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Monotonicity of fuzzy rule bases: On differences between graded and non-graded approaches

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Abstract

This contribution provides an exhaustive explanation of the graded approach to the monotonicity property (of a fuzzy relation) and its non-graded version. Our framework is algebraic. We present a wide variety of examples to support our claims relating to the well defined notion and suitability for fuzzy rule base applications and moreover, to improve the readability of the following mathematical formulae.

Keywords: Fuzzy relations, Monotonicity, Functionality, Fuzzy Class Theory

1. Monotone fuzzy rule bases

The notion of monotonicity of fuzzy relations has been introduced in [1] and it is closely related to the notion of fuzzy function [2]. More precisely, the logical structure of the definition of the monotonicity is identical with the one given for the functionality and as shown in [1]. And hence, the results known in one field can be transferred to the second if appropriate requirements are met.

Historically, monotonicity was connected with fuzzy rule bases (FRBs) only. Monotone fuzzy rule bases that express monotone dependencies are often found in practical applications e.g. [3, 4]. In the center of attention, there are problems with non-monotone outputs (in a classical sense of non-monotone function) of a monotone FRB. These problems are closely related to the chosen model of a fuzzy rule base as well as to the chosen defuzzification method (not considered within this contribution). There are various lines of research following particular models of a monotone FRB. For example, *Mamdani-Assilian* models of monotone FRBs were investigated in [5, 6, 7, 8] where it has been observed that not always they lead to monotone resulting functions. Another line relates to the *implicative* (or alternatively called *gradual* [9]) models of FRBs, see [10, 4].

Let us consider that a fuzzy rule base \mathcal{R} is simply constituted of a collection of pairs of fuzzy sets $\mathcal{R} = \{A_i, B_i\}_{i \in I}$, where each $A_i \subseteq X_1$, $B_i \subseteq X_2$ and I is some finite index set. The choice of a model of \mathcal{R} is reflected in the resulting fuzzy relation R .

The main idea of the following definition of the monotone FRB is to approximately describe an

ideal monotone dependency using a finite number of fuzzy rules.

Definition 1 Let \leq_j be a crisp ordering of fuzzy sets on X_j for each $j = 1, 2$. A rule base \mathcal{R} is said to be monotone if for all $i, j \in I$:

$$\text{if } A_i \leq_1 A_j \text{ then } B_i \leq_2 B_j. \quad (1)$$

Remark that in the above definition

- no model of the FRB is assumed;
- no defuzzification is employed to the considered FRB;
- and therefore, no resulting function is taken into account.

It means that this definition reflects only a dependency between “ordered” couples $\{A_i, A_j\}$ and $\{B_i, B_j\}$.

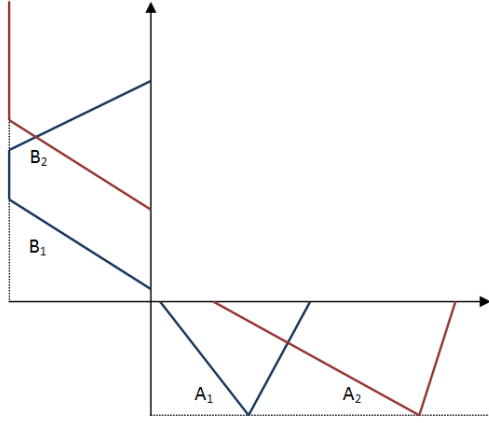
Moreover, this definition of monotone FRB is non-graded (or crisp). What does it mean?

