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***Abstract*—In this contribution, we will introduce a notion of monotonicity of a fuzzy relation as a graded property. We will study its behavior w.r.t. fuzzy set operations and show relationship between monotone crisp functions and monotone fuzzy relations. Finally, a connection to ~~a~~ special models of a monotone rule-base system will be provided.**

I. Introduction

A standard approach of classical mathematics to determination of special objects within a larger collection of objects remains mainly the same for the traditional theory of fuzzy sets. For example, functions constitute a subset of all relations that is determined by a special condition. Analogously, in the traditional fuzzy set theory, fuzzy relations that are functional are determined perfectly within all fuzzy relations by the functionality property. Freely speaking, the traditional fuzzy set theoretical approach works with crisp sets of “fuzzy objects”. Fuzzy sets are identified with their membership functions, i.e., classical objects that are further handled using tools of classical mathematics and results are interpreted there as well.

In our opinion this approach is too restrictive and it needs a paradigmatic shift:

from the Crispness to the Fuzziness of sets or classes collecting fuzzy sets.

It should be done in accordance with a higher-order fuzzy set theory which has its roots in works of Höhle (see [1]), Gottwald (see [2], [3]); and it was elaborated by Novák (see [4] for the Fuzzy Type Theory) and independently by Běhounek and Cintula in [5] as Fuzzy Class Theory. In our work, we stem from the notion of fuzzy class [5] that naturally generalizes the classical notion of a class. Moreover, the fuzzy class theory provides tools for classical writings with graded interpretation. Hence, all notions are developed as in the classical mathematics but the used background logic makes the difference.

Here, we will follow up a methodology of fuzzy mathematics postulated in [6] which has been implemented in works on fuzzy relations [7], fuzzy relational compositions [8] and fuzzy functions [9]. Here, we will mainly use the last referred publication on fuzzy functions and refine results for our purposes. In [9], the notion of fuzzy function is based on two (graded) properties where one of them – the functionality property – accommodates in its structure our intuitive requirements on the monotonicity. More precisely, we can express the property of monotonicity by means of a formula that apply for the functionality property with different predicates. Therefore, many results follow directly as corollaries. We only have to be aware of right assumptions because not all assumptions that lead to a particular result for the functionality property are suitable for the monotonicity.

In this contribution, we will mainly put our attention to the explanation of the graded monotonicity. Over illustrative examples, we show that it is well defined and various theoretical approaches can be interpreted in our framework. Indeed, the whole classical mathematics can be internalized in the fuzzy class theory due to the nature of the used logical connectives that turn to classical ones whenever applied on crisp inputs.

Additionally, we will provide a relationship (in its graded form[1]) between monotone fuzzy relations of a special form and crisp monotone functions. By the construction used in this result, we obtain a large class (fuzzy) of non-trivial examples of monotone fuzzy relations that are not necessarily fuzzy functions. Finally, we will justify usefulness and provide a basic motivation for our work on the example that shows a connection between our notion of monotonicity and the notion of monotone fuzzy rule-base applied in practice.

The paper will be organized in the following way:

- First we introduce our logical framework, basic notions and overview related results.
- The following section will be devoted to the monotonicity notion.
- And finally, Section V will provide a brief insight into the field of monotone fuzzy rule-bases and show

[1] Graded form of the result means graded theorem, which is a provable formula in FCT. For the detailed explanation of this notion, we refer to [6], [10]

its connection and generalization using our approach based on graded monotonicity.

II. Logical framework

For this contribution, tools will remain mainly the same as in [9]. We will recall only basic necessary requirements.

Let us work in the framework of fuzzy class theory (FCT) [5], which is a schematic extension of a background logic (that contain crisp equality $=$ and Baaz-delta Δ) by the comprehension and extensionality axioms. Provability in FCT will be denoted simply by the same shortcut in front of $\vdash$ or it will be explicitly written. The background logic may be various (so that completeness theorem is valid for FCT) according to our actual requirements. In our case, the weakest background logic will be a many-sorted first order involutive monoidal t-norm based logic (IMTL) introduced by Esteva and Godo in [11] and we will deal only in FCTover this logic throughout the whole text.

The language of FCT contains variables for atomic individuals (lowercase letters $x, y, \dots$), variables for fuzzy classes of atomic individuals (uppercase letters $A, B, \dots$), variables for fuzzy classes of fuzzy classes of individuals (calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$), which are also called fuzzy classes of the second order, for fuzzy classes of the third order, etc. Moreover, it has the following set of basic connectives $\{\&, \to, \wedge\}$, involutive negation[2] $\neg$, the quantifier $\forall$, truth constants $\top, \bot$ and variables of the specific sorts.

