University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

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Research report No. 150

2010

Submitted/to appear:

Information Sciences

Supported by:

Institutional Research Plan MSM 6198898701 and Grant IAA108270901 of the GA AV ČR.

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Fuzzy measures and integrals defined on algebras of fuzzy subsets[☆]

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Abstract

This paper presents basic notions about fuzzy measures over algebras of fuzzy subsets of a fuzzy set. It also presents basic ideas on fuzzy integrals defined using these fuzzy measures. Definitions of new types of fuzzy measures and integrals are motivated by our research on generalized quantifiers. Several useful properties of fuzzy measures and fuzzy integrals are stated and proved. Definitions presented in this paper and its results will be employed in subsequent papers on generalized quantifiers defined using this type of fuzzy integral.

Keywords: Fuzzy measure, Fuzzy integral, Fuzzy quantifier

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1. Introduction

This paper provides tools for our ongoing study of semantic properties of fuzzy quantifiers generated by fuzzy measures. Our investigations of this topic began with [2]. We investigated fuzzy quantifiers[1] of type $\langle 1 \rangle$, which are denotations of important noun phrases of natural language, e.g., "something" in "Something is broken.", "everyone" in "Everyone likes Bob.", "nobody" in "Nobody knows everything.", etc. The classical logical quantifiers "for all" and "there exists" also belong to this type. A natural extension of this research is to study quantifiers of type $\langle 1, 1 \rangle$ (e.g. "every" in "Every book has leaves.", "most" in "Most birds fly."), which take two arguments. These type $\langle 1, 1 \rangle$ quantifiers are claimed to be the most important from the point of view of natural language semantics [15].

First, let us answer two questions: Why do we need *fuzzy* quantifiers? Why do we need fuzzy measures and integrals to define them? When we think about the definitions and properties of generalized quantifiers (e.g., *many*, *a few* and

[☆]This paper has been partially supported by the Institutional Research Plan MSM 6198898701 and Grant IAA108270901 of the GA AV ČR.

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[1]Precisely, **L**-fuzzy quantifiers, where **L** denotes a complete residuated lattice.

others), we feel that their truth values should not change abruptly if we gradually change cardinalities of corresponding sets of objects. Consider, for example, the sentence "Many people read books." If the number of people who read books increases by 1, it would be very strange if the truth value of this sentence changed from false to true. Inevitably, researchers started to consider more than two truth values in this context, and so-called *fuzzy quantifiers* emerged.

The semantic interpretation of many generalized quantifiers is often connected to the measurement of "size" of the sets of concern. Consider, e.g., the quantifier "many". The truth value of the proposition "many books have a red cover" clearly depends on the "size" of the set of red books. It is therefore natural to consider measures (and integrals) of (fuzzy) sets as natural tools for the modeling of important classes of monotonically nondecreasing and monotonically nonincreasing generalized quantifiers.

Fuzzy measures and integrals ([18], see also [5, 10, 11]) are important tools for comparing classical or fuzzy sets with respect to their size. Usually, fuzzy measures are set functions defined on some algebra of sets that are monotone with respect to inclusion and that assign zero to the empty set. In this contribution, fuzzy measures are defined on algebras of fuzzy sets (measure spaces). In general, they attain values from a complete residuated lattice.

Contrary to the usual definitions of fuzzy integrals, these integrals can be used to integrate all fuzzy sets. Indeed, they can integrate fuzzy sets that are not standardly measurable with respect to the fuzzy measure space used. This property enables us to introduce fuzzy quantifiers over spaces of all fuzzy sets, not only over spaces of all measurable fuzzy sets.

When we started to investigate fuzzy quantifiers of type $\langle 1, 1\rangle$ determined by fuzzy measures, it proved advantageous to work with the definition of fuzzy measures and integrals different of those in [2], where algebras of fuzzy sets were defined as systems of fuzzy subsets of a crisp set that fulfill certain conditions. The first argument of $\langle 1, 1\rangle$ quantifier is called *restriction* and the second one is called *scope* (i.e., in "Every book has leaves", "to be a book" is the restriction and "to have leaves" is the scope). It is natural to think of the restriction as a *new universe* for the quantifier (in our example, for the determination of the truth value of this sentence, only those objects fulfilling the restriction condition are important, i.e., books). Because we are working with fuzzy subsets of some universe M, the restriction is represented by a *fuzzy* set, and we in effect should in effect be able to define quantifiers on *fuzzy universes*. In this paper, we therefore define a new type of fuzzy measure spaces defined on algebras of subsets of a *fuzzy* set A. We further define fuzzy integrals on these fuzzy measure spaces (see Section 3).

Let us try to derive a formula for the computation of the truth value of a quantified statement with generalized quantifier "many", e.g., "Many sportsmen are tall". Informally, one possibility is to search for a fuzzy subset of the fuzzy set of sportsmen such that this fuzzy set is big (i.e., its measure is as big as possible), and, for all elements x from its support it holds that if x is a sportsman, then x

is tall. We can express this informal idea by the following second-order formula[2]:

$$\mathsf{many}(Sp, Ta) := (\exists Y \in \mathcal{F}^{-}_{Sp})(\forall x \in \mathrm{Supp}(Y)(\mu(Y)\&(Sp(x) \Rightarrow Ta(x)))) \quad (1)$$

where Sp and Ta denote fuzzy sets of sportsmen and tall people, respectively, $\mathcal{F}^{-}_{Sp}$ is the set of all non-empty fuzzy subsets of fuzzy set Sp, $\mathrm{Supp}(Y)$ is the support of a fuzzy set Y and μ denotes a measure of fuzzy sets. If we derive the semantic counterpart of this formula, we obtain

$$||\mathsf{many}(Sp, Ta)|| = \bigvee_{Y \in \mathcal{F}^{-}_{Sp}} \bigwedge_{m \in \mathrm{Supp}(Y)} (\mu(Y) \otimes (Sp(m) \to Ta(m)) \quad (2)$$

where $\otimes$ is a left-continuous t-norm modeling the conjunction and $\to$ is the corresponding residuum from the residuated lattice modeling the implication. This formula is the motivation for the form of the fuzzy integral in Definition 4.1,[3] where we integrate some general function $f(m)$ instead of a particular case of $(Sp \to Ta)(m)$ here. However, the integral in Definition 4.1 is more general than that in (2) in two ways. First, we should generally not consider *all* fuzzy subsets of the fuzzy set of sportsmen, but only those that form an *algebra of fuzzy subsets*[4]. These algebras are defined and studied in Section 3.2. Second, we can use the operation of minimum $\wedge$ from a residuated lattice in (2) instead of $\otimes$.

It turns out that when we are computing values of our integrals, it is not necessary in many cases to go through all fuzzy sets from the corresponding algebra of fuzzy sets. As results in Section 4.1 show, under certain conditions it is sufficient to consider algebras of sets that cover all (crisp) supports of fuzzy sets from that algebra. This fact is chiefly important from the practical point of view, since the computation with sets is much more efficient than the corresponding computation with fuzzy sets.

Many definitions and results in this paper are motivated by our need to use them in our study of semantic properties of generalized quantifiers. For example, the notion of cardinal fuzzy measure space (Section 3.4) is important for the investigation of quantifiers that fulfill so-called permutation invariance (PI). Furthermore, definitions of isomorphisms of fuzzy measure spaces (Section 3.3) are vital for quantifiers that fulfill isomorphism invariance (ISOM).

The paper is structured as follows. In Section 2, we introduce necessary concepts from the theory of residuated lattices and fuzzy sets. In Section 3.1, we define fuzzy measurable spaces as subsets of the system of all fuzzy subsets of a given fuzzy set A that are closed under unions and differences from

[2] We slightly abuse notation by using identical symbols for fuzzy sets on the semantical level and for variables denoting them on the level of formal syntax, to make the formula as transparent as possible.

[3] Note that a similar idea is used for the definition of so-called fuzzy intermediate quantifiers in [12].

[4] It goes without saying that the set of all fuzzy subsets of A should be one of examples of such an algebra.

A. Section 3.2 contains the definition of fuzzy measures as monotone mappings from a fuzzy measurable space to a residuated lattice. We also define an inner fuzzy measure. In Section 3.3, we define isomorphisms between fuzzy measurable spaces and then define isomorphisms between fuzzy measure spaces. We also define the notion of systems of fuzzy measure spaces closed under isomorphisms. The next Section 3.4 presents a definition of cardinal fuzzy measure spaces and describes their properties. Here, fuzzy measures are invariant under automorphisms on the domain of fuzzy set A.

We continue by defining fuzzy integral in Section 4. In Section 4.1, we investigate conditions under which our integral can be expressed using only crisp sets. Then, in Section 4.2, we outline the relationship between our integral and the well-known fuzzy (Sugeno) integral. Section 4.3 presents a variety of properties of our fuzzy integral. Section 4.4 demonstrates a strong convergence theorem for our fuzzy integral. Finally, Section 5 offers concluding remarks and suggests directions for further research.

In subsequent papers, we will define fuzzy quantifiers of type $\langle 1, 1\rangle$ defined using fuzzy measures and integrals introduced in this paper, and we will study semantic properties of these quantifiers (properties such as PI, ISOM, CONS, see [3, 7], see also [2] for analogous properties of type $\langle 1\rangle$ quantifiers).

2. Preliminaries

2.1. Structures of truth values

In this paper, we suppose that the structure of truth values is a *complete residuated lattice*, i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \bot, \top\rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \bot, \top\rangle$ is a complete lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, $\langle L, \otimes, \top\rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{3}$$

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. A residuated lattice is *divisible*, if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$, and satisfies the *law of double negation*, if $(a \rightarrow \bot) \rightarrow \bot = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. For other information about residuated lattices, we refer to [1, 13].

Example 2.1. It is easy to prove (e.g., [6]) that the algebra

$$\mathbf{L}_{\mathrm{T}} = \langle [0, 1], \min, \max, T, \rightarrow_T, 0, 1\rangle,$$

where T is a left continuous t-norm [9] and $a \rightarrow_T b = \bigvee\{c \in [0, 1] \mid T(a, c) \leq b\}$, defines the residuum, is a complete residuated lattice. In this paper, we will refer to complete residuated lattices determined by the Łukasiewicz t-norm and nilpotent minimum, i.e.,

$$T_Ł(a,b) = \max(a+b-1,0),$$
$$T_{nM}(a,b) = \begin{cases} 0, & \text{if } a+b \leq 1, \\ \min(a,b), & \text{otherwise,} \end{cases}$$

respectively. Their residua are as follows:

$$a \to_Ł b = \min(1, 1-a+b),$$
$$a \to_{nM} b = \begin{cases} 1, & \text{if } a \leq b, \\ \max(1-a,b), & \text{otherwise.} \end{cases}$$

In the first case, the complete residuated lattice will be denoted by $\mathbf{L}_Ł$. Note that $\mathbf{L}_Ł$ is a complete MV-algebra called an *Lukasiewicz algebra* (*on* $[0,1]$), where, for example, the distributivity of $\otimes$ over $\wedge$ is satisfied.[5] The residuated lattice determined by the nilpotent minimum is an example of a residuated lattice in which the above-mentioned distributivity fails.

Example 2.2. Let $a, b \in [0,\infty]$ be such that $a < b$. One checks easily that $\mathbf{L}_{[a,b]} = \langle [a,b], \min, \max, \min, \to, a, b\rangle$, where

$$c \to d = \begin{cases} b, & \text{if } c \leq d, \\ d, & \text{otherwise,} \end{cases} \tag{4}$$

is a complete residuated lattice. Note that $\mathbf{L}_{[a,b]}$ is a special example of a more general residuated lattice called a *Heyting algebra*.[6]

Let us define two additional operations for all $a, b \in L$:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a) \tag{5}$$
$$\neg a = a \to \bot \tag{6}$$

which are called the biresiduum and negation, respectively.

2.2. Fuzzy sets

Let $\mathbf{L} = \langle L, \wedge, \vee, \to, \otimes, \bot, \top\rangle$ be a complete residuated lattice and M be a universe of discourse (possibly empty[7]). A mapping $A : M \to L$ is called a *fuzzy set on* M.[8] A value $A(m)$ is called a *membership degree of* m *in the fuzzy set* A.

[5] Here we mean that $\bigwedge_{i\in I}(a \otimes b_i) = a \otimes \bigwedge_{i\in I} b_i$ holds.

[6] A Heyting algebra is a residuated lattice with $\otimes = \wedge$.

[7] Although it is odd-looking, the empty universe of discourse can be important in the theory of fuzzy quantifiers, for example, when we define relativizations of fuzzy quantifiers (e.g., [7] or [2] for an explanation).

[8] In many papers (e.g., [1]), a mapping $A : M \to L$ is called **L**-fuzzy set or **L**-fuzzy subset on M. Since we will always deal with a fixed complete residuated lattice in the following text, we suppose that the denotation "fuzzy set" without a reference to the considered residuated lattice is sufficient.

The set of all fuzzy sets on M is denoted by $\mathcal{F}(M)$. Obviously, if $M = \emptyset$, then the empty mapping $\emptyset$ is the unique fuzzy set on $\emptyset$ and thus $\mathcal{F}(\emptyset) = \{\emptyset\}$. A fuzzy set A on M is called *crisp* if there is a subset Z of M such that $A = 1_Z$, where 1_Z denotes the characteristic function of Z. Particularly, $1_\emptyset$ denotes the empty fuzzy set on M, i.e., $1_\emptyset(m) = \bot$ for any $m \in M$. This convention will also be used for $M = \emptyset$.[9] The set of all crisp fuzzy sets on M is denoted by $\mathcal{P}(M)$. A fuzzy set A is said to be *constant* if there is $c \in L$ such that $A(m) = c$ for any $m \in M$. For simplicity, a constant fuzzy set is denoted by the corresponding element of L, e.g., a, b, c.[10]

Let us denote $\mathrm{Supp}(A) = \{m \mid m \in M \ \&\ A(m) > \bot\}$ and $\mathrm{Core}(A) = \{m \mid m \in M \ \&\ A(m) = \top\}$ the *support* and the *core* of a fuzzy set A, respectively. Obviously, Supp and Core are natural mappings from $\mathcal{F}(M)$ to $\mathcal{P}(M)$ with $\mathrm{Supp}(1_Z) = \mathrm{Core}(1_Z) = Z$ for any crisp fuzzy set. A fuzzy set A is called *normal*, if $\mathrm{Core}(A) \neq \emptyset$. Let $A \in \mathcal{F}(M)$ and Z be a set (not necessary a subset of M). Then $A \restriction Z$ denotes the common restriction of $A : M \to L$ to the set Z. If we want to write an explicit expression for a fuzzy set with finite support, we use the following notation:

$$A = \left\{{}^{a_1}/u_1, \ldots, {}^{a_n}/u_n\right\}, \tag{7}$$

where $u_1, \ldots, u_n \in M$ are elements that are assigned membership degrees $a_1, \ldots, a_n \in L \setminus \{\bot\}$. Note that elements with membership degree $\bot$ are not included in (7).