2. Graded and non-graded properties

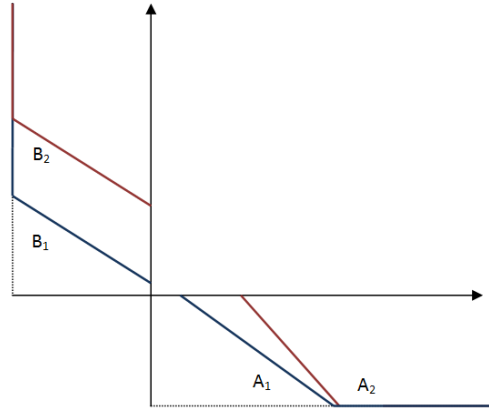
Traditionally, we are used to deal with properties that are uniquely determined (they are either true or false). Although the great development in the theory of fuzzy sets and fuzzy logic has been made there exist only a few works that enjoy the full beauty of the graded world. It is caused by the following facts:

- First, an appropriate tool for handling higher order objects such as fuzzy sets of fuzzy sets, or expressing graded properties of such complex objects has been proposed recently [11, 12].
- And since there was no simple unified way how to generalize crisp properties of fuzzy sets to fuzzy (graded) ones, an usefulness of the graded approach to such properties has not been generally accepted yet.

Exemplary contributions that provide a systematic study of particular graded properties and show results in the form of graded theorems developed within FCT are [13, 14, 15, 2, 1]. There, a motivation for using graded properties is also explained. Generally, there was need of having an opportunity to use some methods that were originally developed for a class of objects that fulfill some crisp property



(a) \mathcal{R}_1 with various types of fuzzy sets



(b) \mathcal{R}_2 with right-open fuzzy intervals

Figure 1: Examples of monotone FRB.

(one or more), for a wider class of objects that violates the given crisp property but they satisfy a graded variant of this property in a reasonable high degree.

Let us demonstrate it on Definition 1: we consider a FRB and we examine its monotonicity classically – using classical mathematical logic. We first introduce an appropriate partial order \leq_f on fuzzy sets based on the ordering of their α -cuts. For a fuzzy set C on a universe U , its α -cut $[C]_\alpha$ is the subset of U defined as

$$[C]_\alpha = \{x \in U \mid C(x) \geq \alpha\}, \text{ for any } \alpha \in]0, 1].$$

Definition 2 Let U be a linear vector space over the real numbers \mathbb{R} . Given two convex¹ fuzzy sets C_1 and C_2 on a universe $U \subseteq \mathbb{R}$, we say that $C_1 \leq_f C_2$ if for any $\alpha \in]0, 1]$ it holds that

$$\inf[C_1]_\alpha \leq \inf[C_2]_\alpha \text{ and } \sup[C_1]_\alpha \leq \sup[C_2]_\alpha.$$

On Figure 1, we see two different examples of monotone FRB in the sense of Definition 1, i.e., where $\leq_{1,2}$ are identical with \leq_f .

¹A fuzzy set $A \in \mathcal{F}(U)$ is called *convex* if the inequality

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq A(x_1) \wedge A(x_2)$$

holds for all $x_1, x_2 \in U$ and all $\lambda \in [0, 1]$.

For the sequel, assume a residuated lattice

$$\mathcal{L} = \langle L, \&, \rightarrow, \wedge, \vee, 0, 1 \rangle, \quad (2)$$

a structure for truth values with the following table providing a possible reading using a natural language:

Reading	Connective
IF ... THEN ...	$\dots \rightarrow \dots$
... AND ...	$\dots \& \dots$
... AND ...	$\dots \wedge \dots$
... OR ...	$\dots \vee \dots$

In the process of fuzzification of (1), we replace crisp orderings $\leq_{1,2}$ by fuzzy orderings $\preceq_{1,2}$ first. And consequently, the non-graded formalism has to be exchanged by graded one. The result is following:

$$(A_i \preceq_1 A_j) \rightarrow (B_i \preceq_2 B_j), \quad (3)$$

which can be read as

$$\text{IF } (A_i \preceq_1 A_j) \text{ THEN } (B_i \preceq_2 B_j).$$

By (3), we can immediately calculate the degree in which two couples of fuzzy sets preserve the given fuzzy orderings or, in other words, how much two fuzzy rules from FRB \mathcal{R} are monotone. Hence, we call it *graded monotonicity* for particular couples of fuzzy sets (or fuzzy rules represented by) $\{A_i, B_i\}$ and $\{A_j, B_j\}$.