Standardly, we introduce the following connectives and quantifier:

$$x \vee y \equiv_{df} ((x \to y) \to y) \wedge ((y \to x) \to x), \tag{1}$$

$$x \oplus y \equiv_{df} \neg(\neg x \,\&\, \neg y), \tag{2}$$

$$x \leftrightarrow y \equiv_{df} (x \to y) \,\&\, (y \to x), \tag{3}$$

$$(\exists x)\varphi \equiv_{df} \neg(\forall x)\neg\varphi. \tag{4}$$

IMTL extends MTL by the following schemata of axioms:

$$\text{(INV)} \quad \neg\neg\varphi \to \varphi.$$

For a reduction of used parenthesis in the following formulae, we introduce a priority of the above connectives and predicates:

1) Predicate $\in$ has the highest priority.
2) $\{\&, \oplus, \vee, \wedge, \leftrightarrow\}$ have the same priority.
3) $\to$ has the lowest priority from the set of all connectives.

Interpretation of the connectives is given by the corresponding operations $\{*, \to_*, \wedge, \vee, \sim\}$, and the constants $\bot, \top$ are interpreted as $\mathbf{0}, \mathbf{1}$, respectively, which together form an IMTL-algebra denoted by $\mathcal{L}$.

An $\mathcal{L}$-structure with a support L for the language J is of the form

$$\mathcal{M} = \langle (X_i)_{for\, s_i}, (r_P)_{P-predicate}, (m_c)_{c-constant} \rangle,$$

[2]Mostly, $\neg$ is reserved for a residual negation and $\sim$ is a usual symbol for the involutive negation. In this paper, we work only with the involutive negation hence, there is no danger of confusion between these two various notations.

$i = 1, 2, \dots, n$, where each X_i is a non-empty set of objects, r_P is an L-fuzzy relation of the respective type and m_c belongs to X_i provided that c is of the type s_i.

A. Basic notions and notations

Let F be a predicate of the type (s_1, s_2), $\preccurlyeq_1, \approx_1$ be predicates of the type (s_1, s_1) and $\preccurlyeq_2, \approx_2$ be predicates of the type (s_2, s_2), i.e. they are interpreted as fuzzy relations $\mathbf{F} \subsetapprox X_1 \times X_2, \boldsymbol{\preccurlyeq_1}, \boldsymbol{\approx_1} \subsetapprox X_1 \times X_1, \boldsymbol{\preccurlyeq_2}, \boldsymbol{\approx_2} \subsetapprox X_2 \times X_2$, respectively. For the better orientation, we will use the same terminology on the syntax as well as on the semantical level. Moreover, we will omit the specification of sorts, whenever it will be clear from the concept.

Notice that we strictly distinguish between syntax and semantics. But without misinterpretation, we can simultaneously use the notion of fuzzy relation and fuzzy set over the respective universe on both levels.

In order to be able to deal with fuzzy relations, we will further assume that the language of FCT contains an apparatus for forming tuples of objects and accessing their components. In what follows, the usual abbreviations of the form

$$\{\langle x_1, \dots, x_n \rangle \mid \phi\}$$

will be used for

$$\{z \mid (\exists x_1) \dots (\exists x_n)(z = \langle x_1, \dots, x_n \rangle \,\&\, \phi)\}.$$

Let R, S be fuzzy relations of the type (s_1, s_2). Then we define the following properties:

- Reflexivity:
$$\mathrm{Refl}(R) \equiv_{df} (\forall x)(\langle x, x \rangle \in R)$$
- Symmetry:
$$\mathrm{Sym}(R) \equiv_{df} (\forall x, y)(\langle x, y \rangle \in R \to \langle y, x \rangle \in R)$$
- Transitivity:
$$\mathrm{Trans}(R) \equiv_{df} (\forall x, y, z) \\ (\langle x, y \rangle \in R \,\&\, \langle y, z \rangle \in T \to \langle x, z \rangle \in R)$$
- Similarity:
$$\mathrm{Sim}(R) \equiv_{df} \mathrm{Refl}(R) \,\&\, \mathrm{Sym}(R) \,\&\, \mathrm{Trans}(R)$$
- Subsethood:
$$R \subseteq S \equiv_{df} (\forall x, y)(\langle x, y \rangle \in R \to \langle x, y \rangle \in S)$$
- Set-similarity:
$$R \approx S \equiv_{df} (R \subseteq S) \,\&\, (S \subseteq R)$$
- Totality:
$$\mathrm{Tot}(R) \equiv_{df} (\forall x \exists y)(\langle x, y \rangle \in R)$$