Let $\{A_i \mid i \in I\}$ be a non-empty family of fuzzy sets on M. Then the *union of A_i* is defined by

$$\left(\bigcup_{i\in I} A_i\right)(m) = \bigvee_{i \in I} A_i(m) \tag{8}$$

for any $m \in M$ and the *intersection of A_i* is defined by

$$\left(\bigcap_{i\in I} A_i\right)(m) = \bigwedge_{i \in I} A_i(m) \tag{9}$$

for any $m \in M$. Let A, B be fuzzy sets on M. The *difference* of A and B is a fuzzy set $A \setminus B$ on M defined by

$$(A \setminus B)(m) = A(m) \otimes (B(m) \to \bot) = A(m) \otimes \neg B(m) \tag{10}$$

for any $m \in M$ and the *complement* of A is a fuzzy set $\overline{A} = 1_M \setminus A$. Finally, an extension of the operations $\otimes$ and $\to$ on L to the operations on $\mathcal{F}(M)$ is given

[9] Sometimes, for the sake of better readability of text, we will write X, Y, Z instead of $1_X, 1_Y, 1_Z$.

[10] We suppose that the meaning of this symbol will be unmistakable from the context, that is, it should be clear when an element of L is considered and when a constant fuzzy set is assumed.

by

$$(A \otimes B)(m) = A(m) \otimes B(m) \quad \text{and} \quad (A \to B)(m) = A(m) \to B(m) \qquad (11)$$

for any $A, B \in \mathcal{F}(M)$ and $m \in M$, respectively.

Since we can deal with fuzzy sets over different universes,[11] let us introduce the ordering relation between fuzzy sets on arbitrary universes. We say that a fuzzy set A on M is an *fuzzy subset of* a fuzzy set B on M' and denote it by $A \subseteq B$, if $A(m) \leq B(m)$ for any $m \in M$. One can see that $A \subseteq B$ implies $\mathrm{Supp}(A) \subseteq \mathrm{Supp}(B)$. Nevertheless, $M \not\subseteq M'$ in general. The set of all fuzzy subsets of A on M is denoted by $\mathcal{F}(A)$, i.e.,

$$\mathcal{F}(A) = \{B \mid B \in \mathcal{F}(M) \text{ and } B \subseteq A\}. \qquad (12)$$

Obviously, $\mathcal{F}(M) = \mathcal{F}(1_M)$. Note that a family of all fuzzy subsets of A (not necessary from $\mathcal{F}(M)$) is a proper class because this family is constructed over a proper class of all sets. Furthermore, we say that a fuzzy set A on M is *equal* to a fuzzy set B on M' and denote it by $A = B$ if $A \subseteq B$ and $B \subseteq A$. Obviously, if A and B are fuzzy sets on the same universe M, then the proposed relations coincide with the common definitions of ordering and equality relations.

Let $f : M \to M'$ be a mapping. A mapping $f^{\to} : \mathcal{F}(M) \to \mathcal{F}(M')$ defined by $f^{\to}(A)(m) = \bigvee_{m' \in f^{-1}(m)} A(m')$ is called the *fuzzy extension* of the mapping f. Obviously, if f is a bijective mapping, then $f^{\to}(A)(f(m)) = A(m)$ for any $m \in M$ or $f^{\to}(A)(m) = A(f^{-1}(m))$ for any $m \in M'$. Note that if $M = M' = \emptyset$, then $\emptyset : \emptyset \to \emptyset$ is the unique bijective mapping here. This empty mapping determines the unique mapping $\emptyset^{\to} : \mathcal{F}(\emptyset) \to \mathcal{F}(\emptyset)$ assigning $1_\emptyset$ to $1_\emptyset$.

3. Fuzzy measures

3.1. Fuzzy measure spaces

Let us consider algebras of fuzzy sets as a base for defining fuzzy measures of fuzzy sets. Contrary to the classical definition of an algebra of sets or of fuzzy sets (e.g., [5, 8, 18, 14, 19]) we consider an algebra of fuzzy sets that are subsets of a given fuzzy set.

Definition 3.1. Let A be a non-empty fuzzy set on M. A subset $\mathcal{F}$ of $\mathcal{F}(A)$ is an *algebra of fuzzy sets on* A, if the following conditions are satisfied

(i) $1_\emptyset, A \in \mathcal{F}$,

(ii) if $X \in \mathcal{F}$, then $A \setminus X \in \mathcal{F}$,

(iii) if $X, Y \in \mathcal{F}$, then $X \cup Y \in \mathcal{F}$.

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[11] The presumption of different universes is mainly important in the investigation of the semantic properties of generalized (fuzzy) quantifiers as the isomorphism invariance or extension (for the definition we refer to [4, 7, 15]. In this paper, one may suppose that a universe of discourse is given.

We denote $\mathbf{Alg}(A)$ the set of all algebras of fuzzy sets on A. A pair $(A, \mathcal{F})$ is called a *fuzzy measurable space* (on A), if $\mathcal{F}$ is an algebra of fuzzy sets on A. Let $(A, \mathcal{F})$ be a fuzzy measurable space and $X \in \mathcal{F}(A)$. We say that X is *$\mathcal{F}$-measurable* if $X \in \mathcal{F}$. To refer to the universe of discourse M of A, we will sometimes write $\mathrm{Dom}(A)$ instead of M.

Remark 3.1. For a non-empty set M, it is easy to check that the proposed definition of an algebra of fuzzy sets on 1_M coincides with the standard definition of an algebra of fuzzy sets on M (see [8, 16, 20] or Appendix E in [18]).

We now present some examples of algebras of fuzzy sets on a non-empty fuzzy set A.

Example 3.2. $\{1_\emptyset, A\}$ and $\mathcal{F}(A)$ are trivial examples of algebras of fuzzy sets on A.

Example 3.3. Let us say that a fuzzy set A on M is a *simple fuzzy set* on M if there exists a family of sets $\{M_i \mid i = 1, \dots, n\}$ such that $\bigcup_{i=1}^n M_i = M$, $M_i \neq M_j$ for any $i \neq j$ and $A(m) = A(m')$ holds for each $m, m' \in M_i$, where $i = 1, \dots, n$. Obviously, the set of all simple fuzzy sets on M is an algebra of fuzzy sets on 1_M.

Example 3.4. Let $M = [0, 1]$ and $\mathbf{L}_\text{Ł}$ be the Łukasiewicz algebra (Example 2.1). Then the set of all continuous mappings $A : M \to L$ is an algebra of fuzzy sets on 1_M.[12]

Example 3.5. Let A be a fuzzy set on M and τ_M be a topology on M. Let $\mathcal{B} = \{A \upharpoonright X \mid X \in \tau_M\}$ be the set of all restrictions of A to all open sets from τ_M. For example, if the open sets are open real intervals and $\mathrm{Supp}(A)$ is a real interval, one may consider the restrictions of A as "open fuzzy subintervals" of A. The set containing all complements and finite unions over $\mathcal{B}$ is an algebra of fuzzy sets on A. Obviously, this generated algebra is the smallest algebra of fuzzy sets containing all fuzzy sets of $\mathcal{B}$.

Example 3.6. Let $\mathbf{L}_\text{Ł}$ be the Łukasiewicz algebra. Let $M = \{a, b\}$ and $A = \{{}^{0.7}/a, {}^{0.6}/b\}$. Then

$$\mathcal{F} = \{1_\emptyset, \{{}^{0.1}/a\}, \{{}^{0.6}/a\}, \{{}^{0.1}/a, {}^{0.6}/b\}, \{{}^{0.6}/a, {}^{0.6}/b\}, A\} \tag{13}$$

is the smallest algebra that is the superset of $\{1_\emptyset, \{{}^{0.6}/a\}, A\}$. For example,

$$(A \setminus \{{}^{0.6}/a\})(a) = 0.7 \otimes (0.6 \to 0) = 0.7 + 0.4 - 1 = 0.1,$$
$$(A \setminus \{{}^{0.6}/a\})(b) = 0.6 \otimes (0 \to 0) = 0.6 + 1 - 1 = 0.6.$$

[12]Note that the set of all continuous mappings need not be an algebra of fuzzy sets for other residuated lattices determined by left continuous t-norms because the negation is not a continuous operation in general.

Hence, we obtain $A\backslash\{{}^{0.6}/a\} = \{{}^{0.1}/a, {}^{0.6}/b\}$. Consider the set $\mathcal{P}$ of all supports of fuzzy sets from $\mathcal{F}$, i.e.,

$$\mathcal{P} = \{\emptyset, \{a\}, \{a, b\}\}.$$

We can see that $\mathcal{P}$ is not an algebra of sets in the usual sense, since $\{b\} \notin \mathcal{P}$.

In contrast to the corresponding property of the algebra of sets, the axioms of the algebra of fuzzy sets do not guarantee that the intersection of two $\mathcal{F}$-measurable fuzzy sets is also an $\mathcal{F}$-measurable fuzzy set, i.e., there can be $C, D \in \mathcal{F}$ with $C \cap D \notin \mathcal{F}$, in general.

Example 3.7. Let $\mathbf{L}_{[0,1]}$ be the algebra from Example 2.2. Let $M = \{a, b\}$ and $A = \{{}^{1}/a, {}^{0.5}/b\}$. Then

$$\mathcal{F} = \{1_\emptyset, \{{}^{1}/a\}, \{{}^{0.5}/b\}, \{{}^{0.5}/a, {}^{0.5}/b, \}, A\} \tag{14}$$

is an algebra of fuzzy sets in the sense of Definition 3.1. However, the following fuzzy set

$$\{{}^{0.5}/a, {}^{0.5}/b\} \cap \{{}^{1}/a\} = \{{}^{0.5}/a\}$$

is not a member of $\mathcal{F}$. Hence, $\mathcal{F}$ is not closed under intersections.

Definition 3.2. We say that an algebra $\mathcal{F}$ of fuzzy sets on A is *closed under intersection*, if $C \cap D \in \mathcal{F}$, whenever $C, D \in \mathcal{F}$.

Obviously, each algebra of fuzzy sets can be extended naturally to an algebra closed under intersection. Under some conditions, however, an algebra of fuzzy sets is already closed under intersection, as the following statement demonstrates.

Theorem 3.1. *Let* $\mathbf{L}$ *be an MV-algebra. Then each algebra of fuzzy sets on* A *is closed under intersection.*

Proof. Let $\mathbf{L}$ be an MV-algebra, A be a non-empty fuzzy set on M, $\mathcal{F}$ be an algebra of fuzzy sets on A and $C, D \in \mathcal{F}$. First, let us note that, under the above-mentioned conditions, $A \setminus (A \setminus X) = X$. In fact, we have

$$(A \setminus (A \setminus X))(m) = A(m) \otimes \neg(A(m) \otimes \neg X(m)) =$$
$$A(m) \otimes (A(m) \to X(m)) = A(m) \wedge X(m) = X(m)$$

for any $m \in M$. Furthermore, we have $(A \setminus C) \cup (A \setminus D) = A \setminus (C \cap D)$. In fact, we may write

$$((A \setminus C) \cup (A \setminus D))(m) = (A(m) \otimes \neg C(m)) \vee (A(m) \otimes \neg D(m)) =$$
$$A(m) \otimes (\neg C(m) \vee \neg D(m)) = A(m) \otimes \neg(C(m) \wedge D(m)) = (A \setminus (C \cap D))(m)$$

for any $m \in M$. Since $A\setminus((A\setminus C)\cup(A\setminus D)) = C\cap D$ and $A\setminus((A\setminus C)\cup(A\setminus D)) \in \mathcal{F}$, we obtain $C \cap D \in \mathcal{F}$. $\square$

Remark 3.8. If $A = 1_{M'}$ and $M' \subseteq M$ in the presumptions of the previous theorem, it is sufficient to suppose that $\mathbf{L}$ is a residuated lattice satisfying the law of double relation because of $(A \setminus X)(m) = \neg X(m)$.

Example 3.9. Algebras of fuzzy sets introduced in Examples 3.2, 3.3 and 3.4 are closed under intersection. The satisfaction of this property in the third example of an algebra follows immediately from Theorem 3.1.

Example 3.10. Let $\mathbf{L}_{[0,1]}$ be the Heyting algebra (see Example 2.2), $M = (-\infty, \infty)$ and τ_M be the common topology on the set of real numbers. Let $A : M \to [0,1]$ be an arbitrary non-empty fuzzy set and $\mathcal{B}$ be the set of all restrictions of A to open sets from τ_M (see Example 3.5). Since the negation is very strong in the Heyting case, one can simply prove that $A \setminus X \in \mathcal{B}$, $A \setminus (A \setminus X) = X$ and $X \cup Y \in \mathcal{B}$ for any $X, Y \in \mathcal{B}$. Hence, $\mathcal{B}$ is an algebra of fuzzy sets on A and, moreover, it is closed under intersection.

3.2. Fuzzy measures

Let us introduce the concept of fuzzy measure as follows. The definition is a modification of the definition of the normed measure with respect to truth values (e.g., [5, 11]).

Definition 3.3. Let $(A, \mathcal{F})$ be a fuzzy measurable space. A mapping $\mu : \mathcal{F} \to L$ is called a *fuzzy measure* on $(A, \mathcal{F})$, if

(i) $\mu(1_\emptyset) = \bot$ and $\mu(A) = \top$,

(ii) if $B, C \in \mathcal{M}$ such that $B \subseteq C$, then $\mu(B) \leq \mu(C)$.

A triplet $(A, \mathcal{F}, \mu)$ is called a *fuzzy measure space*, if $(A, \mathcal{F})$ is a fuzzy measurable space and μ is a fuzzy measure on $(A, \mathcal{F})$. We denote $\mathbf{Fms}(M)$ the class of all fuzzy measurable spaces defined on a non-empty universe M, i.e., a fuzzy measure space $(A, \mathcal{F}, \mu)$ belongs to $\mathbf{Fms}(M)$, if $A \in \mathcal{F}(M)$. It is easy to see that $\mathbf{Fms}(M) \subseteq \mathbf{Fms}(M')$, whenever $M \subseteq M'$. In the following text, a fuzzy measure space $(A, \mathcal{F}, \mu)$ may be briefly denoted by the same bold capital letter as the fuzzy set A on which the algebra is defined, i.e., $\mathbf{A} = (A, \mathcal{F}, \mu)$.

Example 3.11. Let $\mathbf{L}_{\mathrm{T}}$ be an algebra from Example 2.1, where T is a continuous t-norm. Let M be a finite non-empty set and $A \in \mathcal{F}(M)$, $A \neq 1_\emptyset$. Then we can define fuzzy measure space $\mathbf{A}^\sigma = (A, \mathcal{F}(A), \mu^\sigma)$, where

$$\mu^\sigma(X) = \frac{\sum_{m \in M} X(m)}{\sum_{m \in M} A(m)}$$

for all $X \in \mathcal{F}(A)$. For motivation and more general definition of this fuzzy measure, we refer to Example 3.5 of our previous paper [2].

