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# Graded fuzzy rules for functional dependencies:

Normal form based formalization

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**Abstract:** We will introduce graded fuzzy rules that are formalized using a new type of normal form. We will show its approximation properties and a relationship to the existing normal forms.

## 1 Introduction to Graded Fuzzy Rules

In [1] (a complete study in [2]), graded fuzzy rules have been introduced as a generalization of classical fuzzy rules. Their formalizations were performed by Perfilieva's *normal forms* [3] (for more details see e.g. [4, 5, 6, 7]). The normal form approximates a given fuzzy relation from above or below providing that this relation is extensional. A quality of the approximation is expressed in terms of a similarity (equivalence) relation between the original extensional fuzzy relation and its normal form.

Let us explain a way of generalization from the notion of classical fuzzy rules to graded fuzzy rules. Classical fuzzy rules take one of the following form [8, 9, 10, 11, 12]:

$$\begin{array}{llcl} \texttt{Dis}: & D_1 & \textsf{OR} \ldots \textsf{OR} & D_k, \\ \texttt{Con}: & C_1 & \textsf{AND} \ldots \textsf{AND} & C_k, \end{array}$$

where AND (OR) is represented by conjunction (disjunction), and $D_i, C_i,\ i \in I = \{1, \ldots, k\}$, are specific linguistic expressions usually assumed to be as follows:

$$\begin{array}{l} D_i \equiv_{df} (x \in A_i) \textsf{ and } (y \in B_i), \\ C_i \equiv_{df} \textsf{If } (x \in A_i) \textsf{ then } (y \in B_i), \end{array}$$

where each $A_i \underset{\sim}{\subset} X (B_i \underset{\sim}{\subset} Y)$ is represented by a fuzzy set over the respective domain that is denoted by the same symbol and moreover, "$x \in A$" is computed as $A(x)$. "and" relates to a strong conjunction & and "If . . . then" relates to an implication $\to$ of the suitably chosen logic and these connectives are interpreted in a structure for truth values by $*$ and $\to_*$ (e.g. a left continuous t-norm and its residuation), respectively.

**Convention:** If we will deal with classical fuzzy rules we will mean the above interpretation.

As stated in [1], there may exist a certain amount of dubiousness over a fuzzy rule. This can be implemented using a degree that equips the respective rule and together they form a *graded fuzzy rule*. Hence, the modified expressions inside `Dis` and `Con` can be visualized as in [2] (using the same notation as for evaluated formulae [11])

$$\begin{array}{rl} \mathcal{D}_i \equiv_{df} & {}^{f_i\succeq}/(x \in A_i \textsf{ and } y \in B_i), \\ \mathcal{D}'_i \equiv_{df} & {}^{f_i\preceq}/(x \in A_i \textsf{ and } y \in B_i), \end{array}$$

and

$$\begin{array}{rl} \mathcal{C}_i \equiv_{df} & {}^{f_i\preceq}/(\textsf{If } x \in A_i \textsf{ then } y \in B_i), \\ \mathcal{C}'_i \equiv_{df} & {}^{f_i\succeq}/(\textsf{If } x \in A_i \textsf{ then } y \in B_i), \end{array}$$

for all $i \in I$, respectively, and $f_1, \ldots, f_k$ take values from the structure for truth values. There, the particular graded fuzzy rule ${}^{f\succeq}/[x, y] \in R$ reads as

"$[x, y]$ are related by $R$ at most to the degree $f$",

analogously, ${}^{f\preceq}/A$ reads as

"$[x, y]$ are related by $R$ at least to the degree $f$",

In [2], `Dis` with $\mathcal{D}_i, i \in I$, has been formalized using the disjunctive normal form, `Dis` with $\mathcal{D}'_i, i \in I$, using the conjunctive normal form. Up to the author's knowledge, `Con` with $\mathcal{C}_i, i \in I$, and `Con` with $\mathcal{C}'_i, i \in I, i \in I$, have not been elaborated yet. In this contribution, we will take into account only `Con` with $\mathcal{C}_i, i \in I$, and it will be formalized using a new type of normal form. This new normal form is an alternative to the existing Perfilieva's conjunctive normal form and it will be shown that it has analogous approximation properties.