Moreover, the following set operations can be introduced for fuzzy sets A, B of the same type:

$A \cup B =_{df} \{x \mid x \in A \oplus x \in B\}$	strong union
$A \cap B =_{df} \{x \mid x \in A \,\&\, x \in B\}$	strong intersec.
$A \sqcup B =_{df} \{x \mid x \in A \vee x \in B\}$	union
$A \sqcap B =_{df} \{x \mid x \in A \wedge x \in B\}$	intersection

We will additionally deal with relational compositions defined using a class notation. A systematic study can be find in [12]. We will use three basic relational compositions.

- sup-T composition:

$$R \circ S =_{df} \{\langle x,y\rangle \mid (\exists z)(\langle x,z\rangle \in R \,\&\, \langle z,y\rangle \in S)\}$$

- BK-subproduct:

$$R \lhd S =_{df} \{\langle x,y\rangle \mid (\forall z)(\langle x,z\rangle \in R \rightarrow \langle z,y\rangle \in S)\}$$

- BK-superproduct:

$$R \rhd S =_{df} \{\langle x,y\rangle \mid (\forall z)(\langle x,z\rangle \in R \leftarrow \langle z,y\rangle \in S)\}$$

III. Monotonicity of fuzzy relations

Monotonicity is always joined with some (pre)orderings[3] allowing comparison of elements of input and output universes. Therefore, at the beginning, we fix symbols $\preccurlyeq_1, \preccurlyeq_2$ (each $\preccurlyeq_i$ be a binary fuzzy relation on X_i^2) for that purpose. Moreover, assume a binary fuzzy relation F on $X_1 \times X_2$. Our definition will reflect the following intuitive expectation on the monotonicity property of some fuzzy relation:

"If x is smaller than x' AND x is related with y AND x' is related with y' THEN y is smaller than y' "

The language of FCT can carry out gradually each part of the above statement:

x is smaller than x'	$x \preccurlyeq_1 x'$
y is smaller than y'	$y \preccurlyeq_2 y'$
x is related with y	$\langle x,y\rangle \in F$
y is smaller than y'	$\langle x',y'\rangle \in F$
If ... then ...	$\ldots \rightarrow \ldots$
... and ...	$\ldots \,\&\, \ldots$

Hence, we define the property of monotonicity of a fuzzy relation F as follows:

$$\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \equiv_{df} (\forall x,x',y,y') \\ [(x \preccurlyeq_1 x') \,\&\, \langle x,y\rangle \in F \,\&\, \langle x',y'\rangle \in F \rightarrow (y \preccurlyeq_2 y')] \quad (5)$$

If $\vdash \mathrm{Mon}_{\preccurlyeq_{1,2}}(F)$ then we say that F is monotone to the degree 1 w.r.t. $\preccurlyeq_{1,2}$. Or shortly, 1–monotone and then we omit the specification of $\preccurlyeq_{1,2}$ if it is clear from the content.

Convention 1 *We will use the analogous notation generally for various properties.*

Let a property "Prop" of a predicate P be represented by the symbol $\mathrm{Prop}(P)$. If $\vdash \mathrm{Prop}(P)$ then we say that P is 1-Prop.

Notice, that a priori degrees are not parts of the syntax and they are computed for a particular interpretation. But it can be symbolically added for a visual illustration of their semantical behavior. We can freely read our definition together with the degree:

"If $(x \preccurlyeq_1 x')$ and $\langle x,y\rangle \in F$ and $\langle x',y'\rangle \in F$ then $(y \preccurlyeq_2 y')$ for all x,x',y,y'.
Then F is monotone w.r.t. $\preccurlyeq_{1,2}$ to the degree a,"

where a is computed in the following way:

$$a = \bigwedge_{\substack{x,x'\in X_1 \\ y,y'\in X_2}} [(\boldsymbol{\preccurlyeq_1}(x,x') * \mathbf{F}(x,y) * \mathbf{F}(x',y')) \rightarrow_* \boldsymbol{\preccurlyeq_2}(y,y')]$$