Example 3.12. Let $(1_M, \mathcal{M})$ be a fuzzy measurable space of continuous functions introduced in Example 3.4. Then

$$\mu(A) = \int_0^1 A(x)\ dx$$

defines a fuzzy measure on $(1_M, \mathcal{F})$.

Example 3.13. Let $(A, \mathcal{F})$ be a measurable space and A be a normal fuzzy set, i.e., $\mathrm{Core}(A) \neq \emptyset$. A fuzzy measure on $(A, \mathcal{F})$ can be defined by

$$\mu(X) = \bigvee_{m \in \mathrm{Supp}(A)} X(m).$$

A useful tool for defining new measures on a fuzzy measurable space is to use nondecreasing mappings on L, as the following lemma shows.

Lemma 3.2. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space and $g : L \to L$ be a nondecreasing mapping such that $g(\bot) = \bot$ and $g(\top) = \top$. Then*

$$(A, \mathcal{F}, g \circ \mu)$$

is a fuzzy measure space.

Proof. Denote $\mu_g = g \circ \mu$. $\mu_g(1_\emptyset) = \bot$ and $\mu_g(A) = \top$ are immediate. Because g is a nondecreasing mapping, it follows that if $X \subseteq Y$, then $\mu_g(X) \leq \mu_g(Y)$ for all $X, Y \in \mathcal{F}$. Hence, $g \circ \mu$ is a fuzzy measure on $(A, \mathcal{F})$. □

It is sometimes useful to extend a fuzzy measure to a new fuzzy measurable space, since we may have a problem defining it directly. One of the well-known approaches for measures is to establish an inner (or outer) measure, so that the values of the inner (outer) measure are obtained approximately by using known values of fuzzy subsets (or supersets). In the following paragraphs, we will introduce an inner measure for the fuzzy case.

Let $(A, \mathcal{F})$ be a fuzzy measurable space and $X \in \mathcal{F}(M)$. Denote $\mathcal{F}_X$ the set of all $\mathcal{F}$-measurable sets that are contained in X, i.e.,

$$\mathcal{F}_X = \{B \mid B \in \mathcal{F} \text{ and } B \subseteq X\}. \tag{15}$$

Note that $1_\emptyset \in \mathcal{F}_X$ for each $X \in \mathcal{F}(M)$ and if X is an $\mathcal{F}$-measurable set, then also $X \in \mathcal{F}_X$. If $X = A$, then we will simply write $\mathcal{F}$ instead of $\mathcal{F}_A$. The following theorem defines *inner fuzzy measure on $(M, \mathcal{F}')$ determined by fuzzy measure* μ on some fuzzy measurable space $(M, \mathcal{F})$, where $\mathcal{F} \subseteq \mathcal{F}'$.

Theorem 3.3. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space and $\mathcal{F}' \in \mathbf{Alg}(A)$ such that $\mathcal{F} \subseteq \mathcal{F}'$. A mapping $\mu^* : \mathcal{F}' \to L$ defined by*

$$\mu^*(X) = \bigvee_{B \in \mathcal{F}_X} \mu(B) \tag{16}$$

is a fuzzy measure on the fuzzy measurable space $(A, \mathcal{F}')$. Moreover, μ^* and μ coincide on $\mathcal{F}$. We say that μ^* is the inner fuzzy measure on $(A, \mathcal{F}')$ determined by μ*.*

Proof. Clearly, $\mu^*(X) = \mu(X)$ for any $X \in \mathcal{F}$. Hence, $\mu^*(1_\emptyset) = \bot$ and $\mu^*(A) = \top$. Moreover, $\mathcal{F}_X \subseteq \mathcal{F}_Y$, whenever $X \subseteq Y$, and thus $\mu^*(X) \leq \mu^*(Y)$ for any $X, Y \in \mathcal{F}'$ with $X \subseteq Y$. Hence, μ^* is a fuzzy measure on $(A, \mathcal{F}')$ which coincides with μ on $\mathcal{F}$. □

Example 3.14. Let $(A, \mathcal{F}, \mu)$ be the fuzzy measure space where $\mathcal{F} = \{1_\emptyset, A\}$ and $\mathcal{F}' = \mathcal{F}(A)$ (the set of all fuzzy subsets of A). Then

$$\mu^*(X) = \bigvee_{B \in \mathcal{F}_X} \mu(B) = \begin{cases} \top, & X = A, \\ \bot, & \text{otherwise}, \end{cases}$$

since $\mathcal{F}_X$ is either $\{1_\emptyset\}$ or $\{1_\emptyset, A\} = \mathcal{F}$, $\mu(1_\emptyset) = \bot$ and $\mu(A) = \top$.

3.3. Isomorphisms between fuzzy measurable spaces and between fuzzy measure spaces

In our investigation of semantic properties of fuzzy quantifiers defined by fuzzy measures and integrals, we need to construct isomorphisms between fuzzy measurable and measure spaces. This paragraph gives a suitable terminology and useful results.

Definition 3.4. Let $(A, \mathcal{F})$ and $(B, \mathcal{G})$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{F} \to \mathcal{G}$ is an *isomorphism between* $(A, \mathcal{F})$ *and* $(B, \mathcal{G})$, if

(i) g is a bijective mapping with $g(1_\emptyset) = 1_\emptyset$,

(ii) $g(X \cup Y) = g(X) \cup g(Y)$ and $g(A \setminus X) = A' \setminus g(X)$ hold for any $X, Y \in \mathcal{F}$,

(iii) there exists a bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ with $X(m) = g(X)(f(m))$ for any $X \in \mathcal{F}$ and $m \in \mathrm{Dom}(A)$.

Theorem 3.4. *Let $(A, \mathcal{F})$, $(B, \mathcal{G})$ be fuzzy measurable spaces and $g : \mathcal{F} \to \mathcal{G}$ be a surjective mapping. Then g is an isomorphism between $(A, \mathcal{F})$ and $(B, \mathcal{G})$ if and only if there exists a bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ such that $g = f^{\to} \upharpoonright \mathcal{F}$.*

Proof. First, let $g : \mathcal{F} \to \mathcal{G}$ be an isomorphism of spaces $(A, \mathcal{F})$ and $(B, \mathcal{G})$. Then there exists a bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ (according to (iii)) such that $X(m) = g(X)(f(m))$ holds for any $m \in \mathrm{Dom}(A)$. Since $f^{\to}(X)(f(m)) = X(m) = g(X)(f(m))$ for any $m \in \mathrm{Dom}(A)$, then clearly $f^{\to}(X) = g(X)$ for any $X \in \mathcal{F}$ and thus $g = f^{\to}$.

Conversely, let $g : \mathcal{F} \to \mathcal{G}$ be a surjective mapping such that $g = f^{\to}$ for some bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$. Let us suppose that $g(X) = g(Y)$ for some $X, Y \in \mathcal{F}$. Since f is a bijective mapping of $\mathrm{Dom}(A)$ onto $\mathrm{Dom}(B)$, then we have $g(X)(m) = f^{\to}(X)(m) = X(f^{-1}(m)) = Y(f^{-1}(m)) = f^{\to}(Y)(m) = g(Y)(m)$ for any $m \in \mathrm{Dom}(A)$ and thus $X = Y$. Hence, g is a bijective mapping. Furthermore, $g(1_\emptyset)(m) = f^{\to}(1_\emptyset)(m) = 1_\emptyset(f^{-1}(m)) = \bot$ holds for all $m \in \mathrm{Dom}(B)$. Hence, $g(1_\emptyset) = 1_\emptyset$ and (i) is proved. If $X, Y \in \mathcal{F}$, then $g(X \cup Y)(m) = f^{\to}(X \cup Y)(m) = (X \cup Y)(f^{-1}(m)) = X(f^{-1}(m)) \vee$

$Y(f^{-1}(m)) = f^{\to}(X)(m) \vee f^{\to}(Y)(m) = (g(X) \cup g(Y))(m)$ holds for all $m \in \mathcal{G}$. Let $X \in \mathcal{F}$, then $g(A \setminus X)(m) = f^{\to}(A \setminus X) = (A \setminus X)(f^{-1}(m)) = A(f^{-1}(m)) \otimes (X(f^{-1}(m)) \to \bot) = B(m) \otimes (f^{\to}(X)(m) \to \bot) = B(m) \otimes (g(X)(m) \to \bot) = (B \setminus g(X))(m)$ holds for any $X \in \mathcal{F}$ and $m \in \mathrm{Dom}(A)$. Hence, (ii) is proved. Since f is a bijective mapping with $g(X)(f(m)) = f^{\to}(X)(f(m)) = X(m)$, then (iii) is also true and the proof is finished. □

Definition 3.5. Let $(A, \mathcal{F}, \mu)$ and $(B, \mathcal{G}, \mu')$ be fuzzy measure spaces. We say that a mapping $g : \mathcal{F} \to \mathcal{G}$ is an *isomorphism between* $(A, \mathcal{F}, \mu)$ *and* $(B, \mathcal{G}, \mu')$, if

(i) g is an isomorphism between $(A, \mathcal{F})$ and $(B, \mathcal{G})$,

(ii) $\mu(X) = \mu'(g(X))$ for any $X \in \mathcal{F}$.

The notion of isomorphism between fuzzy measure spaces can be formulated in an alternative manner, using a bijection between supports of fuzzy sets on which fuzzy measurable spaces are defined. Note that this approach could be profitable if there is a significant difference between the cardinality of domains and supports of fuzzy sets, e.g., if domains are infinite and supports are of finite cardinality.

Lemma 3.5. *Let* $(A_1, \mathcal{F}_1, \mu_1)$ *and* $(A_2, \mathcal{F}_2, \mu_2)$ *be isomorphic fuzzy measure spaces. If we put* $B_i = A_i \upharpoonright \mathrm{Supp}(A_i)$, $\mathcal{G}_i = \{X_i \upharpoonright \mathrm{Supp}(A_i) \mid X_i \in \mathcal{F}_i\}$ *and* $\mu'_i(X_i \upharpoonright \mathrm{Supp}(A_i)) = \mu_i(X_i)$, *where* $X_i \in \mathcal{F}_i$, *for* $i = 1, 2$, *then the triplets* $(B_1, \mathcal{G}_1, \mu'_1)$ *and* $(B_2, \mathcal{G}_2, \mu'_2)$ *are isomorphic fuzzy measure spaces.*

Proof. This result is a straightforward consequence of the definition of isomorphic fuzzy measure spaces. □

If g is an isomorphism between fuzzy measure spaces $\mathbf{A} = (A, \mathcal{F}, \mu)$ and $\mathbf{B} = (B, \mathcal{G}, \mu')$, then we will write $g(A, \mathcal{F}, \mu) = (B, \mathcal{G}, \mu')$ or shortly $g(\mathbf{A}) = \mathbf{B}$. If an isomorphism g between $\mathbf{A}$ and $\mathbf{B}$ is determined by a bijective mapping $f : \mathrm{Dom}(A) \to \mathrm{Dom}(B)$ (see Theorem 3.4), then we will write $f^{\to}(\mathbf{A}) = \mathbf{B}$.

We say that a system $\mathcal{A}$ of fuzzy measure spaces from $\mathbf{Fms}(M)$ is *closed under isomorphisms in* $\mathbf{Fms}(M)$ if the following holds: if $\mathbf{A} \in \mathcal{A}$ and $\mathbf{B} \in \mathbf{Fms}(M)$ are isomorphic, then $\mathbf{B} \in \mathcal{A}$. In the following text, we will, for the sake of simplicity, omit the term "under isomorphisms" in "closed under isomorphisms" and say only "closed system of fuzzy measure spaces in $\mathbf{Fms}(M)$". Note that there are closed systems of fuzzy measure spaces containing non-isomorphic fuzzy measure spaces. If a system $\mathcal{A}$ of mutually isomorphic fuzzy measure spaces in $\mathbf{Fms}(M)$ is closed, then we say that $\mathcal{A}$ is *closed system of mutually isomorphic fuzzy measure spaces in* $\mathbf{Fms}(M)$. Obviously, each closed system is a union of closed systems of mutually isomorphic fuzzy measure spaces.

Lemma 3.6. *A system* $\mathcal{A}$ *of fuzzy measure spaces in* $\mathbf{Fms}(M)$ *is closed if and only if* $f^{\to}(\mathbf{A}) \in \mathcal{A}$ *for any* $\mathbf{A} \in \mathcal{A}$ *and any permutation* f *on* $\mathrm{Dom}(A)$.

Proof. This result is a straightforward consequence of the definition of closedness under isomorphisms and Theorem 3.4. □

3.4. Cardinal fuzzy measure spaces

An important class of fuzzy measure spaces which will be used in our theory of fuzzy quantifiers is the class of the fuzzy measure spaces being invariant under automorphisms in the following sense.

Definition 3.6. We say that $(A, \mathcal{F}, \mu)$ is a *cardinal* fuzzy measure space if

(i) if $X \in \mathcal{F}$, then $f^{\rightarrow}(X) \in \mathcal{F}$,

(ii) $\mu(X) = \mu(f^{\rightarrow}(X))$

hold for any $X \in \mathcal{F}$ and for any permutation f on $\mathrm{Dom}(A)$.

Remark 3.15. The denotation "cardinal" in the previous definition means that these measures are invariant under the same cardinality of fuzzy sets. We can say that two fuzzy sets X, Y have the same cardinality if there exists a permutation on $\mathrm{Dom}(A)$ such that $f^{\rightarrow}(X) = Y$. Note that the condition (i) is not that important from a the general point of view. However, it is important for our considerations of generalized quantifiers.[13]

Lemma 3.7. *A set* $\{\mathbf{A}\}$ *forms a closed system of fuzzy measure spaces in* $\mathbf{Fms}(M)$ *if and only if* $\mathbf{A}$ *is a cardinal fuzzy measure space.*

Proof. Obvious. □

Lemma 3.8. *If* $\mathbf{A}$ *is a cardinal fuzzy measure space, then* A *is a constant fuzzy set.*

Proof. If A is not a constant fuzzy set, then clearly there exists a permutation f on $\mathrm{Dom}(A)$ such that $A(m) \neq f^{\rightarrow}(A)(f(m))$. This assertion contradicts the presumption of the lemma. □

Example 3.16. Trivial examples of cardinal fuzzy measure spaces in $\mathbf{Fms}(M)$ are a trivial fuzzy measure space $(A, \{1_\emptyset, A\}, \mu)$, where A is a constant fuzzy set in $\mathcal{F}(M)$, and a fuzzy measure space $(A, \mathcal{F}(A), \mu)$, where $\mu(A) = \mu(B)$ whenever there exists a permutation f on $\mathrm{Dom}(A)$ with $f^{\rightarrow}(A) = B$. Further examples can be derived from the second fuzzy measure space in a way such that $\mathcal{F}(A)$ is replaced by $\mathcal{F}$ of all fuzzy sets from $\mathcal{F}(A)$ different from the constant fuzzy sets taking a value $c \in]\bot, a[$ (if $c \neq \neg\neg c$) or values $c, \neg c \in]\bot, a[$ (if $c = \neg\neg c$), where $a = A(m)$ for any $m \in \mathrm{Dom}(A)$.[14]

Lemma 3.9. *Let* $\mathcal{A} = \{(A_i, \mathcal{F}_i, \mu_i) \mid i \in I\}$ *be a closed system of fuzzy measure spaces in* $\mathbf{Fms}(M)$*. Put* $A = \bigcup_{i \in I} A_i$*, let* $\mathcal{F}$ *be the least algebra of fuzzy sets containing* $\bigcup_{i \in I} \mathcal{F}_i$ *and*

$$\mu(X) = \bigvee_{i \in I} \bigvee_{Y \in \mathcal{F}_{i,X}} \mu_i(Y), \tag{17}$$

[13] Perhaps a better denotation could be a "closed cardinal fuzzy measure space".