## 2 Formalization using Perfilieva's Normal Forms

As it was mentioned above, the first two graded fuzzy rules are formalized using Perfilieva's normal forms. Let us provide the original definition [3] and further explain a connection to graded fuzzy rules.

For the sequel, let $X, Y$ be nonempty sets of objects, $I$ be as above and

$$\mathcal{L} = \langle L, *, \to_*, \wedge, \vee, 0, 1\rangle,$$

be a complete residuated lattice. Moreover, let $F \underset{\sim}{\subset} X \times Y$ (i.e. a fuzzy relation on $X \times Y$), $\approx_1 \underset{\sim}{\subset} X^2, \approx_2 \underset{\sim}{\subset} Y^2$ and $\bar{x} = [x, y],\ \bar{x}' = [x', y'], c_i \in X, d_i \in Y$ for all $i \in I$.

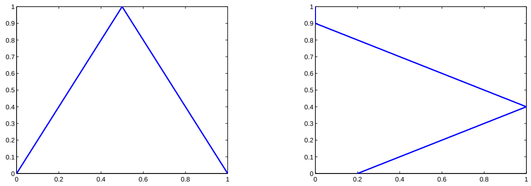
**Definition 1** *The* disjunctive normal form *(*DNF *for short) for a fuzzy relation $F$ is*

$$\mathrm{DNF}_F(\bar{x}) =_{df} \bigvee_{i \in I} [(c_i \approx_1 x) * (d_i \approx_2 y) * F(c_i, d_i)], \quad (1)$$

*The* conjunctive normal form *(*CNF *for short) for $F$ is given by*

$$\mathrm{CNF}_F(\bar{x}) =_{df} \bigwedge_{i \in I} [((x \approx_1 c_i) * (y \approx_2 d_i)) \to_* F(c_i, d_i)]. \quad (2)$$

Let us explain a connection to graded fuzzy rules: for each $i \in I$, let $c_i \approx_1 x$ and $x \approx_1 c_i$ stand for $x \in A_i$ in $\mathcal{D}_i$ and $\mathcal{D}'_i$, respectively. Analogously, let $d_i \approx_2 y$ and $y \approx_2 d_i$ stand for $y \in B_i$ in $\mathcal{D}_i$ and $\mathcal{D}'_i$, respectively. Usually, $\approx_1, \approx_2$ are used to express neighbourhood of the fixed nodes, in our case,



(a) $A \subsetsim X$ (b) $B \subsetsim Y$

Fig. 1: Fuzzy sets for illustration of (graded) fuzzy rules.

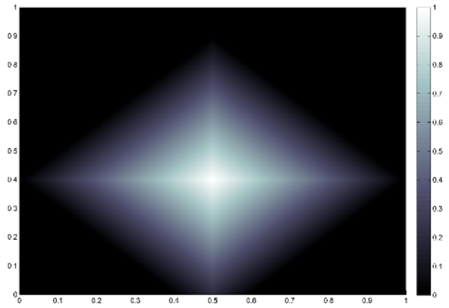


Fig. 2: Classical fuzzy rule $(x \in A)$ and $(y \in B)$

$c_i, d_i$, respectively. But generally, they can take various forms due to our actual requirements. Moreover, let $f_i = F(c_i, d_i)$ for each $i \in I$. Hence, (1) formalizes Dis with $\mathcal{D}_i$ and (2) formalizes Dis with $\mathcal{D}'_i, i \in I$.

The following sequence of figures visualize one particular fuzzy rule – classical and graded; with the input fuzzy sets $A, B$ depicted on Figure 1 over the standard Łukasiewicz algebra $\mathcal{L}$. First, the classical fuzzy rule interpreted as $A(x) * B(y)$ is drawn on Figure 2 and the graded fuzzy rule interpreted as

$$A(x) * B(y) * 0.8$$

can be seen on Figure 3. There, we see that the operation of conjunction $*$ applied on the relation $(A(x) * B(y))$ and the degree 0.8 exhibits as a shift operator and the final fuzzy relation does not exceed this degree, i.e. $A(x) * B(y) * 0.8 \leq 0.8$ (follows from $a * b \leq b$) and it is interpreted as

$$^{0.8\succeq}/(x \in A_i \text{ and } y \in B_i),$$

in this particular case.

