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Generalized Implicative Model of a Fuzzy Rule Base and its Properties

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***Abstract*—In this contribution, an implicative variant of the conjunctive normal form will be recalled and further studied. This normal form is an alternative to the Perfilieva's conjunctive normal form. It will be shown that it is a suitable model for a particular case of graded fuzzy rules introduced as a generalization of classical fuzzy rules. Moreover, approximation properties of the implicative variant of the conjunctive normal form provide a view on a class of fuzzy relations that can be "efficiently" approximated using this normal form. Newly, a suitable inference rule the graded rules formalized using the implicative variant of the conjunctive normal form will be introduced and analyzed. Results in this field extend the theory of approximate reasoning as well as the theory of fuzzy functions.**

***Keywords*-Graded fuzzy rules; Normal forms; Approximate reasoning; Fuzzy functions; Fuzzy control;**

I. Introduction

The Implicative model [1], [2], [3] of fuzzy rules is present from the birth of the theory fuzzy rules. Unfortunately, they are used very rarely in practical applications and the Mamdani-Assilian model [4], [5] completely dominates the field [6]. Among others, the cause of this fact lies mainly in a visual simplicity of the Mamdani-Assilian model, more precisely, a vivid graphical representation by the membership function interpreting this model. Theoretical results (see e.g. [2]) explain why is it so

- the Mamdani-Assilian model suites generally for a relation dependency;
- the Implicative model works well only for a functional dependency.

More precisely, the extensionality is required from a relation that we approximate by the Mamdani-Assilian model. While in the case of the Implicative model, we require additionally the functionality. Provided that the requirements are fulfilled we can expect a well behaving model in the following sense: the Mamdani-Assilian model provides approximation of an ideal fuzzy relation from below and the Implicative model from the above and moreover, it can be done with an arbitrary precision.

In this contribution, we present theoretical results relating to Implicative model of fuzzy rules that show for what kind of dependencies it will works properly and which type of inference has to be used to receive a desirable output. For this purpose, we will recall formalizations of *graded fuzzy rules* (introduced in [7] as a generalization of the known fuzzy rules) by fuzzy relations in three different forms known as normal forms. Two of them has been introduced by I. Perfilieva in [8] and therefore we denote them Perfilieva's normal forms. The remaining normal form has been introduced in [9] as a generalization of the Implicative model and it will be called *implicative normal form*. These formalizations allow to involve additional imprecision, uncertainty or vagueness related to each fuzzy rule in the form of particular degrees attached to the respective fragment of the normal form. Interpretation using the natural language is explained further in the text or we can refer to [7], [10], [9] for more details on this problematic.

It has been shown [11] that specially chosen normal forms can serve as the best approximating fuzzy relations. In the following we will not touch this problematic and we focus mainly on the explaining differences between the traditional approach to fuzzy rules represented by Hájek's work [2] and the graded approach to fuzzy rules [10]. Section IV will be devoted to implicative normal forms and their known properties. Newly, we will present in Section V properties related to approximate reasoning with graded fuzzy rules formalized by the implicative normal form.

II. Hájek's Approach to Fuzzy Control

One of the original approaches for a fuzzy rule base construction generally works only for the so called *positive samples* w.r.t. some (binary) fuzzy relation F, i.e. $(c,d) \in X \times Y$ is called the positive sample w.r.t. F if $F(c,d) = 1$. The definition can be extended for sets in the following way:

$$S = \{(c_i,d_i) |\, c_i \in X, d_i \in Y, i \in I = \{1,2,\ldots,n\}\} \qquad (1)$$

is a set of the positive samples w.r.t. F if each (c_i,d_i) is the positive sample w.r.t. F. Analogously, we say that S is a set of the negative samples w.r.t. F if for each $i \in I$: $F(c_i,d_i) = 0$ (or $\neg F(c_i,d_i) = 1$, where $\neg x = x \to 0$ and $\to$ stands for some fuzzy implication that is usually interpreted as the residuum to a t-norm).