[14] The notation $]\bot, a[$ means the interval of all values b from L for which $\bot < b < a$.

where $\mathcal{F}_{i,X} = \{Y \mid Y \in \mathcal{F}_i \text{ and } Y \subseteq X\}$. Then $(A, \mathcal{F}, \mu)$ is a cardinal fuzzy measure space.

Proof. Denote $\mathbf{A}_i = (A_i, \mathcal{F}_i, \mu_i)$. A simple consequence of the definition of algebra $\mathcal{F}$ and of fuzzy measure μ is the fact that $\mathbf{A} = (A, \mathcal{F}, \mu)$ is a fuzzy measure space. Note that $\mu(X) \neq \mu_i(X)$ for some $X \in \mathcal{F}_i$ in general. Since each element of $\mathcal{F}$ is obtained using a finite number of operations applied on some elements of $\bigcup_{i\in I} \mathcal{F}_i$ and $f^{\rightarrow}(\mathbf{A}_i) \in \mathcal{A}$ holds for any $i \in I$, then $f^{\rightarrow}(X) \in \mathcal{F}$, whenever $X \in \mathcal{F}$. Let f be any permutation on $\mathrm{Dom}(A)$. Since the set $\mathcal{A}$ is closed under isomorphisms, then to each isomorphism $f^{\rightarrow}$ there is a mapping $h : I \to I$ such that $f^{\rightarrow}(\mathbf{A}_i) = \mathbf{A}_{h(i)}$. Obviously, $X \in \mathcal{F}_i$ implies $f^{\rightarrow}(X) \in \mathcal{F}_{h(i)}$. Moreover, if $X \subseteq Y$, then evidently $f^{\rightarrow}(X) \subseteq f^{\rightarrow}(Y)$. Hence, $f^{\rightarrow}$ is a mapping of $\mathcal{F}_{i,X}$ to $\mathcal{F}_{h(i),f^{\rightarrow}(X)}$ for any $X \in \mathcal{F}$ and we can write

$$\mu(X) = \bigvee_{i\in I} \bigvee_{Y\in\mathcal{F}_{i,X}} \mu_i(Y) = \bigvee_{i\in I} \bigvee_{f^{\rightarrow}(Y)\in\mathcal{F}_{h(i),f^{\rightarrow}(X)}} \mu_{h(i)}(f^{\rightarrow}(Y)) \leq \bigvee_{i\in I} \bigvee_{Z\in\mathcal{F}_{i,f^{\rightarrow}(X)}} \mu_i(Z) = \mu(f^{\rightarrow}(X)).$$

We have thus shown that $\mu(X) \leq \mu(f^{\rightarrow}(X))$ for any permutation f on $\mathrm{Dom}(A)$ and $X \in \mathcal{F}$. On the other hand, it is sufficient to consider $Y = f^{\rightarrow}(X)$ and $g = f^{-1}$. Then $\mu(f^{\rightarrow}(X)) = \mu(Y) \leq \mu(g^{\rightarrow}(Y)) = \mu(g^{\rightarrow}(f^{\rightarrow}(X))) = \mu(X)$, where clearly $g^{\rightarrow}(f^{\rightarrow}(X))(m) = f^{\rightarrow}(X)(g^{-1}(m)) = f^{\rightarrow}(X)(f(m)) = X(m)$ for any $m \in \mathrm{Dom}(A_i)$. Hence, we obtain the equality of measures under isomorphisms and $\mathbf{A}$ is a cardinal fuzzy measure space. □

4. ⊙-fuzzy integral

In this section, we will introduce a new type of fuzzy integral that can be defined on an arbitrary fuzzy measure space $(A, \mathcal{F}, \mu)$. The integrated functions are fuzzy sets on $\mathrm{Dom}(A)$. To keep the notation of integrals the same as in the classical measure theory, we prefer, in this section, to use denotations f, g for the integrated functions instead of X, Y. Nevertheless, we will deal with them as with fuzzy sets. For example, $f \cap g$ denotes the intersection of fuzzy sets. This integral is defined over a general operation $\odot$ which substitutes for one of the operations $\wedge$, $\otimes$, i.e., $\odot \in \{\wedge, \otimes\}$.

Definition 4.1. Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space with $M = \mathrm{Dom}(A)$, $f : M \to L$ and X be an $\mathcal{F}$-measurable fuzzy set. The *$\odot$-fuzzy integral of f on X* is given by

$$\int_X^{\odot} f \, d\mu = \bigvee_{Y\in\mathcal{F}_X^-} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m) \odot \mu(Y)), \tag{18}$$

where $\mathcal{F}_X^- = \mathcal{F}_X \setminus \{1_\emptyset\}$. If $X = A$, then we write $\int^{\odot} f \, d\mu$.

A trivial consequence of the definition of the $\odot$-integral is the following useful lemma.

Lemma 4.1. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space with $M = \mathrm{Dom}(A)$, $f : M \to L$ and X be an $\mathcal{F}$-measurable fuzzy set. Then*

$$\int_X^{\odot} f \; d\mu = \int_X^{\odot} f \restriction Z \; d\mu, \tag{19}$$

for any set Z such that $\mathrm{Supp}(X) \subseteq Z \subseteq M$.

4.1. Representation of $\odot$-fuzzy integral using crisp sets

One can note that the definition of the $\odot$-integral may be simplified, since the membership functions of fuzzy sets Y, $Y \in \mathcal{F}_X^-$ are not explicitly used in the computation of the fuzzy integral (instead, only the extent of their support is used) and the fuzzy measure is a nondecreasing mapping. Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space and $X \in \mathcal{F}$ be an $\mathcal{F}$-measurable set. Denote by $R(X)$ the union of all fuzzy sets from $\mathcal{F}$ with the same support as X, i.e.,

$$R(X) = \bigcup \{Y \mid Y \in \mathcal{F} \text{ and } \mathrm{Supp}(Y) = \mathrm{Supp}(X)\}, \tag{20}$$

and say that the algebra $\mathcal{F}$ is *R-complete* if $R(X) \in \mathcal{F}$ for any $X \in \mathcal{F}$. According to the definition of an algebra of fuzzy sets, each finite algebra is R-complete.

Lemma 4.2. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space where $\mathcal{F}$ is R-complete. Then*

$$\int_X^{\odot} f \; d\mu = \bigvee_{Y \in \mathcal{R}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)), \tag{21}$$

where $\mathcal{R}_X^- = \{Y \mid Y \in \mathcal{F}_X^-$ and $R(Y) = Y\}$.

Proof. This result is a straightforward consequence of the monotonicity of a fuzzy measure μ. $\square$

Putting

$$\mathcal{P}_X^- = \{\mathrm{Supp}(Y) \mid Y \in \mathcal{F}_X^-\}, \tag{22}$$

then $\mathrm{Supp} : \mathcal{R}_X^- \to \mathcal{P}_X^-$ defined in a natural way is a bijection. If we define $\mu' : \mathcal{P}_X^- \to L$ by $\mu'(Y) = \mu(\mathrm{Supp}^{-1}(Y))$, then replacing $\mathcal{R}_X^-$ and μ in (21) by $\mathcal{P}_X^-$ and μ', respectively, we obtain the same value of $\int_X^{\odot} f \; d\mu$. Hence, if $\mathcal{F}$ is R-complete, then we may express our integral locally in terms of sets. A natural question arising from this representation is to ask whether the integral can be expressed in terms of sets independently on the structure of truth values used and fuzzy set X over which the integral is computed. More precisely, when to a given fuzzy measure space $(A, \mathcal{F}, \mu)$ there exists a corresponding measure space $(M, \mathcal{P}, \mu')$ with $\mathrm{Supp}(A) = M$ and $\mathcal{P} \subseteq \mathcal{P}(M)$ such that

$$\int_X^{\odot} f \, d\mu = \int_{\mathrm{Supp}(X)}^{\odot} f \, d\mu' \tag{23}$$

for any $X \in \mathcal{F}$ and $f : M \to L$. Note that an analogous idea has been investigated in [20] (see also Appendix E in [18]). A similar result can be found in the following paragraph, where a relation between our fuzzy integral and the Sugeno integral is investigated (cf. Transformation Theorem I. in [20] (or [18], p. 236), and our Corollary 4.8). In this paper, however, we consider a different situation.

To find a full characterization of the equality of integrals is practically impossible, because any result of this sort would depend strongly on the choice of the operation $\odot$ and on the properties of residuated lattices. The following two examples demonstrate the nature of this difficulty. The first example shows that the R-completeness of an algebra $\mathcal{F}$ is not a necessary condition for obtaining the equality (23). The second example illustrates the situation in which, if we change only the structure of truth values, we need the R-completeness to guarantee the equality of integrals in (23).

Example 4.1. Let $\mathbf{L}_{\mathrm{Ł}}$ be the Łukasiewicz algebra, $M = [0, 1]$ and $A_z, B_z : M \to [0, 1]$ be defined by

$$A_z(m) = \begin{cases} z, & \text{if } m \in [0, 0.5[, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_z(m) = \begin{cases} z, & \text{if } m \in [0.5, 1], \\ 0, & \text{otherwise.} \end{cases}$$

for any $z \in]0, 1[$. Put $\mathcal{X} = \{A_z, B_z \mid z \in]0, 1[\}$ be the set of all fuzzy sets defined above and denote $\mathcal{F}$ the smallest algebra of fuzzy sets on M containing $\mathcal{X}$. Let us put $\mu(X) = \int_0^1 X \, dx$. It is easy to see that $\mu(X) = \frac{1}{2}(z + z')$, whenever $X = A_z \cup B_{z'}$ and $A_z \cap B_{z'} = 1_\emptyset$. Then $(M, \mathcal{F}, \mu)$ is a measure space on M. Let us consider the common partial ordering of fuzzy sets on M, i.e., $A \subseteq B$, if $A(m) \le B(m)$ for any $m \in M$. Obviously, $A_z \subset 1_{[0,0.5[}$, $B_z \subset 1_{[0.5,1]}$, $\neg A_z \parallel 1_{[0,0.5[}$ and $\neg B_z \parallel 1_{[0.5,1]}$ for any $z \in]0, 1[$. Since $1_{[0,0.5[} \cap 1_{[0.5,1]} = 1_\emptyset$ and $M \setminus (M \setminus A) = A$ (see the proof of Theorem 3.1) for any fuzzy sets constructed from elements of $\mathcal{X}$ using a finite application of operations $\cup$ and $\setminus$, one may see that $1_{[0,0.5[}, 1_{[0.5,1]} \notin \mathcal{F}$. Hence, $\mathcal{F}$ is not R-complete. Moreover, the set of all supports of fuzzy sets of the algebra $\mathcal{F}$ is

$$\mathcal{P} = [\emptyset, [0, 0.5[, [0.5, 1], M\}.$$

Obviously, $\mathcal{P}$ is an algebra of sets. Define $\mu'(\emptyset) = 0$, $\mu'([0, 0.5[) = \mu'([0.5, 1]) =$

0.5 and $\mu'(M) = 1$.[15] Note that

$$\mu'([0, 0.5[) = \bigvee_{Y \in \mathcal{F}, Y \subseteq 1_{[0,0.5[}} \mu(Y) = 0.5$$

and analogously for $\mu'([0.5, 1])$. Obviously, $(M, \mathcal{P}, \mu')$ is a measurable space and, moreover, we have

$$\int_X^{\otimes} f \; d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \otimes \mu(Y)) =$$

$$\bigvee_{Y \in \mathcal{F}_X^-} ((\bigwedge_{m \in \mathrm{Supp}(Y)} f(m)) \otimes \bigvee_{\substack{Y' \in \mathcal{F}_X^- \\ \mathrm{Supp}(Y') = \mathrm{Supp}(Y)}} \mu(Y)) =$$

$$\bigvee_{Z \in \mathcal{P}^-_{\mathrm{Supp}(X)}} \bigwedge_{m \in Z} (f(m) \otimes \mu'(Z)) = \int_{\mathrm{Supp}(X)}^{\otimes} f \; d\mu',$$

where $\mathcal{P}^-_{\mathrm{Supp}(X)} = \{Z \mid Z \in \mathcal{P} \text{ and } Z \subseteq \mathrm{Supp}(X)\} \backslash \{\emptyset\}$ and the properties of the Łukasiewicz algebra are applied. Hence, one can see that the R-completeness of the algebra $\mathcal{F}$ is not a necessary condition for the equality of integrals.

Example 4.2. Let us now consider the complete residuated lattice determined by the nilpotent minimum. As we have noted in Example 2.1, this residuated lattice does not satisfy the distributivity law for $\otimes$ and $\wedge$. Let us consider the fuzzy measure space $(A, \mathcal{F}, \mu)$ and the measure space $(M, \mathcal{P}, \mu')$ defined in the previous example.[16] Furthermore, let $f : M \to [0, 1]$ be defined by

$$f(m) = \begin{cases} 1 - m, & \text{if } m \in [0, 0.5[, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int^{\otimes} f \; d\mu = \bigvee_{Y \in \mathcal{F}^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \otimes \mu(Y)) = 0,$$

since for any $Y \in \mathcal{F}^-$ there exists $m \in \mathrm{Supp}(Y)$ for which $f(m) \otimes \mu(Y) = 0$ (e.g., if $\mathrm{Supp}(Y) = [0, 0.5[$ and $\mu(Y) = 0.4$, then it is sufficient to consider $m = 0.45$).

[15] Note that we use $\emptyset, [0, 0.5[, [0.5, 1], M$ instead of $1_\emptyset, 1_{[0,0.5[}, 1_{[0.5,1]}, 1_M$ to denote the crisp fuzzy sets here.

[16] According to the definition of residuum for the nilpotent minimum (see Example 2.1), one can see that $\neg a = 1 - a$. This negation thus coincides with the negation in the Łukasiewicz algebra. Hence, we may suppose the same fuzzy measure and measure space as in Example 4.1.