Dually, a negation (defined as $\neg x = x \rightarrow_* 0$) of the same classical fuzzy rule, i.e. $A(x) * B(y) \rightarrow_* 0$, is drawn on Figure 4 and a graded fuzzy rule interpreted as $A(x) * B(y) \rightarrow_* 0.6$ is on Figure 5. Now we have that the implication $\rightarrow_*$ with the degree on the right input position exhibits as a rotation and shift operator. It means that the final fuzzy relation fulfills $A(x) * B(y) \rightarrow_* 0.6 \geq 0.6$ (follows from $b \leq a \rightarrow_* b$) and it is interpreted as

$$^{0.6\preceq}/(x \in A_i \text{ and } y \in B_i).$$

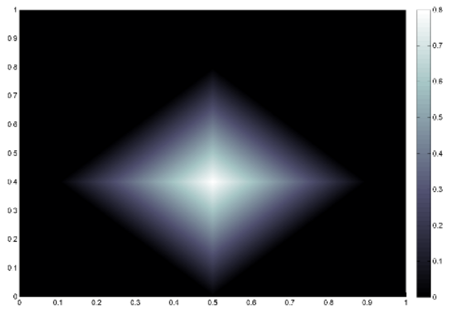


Fig. 3: Graded fuzzy rule $^{0.8\succeq}/(x \in A \text{ and } y \in B)$

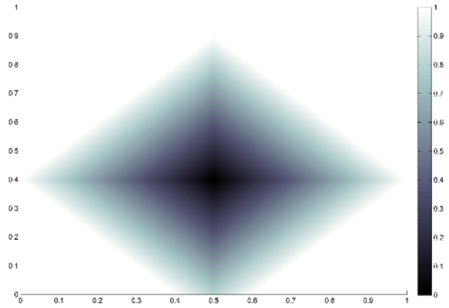


Fig. 4: Negation of the classical fuzzy rule $(x \in A)$ and $(y \in B)$

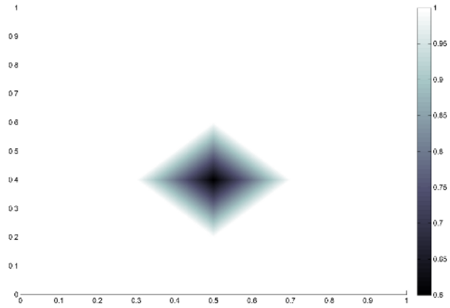


Fig. 5: Graded fuzzy rule $^{0.6\preceq}/(x \in A \text{ and } y \in B)$

The above normal forms were originally assumed to deal with extensional fuzzy relations.

**Definition 2** *$F$ is extensional w.r.t. $\approx_1, \approx_2$ if*

$$x \approx_1 x' * y \approx_2 y' * F(\bar{x}) \leq F(\bar{x}'), \tag{3}$$

*is valid for all $x, x' \in X$ and $y, y' \in Y$.*

**Theorem 3** *[3, 5] Let us consider an extensional fuzzy relation $F$ in the above defined sense. Then*

- $\mathrm{DNF}_F$ *lies below $F$ and* $\mathrm{CNF}_F$ *is above;*
- *a precision of the approximation expressed in terms of the equivalence $\leftrightarrow_*$ (bi-residual operation) depends on the distribution of $(c_i, d_i)$'s over $X \times Y$:*

$$\bigvee_{i \in I} [(c_i \approx_1 x) * (x \approx_1 c_i) * (d_i \approx_2 y) * (y \approx_2 d_i)] \leq \mathrm{DNF}_F(\bar{x}) \leftrightarrow_* F(\bar{x}), \tag{4}$$

*and the same estimation is valid for* $\mathrm{CNF}_F(\bar{x}) \leftrightarrow_* F(\bar{x})$.

In [2], a graded approach [13] to modeling graded fuzzy rules and to approximate reasoning with such rules has been elaborated. For this purpose, the extensionality was generalized to the graded extensionality, which can be translated using the algebraic operation in the following definition:

**Definition 4** *A relation $F$ is said to be $a$-extensional w.r.t. $\approx_1, \approx_2$ if*

$$a = \bigwedge_{x,x' \in X, y,y' \in Y} [[(x \approx_1 x') * (y \approx_2 y') * F(\bar{x})] \rightarrow_* F(\bar{x}')]. \quad (5)$$

*We will shortly denote the right side of the above equality by* $\mathrm{Ext}_{\approx_1,\approx_2} F$.