In [2], Hájek used a set of the positive samples to create a fuzzy rule base using an indistinguishability (or similarity) relation. This procedure can be interpreted from the algebraic point of view it in the following way:

- Requirements:
  1) For the sequel, let X,Y be nonempty sets of objects, I be as above and
  $$\mathscr{L} = \langle L, *, \to_*, \wedge, \vee, 0, 1 \rangle,$$
  be a complete residuated lattice.
  2) Moreover, let $F \underset{\sim}{\subset} X \times Y$ (i.e. a fuzzy relation on $X \times Y$), $\approx_1 \underset{\sim}{\subset} X^2, \approx_2 \underset{\sim}{\subset} Y^2$ be similarity relations and $\bar{x} = [x,y]$, $\bar{x}' = [x',y']$;
  3) S be a set in the form (1) of the positive samples w.r.t. F.
- Fuzzy rules are defined as
$$\mathrm{Mamd}(x,y) =_{df} \bigvee_{i \in I} ((x \approx_1 c_i) * (y \approx_2 d_i)),$$
$$\mathrm{Rules}(x,y) =_{df} \bigwedge_{i \in I} ((x \approx_1 c_i) \to_* (y \approx_2 d_i)).$$
- Properties:
  1) If F is extensional then $\mathrm{Mamd} \subseteq F = 1$, where $A \subseteq B \equiv_{df} \bigwedge_{x \in X} (A(x) \to_* B(x))$ for the unary fuzzy relations A,B and analogously for n-ary fuzzy relations.
  2) If F is functional (for the definition see the following section) then $F \subseteq \mathrm{Rules} = 1$.
  3) If F is extensional (for the definition see the following section) and functional then
  $$\bigwedge_{x \in X} \bigvee_{i \in I} (x \approx_1 c_i) * (x \approx_1 c_i) \leq (\mathrm{Rules} \subseteq \mathrm{Mamd}), \tag{2}$$
  which consequently gives us the following estimations
  $$\bigwedge_{x \in X} \bigvee_{i \in I} (x \approx_1 c_i) * (x \approx_1 c_i) \leq \begin{array}{l} (F \approx \mathrm{Mamd}), \\ (F \approx \mathrm{Rules}), \end{array} \tag{3}$$
  where $F \approx S \equiv_{df} (F \subseteq S) * (S \subseteq F)$.

In the original source, Hájek investigated also properties of fuzzy control based on Mamd and $(\max, *)$ compositional rule of inference.

III. Graded Fuzzy Rules Formalized by Perflieva's Normal Forms

Graded fuzzy rules (introduced in [7] and further elaborated in [10]) were motivated by a need of improvement of approximation using fuzzy rules. Graded fuzzy rules were originally formalized using Perfilieva's normal forms [8]. Let us provide the original definition [8] and further explain a connection to graded fuzzy rules.

For a generality, we do not put any requirements on $\approx_{1(2)}$, they are assume to be arbitrary binary fuzzy relations in the sequel.

Definition 1 *The* disjunctive normal form *(*DNF *for short) for a fuzzy relation F is*
$$\mathrm{DNF}_F(\bar{x}) =_{df} \bigvee_{i \in I} [(c_i \approx_1 x) * (d_i \approx_2 y) * F(c_i, d_i)], \tag{4}$$
The conjunctive normal form *(*CNF *for short) for F is given by*
$$\mathrm{CNF}_F(\bar{x}) =_{df} \bigwedge_{i \in I} [((x \approx_1 c_i) * (y \approx_2 d_i)) \to_* F(c_i, d_i)]. \tag{5}$$

As stated in [10], [7], an ambiguity or uncertainty over a certain fuzzy rule can be can be implemented using a degree that equips the respective rule and together they form the so called *graded fuzzy rule*. A visualization of a collection of graded fuzzy rules is the following:

$$\begin{array}{ll} {}^{f_1 \succeq}/(x \in A_1 \text{ and } y \in B_1) & \text{OR} \\ \dots & \text{OR} \\ {}^{f_n \succeq}/(x \in A_n \text{ and } y \in B_n), & \end{array}$$

for the case of DNF and

$$\begin{array}{ll} {}^{f_1 \preceq}/(x \in A_1 \text{ and } y \in B_1) & \text{AND} \\ \dots & \text{AND} \\ {}^{f_n \preceq}/(x \in A_n \text{ and } y \in B_n), & \end{array}$$

for the case of CNF, where

Symbol	Interpretation of
f_i	$F(c_i, d_i)$
$x \in A_i$	$(x \approx_1 c_i)$
$y \in B_i$	$(y \approx_2 d_i)$
$*$	and
$\vee$	OR
$\wedge$	AND