Now, let us explain a relationship between the monotonicity and functionality properties. If we replace $\preccurlyeq_{1,2}$ by $\approx_{1,2}$ then the definition of monotonicity turns directly into the definition of functionality, i.e.

$$\mathrm{Func}_{\approx_{1,2}}(F) \equiv_{df} (\forall x,x',y,y') \\ [(x \approx_1 x') \,\&\, (\langle x,y\rangle \in F) \,\&\, (\langle x',y'\rangle \in F) \rightarrow (y \approx_2 y')]$$

This definition is a straightforward translation (for the details see [9]) of the classical definition of the functionality property of a crisp relation F:

$$\text{If } (\langle x,y\rangle \in F) \text{ and } (\langle x,y'\rangle \in F) \text{ then } y = y', \quad (6)$$

whereas monotonicity of a crisp binary relation is not a usual notion. Here we stem only from an intuitive concept described at the beginning of this section, which turns to be very natural. Indeed, classical mathematics deals mostly with monotone functions for which our definition applies as well:

$$\mathrm{Mon}_{\preccurlyeq_{1,2}}(y = f(x)) \equiv_{df} (\forall x,x',y,y') \\ [(x \preccurlyeq_1 x') \,\&\, (y = f(x)) \,\&\, (y' = f(x')) \rightarrow (y \preccurlyeq_2 y')], \quad (7)$$

which can be simplified using axioms of $=$ into the following form

$$(\forall x,x',y,y')[(x \preccurlyeq_1 x') \rightarrow (f(x) \preccurlyeq_2 f(x'))], \quad (8)$$

which will be denoted by $\mathrm{Mon}_{\preccurlyeq_{1,2}}(f)$ and we will speak about the monotonicity of f w.r.t. $\preccurlyeq_{1,2}$.

Using $\mathrm{Mon}_{\preccurlyeq_{1,2}}(f)$, we can compute degrees of monotonicity of the crisp function f w.r.t. (fuzzy orderings) $\preccurlyeq_{1,2}$ and in the special case of the classical crisp orderings $\leq$, we obtain the classical definition of monotonicity of a function (i.e. it assigns only values $\mathbf{0}$ and $\mathbf{1}$). In this sense, our definition corresponds with the classical notion.

The following list of examples will address the monotonicity for various cases of functions or relations that are either crisp or fuzzy. The main aim is to provide a practical insight to the problems.

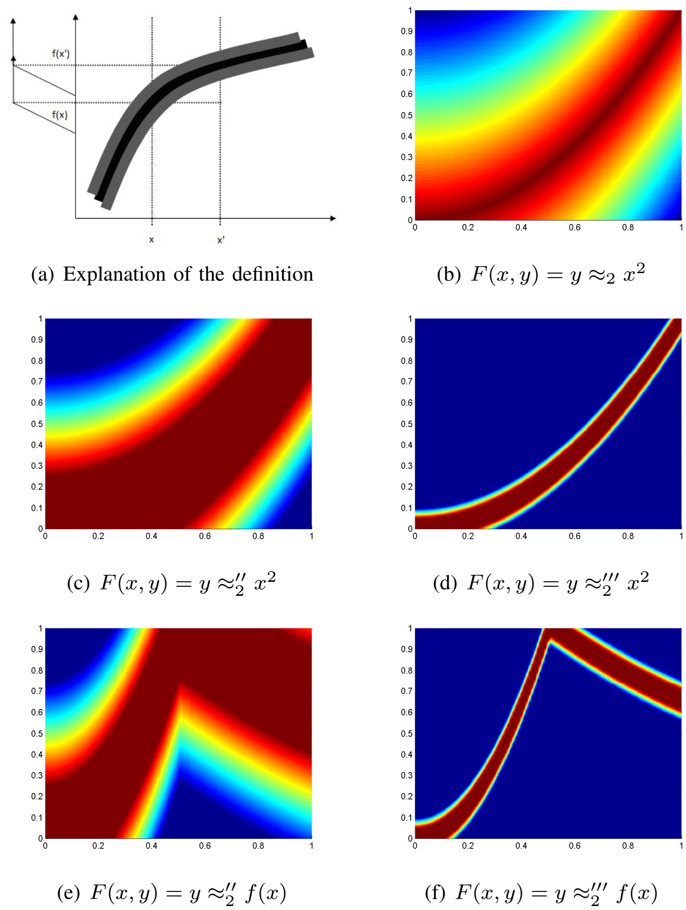
Example 2 *Let $f(x) = x^2$ on $[0,1]$ and*