Whenever we investigate (absolute) truth of (3) we put it equal to 1 and obtain that it is fulfilled if and only if

$$(A_i \preceq_1 A_j) \leq (B_i \preceq_2 B_j), \quad (4)$$

that is exactly the *non-graded monotonicity* for the particular couples of fuzzy sets $\{A_i, B_i\}$ and $\{A_j, B_j\}$. In this case, we have that the non-graded monotonicity do not exhibit degrees and it is either true or false. Moreover, it is a special case of the graded monotonicity.

Inspecting Definition 1, there is still some work to be done, namely, quantification over indexes i, j , which is implemented by means of \bigwedge . Consequently, the whole Definition 1 is translated to

$$\bigwedge_{i,j \in I} [(A_i \preceq_1 A_j) \rightarrow (B_i \preceq_2 B_j)], \quad (5)$$

denoted by $\text{MonRB}_{\preceq_{1,2}}(\mathcal{R})$. That is called *graded monotonicity* of a FRB \mathcal{R} , where evaluation of (5) gives a degree of this property. Using the following language:

Reading	Operation
A is smaller than A'	$A \preceq_1 A'$
B is smaller than B'	$B \preceq_2 B'$
FOR ALL $i \dots$	$\bigwedge_i \dots$

we can read (5) as

“FOR ALL $i, j \in I$
 [IF (A_i is smaller than A_j) THEN
 (B_i is smaller than B_j)] ”

Finally, let us provide a formal definition of graded monotonicity that directly involves degrees.

Definition 3 Let \mathcal{L} be a residuated lattice of the form (2), $\preceq_{1(2)}$ be a binary fuzzy relation over a non-empty set $X_{1(2)}$ and \mathcal{R} be a FRB. If

$$a = \bigwedge_{i,j \in I} [(A_i \preceq_1 A_j) \rightarrow (B_i \preceq_2 B_j)], \quad (6)$$

then we say that \mathcal{R} is a -monotone w.r.t. $\preceq_{1,2}$.

Let us comment on the above definition:

- Assuming a class of FRBs, e.g. $\{\mathcal{R}_k\}_k$, we distinguish a fuzzy sub-class of monotone FRBs, where we assign to each FRB \mathcal{R}_k a particular degree a_k by means of (6), i.e., $a_k = \text{MonRB}_{\preceq_{1,2}}(\mathcal{R}_k)$.
- The degree of monotonicity can be evaluated for an arbitrary \mathcal{R} .
- If $a = 1$ for \mathcal{R} , we can say \mathcal{R} is monotone and if $a = 0$, we can safely say \mathcal{R} is not monotone.
- The definition deals with degrees. We may proceed directly without specifying degrees just by processing the syntax of a specific fuzzy logic. This way has been elaborated in [1].
- The crisp $\leq_{1,2}$ are also allowed as a special case of fuzzy relations $\preceq_{1,2}$. Then $a \in \{0, 1\}$ and evaluation of this property produces the same result as checking the classical monotonicity of \mathcal{R} given by Definition 1. It is $a = 1$ for monotone \mathcal{R} in the sense of Definition 1 and $a = 0$ for non-monotone \mathcal{R} .