We can read a particular graded fuzzy rule ${}^{f \succeq}/[x,y] \in R$ as

"$[x,y]$ are related by R at most to the degree f",

analogously, ${}^{f \preceq}/A$ reads as

"$[x,y]$ are related by R at least to the degree f",

Let us explain why "at least" and "at most" are used in the reading of graded fuzzy rules by analyzing one particular graded rule for , i.e. the one particular fragment of the normal form:

1) In the case of DNF, we conclude that the operation of conjunction $*$ applied on the relation $\underbrace{[(c_i \approx_1 x) * (d_i \approx_2 y)]}_{R(x,y)}$ and the degree $F(c_i, d_i)$ exhibits as a kind of shift or resize operator. The reason for this claim is that the final fuzzy relation does not exceed this degree, i.e. $R(x,y) * F(c_i, d_i) \leq F(c_i, d_i)$ (it follows from $a * b \leq b$). This fact is distinguished by the symbol $\succeq$ that we read as "at most" in the associated graded fuzzy rule.

2) In the case of CNF, we have a different situation. The implication $\rightarrow_*$ with the degree on the right input position behaves as a rotation and shift (or resize) operator. It means that the final fuzzy relation fulfills $[((x \approx_1 c_i) * (y \approx_2 d_i)) \rightarrow_* F(c_i, d_i)] \geq F(c_i, d_i)$ (follows from $b \leq a \rightarrow_* b$). Analogously, we distinguish this fact by the symbol $\preceq$ in the graded fuzzy rule and we read it as "at least".

As shown in [8], [12] (generalized to non-symmetric $\approx_{1,2}$ in [13] and extended to graded theorems in [10]) graded fuzzy rules based on Perfilieva's normal forms are suitable for extensional fuzzy relations.

Definition 2 *F is extensional w.r.t. $\approx_1, \approx_2$ if*

$$x \approx_1 x' * y \approx_2 y' * F(\bar{x}) \leq F(\bar{x}'), \tag{6}$$

is valid for all $x, x' \in X$ and $y, y' \in Y$.

Theorem 3 *[8], [11] Let us consider an extensional fuzzy relation F in the above defined sense. Then*

- DNF_F *lies below F and* CNF_F *is above;*
- *a precision of the approximation expressed in terms of the equivalence $\leftrightarrow_*$ (bi-residual operation) depends on the distribution of (c_i, d_i)'s over $X \times Y$:*

$$\bigvee_{i \in I} [(c_i \approx_1 x) * (x \approx_1 c_i) * (d_i \approx_2 y) * (y \approx_2 d_i)] \leq \mathrm{DNF}_F(\bar{x}) \leftrightarrow_* F(\bar{x}), \tag{7}$$

and the same estimation is valid for $\mathrm{CNF}_F(\bar{x}) \leftrightarrow_* F(\bar{x})$.

These results can be extended to graded properties so that also partially extensional relations can be allowed. First, let us recall the graded extensionality taken from [10] and translated into our algebraic framework.

Definition 4 *A relation F is said to be a-extensional w.r.t. $\approx_1, \approx_2$ if*

$$a = \bigwedge_{x, x' \in X, y, y' \in Y} [[(x \approx_1 x') * (y \approx_2 y') * F(\bar{x})] \rightarrow_* F(\bar{x}')], \tag{8}$$

where $a \in L$. We will shortly denote the right side of the above equality by $\mathrm{Ext}_{\approx_1, \approx_2} F$.