$$F(x,y) =_{df} (y = f(x)).$$

- *F is obviously 1-monotone fuzzy relation w.r.t. $\leq, \leq$.*
- *It is also 1-monotone fuzzy relation w.r.t. $\preccurlyeq_1, \rightarrow$, where*

$$x \preccurlyeq_1 y =_{df} x^2 \rightarrow y^2.$$

[3] In the following text, we will often speak about some orderings or similarities. But on the syntax level, we do not put any requirements to the related symbols. When we say e.g. $\preccurlyeq$ is ordering, we mean that it is usually interpreted as ordering.



(a) Explanation of the definition
(b) $F(x,y) = y \approx_2 x^2$
(c) $F(x,y) = y \approx_2'' x^2$
(d) $F(x,y) = y \approx_2''' x^2$
(e) $F(x,y) = y \approx_2'' f(x)$
(f) $F(x,y) = y \approx_2''' f(x)$

Fig. 1. Examples of monotone fuzzy relations.

- *But it is not* 1*-monotone fuzzy relation w.r.t.* $\to, \to$.

Example 3 *Figure 1(a) depicts the case of* $F(x,y) = y \approx_2 f(x)$*, which we can assume as a typical representative of monotone fuzzy relations whenever* f *is monotone.* F *should be monotone w.r.t.* $\preccurlyeq_{1,2}$ *and then we can declare that elements smaller than* $f(x)$ *should be smaller than elements bigger than* $f(x')$*, formally:*

$$[(x \preccurlyeq_1 x') \,\&\, (y \approx_2 f(x)) \,\&\, (y' \approx_2 f(x'))] \to (y \preccurlyeq_2 y'). \quad (9)$$

As a special case, we can assume a binary fuzzy relation on truth values defined by $F(x,y) = y \leftrightarrow x^2$*, additionally define* $x \preccurlyeq_1 y = x^2 \to y^2$ *and take* $\preccurlyeq_2 \equiv_{df} \to$*. Then we obtain that* F *(depicted on Figure 1(b)) is* 1*-monotone.*

Example 4 *Whenever, only* $\preccurlyeq_2$ *is at the disposal, for an arbitrary* f *(which is onto* X*) we can define a binary fuzzy relation on* X^2 *as follows:*

$$F(x,y) = (y \preccurlyeq_2 f(x)) \,\&\, (f(x) \preccurlyeq_2 y),$$

and additionally, define $x \preccurlyeq_1 y = f(x) \preccurlyeq_2 f(y)$*. Then we obtain that if* $\preccurlyeq_2$ *is transitive (and we need this requirement twice) then* F *is monotone, i.e.*

$$\vdash (\mathrm{Trans}(\preccurlyeq_2))^2 \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(F)\,.$$

Example 5 *Let* $X, Y = [0,1]$ *and* $\mathcal{L}$ *be the standard Łukasiewicz algebra.*

- *Let*

$$x \approx_1'' y =_{df} (1 \wedge (1.5 - |x^2 - y^2|)) \vee 0,$$
$$x \approx_2'' y =_{df} (1 \wedge (1.5 - |x - y|)) \vee 0.$$

  Then $F(x,y) = y \approx_2'' x^2$ *depicted on Figure 1(c) is* 1*-monotone w.r.t.* $\preccurlyeq_{1,2}''$ *and* 1*-functional w.r.t.* $\approx_{1,2}''$ *defined by*

$$x \preccurlyeq_1'' y =_{df} \begin{cases} 1, & x \le y; \\ 0 \vee (1 \wedge (1.5 + (y^2 - x^2))), & \textit{otherwise;} \end{cases}$$
$$x \preccurlyeq_2'' y =_{df} \begin{cases} 1, & x \le y; \\ 0 \vee (1 \wedge (1.5 + (y - x))), & \textit{otherwise.} \end{cases}$$