**Example 5** *Let $\mathcal{L}$ be the standard Łukasiewicz algebra and*

$$F(\bar{x}) =_{df} y \leftrightarrow_* (x \cdot x)$$

*on $M^2$, where $M = \{0.05 \cdot k | \; k = 0, 1, \ldots, 20\}$. Then $F$ is extensional w.r.t. $\approx_1, \approx_2$ defined as*

$$x \approx_1 y =_{df} (x \leftrightarrow_* y) * (x \leftrightarrow_* y),$$
$$x \approx_2 y =_{df} x \leftrightarrow_* y.$$

*In other words, it is*

$$1\text{-extensional w.r.t. } \approx_1, \approx_2.$$

*But if we change $\approx_1$ to $\leftrightarrow_*$ then we obtain that*

$$F \text{ is } 0.75\text{-extensional w.r.t. } \approx_2, \approx_2.$$

*And even worse, if we take the original $\approx_1$ and change $\approx_2$ to $\approx_1$ then we have that*

$$F \text{ is } 0.19\text{-extensional w.r.t. } \approx_1, \approx_1.$$

This approach allows also not completely extensional relations and the following properties can be shown.

**Theorem 6** *[2]*

- *The degree of extensionality estimates the inclusion of* $\mathrm{DNF}_F$ *and $F$ as well as $F$ and* $\mathrm{CNF}_F$*:*

$$\mathrm{Ext}_{\approx_1,\approx_2} F \leq \bigwedge_{x \in X, y \in Y} (\mathrm{DNF}_F(\bar{x}) \rightarrow_* F(\bar{x})),$$
$$\mathrm{Ext}_{\approx_1,\approx_2} F \leq \bigwedge_{x \in X, y \in Y} (F(\bar{x}) \rightarrow_* \mathrm{CNF}_F(\bar{x})).$$

- *A general estimation for the precision of the approximation using normal forms is valid in the following form:*

$$\mathrm{Ext}_{\approx_1,\approx_2} F * \bigwedge_{x \in X, y \in Y} C(\bar{x}) \leq \bigwedge_{x \in X, y \in Y} (\mathrm{DNF}_F(\bar{x}) \leftrightarrow_* F(\bar{x})),$$

*where $C(\bar{x})$ denotes the left side of the inequality (4). The same estimation holds for* $\mathrm{CNF}_F$.

All the above results can be found in [2] in the form of special logical formulae – graded theorems. Remark that the adjective "graded" is used for something behaving gradually which can be internalized using truth degrees (grades). In this text, we use the following notions:

- *Graded fuzzy rules* – fuzzy rules modified using truth degrees;
- *Graded property* – a property of an object that comes with degrees, e.g. $a$-extensionality, graded properties of fuzzy relations used e.g. in [14] or [15];
- *Graded approach* – an approach enabling to handle graded properties so that we do not lose information coded by grades, see e.g. [13, 16];
- *Graded theorems* – theorems proved on the bases of the graded approach, i.e. only with graded properties. For example, the last inequality above is an algebraic interpretation of the respective graded theorem. The detailed explanation can be found in [2].

## 3 CONJUNCTIVE NORMAL FORM – IMPLICATIVE VARIANT

In this section, we are going to introduce a new normal form and to show its properties.

**Definition 7** *The* conjunctive normal form – implicative variant *(*INF *for short) for a fuzzy relation $F$ is*

$$\mathrm{INF}_F(\bar{x}) =_{df} \bigwedge_{i \in I} [F(c_i, d_i) \rightarrow_* [(c_i \approx_1 x) \rightarrow_* (d_i \approx_2 y)]]. \quad (6)$$

*For the sake of brevity and in order to simplify the distinction between the conjunctive normal forms, we will call the* conjunctive normal form – implicative variant *as the* implicative normal form.

Since generally $\approx_1, \approx_2$ do not need to be symmetric we can introduce also a variant of the above defined INF by juxtaposition of the variables and constants

$$\mathrm{INF}'_F(\bar{x}) =_{df} \bigwedge_{i \in I} [F(c_i, d_i) \rightarrow_* [(x \approx_1 c_i) \rightarrow_* (y \approx_2 d_i)]]. \quad (7)$$

Apparently, most of the results are valid for both implicative normal forms therefore, we will deal only with INF in the sequel and we will explain differences where it will be necessary.