Observe that using the classical definition of extensionality we determine a crisp class of fuzzy relations that are 1-extensional. a-extensionality also determines a crisp class of fuzzy relations that are a-extensional. But truth values computed by $\mathrm{Ext}_{\approx_1, \approx_2} F$ for all fuzzy relations over the universe of the discourse define the fuzzy class of fuzzy relations that are extensional.

Example 5 *Let $\mathscr{L}$ be the standard Łukasiewicz algebra and*

$$F(\bar{x}) =_{df} y \leftrightarrow_* (x \cdot x)$$

on M^2, where $M = \{0.05 \cdot k|\ k = 0, 1, \ldots, 20\}$. Then F is extensional w.r.t. $\approx_1, \approx_2$ defined as

$$x \approx_1 y =_{df} (x \leftrightarrow_* y) * (x \leftrightarrow_* y),$$
$$x \approx_2 y =_{df} x \leftrightarrow_* y.$$

In the terms of graded extensionality, it is

$$1\text{-extensional w.r.t. } \approx_1, \approx_2.$$

But if we change $\approx_1$ to $\leftrightarrow_*$ then we obtain that

$$F \text{ is } 0.75\text{-extensional w.r.t. } \approx_2, \approx_2.$$

And even worse, if we take the original $\approx_1$ and change $\approx_2$ to $\approx_1$ then we have that

$$F \text{ is } 0.19\text{-extensional w.r.t. } \approx_1, \approx_1.$$

The following theorem summarizes properties of approximation using the normal forms in the graded style that allows also not completely extensional relations and in this case, it provides information about the resulting quality of the approximation by estimation of degrees of the required properties.

Theorem 6 *[10]Let F be a-extensional. Then*

- *Subsethood:* $a \leq (\mathrm{DNF}_F \subseteq F)$, $a \leq (F \subseteq \mathrm{CNF}_F)$.
- *Estimation for a precision of the approximation:*

$$a * \bigwedge_{x \in X, y \in Y} C(\bar{x}) \leq (\mathrm{DNF}_F \approx F),$$

where $C(\bar{x})$ denotes the left side of the inequality (7). The same estimation holds for CNF_F.

The first two inequalities say that the degree of extensionality estimates the degree of inclusion of DNF_F in F and F in CNF_F, respectively. The last inequality provides the lower estimation of the quality of approximation using the normal form that is generated by the degree of extensionality and the degree of lets say the good covering expressed by $\bigwedge_{x \in X, y \in Y} C(\bar{x})$. Informally speaking, higher the degrees of requirements (extensionality and good covering) better is the resulting approximation.

IV. Graded Fuzzy Rules Formalized by the Implicative Normal Form

In this section, we are going to recall an implicative variant of the conjunctive normal form (implicative normal form for short) introduced in [9] and to show its properties. This normal form differs from the Perfilieva's conjunctive normal form and it has been introduced for a distinct purpose. While the Perfilieva's conjunctive normal form is suitable for extensional fuzzy relations and negative samples, the implicative normal form has been aimed for functional fuzzy relations and positive samples.

Definition 7 *The* conjunctive normal form – implicative variant *(*INF *for short) for a fuzzy relation F is*

$$\mathrm{INF}_F(\bar{x}) =_{df} \bigwedge_{i\in I}[F(c_i,d_i) \to_* [(c_i \approx_1 x) \to_* (d_i \approx_2 y)]]. \quad (9)$$

For the sake of brevity and in order to simplify the distinction between the conjunctive normal forms, we will call the conjunctive normal form – implicative variant *as the* implicative normal form.

Generally for non-symmetric $\approx_1, \approx_2$, we can introduce also a variant of the above defined INF by juxtaposition of the variables and constants

$$\mathrm{INF}'_F(\bar{x}) =_{df} \bigwedge_{i\in I}[F(c_i,d_i) \to_* [(x \approx_1 c_i) \to_* (y \approx_2 d_i)]]. \quad (10)$$

Obviously, most of the results are valid for both implicative normal forms therefore, we will deal only with INF in the sequel and we will explain differences where it will be necessary.