Now, let us investigate the absolute truth of (6):

$$\bigwedge_{i,j \in I} (A_i \preceq_1 A_j) \rightarrow (B_i \preceq_2 B_j) = 1,$$

it follows that none of the implications can be smaller than 1

$$(A_i \preceq_1 A_j) \rightarrow (B_i \preceq_2 B_j) = 1,$$

for all $i, j \in I$, and equivalently

$$(A_i \preceq_1 A_j) \leq (B_i \preceq_2 B_j), \quad (7)$$

for all $i, j \in I$, which are the principal inequalities for the definition of the non-graded monotonicity of \mathcal{R} . Let us summarize it in the following definition:

Definition 4 Let $\mathcal{L}, \preceq_{1(2)}$ and \mathcal{R} be as in Definition 3. If for each $i, j \in I$

$$(A_i \preceq_1 A_j) \leq (B_i \preceq_2 B_j) \quad (8)$$

is valid then we say that \mathcal{R} fulfills non-graded monotonicity (or it is non-graded monotone) w.r.t. $\preceq_{1,2}$.

Example 5 Let us consider an example of a fuzzy rule base \mathcal{R} consisting of two fuzzy rules $\{A_i, B_i\}$, $i = 1, 2$, that is monotone in the sense of (8). First of all, we have to define fuzzy orderings $\preceq_{1,2}$ of fuzzy sets. For this, we adopt the definition introduced by U. Bodenhofer et al. (see [16, 17]) that defines the fuzzy order $\preceq: \mathcal{F}^2(U) \rightarrow [0, 1]$ of two fuzzy sets $C, D \in \mathcal{F}(U)$ as follows:

$$C \preceq D = \bigwedge_{u \in U} [(ATL(D)(u) \rightarrow ATL(C)(u)) \& (ATM(C)(u) \rightarrow ATM(D)(u))] \quad (9)$$

where ATL, ATM denote at-least and at-most modifiers [18], respectively, that are given as follows:

$$ATL(C) = \bigvee (C(t) \& t \leq u),$$

$$ATM(C) = \bigvee (C(t) \& u \leq t)$$

for arbitrary $C \in \mathcal{F}(U)$. The symbol \leq generally denotes a $T - E -$ ordering, i.e., a fuzzy ordering with respect to a t -norm T and E is a fuzzy equivalence relation.

For this example, it is sufficient to consider \leq to be equivalent to the ordering of real numbers \leq and hence:

$$ATL(C) = \bigvee_{t \leq u} C(t),$$

$$ATM(C) = \bigvee_{u \leq t} C(t).$$

Further, let the standard Łukasiewicz algebra \mathcal{L} be the underlying structure for truth values, $X_1 = X_2 = [0, 4]$ and the involved fuzzy sets be given as follows:

$$A_1 = \begin{cases} (x - 1) \vee 0, & x \in [1, 2], \\ (3 - x) \vee 0, & x \in [2, 3], \end{cases}$$

$$A_2 = \begin{cases} (x - 2) \vee 0, & x \in [2, 3], \\ (4 - x) \vee 0, & x \in [3, 4], \end{cases}$$

$$B_1 = \begin{cases} (y - 1) \vee 0, & y \in [1, 2], \\ (3 - y) \vee 0, & y \in [2, 3], \end{cases}$$

$$B_2 = \begin{cases} (y - 2) \vee 0, & y \in [2, 3], \\ (4 - y) \vee 0, & y \in [3, 4]. \end{cases}$$

It is easy to show that $A_1 \preceq A_2 = 1$, $A_2 \preceq A_1 = 0$ and that also $B_1 \preceq B_2 = 1$, $B_2 \preceq B_1 = 0$ and hence, that

$$(A_1 \preceq A_2) \rightarrow (B_1 \rightarrow B_2) = 1,$$

$$(A_2 \preceq A_1) \rightarrow (B_2 \rightarrow B_1) = 1$$

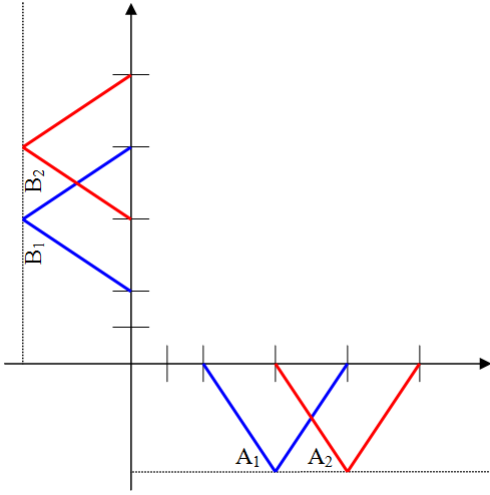


Figure 2: Non-graded monotone fuzzy rule base \mathcal{R} from Example 5.