On the other side, we may write

$$\int^{\otimes} f \, d\mu' = \bigvee_{Y \in \mathcal{P}^-} \bigwedge_{m \in Y} (f(m) \otimes \mu(Y)) \geq$$

$$\bigwedge_{m \in [0,0.5[} (f(m) \otimes \mu([0,0.5[)) = \bigwedge_{m \in [0,0.5[} (f(m) \otimes 0.5) = \bigwedge_{m \in [0,0.5[} 0.5 = 0.5,$$

since $1 - m + 0.5 > 1$ for any $m \in [0, 0.5[$. Hence, the fuzzy integral cannot be expressed using the measure space $(M, \mathcal{P}, \mu')$. Note that we obtain the same result for other measure spaces.[17] In fact, it is easy to see that the integrals are not identical for $\mathcal{P} = \{\emptyset, M\}$. Further, if $\mathcal{P}$ contains another set X, then we can simply find a function f' with $\mathrm{Dom}(f') = X$ for which the equality (23) is not satisfied. If $\mu([0, 0.5[) > 0.5$, then f defined above gives

$$\int^{\otimes} f \, d\mu' \geq \bigwedge_{m \in [0,0.5[} \min(1 - m, \mu([0, 0.5[) = 0.5.$$

Here again, the integrals are not identical. Finally, if $\mu([0, 0.5[) < 0.5$, then it is sufficient to define $f'(m) = \min(1, f(m) + a)$, where $a < 0.5 - \mu([0, 0.5[)$. One can simply prove that

$$\int^{\otimes} f' \, d\mu = \bigvee_{\substack{Y \in \mathcal{F}, \\ Y \subseteq 1_{[0,0.5[}}} \mu(Y) = 0.5 >$$

$$0 = \bigwedge_{m \in [0,0.5[} \min(1 - m + a, \mu([0, 0.5[)) = \int^{\otimes} f' \, d\mu'.$$

Note that the equality of the fuzzy integrals could be guaranteed if the algebra $\mathcal{F}$ is R-complete, i.e., when $1_{[0,0.5[}, 1_{[0.5,1]} \in \mathcal{F}$.

To specify a necessary condition under which we can replace a fuzzy measurable space by a (crisp) measurable space in the computation of our fuzzy integral, let us state a natural construction of measurable space from a given fuzzy measurable space. Let $(A, \mathcal{F})$ be a fuzzy measurable space and $M = \mathrm{Supp}(A)$.[18]

Definition 4.2. Let $(A, \mathcal{F})$ be a fuzzy measurable space and $M = \mathrm{Supp}(A)$. We say that an algebra of sets $\mathcal{P}$ on M containing all supports of fuzzy sets from $\mathcal{F}$ is a *crisp representation of the algebra* $\mathcal{F}$, if, for any $X \in \mathcal{P}$, there exists $Y \in \mathcal{F}$ such that

(i) $\mathrm{Supp}(Y) \subseteq X$,

[17] The choice of $(M, \mathcal{P}, \mu')$ seems to be the best choice to replace a fuzzy measure space by a measure space.

[18] Recall that the set of all supports of fuzzy sets from $\mathcal{F}$ need not be an algebra of sets in general (see Example 3.6).

(ii) if $\mathrm{Supp}(Y') \subseteq X$ for $Y' \in \mathcal{F}$, then $Y' \subseteq Y$.

Lemma 4.3. *If there exists a crisp representation of an algebra $\mathcal{F}$, then $\mathcal{F}$ is R-complete.*

Proof. Let $X = \mathrm{Supp}(Z)$ for some $Z \in \mathcal{F}$. According to the presumption, there exists $Y \in \mathcal{F}$ such that, for any $Y' \in \mathcal{F}$ with $\mathrm{Supp}(Y') \subseteq X$ we have $Y' \subseteq Y$. Hence, Y is the supremum of all fuzzy sets of $\mathcal{F}$ whose supports coincide with X and thus the algebra $\mathcal{F}$ is R-complete. □

To show that the opposite implication is not true in general, let us consider an extension of Example 3.6 to the infinite case.

Example 4.3. Let $\mathbf{L}_{\mathrm{L}}$ be the Łukasiewicz algebra. Let $M = \{1, 2, \dots\}$ be the set of all natural numbers and $A = \{{}^{0.7}/1, {}^{0.6}/2, {}^{0.6}/3, \dots\}$. Let

$$A_1 = \{{}^{0.6}/1\},$$

and

$$A_n = \{{}^{0.6}/2, \dots, {}^{0.6}/n\},$$

for $n = 2, 3, \dots$ Clearly, $A_n \subset A_{n+1}$ and $A_n = A_{n+1} \upharpoonright \{2, \dots, n\}$ for any $n \geq 2$. Put $\mathcal{B} = \{A_n \mid n = 1, 2, \dots\}$ and consider the smallest algebra of fuzzy sets $\mathcal{F}$ on A containing $\mathcal{B}$. Note that

$$A \setminus A_1 = \{{}^{0.1}/1, {}^{0.6}/2, \dots\}$$

and

$$A \setminus A_n = \{{}^{0.7}/1, {}^{0.6}/n+1, {}^{0.6}/n+2, \dots\}$$

for any $n = 2, 3, \dots$ Hence, we obtain $Y(1) \in \{0, 0.1, 0.6, 0.7\}$ and $Y(n) \in \{0, 0.6\}$ for any $Y \in \mathcal{F}$ and $n = 2, 3, \dots$ Thus $R(Y) \in \mathcal{F}$, whenever $Y \in \mathcal{F}$, and the algebra $\mathcal{F}$ is R-complete. By an analogous argument to that of Example 3.6, one may show that the set $X = \{2, 3, \dots\}$ is not a support of a fuzzy set in $\mathcal{F}$, for otherwise, $\{{}^{0.7}/1\} \in \mathcal{F}$. However, this is impossible because $\{{}^{0.6}/1\}$ and $\{{}^{0.1}/1\} = A_1 \cap A \setminus A_1$ (see Theorem 3.1) are the unique fuzzy sets of $\mathcal{F}$ having the support $\{1\}$. Let $\mathcal{P}$ be the smallest algebra of sets containing all supports of fuzzy sets from $\mathcal{F}$. Obviously, $X \in \mathcal{P}$. Note that $A_i \cap A \setminus A_1$ either has the empty support or has support in the form $\{m, m+1, \dots, m+k\}$ for suitable natural numbers m and k. Since a finite union of fuzzy sets can be applied, to each fuzzy set Y with its support in X there exists a natural number n such that $Y \subset A_n$. Let us suppose that $\mathcal{P}$ is a crisp representation of the algebra of fuzzy sets $\mathcal{F}$ and that there exists $Y \in \mathcal{F}$ satisfying the conditions (i) and (ii) for X. According to the presumption, $\mathrm{Supp}(Y) \subset X$. Then, however, there exists a natural number n such that $Y \subset A_n$ and the condition (ii) fails. Hence, the algebra $\mathcal{P}$ is not a crisp representation of $\mathcal{F}$. Since $\mathcal{P}$ is the smallest algebra, the same result can be obtained for any algebra of sets containing the set of all supports of fuzzy sets from $\mathcal{F}$.

Remark 4.4. Note that the counterexample above was constructed using an algebra of fuzzy sets that has an infinite number of elements. It is easy to see that the notions of R-completeness and crisp representation are equivalent for algebras with finite numbers of elements.

Theorem 4.4. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space with $M = \mathrm{Supp}(A)$, $\mathcal{P}$ be a crisp representation of $\mathcal{F}$ and $\mu' : \mathcal{P} \to L$ be defined by*

$$\mu'(X) = \bigvee_{\substack{Y \in \mathcal{F} \\ \mathrm{Supp}(Y) \subseteq X}} \mu(Y), \tag{24}$$

Then $(M, \mathcal{P}, \mu')$ is a measure space for which the equality (23) is satisfied.

Proof. A simple consequence of Lemma 4.2 and 4.3 and the fact that $\mu'(X) \geq \mu(Y)$, whenever $\mathrm{Supp}(Y) = X$, is the inequality

$$\int_X^{\odot} f \; d\mu \leq \int_{\mathrm{Supp}(X)}^{\odot} f \; d\mu'.$$

Since $\mathcal{P}$ is a crisp representation of $\mathcal{F}$, to each set Y there exists a fuzzy set $Z \in \mathcal{F}$ satisfying (i) and (ii) of the definition. Denote such a fuzzy set by Z_Y. According to the definition of μ', we have $\mathrm{Supp}(Z_Y) \subseteq Y$ and $\mu(Z_Y) = \mu'(Y)$ for any $Y \in \mathcal{P}$. Hence, we may write

$$\int_{\mathrm{Supp}(X)}^{\odot} f \; d\mu' = \bigvee_{Y \in \mathcal{P}_X^-} \bigwedge_{m \in Y} (f(m) \odot \mu'(Y)) \leq$$
$$\bigvee_{Y \in \mathcal{P}_X^-} \bigwedge_{m \in \mathrm{Supp}(Z_Y)} (f(m) \odot \mu(Z_Y)) \leq$$
$$\bigvee_{Z \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Z)} (f(m) \odot \mu(Z)) = \int_X^{\odot} f \; d\mu$$

and the proof is finished. □

Remark 4.5. According to Remark 4.4 and the fact that each finite algebra of fuzzy sets is R-complete, we can restrict ourselves in the finite case to the algebras of sets and fuzzy measures constructed over these algebras to express an arbitrary $\odot$-fuzzy integral. This restriction is very useful from the practical point of view because all constructions and computations over sets are simpler than in the case of fuzzy sets.

4.2. A relation to the Sugeno integral

To study the relation between our concept of fuzzy integral and the Sugeno integral, let us define

$$F_a = \{m \mid m \in M \text{ and } f(m) \geq a\} \tag{25}$$

for any $a \in L$. The set F_a is an a-cut of the mapping f. Note that a mapping f in the fuzzy integral theory is $\mathcal{F}$-measurable, if $F_a \in \mathcal{F}$ for any $a \in L$ (where $L = [0, \infty]$ or $L = [0, 1]$). Since we do not suppose this type of measurability of mappings in our approach, we have to use the inner fuzzy measure μ^* determined by the original measure μ to derive the measure of fuzzy sets which are not $\mathcal{F}$-measurable, as one can see in the following theorems (cf. [17, 18]).

Theorem 4.5. *Let* $\mathbf{L}$ *be a complete divisible residuated lattice,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Dom}(A)$ *and* $f : M \to L$*. Then*

$$\int_X^\wedge f \; d\mu = \bigvee_{a \in L} (a \wedge \mu^*(X \cap 1_{F_a})), \tag{26}$$

where μ^* *is the inner fuzzy measure on* $(A, \mathcal{F}(A))$ *determined by* μ*.*

Proof. Recall that $\mathcal{F}_Y = \{Z \in \mathcal{F} \mid Z \subseteq Y\}$ and $\mu^*(Y) = \bigvee_{Z \in \mathcal{F}_Y} \mu(Z)$ (see Theorem 3.3). For simplicity, put $Z(a) = X \cap 1_{F_a}$ for any $a \in L$. One can check easily that $\bigwedge_{m \in \mathrm{Supp}(Z(a))} f(m) \geq a$. Note that $\mathrm{Supp}(Z(a)) \subseteq F_a$ and $\bigwedge \emptyset = \top$. From this and from the divisibility of $\mathbf{L}$ (recall that $\bigvee_{i \in I}(a \wedge b_i) = a \wedge \bigvee_{i \in I} b_i$ is true), we may write

$$\bigvee_{a \in L} (a \wedge \mu^*(X \cap \ 1_{F_a})) = \bigvee_{a \in L} (a \wedge \mu^*(Z(a))) \leq$$

$$\bigvee_{a \in L} ((\bigwedge_{m \in \mathrm{Supp}(Z(a))} f(m)) \wedge (\bigvee_{Y \in \mathcal{F}_{Z(a)}} \mu(Y))) =$$

$$\bigvee_{a \in L} \bigvee_{Y \in \mathcal{F}_{Z(a)}} ((\bigwedge_{m \in \mathrm{Supp}(Z(a))} f(m)) \wedge \mu(Y)) \leq$$

$$\bigvee_{a \in L} \bigvee_{Y \in \mathcal{F}_{Z(a)}} ((\bigwedge_{m \in \mathrm{Supp}(Y)} f(m)) \wedge \mu(Y)) =$$

$$\bigvee_{\substack{a \in L \\ Z(a) \neq 1_\emptyset}} \bigvee_{Y \in \mathcal{F}_{Z(a)}} ((\bigwedge_{m \in \mathrm{Supp}(Y)} f(m)) \wedge \mu(Y)) =$$

$$\bigvee_{\substack{a \in L \\ Z(a) \neq 1_\emptyset}} \bigvee_{Y \in \mathcal{F}^-_{Z(a)}} ((\bigwedge_{m \in \mathrm{Supp}(Y)} f(m)) \wedge \mu(Y)) \leq$$

$$\bigvee_{\substack{a \in L \\ Z(a) \neq 1_\emptyset}} \bigvee_{Y \in \mathcal{F}^-_X} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \wedge \mu(Y)) =$$

$$\bigvee_{Y \in \mathcal{F}^-_X} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \wedge \mu(Y)) = \int_X^\wedge f \; d\mu.$$

Note that

$$\bigvee_{\substack{a\in L\\ Z(a)=1_\emptyset}} \bigvee_{Y\in\mathcal{F}_{Z(a)}} ((\bigwedge_{m\in\mathrm{Supp}(Y)} f(m)) \wedge \mu(Y)) = \bot,$$

since $\mathcal{F}_{Z(a)} = \{1_\emptyset\}$ and $\mu(1_\emptyset) = \bot$. Therefore, this part could be omitted in the previous derivation. Furthermore, put $a_Y = \bigwedge_{m\in\mathrm{Supp}(Y)} f(m)$. It is easy to see that $\mathrm{Supp}(Y) \subseteq F_{a_Y} = \{m \mid m \in M \text{ and } f(m) \geq a_Y\}$. Hence, $\mu(Y) \leq \mu(Z(a_Y))$ for any $Y \in \mathcal{F}^-_X$. Then the opposite inequality may be derived as follows

$$\int_X^\wedge f\, d\mu = \bigvee_{Y\in\mathcal{F}^-_X} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m) \wedge \mu(Y)) =$$
$$\bigvee_{Y\in\mathcal{F}^-_X} ((\bigwedge_{m\in\mathrm{Supp}(Y)} f(m)) \wedge \mu(Y)) = \bigvee_{Y\in\mathcal{F}^-_X} (a_Y \wedge \mu(Y)) \leq$$
$$\bigvee_{Y\in\mathcal{F}^-_X} (a_Y \wedge \mu^*(Z(a_Y))) \leq \bigvee_{a\in L} (a \wedge \mu^*(Z(a))) = \bigvee_{a\in L} (a \wedge \mu^*(X \cap 1_{F_a}))$$

and the proof is finished. □

A reader can see from the previous proof that the presumption of divisibility of the residuated lattice is important for manipulations using the inner fuzzy measure μ^*. This measure can be applied to the fuzzy sets $X \cap 1_{F_a}$ that are not $\mathcal{F}$-measurable in general. The following theorem shows that if all fuzzy sets $X \cap 1_{F_a}$ are $\mathcal{F}$-measurable, then the presumption of the divisibility of the residuated lattice is redundant.[19]