  It means that $\vdash \mathrm{Mon}_{\preccurlyeq_{1,2}''}(F)$ *and also* $\vdash \mathrm{Func}_{\approx_{1,2}''}(F)$
- *Let* $x \approx_1''' y =_{df} (1 \wedge (2 - 20|x^2 - y^2|)) \vee 0$ *and* $x \approx_2''' y =_{df} (1 \wedge (2 - 20|x-y|)) \vee 0$*. Then* $F(x,y) = y \approx_2''' x^2$ *depicted on Figure 1(d) is* 1*-monotone w.r.t.* $\preccurlyeq_{1,2}'''$ *and* 1*-functional w.r.t.* $\approx_{1,2}'''$ *defined by*

$$x \preccurlyeq_1''' y =_{df} \begin{cases} 1, & x \le y; \\ 0 \vee (1 \wedge (2 + 20(y^2 - x^2))), & \textit{otherwise;} \end{cases}$$
$$x \preccurlyeq_2''' y =_{df} \begin{cases} 1, & x \le y; \\ 0 \vee (1 \wedge (2 + 20(y - x))), & \textit{otherwise.} \end{cases}$$

- *Analogous cases can be constructed for* $f_1(x) = 4x^2$ *for* $0 \le x \le 0.5$ *and*

$$f_2(x) = \begin{cases} 1.5 - \sin(x), & x > 0.5; \\ 0, & \textit{otherwise.} \end{cases}$$

  It means that each fuzzy relation based on the above functions is 1*-monotone w.r.t. the suitable chosen* $\preccurlyeq_{1,2}$ *(as in the above examples). But obviously, fuzzy relation representing their union, i.e. ,*

$$F(x,y) =_{df} (f_1(x) \approx'' y) \cup (f_2(x) \approx'' y),$$

  depicted on Figure 1(e) (1(f) for the case of $\approx'''$*), is not monotone (neither* 1*-monotone nor partially). It is* 0*-monotone, so we can safely say that* F *is not monotone.*

Let us summarize properties relating to monotonicity that are direct translations of results from [9]. FCT proves the following formulae:

$$\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \,\&\, \mathrm{Mon}_{\preccurlyeq_{2,3}}(S) \to \mathrm{Mon}_{\preccurlyeq_{1,3}}(F \circ S), \quad (10)$$
$$\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \,\&\, \mathrm{Mon}_{\preccurlyeq_{1,2}}(S) \to \mathrm{Mon}_{\preccurlyeq_{1,2}^2}(F \cap S), \quad (11)$$
$$\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \wedge \mathrm{Mon}_{\preccurlyeq_{1,2}}(S) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(F \sqcap S), \quad (12)$$
$$\mathrm{Mon}_{\preccurlyeq_{1,2}}(F \sqcup S) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \wedge \mathrm{Mon}_{\preccurlyeq_{1,2}}(S)\,. \quad (13)$$

Readings of the above results:
(10) – "IF F is monotone w.r.t. $\preccurlyeq_{1,2}$ AND S is monotone w.r.t. $\preccurlyeq_{2,3}$ THEN their composition $F \circ S$ is monotone w.r.t. $\preccurlyeq_{1,3}$."

(11) – "IF F AND S are monotone w.r.t. $\preccurlyeq_{1,2}$ THEN their strong intersection is monotone w.r.t. $\preccurlyeq_{1,2}^2$."
(12) – Analogously to the above reading, we have the following statement:
"IF F and S are monotone THEN their intersection is monotone."
where ... and ... interprets $\wedge$ and the specification is omitted.
(13) – "IF the union of F and S is monotone THEN F is monotone and S is monotone."

We can also show properties of similar fuzzy relations or the subsethood preserving property. FCT proves the following formulae:

$$(S \subseteq F) \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(S)], \qquad (14)$$
$$(F \approx S)^2 \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \leftrightarrow \mathrm{Mon}_{\preccurlyeq_{1,2}}(S)]. \qquad (15)$$

Readings of the above results:
(14) – "IF S is a subset of F AND F is monotone w.r.t. $\preccurlyeq_{1,2}$ THEN S is also monotone w.r.t. $\preccurlyeq_{1,2}$."
(15) – "IF S and F are set-similar AND F is monotone w.r.t. $\preccurlyeq_{1,2}$ THEN S is also monotone w.r.t. $\preccurlyeq_{1,2}$."