Observe that

$$\mathrm{INF}_F(\bar{x}) = \bigwedge_{i \in I} [(c_i \approx_1 x) \rightarrow_* \underbrace{[F(c_i, d_i) \rightarrow_* (d_i \approx_2 y)]}_{B_i}],$$
$$\mathrm{DNF}_F(\bar{x}) = \bigvee_{i \in I} [(c_i \approx_1 x) * \underbrace{[(d_i \approx_2 y) * F(c_i, d_i)]}_{B'_i}],$$

and

$$\mathrm{CNF}_F(\bar{x}) = \bigwedge_{i \in I} [(x \approx_1 c_i) \rightarrow_* \underbrace{[(y \approx_2 d_i) \rightarrow_* F(c_i, d_i)]}_{B''_i}].$$

Hence, all normal forms can be viewed as formalizations of the classical fuzzy rules with appropriately modified antecedent parts:

- $\mathrm{INF}_F$ formalizes `Con` with the consequents $B_i, i \in I$;
- $\mathrm{CNF}_F$ formalizes `Con` with the consequents $B''_i, i \in I$;
- $\mathrm{DNF}_F$ formalizes `Dis` with the consequents $B'_i, i \in I$.

An explanation of the relationship between graded fuzzy rules and INF will come in the next section.

Observe that if we assume symmetric $\approx_1, \approx_2$ then the differences between CNF and INF are on the antecedent parts $B_i$ and $B_i''$. Considering BL-algebra $\mathcal{L}$, we see that $B_i$ equals $B_i'$ only in the cases of $y \in Y$ for which $(d_i \approx_2 y) = F(c_i, d_i)$ (it follows from $a * (a \rightarrow_* b) = a \wedge b$). Therefore, CNF and INF have completely different natures.

Let us return to the example of classical fuzzy rules from the previous section and compare Figure 4 with Figure 6: Figure 4 is represented by the following expression

$$[A(x) * B(y)] \rightarrow_* 0,$$

and based on the assignment given in the previous section, it represents one particular graded fuzzy rule formalized by CNF, i.e.

$$[(x \approx_1 0.5) * (y \approx_2 0.4)] \rightarrow_* 0;$$

while Figure 6 presents the following relation:

$$1 \rightarrow_* [A(x) \rightarrow_* B(y)],$$

which gives INF in the form

$$1 \rightarrow_* [(x \approx_1 0.5) \rightarrow_* (y \approx_2 0.4)].$$

As shown in [17], this type of classical fuzzy rules (we can call them implicative fuzzy rules) are used to approximate functional dependencies with specially chosen samples

$$\mathtt{Samples} = \{[c_i, d_i]\}_{i \in I},$$

for which $c_i \in X, d_i \in Y$ and $F(c_i, d_i) = 1$ for all $i \in I$. There the functionality property was the exact rewriting of the classical set theoretical axiom of functionality and therefore it ignores an indistinguishability of elements on the universe $X$. Due to the full generality, we prefer to use the following definition taken from [18]:

**Definition 8** *A fuzzy relation $F$ is functional w.r.t. $\approx_1, \approx_2$ if*

$$x \approx_1 x' * F(\bar{x}) * F(\bar{x}') \leq y \approx_2 y', \tag{8}$$

*is valid for all $x, x' \in X$ and $y, y' \in Y$.*

**Theorem 9** *Let $F$ be functional w.r.t. $\approx_1, \approx_2$ and $[c_i, d_i] \in$ $\mathtt{Samples}$. Then*

- $\mathrm{INF}_F$ *lies above $F$, i.e.*

$$F(\bar{x}) \leq \mathrm{INF}_F(\bar{x}), \textit{ for all } x \in X, y \in Y;$$

- *additionally, if $F$ is extensional w.r.t. $\approx_1, \approx_2$ then*

$$\bigvee_{i \in I} [(c_i \approx_1 x) * (c_i \approx_1 x)] \leq F(\bar{x}) \leftrightarrow_* \mathrm{INF}_F(\bar{x}),$$

*and in the case of $\mathrm{INF}'_F$, the estimation differs on the left side of the above inequality where we have*

$$\bigvee_{i \in I} [(c_i \approx_1 x) * (x \approx_1 c_i)].$$

Remark that the proofs of the above inequalities do not vary much from the related results in [17], where the requirements are more restrictive.