Theorem 4.6. *Let* $\mathbf{L}$ *be a complete residuated lattice.* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Dom}(A)$ *and* $f : M \to L$ *with* $X \cap 1_{F_a} \in \mathcal{F}$ *for any* $a \in L$. *Then*

$$\int_X^\wedge f\, d\mu = \bigvee_{a\in L} (a \wedge \mu(X \cap 1_{F_a})). \tag{27}$$

Proof. Analogously to the proof of the previous theorem, we may write

$$\bigvee_{a\in L} (a \wedge \mu(X \cap\ 1_{F_a})) \leq \bigvee_{a\in L} (\bigwedge_{m\in\mathrm{Supp}(X\cap\ 1_{F_a})} (f(m) \wedge \mu(X \cap\ 1_{F_a})) \leq$$
$$\bigvee_{Y\in\mathcal{F}^-_X} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m) \wedge \mu(Y)) = \int_X^\wedge f\, d\mu,$$

[19]Note that in classical fuzzy measure theory, a mapping is $\mathcal{F}$-measurable if $X \cap 1_{F_a}$ is measurable for any $a \in L$. Hence, the following theorem gives a direct representation by the fuzzy (Sugeno) integral (cf. Theorem 7.1. in [18]).

since $X \cap\ 1_{F_a}$ is $\mathcal{F}$-measurable and thus $X \cap\ 1_{F_a} \in \mathcal{F}_X^-$ for any $a \in L$ and $X \cap\ 1_{F_a} \neq 1_\emptyset$ (if $X \cap\ 1_{F_a} = 1_\emptyset$, then $\mu(X \cap\ 1_{F_a}) = \bot$ and this value has no use in the computation of the integral). Denote $a(Y) = \bigwedge_{m \in \mathrm{Supp}(Y)} f(m)$. Clearly, $Y \subseteq X \cap 1_{F_{a(Y)}}$ and thus $\mu(Y) \leq \mu(X \cap 1_{F_{a(Y)}})$ for any $Y \in \mathcal{F}_X^-$. Hence, the opposite inequality may be derived as follows

$$\int_X^{\wedge} f\ d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \wedge \mu(Y)) =$$

$$\bigvee_{Y \in \mathcal{F}_X^-} ((\bigwedge_{m \in \mathrm{Supp}(Y)} f(m)) \wedge \mu(Y)) = \bigvee_{Y \in \mathcal{F}_X^-} (a(Y) \wedge \mu(Y)) \leq$$

$$\bigvee_{Y \in \mathcal{F}_X^-} (a(Y) \wedge \mu(X \cap 1_{F_{a(Y)}})) \leq \bigvee_{a \in L} (a \wedge \mu(X \cap 1_{F_a}))$$

and the proof is finished. □

Theorem 4.7. *Let* $\mathbf{L}$ *be a complete MV-algebra,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Dom}(A)$ *and* $f : M \to L$*. Then*

$$\int_X^{\otimes} f\ d\mu = \bigvee_{a \in L} (a \otimes \mu^*(X \cap 1_{F_a})), \tag{28}$$

where μ^* *is the inner fuzzy measure on* $(A, \mathcal{F}(A))$ *determined by* μ*.*

Proof. The reasoning used in the proof for the case of $\wedge$ can be applied here too, and $\bigwedge_{i \in I}(a \otimes b_i) = a \otimes \bigwedge_{i \in I} b_i$ is used. □

Remark 4.6. Unfortunately, the F-measurability of fuzzy sets $X \cap 1_{F_a}$ is not sufficient to derive a result analogous to Theorem 4.6 for the operation $\otimes$, since it does not ensure the distributivity of $\otimes$ over $\bigwedge$. Supposing that $\mathbf{L}$ is a complete MV-algebra and $X \cap 1_{F_a} \in \mathcal{F}$ for any $a \in L$, then $\mu^*(X \cap 1_{F_a})$ may be replaced by $\mu(X \cap 1_{F_a})$ in (28).

As we have mentioned above, many circumstances influence the expressibility of fuzzy integrals using sets. Two examples are described in the following corollaries. We will show that under some conditions, the presumption "to have a crisp representation" is redundant.

Corollary 4.8. *Let* $\mathbf{L}$ *be a complete divisible residuated lattice, let* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Supp}(A)$*, and let* $(M, \mathcal{P}, \mu')$ *be a fuzzy measure space such that* $\mathcal{P}$ *is an algebra of sets containing all supports of fuzzy sets of* $\mathcal{F}$ *and* $\mu'(Y)$ *is defined by (24). Then*

$$\int_X^{\wedge} f\ d\mu = \int_{\mathrm{Supp}(X)}^{\wedge} f\ d\mu' \tag{29}$$

for any $X \in \mathcal{F}$ *and* $f : M \to L$*.*

PROOF. It is sufficient to prove the equality $\mu^*(X\cap 1_{F_a}) = \mu'^*(\mathrm{Supp}(X)\cap F_a)$ for any $a \in L$. Let $a \in L$ be arbitrary. Obviously, $\mathrm{Supp}(X \cap 1_{F_a}) = \mathrm{Supp}(X) \cap F_a$. Then

$$\mu'^*(\mathrm{Supp}(X) \cap F_a) = \bigvee_{\substack{Y\in\mathcal{P}\\ Y\subseteq \mathrm{Supp}(X)\cap F_a}} \mu'(Y) =$$

$$\bigvee_{\substack{Y\in\mathcal{P}\\ Y\subseteq \mathrm{Supp}(X)\cap F_a}} \bigvee_{\substack{Z\in\mathcal{F}\\ \mathrm{Supp}(Z)\subseteq Y}} \mu(Z) = \bigvee_{\substack{Z\in\mathcal{F}\\ \mathrm{Supp}(Z)\subseteq \mathrm{Supp}(X)\cap F_a}} \mu(Z) =$$

$$\bigvee_{\substack{Z\in\mathcal{F}\\ \mathrm{Supp}(Z)\subseteq \mathrm{Supp}(X)\cap F_a}} \mu(Z) = \bigvee_{\substack{Z\in\mathcal{F}\\ Z\subseteq X\cap 1_{F_a}}} \mu(Z) = \mu^*(X \cap 1_{F_a}).$$

□

Corollary 4.9. *Let* **L** *be a complete MV-algebra,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Supp}(A)$. *Let* $(M, \mathcal{P}, \mu')$ *be a fuzzy measure space such that* $\mathcal{P}$ *is an algebra of sets containing all supports of fuzzy sets of* $\mathcal{F}$ *and* $\mu'(Y)$ *is defined by (24). Then*

$$\int_X^{\otimes} f \ d\mu = \int_{\mathrm{Supp}(X)}^{\otimes} f \ d\mu' \tag{30}$$

for any $X \in \mathcal{F}$ *and* $f : M \to L$.

PROOF. This proof is analogous to the proof of the case for $\wedge$. □

4.3. Properties of the $\odot$*-fuzzy integral*

Let us show some properties of the proposed $\odot$-fuzzy integral. The following theorem furnishes conditions under which the $\odot$-fuzzy integral defines a fuzzy measure. Note that a result analogous to that of Theorem 7.14. in [18] cannot be proved because we assume a normed measure.

Theorem 4.10. *Let* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Dom}(A)$. *If* $B \in \mathcal{F}(M)$ *such that* $A \subseteq B$, $\mathcal{G} \in \mathbf{Alg}(B)$ *and* A *is crisp, then* $\nu : \mathcal{G} \to L$ *defined by*

$$\nu(X) = \int^{\odot} X \ d\mu \tag{31}$$

is a fuzzy measure on $(B, \mathcal{G})$.

PROOF. If $X = 1_\emptyset$, then (recall that $\mathcal{F}_A^- = \mathcal{F}^-$)

$$\nu(1_\emptyset) = \bigvee_{Y\in\mathcal{F}^-} \bigwedge_{m\in\mathrm{Supp}(Y)} (1_\emptyset(m) \odot \mu(Y)) = \bigvee_{Y\in\mathcal{F}^-} \bigwedge_{m\in\mathrm{Supp}(Y)} \bot = \bot.$$

Let $X \subseteq Y$ be two $\mathcal{G}$-measurable fuzzy sets. Then

$$\nu(X) = \bigvee_{Z \in \mathcal{F}^-} \bigwedge_{m \in \mathrm{Supp}(Z)} (X(m) \odot \mu(Z)) \leq$$
$$\bigvee_{Z \in \mathcal{F}^-} \bigwedge_{m \in \mathrm{Supp}(Z)} (Y(m) \odot \mu(Z)) = \nu(Y),$$

since $X(m) \leq Y(m)$ for any $m \in M$. Finally, we may write

$$\nu(A) = \bigvee_{Y \in \mathcal{F}^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \odot \mu(Y)) = \bigvee_{Y \in \mathcal{F}^-} (A(m) \odot \mu(Y)) =$$
$$A(m) \odot \mu(A) = \top \odot \top = \top,$$

since A is a crisp fuzzy set. From this and the monotonicity of ν, we obtain that $\nu(B) = \top$ and ν is a fuzzy measure on $(B, \mathcal{G})$. □

Remark 4.7. Note that the presumption of the crispness of the fuzzy set A in the previous theorem is significant, for otherwise we can obtain $\nu(B) < \top$, in general, and ν is not a fuzzy measure according to our definition.

Lemma 4.11. *Let* $\mathbf{L}$ *be a complete linearly ordered lattice. Then*

$$\int_X^{\odot} f \; d\mu \leq a \vee \mu^*(X \cap 1_{F_a}) \tag{32}$$

holds for any $a \in L$.

Proof. For any $a \in L$, we may write

$$\int_X^{\odot} f \; d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) =$$
$$\bigvee_{\substack{Y \in \mathcal{F}_X^- \\ Y \subseteq X \cap 1_{F_a}}} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) \vee \bigvee_{\substack{Y \in \mathcal{F}_X^- \\ Y \not\subseteq X \cap 1_{F_a}}} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) \leq$$
$$\bigvee_{\substack{Y \in \mathcal{F}_X^- \\ Y \subseteq X \cap 1_{F_a}}} \mu(Y) \vee a = \bigvee_{\substack{Y \in \mathcal{F} \\ Y \subseteq X \cap 1_{F_a}}} \mu(Y) \vee a = \mu^*(X \cap 1_{F_a}) \vee a,$$

where the inequality

$$\bigvee_{\substack{Y \in \mathcal{F}_X^- \\ Y \not\subseteq X \cap 1_{F_a}}} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) \leq a$$

follows from the fact that $Y \not\subseteq X \cap 1_{F_a}$ (note that $\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X)$) implies $f(m) < a$ for some $m \in \mathrm{Supp}(Y)$ and thus $\bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) < a$. □

Let $\mathbf{L}$ be a complete residuated lattice and $a \in L$. We say that $\mathbf{L}$ is continuous from below in a if

$$\bigvee_{a' \in [\bot, a[} a' = a, \tag{33}$$

where $[\bot, a[= \{a' \mid a' \in L \text{ and } a' < a\}$.

Lemma 4.12. *Let* $\mathbf{L}$ *be a complete linearly ordered divisible residuated lattice continuous from below in* $\top$. *Then* $\int_X^{\wedge} f \; d\mu = \top$ *if and only if* $\mu^*(X \cap 1_{F_a}) = \top$ *for any* $a \in L \setminus \{\top\}$.

Proof. If $\int_X^{\wedge} f \; d\mu = \top$, then it follows from Lemma 4.11 that

$$a \vee \mu^*(X \cap 1_{F_a}) = \top.$$

Thus, if $a \in L \setminus \{\top\}$, then $\mu^*(X \cap 1_{F_a}) = \top$. If $\mu^*(X \cap 1_{F_a}) = \top$ for any $a \in L \setminus \{\top\}$, then, according to Theorem 4.5 and from the continuity of $\mathbf{L}$ in $\top$, we may write

$$\int_X^{\wedge} f \; d\mu = \bigvee_{a \in L} (a \wedge \mu^*(X \cap 1_{F_a})) \geq$$

$$\bigvee_{a \in L \setminus \{\top\}} (a \wedge \mu^*(X \cap 1_{F_a})) = \bigvee_{a \in L \setminus \{\top\}} (a \wedge \top) = \bigvee_{a \in L \setminus \{\top\}} a = \top$$

and the proof is finished. □

Lemma 4.13. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice continuous from below in* $\top$. *Then* $\int_X^{\otimes} f \; d\mu = \top$ *if and only if* $\mu^*(X \cap 1_{F_a}) = \top$ *for any* $a \in L \setminus \{\top\}$.

Proof. In this lemma, the necessity follows by analogy to Lemma 4.11. Let us suppose that $\mu^*(X \cap 1_{F_a}) = \top$ for any $a \in L \setminus \{\top\}$. Denote $a_Y = \bigwedge_{m \in \mathrm{Supp}(Y)} f(m)$. Then

$$\int_X^{\otimes} f \; d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \otimes \mu(Y)) \geq$$

$$\bigvee_{Y \in \mathcal{F}_X^-} \Big(\Big(\bigwedge_{m \in \mathrm{Supp}(Y)} f(m)\Big) \otimes \mu(Y)\Big) = \bigvee_{Y \in \mathcal{F}_X^-} (a_Y \otimes \mu(Y)) \geq$$

$$\bigvee_{a \in L \setminus \{\top\}} \bigvee_{\substack{Y \in \mathcal{F}_X^- \\ Y \subseteq X \cap 1_{F_a}}} (a \otimes \mu(Y)) = \bigvee_{a \in L \setminus \{\top\}} \Big(a \otimes \bigvee_{\substack{Y \in \mathcal{F}_X^- \\ Y \subseteq X \cap 1_{F_a}}} \mu(Y)\Big) =$$

$$\bigvee_{a \in L \setminus \{\top\}} \Big(a \otimes \bigvee_{\substack{Y \in \mathcal{F} \\ Y \subseteq X \cap 1_{F_a}}} \mu(Y)\Big) = \bigvee_{a \in L \setminus \{\top\}} (a \otimes \mu^*(X \cap 1_{F_a})) =$$

$$\bigvee_{a \in L \setminus \{\top\}} (a \otimes \top) = \top$$

and the proof is finished. □

Lemma 4.14. *Let* **L** *be a complete linearly ordered residuated lattice which is not continuous from below in* $\top$. *Then* $\int_X^{\odot} f\ d\mu = \top$ *if and only if* $\mu^*(X \cap 1_{F_\top}) = \top$.