(14) and (15) produce a long list of corollaries (provable in FCT) for the relational compositions. Because of the space limitations, we will present it only for sup-T composition and BK-subproduct.

$$(S_1 \subseteq S_2) \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(R \circ S_2) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(R \circ S_1)],$$
$$(S_1 \subseteq S_2) \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(R \triangleleft S_1) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(R \triangleleft S_2)],$$
$$(R_1 \subseteq R_2) \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(R_2 \triangleleft S) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(R_1 \triangleleft S)],$$
$$(S_1 \approx S_2)^2 \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(R \circ S_2) \leftrightarrow \mathrm{Mon}_{\preccurlyeq_{1,2}}(R \circ S_1)],$$
$$(S_1 \approx S_2)^2 \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(R \triangleleft S_2) \leftrightarrow \mathrm{Mon}_{\preccurlyeq_{1,2}}(R \triangleleft S_1)],$$
$$(R_1 \approx R_2)^2 \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(R_2 \triangleleft S) \leftrightarrow \mathrm{Mon}_{\preccurlyeq_{1,2}}(R_1 \triangleleft S)].$$

IV. FROM MONOTONE CRISP FUNCTIONS TO MONOTONE FUZZY RELATIONS AND VICE-VERSA

In this section, we will show:

1'st a particular construction of monotone fuzzy relations using monotone crisp functions;
2'nd and next, we will explore the reverse step, where we provide a decomposition method for a monotone fuzzy relation, which produces a monotone crisp function.

The notion of monotonicity used above comes in degrees, i.e., we assume it either in the form (5) or (8).

Let us address the first problem.

Theorem 6 *Let us define*

$$F_f(x, y) \equiv_{df} (y \preccurlyeq_2 f(x))\&(f(x) \preccurlyeq_2 y), \qquad (16)$$

Then FCT *proves*

$$(\mathrm{Trans}(\preccurlyeq_2))^2 \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(f) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(F)_f]. \qquad (17)$$

Reading of the result:
(17) – "IF $\preccurlyeq_2$ is transitive, AND we need this property twice, AND f is monotone w.r.t. $\preccurlyeq_{1,2}$ THEN F is also monotone w.r.t. $\preccurlyeq_{1,2}$."
Instead of stating how many times we need a requirement, we can define a linguistic intensifier, e.g. VERY for the power of 2. Then we can read (17) as follows:
"IF $\preccurlyeq_2$ is VERY transitive AND f is monotone THEN F is also monotone."

Now, let us proceed in a reverse way: we start from a monotone fuzzy relation F and we extract a monotone crisp function f_F. It will be done in such a way that the graph of f_F pases through maximal values of the membership function associated with F. This fact is expressed by formula (18).

Theorem 7 *Let us define a class*

$$\mathrm{Dom}(F) =_{df} \{x \mid (\exists y)\langle x, y\rangle \in F\},$$

which will be called the domain of F. And moreover

$$\mathrm{Def}_{F,f} \equiv_{df} (\forall x)[\langle x, f(x)\rangle \in F \leftrightarrow x \in \mathrm{Dom}(F)]. \qquad (18)$$

Then FCT*proves*

$$(\mathrm{Tot}(F))^2 \;\&\; (\mathrm{Def}_{F,f_F}) \to [\mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \to \mathrm{Mon}_{\preccurlyeq_{1,2}}(f_F)]. \qquad (19)$$

Reading of the result:
(19) "IF F is VERY total AND monotone AND f_F passes through the domain of F THEN f_F is also monotone."

V. MONOTONE FUZZY RULE-BASES

Now, we will focus on a definition of a monotone rule-base[4] and provide a translation of this definition into our framework. In this subsection, we will freely use the language of the traditional fuzzy set theory and our language. We believe that the mathematics, notation and their frameworks used here can be understand without adding more details.

As it is usual habit, a rule-base constitutes of a collection of pairs of fuzzy sets $\mathcal{R} = \{A_i, B_i\}_{i\in I}$, where each $A_i \underset{\sim}{\subset} X_1$, $B_i \underset{\sim}{\subset} X_2$ and I is some index set. The choice of a model of $\mathcal{R}$ is reflected in the resulting fuzzy relation R.

Definition 8 *Let $\leq_j$ be a crisp ordering of fuzzy sets on X_j for each $j = 1, 2$. A rule-base $\mathcal{R}$ is said to be monotone if for all $i, j \in I$:*

$$\text{if } A_i \leq_1 A_j \text{ then } B_i \leq_2 B_j. \qquad (20)$$

We emphasize a fact that this definition relates purely to the given rule-base and it has no connection with the (input–output) processing (or approximate reasoning) at this point. Moreover, this property is crisp. Since both $\leq_{1,2}$ are crisp then the whole definition can be written in our formalism as follows:

$$\mathrm{MonRB}_{\leq_{1,2}}(R) \equiv_{df} \bigwedge_{i,j\in I} [(A_i \leq_1 A_j) \to (B_i \leq_2 B_j)], \qquad (21)$$

[4]For this section, we omit the specification "fuzzy" when talking about fuzzy rule-bases.

and evaluation of this property produces the same result as checking the classical monotonicity of $\mathcal{R}$ given by Definition 8. To receive a fully gradual definition of a monotone rule-base, we can exchange the crisp orderings of fuzzy sets by fuzzy ones. Moreover, observe that Definition 8 expresses the monotonicity of a functional dependency over sets of fuzzy sets.