Now, let us explain a connection with graded fuzzy rules. Let us take into account only a single segment of INF and we analyze its core, i.e. we have $F(c_i,d_i) \to_* [(c_i \approx_1 x) \to_* (d_i \approx_2 y)] = 1$ that is valid if and only if $(c_i \approx_1 x) \to_* (d_i \approx_2 y) \in [F(c_i,d_i),1]$. Therefore we interpret this fuzzy relation in INF as one graded fuzzy rule in the form

$${}^{f_i\preceq}/(\mathsf{If}\ x \in A_i\ \mathsf{then}\ y \in B_i),$$

where the used symbols are interpreted as it was specified in the table in Section III and additionally, "If …then" interprets $\to_*$. We read the above graded fuzzy rule as "(If $x \in A$ then $y \in B$) at least to the degree f_i" with the above explained meaning. Hence, INF formalizes the following collection of graded fuzzy rules:

$$\begin{array}{ll} {}^{f_1\preceq}/(\mathsf{If}\ x \in A_1\ \mathsf{then}\ y \in B_1) & \text{AND} \\ \dots & \text{AND} \\ {}^{f_n\preceq}/(\mathsf{If}\ x \in A_n\ \mathsf{then}\ y \in B_n). & \end{array}$$

As noted at the beginning of this section, the implicative normal forms are suited to approximate functional fuzzy relations (slightly less general version is used in the field of fuzzy control see e.g. [14], [2], [15]). The following definition will be directly in the graded form inspired by [16], where it has been introduced within the formal framework of fuzzy class theory [17].

Definition 8 *A relation F is said to be a-functional w.r.t. $\approx_{1,2}$ if*

$$a = \bigwedge_{x,x'\in X,y,y'\in Y}[[(x \approx_1 x')* F(x,y)*F(x',y')] \to_* (y \approx_2 y')].$$

We will shortly denote the right side of the above equality by $\mathrm{Fun}_{\approx_{1,2}} F$.

Example 9 *In the setting of Example 5, we can compute the following degrees of functionality for F:*

F is 1-functional w.r.t. $\approx_1, \approx_2$;
F is 0.75-functional w.r.t. $\approx_2, \approx_2$;
F is 0.5-functional w.r.t. $\approx_1, \approx_1$;
F is 0.31-functional w.r.t. $\approx_2, \approx_1$.

If we change the background algebraic structure to the standard product algebra then the fist degree of functionality will remain the same and the rest will change to $0.25, 0.05, 0.0125$, *respectively.*

Let us provide the main results taken from [9] relating to properties of an approximation (analogous to the Hajek's results) using the implicative normal forms.

Theorem 10 *If F is a-functional then*

$$a \le \bigwedge_{x\in X,y\in Y}(F(x,y) \to_* \mathrm{INF}(x,y)).$$

It means that the degree of functionality $\mathrm{Fun}_{\approx_{1,2}} F$ is the lower estimation of the inclusion F and INF denoted as $F \subseteq \mathrm{INF}$, i.e. $\mathrm{Fun}_{\approx_{1,2}} F \le F \subseteq \mathrm{INF}$.

Theorem 11 *Let C and C' be given by*

$$C(x) =_{df} \bigvee_{i\in I}[(c_i \approx_1 x)*(c_i \approx_1 x)], \quad (11)$$

$$C'(x) =_{df} \bigvee_{i\in I}[(c_i \approx_1 x)*(x \approx_1 c_i)]. \quad (12)$$

If F is a-functional and b-extensional and moreover, the normal forms are constructed in the nodes taken from the set `Samples` *then*

$$a*b*\bigwedge_{x\in X}C(x) \le \bigwedge_{x\in X,y\in Y}(F(x,y) \leftrightarrow_* \mathrm{INF}(x,y)).$$

The result for INF'_F with symmetric $\approx_2$ is the same inequality with C' instead of C. Moreover, an interpretation of this result is analogous to the one given for Theorem 6.

