm{DNF}_F^D(x,y) = \bigvee_{(p,q)\in D} (p \approx_1 x) * (q \approx_2 y) * F(p,q).$$
It can be proved that $a * b \le \mathrm{INF}^C_{\mathrm{INF}_F^D} \subseteq \mathrm{INF}^C_{\mathrm{DNF}_F^D}$ and $a*b*\bigwedge_{x\in D} P(x) \le \bigwedge_{x\in D}(\mathrm{INF}^C_{\mathrm{DNF}_F^D}(x) \to_* F(x))$, hence $a * b * \bigwedge_{x\in D} P(x) \le \bigwedge_{x\in D}(\mathrm{INF}^C_{\mathrm{INF}_F^D}(x) \to_* F(x))$. The reverse implication follows from (9).

■

The above properties provides the following information: 1'st we approximate a fuzzy relation F given only in some points D using $\mathrm{INF}^C_{\mathrm{INF}_F^D}$ from the above; 2'nd this approximation is the best one such that it is from the above and it has the form of INF; 3'th we have an estimation of the precision of this approximation given in the terms of degrees of extensionality, functionality and good covering for the case of the positive samples D. The general approximation theorem for an arbitrary set of samples D can be obtained but the dependency will include also the functional values of F. To exclude these values from the estimation is a matter of the future research.

Let us provide an illustrative example for this method.

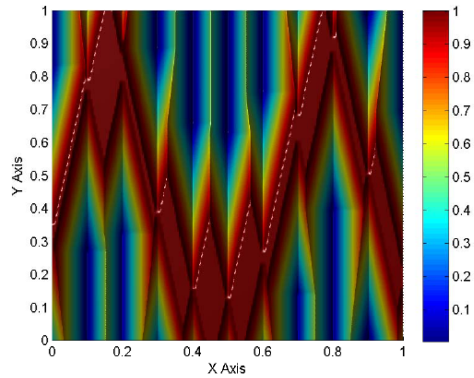


Figure 4. Perfect graded fuzzy rules from Example 13

Example 13 *Let $\mathscr{L}$ be the standard Łukasiewicz algebra, $X = Y = [0,1]$, $S = \{0.005 \cdot k \mid k = 0,\dots,200\}$, and*
$$D = \{(d, f(d)) \mid d \in S\}$$
be a set of the positive samples w.r.t. some $F \mathrel{\tilde{\subseteq}} X \times Y$, where f is defined as follows:
$$f(x) = \exp(\sin(10x))/3 + \mathrm{Rand}(x),$$
depicted on Figure 5 together with nodes
$$C = \{(0.1 \cdot k, g(0.1 \cdot k)) \mid k = 0,\dots,10\},$$
$$g(x) = \exp(\sin(10x))/3,$$
in which we construct $\mathrm{INF}^C_{\mathrm{INF}_F^D}$.

By means of (8), we compute the degrees in which the respective fuzzy rule fits to the given data D and we obtain

$$[\mathrm{INF}_F^D(0,g(0)), \mathrm{INF}_F^D(0.1,g(0.1)), \dots, \mathrm{INF}_F^D(1,g(1))] = [0.9552, 0.9645, 0.9096, 0.9863, 0.9974, 0.9918, 0.9598, 0.9168, 0.9541, 0.9917, 0.9305]$$

Applying (4), we obtain a fuzzy relation depicted on Figure 4 that represents a generalized Implicative model of fuzzy rules that can be interpreted as a collection of perfect graded fuzzy rules:

$$^{0.9863 \preceq}/(\textit{If } x \in A_1 \textit{ then } y \in B_1), \dots,$$
$$^{0.9305 \preceq}/(\textit{If } x \in A_{11} \textit{ then } y \in B_{11}).$$

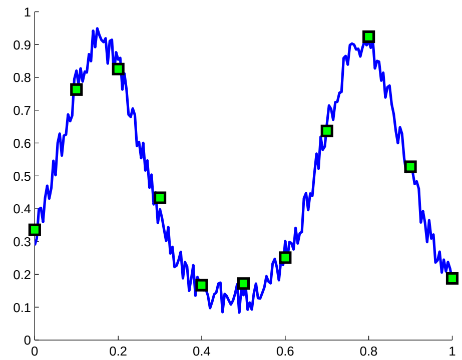
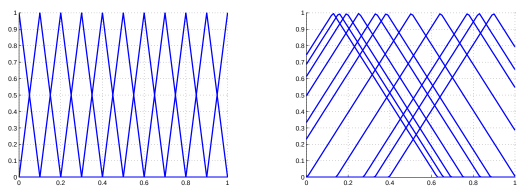


Figure 5. The noisy function f with the nodes C (green dots) in which we construct $\mathrm{INF}^{C}_{\mathrm{INF}^{D}_{F}}$



(a) The input fuzzy sets on X. (b) The output fuzzy sets on Y.

Figure 6. Input and output fuzzy sets from Example 13

V. Conclusions

We have recalled the implicative normal form together with the results showing its suitability for an approximation of dependencies represented by functional and extensional fuzzy relations to provide a construction of a generalized Implicative model that suits the best to the given data. Section IV presents a way of computing the best approximation of the given positive samples representing an ideal fuzzy relation and Theorem 12 comprises of the theoretical results providing the justification and explanation in what sense it is the best. In the author's opinion, these results open a new view on the (generalized) Implicative models of fuzzy rules and moreover, they provide a complete view on the problematic of well setting of a fuzzy rule base system where the fuzzy rules are formalized by the Implicative normal form and indeed justifying its proper behavior.

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