Assuming a mapping f from the set of all fuzzy sets on X_1 denoted by $\mathcal{F}(X_1)$ into the set of all fuzzy sets on X_2 denoted by $\mathcal{F}(X_2)$, i.e., $f : \mathcal{F}(X_1) \mapsto \mathcal{F}(X_2)$. Furthermore, define

$$R_f(A, B) =_{df} \{\langle A, B\rangle \mid B = f(A)\},$$

and consider fuzzy orderings $\lesssim_{1,2}$. Then we obtain that $\mathrm{MonRB}_{\lesssim_{1,2}}(R_f)$ is represented by formula

$$\bigwedge_{i,j\in I} [(A_i \lesssim_1 A_j) \to (f(A_i) \lesssim_2 f(A_j))], \qquad (22)$$

which is exactly the monotonicity of the second-level fuzzy relation[5] R_f (or the second-level (crisp) function[6] f) at fuzzy sets $\{A_i\}_{i\in I}$. Hence, if we take into account formula (8) and transfer to the second-level, we obtain

$$\mathrm{Mon}_{\lesssim_{1,2}}(f) \equiv_{df} \forall(A_1, A_2) \\ [(A_1 \lesssim_1 A_2) \to (f(A_1) \lesssim_2 f(A_2))], \qquad (23)$$

Hence, we directly conclude

$$\vdash \mathrm{Mon}_{\lesssim_{1,2}}(f) \to \mathrm{MonRB}_{\lesssim_{1,2}}(R_f).$$

It means that the monotonicity of a second-level function f implies the monotonicity of the rule-base based on this function. But this not the other way round.

In the above approach, we have shown that the classical approach stemming from Definition 8 can be easily dealt within the graded formalism of FCTby the direct translation to the definition provided in formula (22). However, second-level fuzzy orderings may be too general for practical applications. Therefore, we propose to down-grade from these fuzzy orderings to 1-level fuzzy orderings and use the definition of monotonicity in the form (5). This definition already provides an insight into possible structures of rule-base models approximating an ideal monotone fuzzy relation. The following construction provides one particular view:

$$\vdash \mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \to \\ [F(x,y) \to \bigwedge_{i\in I}(F(c_i, d_i) \to [(x \preccurlyeq_1 c_i) \to (y \preccurlyeq_2 d_i)])], \qquad (24)$$

Considering $F(x, y) =_{df} (y = f(x))$, we obtain

$$\vdash \mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \to \\ [F(x,y) \to \underbrace{\bigwedge_{i\in I}[(x \preccurlyeq_1 c_i) \to (y \preccurlyeq_2 f(c_i))]}_{\mathrm{Rules}_F}]. \qquad (25)$$

[5] It is a fuzzy set over $\mathcal{F}(X_1) \times \mathcal{F}(X_2)$.

[6] It means that the input and output arguments are fuzzy sets.

Hence

$$\vdash \mathrm{Mon}_{\preccurlyeq_{1,2}}(F) \to [F \subseteq \mathrm{Rules}_F]. \qquad (26)$$

We read this formula as follows:
"If F is monotone then Rules_F approximates F from above." Rules_F can be identified with Bodenhofer's "At most" model of a rule-base.

VI. Conclusion

We have generalized the monotonicity property into the monotonicity involving fuzziness (i.e., grades). The definition has been introduced generally for an arbitrary fuzzy relation and also without any requirements ~~put~~ on symbols used for comparison of elements of the respective universe. All results were presented in the form of graded theorems, which enables natural reading. Hence, we were able to describe the results naturally and clearly without any details relat~~ing~~ to the degrees. We point out that degrees play their r~~u~~le on the semantical level in the particular interpretation of all used symbols. Here we see the power of our approach, which can be briefly expressed in the form of the following motto:

"We speak naturally but we think in grades."

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