Proof. If **L** is not continuous in $\top$, then $\bigvee_{a\in L\setminus\{\top\}} a < \top$ and, hence,

$$\int_X^{\odot} f\ d\mu = \top \le \mu^*(X\cap 1_{F_\top}) \vee \bigvee_{\substack{Y\in\mathcal{F}_X^-\\ Y\not\subseteq X\cap 1_{F_\top}}} \bigwedge_{m\in \mathrm{Supp}(Y)} (f(m)\odot\mu(Y)) \le$$

$$\mu^*(X\cap 1_{F_\top}) \vee \bigvee_{a\in L\setminus\{\top\}} a = \mu^*(X\cap 1_{F_\top}).$$

Let $\mu^*(X\cap 1_{F_\top}) = \top$. It is easy to see that $F_\top \neq \emptyset$ and there exists $Y \in \mathcal{F}_X^-$ such that $Y \subseteq X \cap 1_{F_\top}$ (otherwise, $\mu^*(X\cap 1_{F_\top}) = \bot$). Then

$$\int_X^{\odot} f\ d\mu \ge \bigvee_{\substack{Y\in\mathcal{F}_X^-\\ Y\subseteq X\cap 1_{F_\top}}} \bigvee_{m\in \mathrm{Supp}(Y)} (f(m)\odot\mu(Y)) =$$

$$\bigvee_{\substack{Y\in\mathcal{F}_X^-\\ Y\subseteq X\cap 1_{F_\top}}} \mu(Y) = \bigvee_{\substack{Y\in\mathcal{F}\\ Y\subseteq X\cap 1_{F_\top}}} \mu(Y) = \mu^*(X\cap 1_{F_\top}) = \top.$$

□

In the rest of this paragraph, let us suppose that $\mathrm{Dom}(A) = M$.

Theorem 4.15. *Let* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $X, Y \in \mathcal{F}$. *Then*

(i) $\int_{X\cup Y}^{\odot} f\ d\mu \ge \int_X^{\odot} f\ d\mu \vee \int_Y^{\odot} f\ d\mu$,

(ii) $\int_{X\cap Y}^{\odot} f\ d\mu \le \int_X^{\odot} f\ d\mu \wedge \int_Y^{\odot} f\ d\mu$.

Proof. This result follows trivially from $\mathcal{F}_A^- \subseteq \mathcal{F}_B^-$ for any $A \subseteq B$. □

Theorem 4.16. *Let* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space. Then*

(i) $\int_X^{\odot} (f\cap g)\ d\mu \le \int_X^{\odot} f\ d\mu \wedge \int_X^{\odot} g\ d\mu$,

(ii) $\int_X^{\odot} (f\cup g)\ d\mu \ge \int_X^{\odot} f\ d\mu \vee \int_X^{\odot} g\ d\mu$,

(iii) $\int_X^{\odot} (c\otimes f)\ d\mu \ge c\otimes \int_X^{\odot} f\ d\mu$,

(iv) $\int_X^{\wedge} (c\cap f)\ d\mu = c\wedge \int_X^{\wedge} f\ d\mu$,

(v) $\int_X^{\odot} (c\to f)\ d\mu \le c\to \int_X^{\odot} f\ d\mu$,

hold for any $X \in \mathcal{F}$, $f, g : M \to L$ *and* $c \in L$.

Proof. Analogously to the proof of monotonicity of ν in Theorem 4.10, we may show that $I(f) = \int_X^{\odot} f\ d\mu$ is a monotonically nondecreasing function. The statements (i) and (ii) are both simple consequences of this fact.

Let $f : M \to L$ and $c \in L$. Clearly,

$$((c \otimes a_i) \otimes b) \geq c \otimes \bigwedge_{i \in I} (a_i \otimes b)$$

for any $i \in I$ ($I \neq \emptyset$) implies $\bigwedge_{i \in I}((c \otimes a_i) \otimes b) \geq c \otimes \bigwedge_{i \in I}(a_i \otimes b)$ and

$$\bigwedge_{i \in I}((c \otimes a_i) \wedge b) \geq \big(c \otimes \bigwedge_{i \in I} a_i\big) \wedge c \otimes b \geq c \otimes \big(\bigwedge_{i \in I}(a_i \wedge b)\big).$$

Hence, we may write $\bigwedge_{i \in I}(c \otimes a_i) \odot b) \geq c \otimes \bigwedge_{i \in I}(a_i \odot b)$ for $\odot$. Then

$$\int_X^{\odot} (c \otimes f)\ d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \ \bigwedge_{m \in \mathrm{Supp}(Y)} ((c \otimes f(m)) \odot \mu(Y)) \geq$$

$$\bigvee_{Y \in \mathcal{F}_X^-} c \otimes \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m)) \odot \mu(Y)) =$$

$$c \otimes \bigvee_{Y \in \mathcal{F}_X^-} \ \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m)) \odot \mu(Y)) = c \otimes \int_X^{\odot} f\ d\mu$$

and (iii) is proved. The statement (iv) can be proved analogously, where $\bigwedge_{i \in I}((c \wedge a_i) \wedge b) = c \wedge \bigwedge_{i \in I}(a_i \wedge b)$ is used. It is easy to show that $(c \to a) \odot b \leq c \to (a \odot b)$ holds for any $a, b, c \in L$ in each residuated lattice. In fact, for example, we can write for $\odot = \wedge$

$$((c \to a) \wedge b) \otimes c \leq ((c \to a) \otimes c) \wedge (b \otimes c) \leq a \wedge b$$

and thus $(c \to a) \wedge b \leq c \to (a \wedge b)$. Hence, we obtain

$$\int_X^{\odot} (c \to f)\ d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \ \bigwedge_{m \in \mathrm{Supp}(Y)} ((c \to f(m)) \odot \mu(Y)) \leq$$

$$\bigvee_{Y \in \mathcal{F}_X^-} \ \bigwedge_{m \in \mathrm{Supp}(Y)} (c \to (f(m) \odot \mu(Y))) =$$

$$\bigvee_{Y \in \mathcal{F}_X^-} \big(c \to \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y))\big) \leq$$

$$c \to \bigvee_{Y \in \mathcal{F}_X^-} \ \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) = c \to \int_X^{\otimes} f\ d\mu$$

and (v) is proved. □

Theorem 4.17. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space, $c \in L$ and X be a set. Then* *(i)* $\int^{\otimes}(c \odot 1_X)\ d\mu = c \otimes \mu^*(1_X)$, *if* $X \subseteq \mathrm{Dom}(A)$,

(ii) $\int^{\wedge}(c \odot 1_X)\ d\mu \geq c \wedge \mu^*(1_X)$, *if* $X \subseteq \mathrm{Dom}(A)$,

(iii) $\int^{\otimes}(c \odot 1_X)\ d\mu = c \otimes \mu(1_X)$, *if* $1_X \in \mathcal{F}$,

(iv) $\int^{\wedge}(c \odot 1_X)\ d\mu \geq c \wedge \mu(1_X)$, *if* $1_X \in \mathcal{F}$,

(v) $\int^{\otimes} 1_X\ d\mu = \mu(1_X)$, *if* $1_X \in \mathcal{F}$,

(vi) $\int^{\wedge} 1_X\ d \geq \mu(1_X)$, *if* $1_X \in \mathcal{F}$,

(vii) $\int^{\odot} c\ d\mu = c$.

If $\mathbf{L}$ *is an MV-algebra, then all inequalities may be replaced by equalities.*

Proof. Let $X \subseteq \mathrm{Dom}(A)$ be an arbitrary set. Then we have

$$\int^{\otimes}(c \odot 1_X)\ d\mu = \bigvee_{\substack{Y \in \mathcal{F} \\ Y \subseteq 1_X}} \bigwedge_{m \in \mathrm{Supp}(Y)} ((c \odot 1_X(m)) \otimes \mu(Y)) =$$

$$\bigvee_{\substack{Y \in \mathcal{F} \\ Y \subseteq 1_X}} (c \otimes \mu(Y)) = c \otimes \bigvee_{\substack{Y \in \mathcal{F} \\ Y \subseteq 1_X}} \mu(Y) = c \otimes \mu^*(1_X),$$

where $\bigvee_{i \in I}(c \otimes a_i) = c \otimes \bigvee_{i \in I} a_i$ is used. Hence, (i) is proved. The statement (ii) can be proved analogously. Since $\mu^*(A) = \mu(A)$ for any $A \in \mathcal{M}$, then (iii) and (iv) are true. The statements (v) and (vi) are simple consequences of (iii) and (iv), respectively. The last statement follows immediately from (i) and from the fact that $\mu^*(1_{\mathrm{Dom}(A)}) = \mu(A) = \top$. □

Theorem 4.18. *Let* $g^{\rightarrow}$ *be an isomorphism between* $(A, \mathcal{F}, \mu)$ *and* $(B, \mathcal{G}, \nu)$, $f : \mathrm{Dom}(A) \rightarrow L$ *be a mapping and* X *be a* $\mathcal{F}$*-measurable fuzzy set. Then*

$$\int_X^{\odot} f\ d\mu = \int_{g^{\rightarrow}(X)}^{\odot} f \circ g^{-1}\ d\nu. \tag{34}$$

Proof. Obviously, if $X \in \mathcal{F}$, then $g^{\rightarrow}(X) \in \mathcal{G}$. Since $g^{\rightarrow}(X) \subseteq g^{\rightarrow}(Y)$ if and only if $X \subseteq Y$, then one can simply verify that

$$g^{\rightarrow}(\mathcal{F}_X^-) = \mathcal{G}_{g^{\rightarrow}(X)}^-.$$

If $f : M \to L$ is a mapping, then

$$\int_X^{\odot} f\ d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) =$$

$$\bigvee_{g^{\to}(Y) \in \mathcal{G}^-_{g^{\to}(X)}} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \nu(g^{\to}(Y))) =$$

$$\bigvee_{g^{\to}(Y) \in \mathcal{G}^-_{g^{\to}(X)}} \bigwedge_{g(m) \in \mathrm{Supp}(g^{\to}(Y))} (f(g^{-1}(g(m))) \odot \nu(g^{\to}(Y))) =$$

$$\bigvee_{Z \in \mathcal{G}^-_{g^{\to}(X)}} \bigwedge_{m' \in \mathrm{Supp}(Z)} ((f \circ g^{-1})(m') \odot \nu(Z)) = \int^{\otimes}_{g^{\to}(X)} f \circ g^{-1}\ d\nu.$$

□

Corollary 4.19. *Let $(A, \mathcal{F}, \mu)$ be a cardinal fuzzy measure space. Then*

$$\int_X^{\odot} f\ d\mu = \int^{\odot}_{g^{\to}(X)} f \circ g^{-1}\ d\mu \tag{35}$$

for any $X \in \mathcal{F}$, $f : M \to L$ and any permutation g on M.

Proof. This result is a simple consequence of the previous theorem and of the fact that $\mu(X) = \mu(g^{\to}(X))$ holds for any $X \in \mathcal{F}$ in an arbitrary cardinal fuzzy measure space. □

Corollary 4.20. *Let $(A, \mathcal{F}, \mu)$ be a cardinal fuzzy measure space. Then*

$$\int^{\odot} f\ d\mu = \int^{\odot} f \circ g\ d\mu \tag{36}$$

for any $f : M \to L$ and any permutation g on M.

Proof. Let g be a permutation on M and put $h = g^{-1}$. According to Lemma 3.8, A is a constant fuzzy set. Hence, $g^{\to}(A) = h^{\to}(A) = f^{\to}(A)$. By the previous corollary we may write

$$\int^{\odot} f\ d\mu = \int_A^{\odot} f\ d\mu = \int_A^{\odot} f \circ h^{-1}\ d\mu = \int_A^{\odot} f \circ g\ d\mu = \int^{\odot} f \circ g\ d\mu.$$

□

Let us suppose that $(A, \mathcal{F}, \mu)$ is a cardinal fuzzy measure space and define

$$\eta(f) = \int^{\odot} f\ d\mu.$$

The result of the previous corollary says that the mapping η is invariant under permutations on M, i.e., $\eta(f) = \eta(f \circ g)$ holds for any $f : M \to L$ and a permutation g on M. Note that if A is a crisp set, then η is invariant measure under all permutations on M and the triplet $(A, \mathcal{F}(A), \eta)$ is a cardinal measure space (see Theorem 4.10). Thus we proved that the cardinality of a fuzzy measure space is a sufficient condition which ensures the invariance of η under all permutations. An interesting question is whether the cardinality of fuzzy measure space is also a necessary condition. Unfortunately, the answer is negative. It is sufficient to consider the trivial fuzzy measure space $\mathbf{A}$ containing only $1_\emptyset$ and A, where A is not a constant fuzzy set. According to Lemma 3.8, this trivial fuzzy measure space $\mathbf{A}$ is not cardinal (A is not a constant fuzzy set), nevertheless, one can simply prove that (36) is satisfied for any f and g. On the other hand, we can prove the following interesting theorems for the both cases of $\odot$.