which gives

$$\bigwedge_{i,j} [(A_i \preceq A_j) \rightarrow (B_i \preceq B_j)] = 1.$$

So, fuzzy rule base \mathcal{R} is an example of a fuzzy rule base that is non-graded monotone w.r.t. $\preceq_{1,2}$, i.e., it fully obeys (8).

Let us note that the fuzzy rule base in Example 5 was chosen to meet also the original definition of a monotone fuzzy rule base, i.e., to meet Definition 1. But the given fuzzy rule base may be easily modified in order to lose this property but still in such a way that it is monotone (non-graded monotone, to be precise) according to the newly proposed Definition 4.

Example 6 Let us consider a fuzzy rule base \mathcal{R}' consisting of two fuzzy rules $\{A_i, B_i\}$, $i = 1, 2$, that is monotone in the sense of (8) and it is not monotone in the sense of Definition 1. Let us keep all the setting (algebra, fuzzy ordering etc.) from Examples 5.

Then, let the involved fuzzy sets are slightly modified as follows:

$$A_1 = \begin{cases} (x-1) \vee 0, & x \in [1, 2], \\ (3-x) \vee 0, & x \in [2, 3], \end{cases}$$

$$A_2 = \begin{cases} (\frac{x-0.5}{2.5}) \vee 0, & x \in [0.5, 3], \\ (4-x) \vee 0, & x \in [3, 4], \end{cases}$$

$$B_1 = \begin{cases} (y-1) \vee 0, & y \in [1, 2], \\ (3-y) \vee 0, & y \in [2, 3], \end{cases}$$

$$B_2 = \begin{cases} (\frac{y-0.5}{2.5}) \vee 0, & y \in [0.5, 3], \\ (4-y) \vee 0, & y \in [3, 4]. \end{cases}$$

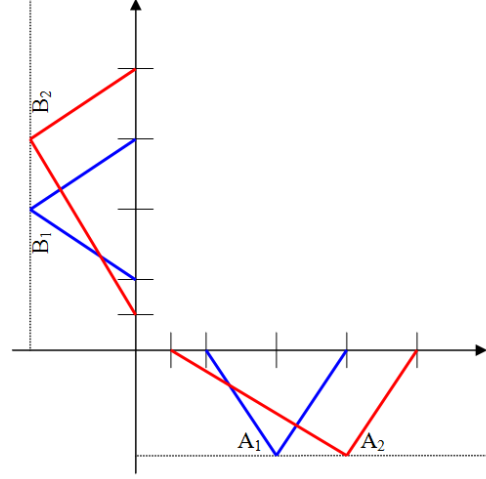


Figure 3: Non-graded monotone fuzzy rule base \mathcal{R}' from Example 6.

Then it can be shown that

$$\bigwedge_{x \in X_1} (\text{ATL}(A_2) \rightarrow \text{ATL}(A_1)) = 0.2 \rightarrow 0 = 0.8,$$

$$\bigwedge_{x \in X_1} (\text{ATM}(A_1) \rightarrow \text{ATM}(A_2)) = 1,$$

$$\bigwedge_{y \in X_2} (\text{ATL}(B_2) \rightarrow \text{ATL}(B_1)) = 0.2 \rightarrow 0 = 0.8,$$

$$\bigwedge_{y \in X_2} (\text{ATM}(B_1) \rightarrow \text{ATM}(B_2)) = 1,$$

and hence that both implications hold:

$$(A_1 \preceq A_2) \rightarrow (B_1 \rightarrow B_2) = 0.8 \rightarrow 1 = 1,$$

$$(A_2 \preceq A_1) \rightarrow (B_2 \rightarrow B_1) = 0.8 \rightarrow 1 = 1$$

which gives

$$\bigwedge_{i,j} [(A_i \preceq A_j) \rightarrow (B_i \preceq B_j)] = 1.$$

So, fuzzy rule base \mathcal{R}' is an example of a fuzzy rule base that is non-graded monotone w.r.t. $\preceq_{1,2}$, although it could not be called monotone if only the original (crisp) Definition 1 was at disposal.

For the completeness of the information the reader is provided with, let us add an example of a FRB that is not non-graded monotone but that is at least a -monotone to some degree a .

Example 7 Let us consider an example of a fuzzy rule base \mathcal{R}'' consisting of two fuzzy rules $\{A_i, B_i\}$, $i = 1, 2$. Let us keep all the setting (algebra, fuzzy ordering etc.) from Examples 5.

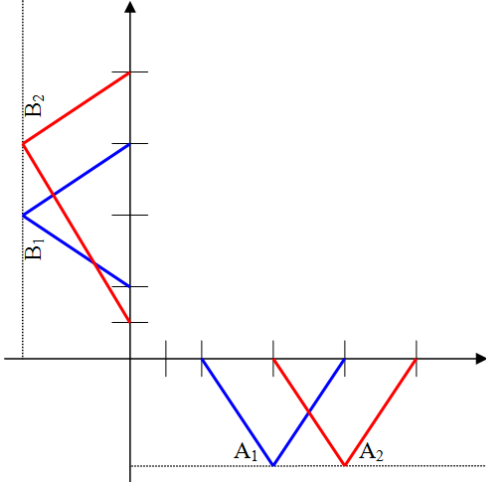


Figure 4: Fuzzy rule base \mathcal{R}'' from Example 7 that is 0.8-monotone.

Let the involved fuzzy sets be given as follows:

$$A_1 = \begin{cases} (x-1) \vee 0, & x \in [1, 2], \\ (3-x) \vee 0, & x \in [2, 3], \end{cases}$$

$$A_2 = \begin{cases} (x-2) \vee 0, & x \in [2, 3], \\ (4-x) \vee 0, & x \in [3, 4], \end{cases}$$

$$B_1 = \begin{cases} (y-1) \vee 0, & y \in [1, 2], \\ (3-y) \vee 0, & y \in [2, 3], \end{cases}$$

$$B_2 = \begin{cases} (\frac{y-0.5}{2.5}) \vee 0, & y \in [0.5, 3], \\ (4-y) \vee 0, & y \in [3, 4]. \end{cases}$$

It is easy to show that $A_1 \preceq A_2 = 1$, $A_2 \preceq A_1 = 0$, $B_2 \preceq B_1 = 0$ and that

$$\bigwedge_{y \in X_2} (\text{ATL}(B_2) \rightarrow \text{ATL}(B_1)) = 0.2 \rightarrow 0 = 0.8.$$

Hence

$$(A_1 \preceq A_2) \rightarrow (B_1 \rightarrow B_2) = 0.8 \quad (*)$$

$$(A_2 \preceq A_1) \rightarrow (B_2 \rightarrow B_1) = 1$$

which gives

$$\bigwedge_{i,j} [(A_i \preceq A_j) \rightarrow (B_i \preceq B_j)] = 0.8.$$

So, fuzzy rule base \mathcal{R}'' is an example of a fuzzy rule base that is not non-graded monotone w.r.t. $\preceq_{1,2}$, i.e., (8) fails by (*). On the other hand, it is monotone at least to some degree, particularly, to the degree 0.8. Hence, \mathcal{R}'' is 0.8-monotone.