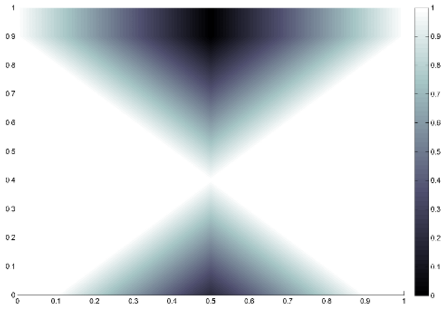


Fig. 6: Classical fuzzy rule $\mathsf{If}$ $(x \in A)$ $\mathsf{then}$ $(y \in B)$ interpreted in the standard Łukasiewicz algebra

## 4 Implicative normal forms as graded fuzzy rules

For the sake of brevity, we use symmetric $\approx_1, \approx_2$ and denote $x \approx_1 c_i$ by $A_i$, $y \approx_1 d_i$ by $B_i$ and $F(c_i, d_i)$ by $F_i$ in the sequel. In the case of Łukasiewicz algebra, the duality of DNF and CNF is quit obvious due to the involutive residual negation $x = \neg\neg x$. We have

$$\neg \left( \bigwedge_i (A_i * B_i \rightarrow_* \neg(f_i)) \right) = \bigvee_i \neg(A_i * B_i \rightarrow_* \neg(f_i)) =$$
$$\bigvee_i (A_i * B_i * \neg(\neg(f_i))) = \bigvee_i (A_i * B_i * f_i).$$

Moreover, the duality of DNF and INF can be observed from the following sequence of equalities:

$$\neg \left( \bigwedge_i (f_i \rightarrow_* (A_i \rightarrow_* \neg(B_i))) \right) =$$
$$= \bigvee_i \neg(f_i \rightarrow_* (A_i \rightarrow_* \neg(B_i)))$$
$$= \bigvee_i (f_i * \neg(A_i \rightarrow_* \neg(B_i)))$$
$$= \bigvee_i (f_i * A_i * \neg(\neg(B_i))) = \bigvee_i (A_i * B_i * f_i).$$

From there, the different nature of CNF and INF is also clear

$$\bigwedge_i (A_i * B_i \rightarrow_* \neg(f_i)) = \bigwedge_i (f_i \rightarrow_* (A_i \rightarrow_* \neg(B_i)))$$

But if we switch to e.g. the standard product algebra, we loose involution and even worse, the residual negation gives only values 0 and 1. Hence, in the case of CNF, we can use only values that are small but not zero. Figure 7 shows an example of a graded fuzzy rule formalized by CNF based on $A, B$ from Figure 1. Observe once more that a higher degree gives lower "confidence" on the particular graded fuzzy rule and the extreme degree 1 implies that the respective interpretation obviously gives 1 everywhere, i.e. $A(x) * B(y) \rightarrow_* 1 = 1$, for all $x \in X, y \in Y$. While INF behaves the other way round , i.e. $1 \rightarrow_* (A(x) \rightarrow_* B(y)) = A(x) \rightarrow_* B(y)$, for all $x \in X, y \in Y$, see Figure 8.

Now, let us concentrate on the interpretation of the implicative normal form as a formalization of graded fuzzy rules. Similarly as in the case of DNF and CNF, degrees and implication work together as modifiers of fuzzy rules in INF. It