Example 12 *From Example 5 and 9, it follows that the precision of approximation of F using* INF *w.r.t. $\approx_1, \approx_2$ depends only on the suitable partition of $X \times Y$ such that $\bigwedge_{x\in X} C(x)$ is as high as possible (it also leads to an optimization problem) in the both cases of the background* *algebra. While e.g. in the case of* $\approx_2, \approx_2$ *and the Łukasiewicz standard algebra, we have that*

$$\underbrace{0.5}_{0.75*0.75} * \bigwedge_{x\in X} C(x) \leq \bigwedge_{x\in X, y\in Y} (F(x,y) \leftrightarrow_* \mathrm{INF}(x,y)).$$

From the sequence of examples, we see that the normal form based approximations and hence generally "fuzzy rule based approximations" are very sensitive to all choices of the input parameters such as the choice of the background algebra, binary fuzzy relations $\approx_1, \approx_2$ number of nodes and their distribution over the respective universe.

V. Consequences of the Implicative Normal Form Based Formalization to Fuzzy Control

From the point of view of fuzzy control, it is important to investigate the outputs of the particular approximate inference (reasoning). It appeared [10], that it is necessary to distinguish between approximate inference using DNF and CNF. Let us briefly describe them together with the new inference rule for INF:

$$R_{\mathrm{DNF}}: \frac{A', \mathrm{DNF}}{B^*}, \quad R_{\mathrm{CNF}}: \frac{A', \mathrm{CNF}}{B^{**}} \quad \text{and } R_{\mathrm{INF}}: \frac{A', \mathrm{INF}}{B'},$$

where

$$B^*(y) =_{df} \bigvee_{x\in X} (A'(x) * \mathrm{DNF}_F(x,y)),$$
$$B^{**}(y) =_{df} \bigwedge_{x\in X} (A'(x) \rightarrow_* \mathrm{CNF}_F(x,y)),$$
$$B'(y) =_{df} \bigvee_{x\in X} (A'(x) * \mathrm{INF}_F(x,y)).$$

We refer to [7], [10] for properties of R_{DNF} and R_{CNF}. Due to the space limitation, we will provide only properties of R_{INF} without proofs.

Theorem 13 *Let*

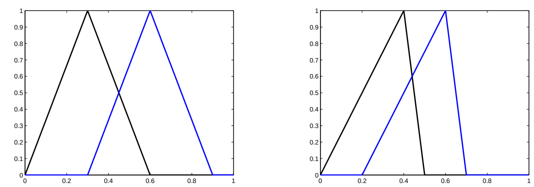
$$A_i(x) =_{df} (x \approx_1 c_i),$$
$$B_i(y) =_{df} F(c_i, d_i) \rightarrow_* (y \approx_2 d_i).$$

- $(A' \subseteq A_i) \rightarrow_* (B' \subseteq B_i)$;
- $\bigvee_{x\in X}(A_i(x)) * (A_i \subseteq A') \rightarrow_* (B_i \subseteq B')$;
- $\bigvee_{x\in X}(A_i(x)) * (A_i \approx A') \rightarrow_* (B_i \approx B')$.

From these properties we see that R_{INF} is very natural approximate inference since it provides an expected output without complicated requirements. Indeed, there is only one extraordinary requirement $\bigvee_{x\in X}(A_i(x))$ that can be characterized as non-emptiness and it can be read as: there exists $x \in A_i$.

Example 14 *As an illustration, let us consider Łukasiewicz standard algebra and two graded fuzzy rules*

$$^{0.9\preceq}/(\mathit{If}\ x \in A_1\ \mathit{then}\ y \in B_1) \quad \text{AND}$$
$$^{0.8\preceq}/(\mathit{If}\ x \in A_2\ \mathit{then}\ y \in B_2),$$



(a) Fuzzy sets on X, A_1 – black line; A_2 – blue line. (b) Fuzzy sets on Y, B_1 – black line; B_2 – blue line