2.1. Monotone fuzzy rule bases and their relationship to monotone functions

For the sequel, assume a crisp function f from the set of all fuzzy sets on X_1 denoted by $\mathcal{F}(X_1)$ into the set of all fuzzy sets on X_2 denoted by $\mathcal{F}(X_2)$, i.e., $f : \mathcal{F}(X_1) \mapsto \mathcal{F}(X_2)$. Furthermore, let $\approx_{1(2)}$ be a similarity on $\mathcal{F}(X_{1(2)})$, $\preceq_{1,2}$ be fuzzy orderings of fuzzy sets.

Specially, a FRB can be viewed as a collection of information about some function. Let us emphasize that in our case this function is from $\mathcal{F}(X_1)$ into $\mathcal{F}(X_2)$. Taking into account Definition 3, we can immediately generalize it to the case of graded monotonicity of such function.

Definition 8 A crisp function $f : \mathcal{F}(X_1) \mapsto \mathcal{F}(X_2)$ is said to be a -monotone w.r.t. $\preceq_{1,2}$ if

$$a = \bigwedge_{X, X' \in \mathcal{F}(X_1)} [(X \preceq_1 X') \rightarrow (f(X) \preceq_2 f(X'))]. \quad (10)$$

We will shortly denote the right side of the above equation by $\text{Mon}_{\preceq_{1,2}}(f)$.

This definition perfectly translates the classical definition of a monotone function to the graded form. $\text{Mon}_{\preceq_{1,2}}(f)$ can be translated as

$$(\text{"FOR ALL } X, X' \text{ [IF } (X \text{ IS SMALLER THAN } X') \text{ THEN } f(X) \text{ IS SMALLER THAN } f(X') \text{"]})$$

Next, we can define the following binary fuzzy relation on $\mathcal{F}(X_1) \times \mathcal{F}(X_2)$:

$$R_f(X, Y) = Y \approx_2 f(X), \quad (11)$$

which is closely related to the notion of the fuzzy function. The known results (see e.g. [19] or [20]) related to that notion can be transferred to higher order objects (such as fuzzy sets of fuzzy sets) and hence, we have that R_f is

- *extensional*:

$$(X \approx_1 X') \& (Y \approx_2 Y') \& R_f(X, Y) \leq R_f(X', Y')$$

is valid for an arbitrary X, X', Y, Y' from the respective domains;

- and *functional*:

$$(X \approx_1 X') \& R_f(X, Y) \& R_f(X', Y') \leq (Y \approx_2 Y')$$

is valid for an arbitrary X, X', Y, Y' from the respective domains.

Let us consider a fuzzy rule base

$$\mathcal{R}_f = \{A_i, B_i\}_{i \in I}$$

consisting of the perfect samples of f . It means that for each $i \in I$: $B_i \approx f(A_i) = 1$. Then it can be proved an interesting inequality (see [1])

$$\text{Mon}_{\preceq_{1,2}}(f) \leq \text{MonRB}_{\preceq_{1,2}}(\mathcal{R}_f).$$

It means that

- the degree of monotonicity of f is smaller or equal than the degree of monotonicity of the rule base based on f ;
- or informally speaking, the monotonicity of f implies the monotonicity of \mathcal{R}_f ;
- but this not the other way round.

3. Conclusion

In this contribution, we have “fuzzified” the monotonicity property of a FRB into the graded one, i.e., a -monotonicity where a represents the particular degree in which a given FRB is monotone. Furthermore, it has been extended to an arbitrary function from fuzzy sets to fuzzy sets. Contrary to the previous work [1] where we worked within a logical framework without a direct interpretation using the degrees of truth, we emphasize here an immediate evaluation of grades. This approach may become useful for readers that are not familiar with a formal logical notational style. And moreover, we have presented examples supporting our way of generalization. In this way, we point out that degrees play their role also in this particular problem relating to FRBs and it is not only an artificial generalization. Finally, the last section showed a connection with functions (with a special domain and range) and foreshadowed the promising area for future work.

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