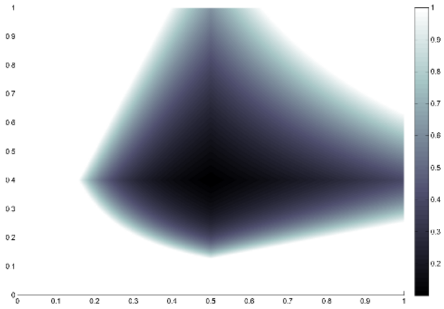


Fig. 7: $A(x) * B(y) \to_* 0.1$, where $*$ is the product t-norm

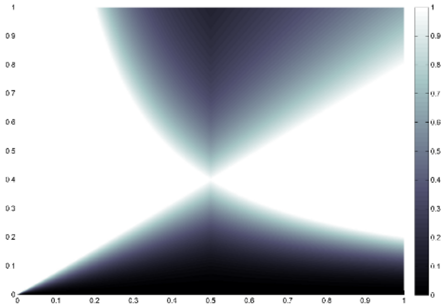


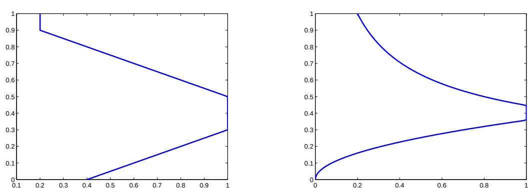
Fig. 8: Classical fuzzy rule $\mathsf{If}\ (x \in A)\ \mathsf{then}\ (y \in B)$ interpreted in the standard product algebra

can be seen from Figure 9 that it is a kind of shift operator applied on the antecedent part $B$, obviously $f \to B(y) = 1$ if and only if $f \leq B(y)$. The whole relation is drawn on Figure 10 for the case of standard Łukasiewicz algebra. We see that the core of this relation is widened comparing to Figure 6. Analogously as for a single fuzzy set, we have that $0.8 \to_* (A(x) \to_* B(y)) = 1$ if and only if $A(x) \to_* B(y) \in [0.8, 1]$. Therefore we interpret this relation in INF as one graded fuzzy rule in the form

$$^{0.8\preceq}/(\mathsf{If}\ x \in A\ \mathsf{then}\ y \in B)$$

that we read as "($\mathsf{If}\ x \in A\ \mathsf{then}\ y \in B$) at least to the degree 0.8" with the above explained meaning. And hence, INF formalizes

$$\mathtt{Con}\text{ with graded fuzzy rules } \mathcal{C}_i, i \in I.$$



(a) $*$ – Łukasiewicz t-norm  (b) $*$ – product t-norm

Fig. 9: $B'(y) = 0.8 \to_* B(y)$ – modified fuzzy set $B \subsetapprox Y$ from Figure 1(b).

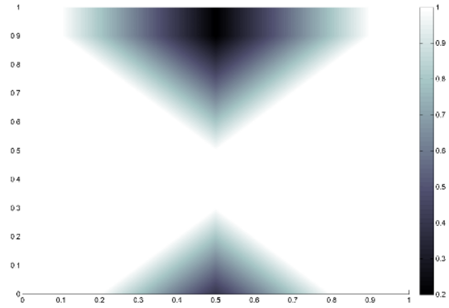


Fig. 10: Graded fuzzy rule $^{0.8\preceq}/\mathsf{If}\ (x \in A)\ \mathsf{then}\ (y \in B)$

## 5 Approximation Properties of the Implicative Normal Form

Let us put our attention to functional fuzzy relations. In this case we can prove the upper approximation property by INF.

**Proposition 10** *If $F$ is functional then $F(\bar{x}) \leq \mathrm{INF}_F(\bar{x})$, for all $x \in X, y \in Y$.*

And also an analogy to the estimation of precision of approximation using DNF and CNF can be shown for INF.

**Proposition 11** *Let us define $C$ and $C'$ as follows:*

$$C(x) =_{df} \bigvee_{i \in I} [(c_i \approx_1 x) * (c_i \approx_1 x)], \tag{9}$$

$$C'(x) =_{df} \bigvee_{i \in I} [(c_i \approx_1 x) * (x \approx_1 c_i)]. \tag{10}$$

*If $F$ is functional and extensional and moreover, the normal forms are constructed in the nodes taken from* `Samples` *then*

$$C(x) \leq \mathrm{INF}_F(\bar{x}) \leftrightarrow_* F(\bar{x}), \tag{11}$$

$$C'(x) \leq \mathrm{INF}'_F(\bar{x}) \leftrightarrow_* F(\bar{x}), \tag{12}$$

*for all $x \in X, y \in Y$, provided that $\approx_2$ is symmetric in (12).*

**Proof:** If $F$ is functional and extensional then

$$\mathrm{DNF}_F(\bar{x}) \leq F(\bar{x}) \leq \begin{matrix} \mathrm{INF}_F(\bar{x}), \\ \mathrm{INF}'_F(\bar{x}), \end{matrix}$$

for all $x \in X, y \in Y$, and moreover it can be proved that

$$C(x) \leq \mathrm{INF}_F(\bar{x}) \to_* \mathrm{DNF}_F(\bar{x}),$$

using $\underbrace{(a \to a)}_{=1} * (a \to b) \leq (a * a) \to_* (a * b)$, and

$$C'(x) \leq \mathrm{INF}'_F(\bar{x}) \to_* \mathrm{DNF}_F(\bar{x}),$$

using $(a' \to a') * (a \to b) \leq (a' * a) \to_* (a' * b)$. Now (11) and (12) follow immediately. QED