Figure 1. Fuzzy sets in the graded fuzzy rules from Example 14

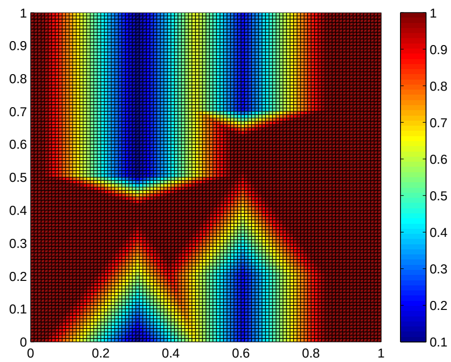
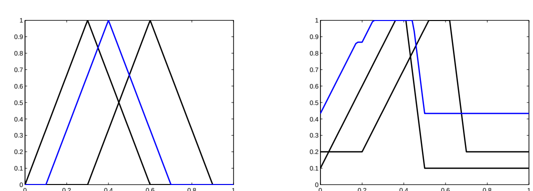


Figure 2. Graded fuzzy rules from Example 14

where A_i, B_i *are (non-symmetric) triangular fuzzy numbers depicted on Figure 1. The resulting fuzzy relation is visualized on Figure 2. The output of the inference* R_{INF} *with the input fuzzy set* A' *(blue line on Figure 3(a), i.e. shifted* A_1*) is a fuzzy set* B' *drawn on Figure 3(b). On these figures, we demonstrate the last inequality in the above theorem:* $(A_1 \approx A') \doteq 0.6667$ *and* $\bigvee_{x\in X}(A_1(x)) = 1$*, hence,* $0.06667 \leq (0.9 \rightarrow_* B_1) \approx B'$*. In the case of the input fuzzy set* A' *identical with* A_1 *(or* A_2*), we obtain exactly* $0.9 \rightarrow_* B_1$ *(*$0.8 \rightarrow_* B_2$*) as the output* B'*.*

The following properties relate to a position of the reconstructed implicative rule $A' \rightarrow_* B'$ and the ideal fuzzy relation F.

Theorem 15 *Assume the notational convention as in Theo-*



(a) A_1, A_2 – black line; A' – blue line. (b) $0.9 \rightarrow_* B_1, 0.8 \rightarrow_* B_2$ – black line; B' – blue line

Figure 3. Inference over the graded fuzzy rules from Example 14

rem 13 and moreover

$$F_h(y) =_{df} \bigvee_{x\in X} F(x,y);$$

- *If F is a-functional then $a \leq F \subseteq (A' \to_* B')$;*
- *If F is a-functional and b-extensional then*

$$a * b * \bigwedge_{x\in X} C(x) \leq (A' \to_* B') \subseteq (A' \to_* F_h).$$

The first inequality shows that the degree of functionality of F estimates the degree of inclusion of F in $A' \to_* B'$. And the second inequality provides an estimation of the reverse inclusion, i.e. the degrees of extensionality, functionality and the good partition estimates inclusion of $A' \to_* B'$ in $A' \to_* F_h$. An open question remains whether it is possible to prove $(A' \to_* B') \subseteq F$.

VI. Conclusions

In this contribution the implicative normal form has been recalled together with the results showing its suitability for an approximation of dependencies represented by functional and extensional fuzzy relations. In Theorem 11, an estimation of the equivalence between the given fuzzy relation and its INF has been provided. The resulting inequality is the same as in the case of DNF and CNF. Which gives us a confidence on the efficiency of INF. Moreover, we have worked purely with the graded notions and all provided results carry the information about the involved degrees. Hence, they widens an applicability of normal forms to partially extensional and partially functional fuzzy relations.

Additionally, a connection between graded fuzzy rules and the implicative normal form was explained. Indeed, graded fuzzy rules can be seen as classical fuzzy rules with the modified antecedent parts (a different modification for each normal form). This view provides an insight into the nature of descriptions formalized by normal forms.

Finally, a suitable inference rule has been introduced and studied. Our choice of the inference rule is supported by the theoretical results mainly by Theorem 13 that explains behavior of an output of this inference w.r.t. a particular fuzzy set on the consequent part of the respective fuzzy rule. This theoretical analysis of the generalized Implicative model of fuzzy rules is aimed to provide a complete view on the problematic of well setting of a fuzzy rule base system and indeed justifying its proper behavior.

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