Compare these estimations with (4): $C$ and $C'$ in (11) and (12), respectively, depend only on the partition of the domain $X$ given by fuzzy sets $c_i \approx_1 x$ and $x \approx_1 c_i$, while the estimations for DNF and CNF are dependent additionally on the partition of $Y$ by $d_i \approx_2 y$ and $y \approx_2 d_i$.

We can generalize the above results by considering the graded functionality given by the following definition.

**Definition 12** *A relation $F$ is said to be $a$-functional w.r.t. $\approx_1, \approx_2$ if*

$$a = \bigwedge_{x,x' \in X, y,y' \in Y} [[(x \approx_1 x') * F(\bar{x}) * F(\bar{x}')] \to_* (y \approx_2 y')].$$

*We will shortly denote the right side of the above equality by* $\mathrm{Fun}_{\approx_1,\approx_2} F$.

Due to the space limitation, we will only provide results without proofs in the sequel. Notice that cores of the proofs remains the same as in the previous cases.

Analogously to the above upper approximation property (see Theorem 9), we have a graded variant of this result.

**Theorem 13** *If $F$ is $a$-functional then*

$$a \leq \bigwedge_{x \in X, y \in Y} (F(\bar{x}) \to_* \mathrm{INF}(\bar{x})).$$

It means that the degree of functionality $\mathrm{Fun}_{\approx_1,\approx_2} F$ is the lower estimation of the inclusion $F$ and INF denoted as $F \subseteq \mathrm{INF}$, i.e. $\mathrm{Fun}_{\approx_1,\approx_2} F \leq F \subseteq \mathrm{INF}$.

And a generalization of Proposition 11 takes the following form:

**Theorem 14** *Let $C$ be given by (9).*

*If $F$ is $a$-functional and $b$-extensional and moreover,* INF *is constructed in the nodes taken from the set* `Samples` *then*

$$a * b * \bigwedge_{x \in X} C(x) \leq \bigwedge_{x \in X, y \in Y} (F(\bar{x}) \leftrightarrow_* \mathrm{INF}(\bar{x})).$$

The same result holds for $\mathrm{INF}'_F$ with symmetric $\approx_2$. It is only necessary to exchanged $C$ by $C'$ (given by (10)) in the above inequality.

Due to the above theorem, we have received an estimation of the equivalence between $F$ and INF for the case of not fully extensional and functional fuzzy relation $F$. The degrees of functionality and extensionality lower the estimation. And hence, smaller are degrees of functionality and extensionality worse is the approximation (or in other words, smaller is the degree of similarity between $F$ and INF).

## 6 Conclusions

In this contribution the implicative normal form (i.e. the conjunctive normal form – implicative variant) has been introduced. The theoretical results show that it is suitable for an approximation of dependencies that can be represented using functional and extensional fuzzy relations. In this case, an estimation of the equivalence between the given fuzzy relation and its INF has been proved in the form of the same inequality as it is valid for DNF and CNF. Which gives us a confidence on the efficiency of INF. Moreover, the graded notions of extensionality and functionality have been provided and the above described results shown in their graded form. These results widens an applicability of normal forms to partially extensional and partially functional fuzzy relations.

Additionally, a connection between graded fuzzy rules was described and the differences between Perfilieva's normal forms and the implicative normal form was explained. Indeed, graded fuzzy rules can be seen as classical fuzzy rules with the modified consequent parts (the different modification for each normal form). This view provides an insight into the nature of descriptions formalized by normal forms: we saw that the disjunctive and implicative normal forms are suitable for "positive" descriptions while the conjunctive normal form serves well for "negative" descriptions. In this sense, we have here two complementary descriptions of the given dependency – formalized by DNF and CNF; and formalized by DNF and INF, respectively.

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