Theorem 4.21. *Let* $\mathbf{L}$ *be a divisible complete residuated lattice and* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space such that*

$$\int^{\wedge} f \, d\mu = \int^{\wedge} f \circ g \, d\mu \tag{37}$$

holds for any $f : M \to L$ *and any permutation* g *on* M*. Then there exists a cardinal fuzzy measure space* $(B, \mathcal{G}, \nu)$ *such that* $\mathcal{F} \subseteq \mathcal{G}$ *and*

$$\int^{\wedge} f \, d\mu = \int^{\wedge} f \, d\nu. \tag{38}$$

for any $f : M \to L$ *and any permutation* g *on* M*.*

Proof. Let $\mathbf{A} = (A, \mathcal{F}, \mu)$ be a fuzzy measure space and (37) be satisfied for any $f : M \to L$ and any permutation g on M. Put $\mathcal{A} = \{g^{\rightarrow}(\mathbf{A}) \mid g \in \mathrm{Perm}(M)\}$, where $\mathrm{Perm}(M)$ denotes the set of all permutations on M. Denote $g^{\rightarrow}(\mathbf{A}) = (A_g, \mathcal{F}_g, \mu_g)$, where clearly $A_g = g^{\rightarrow}(A)$, $\mathcal{F}_g = \{g^{\rightarrow}(X) \mid X \in \mathcal{F}\}$ and $\mu_g(g^{\rightarrow}(X)) = \mu(X)$. According to Lemma 3.6, $\mathcal{A}$ is a closed system of fuzzy measure spaces in $\mathbf{Fms}(M)$. Define a cardinal fuzzy measure space $\mathbf{B} = (B, \mathcal{G}, \nu)$ from $\mathcal{A}$ as in Lemma 3.9, where we may write

$$\nu(X) = \bigvee_{g \in \mathrm{Perm}(M)} \bigvee_{Y \in \mathcal{F}_{g,X}} \mu_g(Y). \tag{39}$$

Let us write $\int_A^{\wedge} f \, d\mu$ (and analogously for $\int_B^{\wedge} f \, d\nu$) instead of only $\int^{\wedge} f \, d\mu$ in the following part to ensure a better readability of the text. Since $\mathcal{F} \subseteq \mathcal{G}$ and $\mu(X) \leq \nu(X)$, then it trivially holds that

$$\int_A^{\wedge} f \, d\mu \leq \int_B^{\wedge} f \, d\nu.$$

On the other hand, using the divisibility of the residuated lattice, due to Corollary 4.19 and according to the presumption on the fuzzy measure space $\mathbf{A}$, we may write

$$\int_B^{\wedge} f\ d\nu = \bigvee_{Y\in\mathcal{G}^-} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m)\wedge \nu(Y)) =$$

$$\bigvee_{Y\in\mathcal{G}^-} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m)\wedge \bigvee_{g\in\mathrm{Perm}(M)} \bigvee_{Z\in\mathcal{F}_{g,Y}} \mu_g(Z)) =$$

$$\bigvee_{Y\in\mathcal{G}^-} (\bigwedge_{m\in\mathrm{Supp}(Y)} f(m))\wedge \bigvee_{g\in\mathrm{Perm}(M)} \bigvee_{Z\in\mathcal{F}_{g,Y}} \mu_g(Z)) =$$

$$\bigvee_{Y\in\mathcal{G}^-} \bigvee_{g\in\mathrm{Perm}(M)} \bigvee_{Z\in\mathcal{F}_{g,Y}} ((\bigwedge_{m\in\mathrm{Supp}(Y)} f(m))\wedge \mu_g(Z)) =$$

$$\bigvee_{Y\in\mathcal{G}^-} \bigvee_{g\in\mathrm{Perm}(M)} \bigvee_{Z\in\mathcal{F}_{g,Y}} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m)\wedge \mu_g(Z)) \leq$$

$$\bigvee_{g\in\mathrm{Perm}(M)} \bigvee_{Z\in\mathcal{F}_g} \bigwedge_{m\in\mathrm{Supp}(Y)} (f(m)\wedge \mu_g(Z)) =$$

$$\bigvee_{g\in\mathrm{Perm}(M)} \int_{g^{\to}(A)}^{\wedge} f\ d\mu_g = \bigvee_{g\in\mathrm{Perm}(M)} \int_A^{\wedge} f\circ g^{-1}\ d\mu =$$

$$\bigvee_{g\in\mathrm{Perm}(M)} \int_A^{\wedge} f\ d\mu = \int_A^{\wedge} f\ d\mu.$$

Hence, we obtain the equality (38) and the proof is finished. □

Theorem 4.22. *Let* $\mathbf{L}$ *be a complete MV-algebra and* $(A,\mathcal{F},\mu)$ *be a fuzzy measure space such that*

$$\int^{\otimes} f\ d\mu = \int^{\otimes} f\circ g\ d\mu \tag{40}$$

holds for any $f : M \to L$ *and any permutation* g *on* M*. Then there exists a cardinal fuzzy measure space* $(B,\mathcal{G},\nu)$ *such that* $\mathcal{F}\subseteq\mathcal{G}$ *and*

$$\int^{\otimes} f\ d\mu = \int^{\otimes} f\ d\nu. \tag{41}$$

for any $f : M \to L$ *and any permutation* g *on* M*.*

Proof. This proof is analogous to the proof of the case of $\wedge$. □

Remark 4.8. Considering the mentioned trivial fuzzy measure space $(A,\mathcal{F},\mu)$, where A is different from a constant fuzzy set and B is the fuzzy set used in the proof of Theorem 4.21 (and defined in Lemma 3.9), then $(B,\{1_\emptyset,B\},\nu)$ is the least fuzzy measure space which is cardinal and satisfies (41).[20] Hence, one can see that the cardinal fuzzy measure space constructed in the previous proof does not give, in general, the least cardinal fuzzy measure space which satisfies (41).

[20] Here, we consider that $(B,\mathcal{G},\nu)$ *is less than or equal to* $(B,\mathcal{G}',\nu)$, if $\mathcal{G}\subseteq\mathcal{G}'$.

4.4. Convergence theorem for $\odot$-fuzzy integrals

Fuzzy integral theory offers many results on the various types of convergence of fuzzy integrals. Two examples of such results are the Lebesgue's dominated convergence theorem and Fatou's lemma. Perhaps all results use some form of the continuity of fuzzy measures. Moreover, the convergence is usually studied in a useful subset of real numbers which have many nice properties. In our case, when fuzzy measures are not continuous and a complete residuated lattice need not be, for example, dense[21] or linearly ordered, in general, it is naturally very difficult to propose theorems analogous to the standard ones. If we concede the presumptions of continuity and null-additivity of fuzzy measures, density of complete residuated lattices (especially, of MV-algebras) and measurability of mappings $f : M \to L$, then, for example, the Lebesgue's dominated convergence theorem or Fatou's lemma can be proved (using Theorems 4.5 and 4.7). Note that the proof for $\odot = \wedge$ can be carried out analogously to the proof of Theorem 7.5 in [18]. The proof for $\odot = \otimes$ is much more complicated, since the idempotency of this operation cannot be used.[22] A similar question has been investigated in [21], where $\otimes$ is a generalized t-norm defined on $[0, \infty]$. However, this question is beyond the scope of this paper, since very careful preparation would be needed to perform the analysis. In the following paragraphs, we will present only one result on the convergence of integrals. This result will assume strong convergence of mappings. It also holds for non-continuous fuzzy measures and for non-dense residuated lattices.

Let us start with the convergence of values of a residuated lattice. Here, we replace the notion of the absolute difference of two values by their similarity, naturally represented by the biresiduum. Of course, the biresiduum of two identical elements is equal to one. Hence, we have to use the opposite sign of inequality (namely ">") in (42) in the following definition.[23]

Definition 4.3. Let $\{a_n\} \subset L$ be a sequence of elements and $b \in L$. We say that $a_1, a_2, \dots$ *converges to* b, if for any $a \in L$, $a < \top$, there exists a natural number n_0 such that

$$a_n \leftrightarrow b > a \tag{42}$$

for any $n > n_0$.

We will write $a_n \to b$, if the sequence $a_1, a_2, \dots$ converges to b. To investigate the convergence of the proposed fuzzy integral we cannot use the standard definition based on the pointwise convergence of mappings, as the following example shows.

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[21] We say that a lattice is dense if for any $a < b$ in L there is $c \in L$ with $a < c < b$.

[22] It means that $a \wedge b = b$ for $a \geq b$, but $a \otimes b \leq b$ for $a \geq b$ in general.

[23] A standard definition uses some sort of a distance for determining whether a_n is close to b. Therefore, it uses "<" instead.

Example 4.9. Let $N = \{1, 2, \dots\}$ be the set of all natural numbers and $f_n : N \to [0, 1]$ be defined by

$$f_n(m) = \min\left(1, \frac{1}{2^{n-m}}\right) \tag{43}$$

for any $m \in N$. It is easy to see that $f_n(m) = 1$ for any $m \geq n$ and $f_{n+1} \subset f_n$ for any n, i.e., $f_1, f_2, \dots$ is a non-increasing sequence. Let **L** be the Łukasiewicz algebra (i.e., $b \leftrightarrow c = 1-|b-c|$) and $a < 1$ be an arbitrary element of L. One can see that, for any $m \in N$, there exists n_0 such that $f_n(m) \leftrightarrow 0 = 1 - f_n(m) > a$ for any $n > n_0$ (this follows from the fact that $\lim_{n\to\infty} f_n(m) = 0$ for any m). Thus $f_1, f_2, \dots$ pointwise converges to f in the sense of Definition 4.3, where $f(m) = 0$ for any $m \in N$. Let $(N, \mathcal{P}(N), \mu)$ be a fuzzy measure space, where

$$\mu(Y) = \begin{cases} 0, & \text{if } Y \text{ is finite,} \\ 0.5, & \text{if } Y \text{ is infinite and } Y \neq N, \\ 1, & \text{if } Y = N. \end{cases} \tag{44}$$

Since $f(m) = 0$, then $\int^{\wedge} f \; d\mu = 0$ (see (vii) in Theorem 4.17). On the other hand, for any $n \in N$, we may construct the set $F_n = \{m \mid m \geq n\}$ of values of N for which $f(m) = 1$. Evidently, the set F_n is infinite for any m and $F_1 = N$. One can simply prove that $\int^{\wedge} f_1 \; d\mu = 1$ and $\int^{\wedge} f_n \; d\mu = 0.5$ for any $n > 1$. Hence, we obtain that the values of integrals converge to 0.5 and thus $\int^{\wedge} f_n \; d\mu \nrightarrow \int^{\wedge} f \; d\mu$.

Since we deal with a very general definition of fuzzy measure space (e.g., the continuity of the measure is not supposed), the pointwise convergence of mappings is too weak to ensure the convergence of integrals. From the example above, one can see that the convergence of mappings is too "slow" and to each $0 < a < \top$ there is no n_0 such that $f_n(m) \leftrightarrow 0 > a$ is satisfied for any $m \in N$ and $n \geq n_0$. This observation motivates us to investigate the convergence theorem for sequences of mappings that converge "rapidly" to given mappings. Such convergence will be called strong. A precise formulation is given in the following definition.

Definition 4.4. Let $\{f_n\} \subset \mathcal{F}(M)$ be a sequence of mappings and $f \in \mathcal{F}(M)$. We say that $f_1, f_2, \dots$ *strongly converges to* f, if for any $a \in L$, $a < \top$, there exists a natural number n_0 such that

$$f_n(m) \leftrightarrow f(m) > a \tag{45}$$

for any $m \in M$ and $n > n_0$.

Analogously, we will write $f_n \to f$, when the sequence $f_1, f_2, \dots$ strongly converges to f. Now, let us define

$$f \leftrightarrow g = \bigwedge_{m \in M} (f(m) \leftrightarrow g(m)) \tag{46}$$

for any mappings $f, g \in \mathcal{F}(M)$. Then a relation between the closeness of functions f, g and the closeness of values of their integrals can be expressed as follows.

Lemma 4.23. *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : \mathrm{Dom}(A) \to L$. Then*

$$\int_X^{\odot} f\, d\mu \leftrightarrow \int_X^{\odot} g\, d\mu \geq f \leftrightarrow g. \tag{47}$$

Proof. We may write

$$\int_X^{\odot} f\, d\mu \leftrightarrow \int_X^{\odot} g\, d\mu =$$

$$\bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) \leftrightarrow \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (g(m) \odot \mu(Y)) \geq$$

$$\bigwedge_{Y \in \mathcal{F}_X^-} (\bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \odot \mu(Y)) \leftrightarrow \bigwedge_{m \in \mathrm{Supp}(Y)} (g(m) \odot \mu(Y))) \geq$$

$$\bigwedge_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} ((f(m) \odot \mu(Y)) \leftrightarrow (g(m) \odot \mu(Y))) \geq$$

$$\bigwedge_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in M} (f(m) \odot \mu(Y)) \leftrightarrow (g(m) \odot \mu(Y)) \geq$$

$$\bigwedge_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in M} ((f(m) \leftrightarrow g(m)) \odot (\mu(Y) \leftrightarrow \mu(Y)))$$

$$\bigwedge_{m \in M} ((f(m) \leftrightarrow g(m)) \odot \top = f \leftrightarrow g,$$

where we use $\bigwedge_{i \in I}(a_i \leftrightarrow b_i) \leq (\bigvee_{i \in I} a_i) \leftrightarrow (\bigvee_{i \in I} b_i)$, $\bigwedge_{i \in I}(a_i \leftrightarrow b_i) \leq (\bigwedge_{i \in I} a_i) \leftrightarrow (\bigwedge_{i \in I} b_i)$ and $(a_1 \leftrightarrow b_1) \odot (a_2 \leftrightarrow b_2) \leq (a_1 \odot a_2) \leftrightarrow (b_1 \odot b_2)$ for $\odot \in \{\wedge, \otimes\}$ holding in each complete residuated lattice. □

The following theorem shows a basic type of convergence of fuzzy integrals for strongly convergent sequences of mappings.

Theorem 4.24. *If $f_n \to f$, then $\int_X^{\odot} f_n\, d\mu \to \int_X^{\odot} f\, d\mu$.*

Proof. Let $f_1, f_2, \ldots$ be a sequence with $f_n \to f$ and $a \in L$ with $a < \top$. We must distinguish two cases. First, let us suppose that there is no element $a' \in L$ for which $a < a' < \top$. Since $f_n \to f$, then there exists n_0 such that $f_n(m) \leftrightarrow f(m) > a$ for any $m \in M$ and $n > n_0$. A simple consequence of the presumption is

$$f_n(m) \leftrightarrow f(m) = \top \tag{48}$$

for any $m \in M$ and $n > n_0$. Hence, $f_n \leftrightarrow f = \top$ for any $n > n_0$ (see the definition (46)). According to Lemma 4.23, we have

$$\int_X^{\odot} f_n \, d\mu \leftrightarrow \int_X^{\odot} f \, d\mu \geq f_n \leftrightarrow f = \top > a$$

for any $n > n_0$. Hence, $\int_X^{\odot} f_n \, d\mu \to \int_X^{\odot} f \, d\mu$. Now, let us suppose that there exists $a' \in L$ such that $a < a' < \top$. Then to the value a' there is n_0 such that $f_n(m) \leftrightarrow f(m) > a'$ for any $m \in M$ and $n > n_0$. We obtain $f_n \leftrightarrow f \geq a'$ for any $n > n_0$ and, according to Lemma 4.23, we may write

$$\int_X^{\odot} f_n \, d\mu \leftrightarrow \int_X^{\odot} f \, d\mu \geq f_n \leftrightarrow f \geq a' > a$$

for any $n > n_0$. Hence, again $\int_X^{\odot} f_n \, d\mu \to \int_X^{\odot} f \, d\mu$ and the proof is finished. □

5. Conclusion

In this paper, we introduced a new type of fuzzy integral defined on an algebra of fuzzy subsets of a given fuzzy set. This definition was motivated by our research on generalized quantifiers, where such a definition turned out to be useful and natural. We studied this integral from various viewpoints, showed some of its properties and investigated how it is possible to express it using crisp sets, which allow calculations of values of these integrals to be much more effective. We also studied its relation to the Sugeno integral and, finally, presented one type of convergence theorem using the notion of strong convergence of a sequence of mappings.

In a series of subsequent papers, we will define a new way of modeling quantifiers of type $\langle 1, 1 \rangle$ using the fuzzy integral introduced here. We will also study semantic properties of these fuzzy quantifiers: permutation invariance, conservativity, extension, monotonicity, etc. Investigation of these properties will extensively use properties of fuzzy integrals introduced and proved in this paper. We believe that this class of fuzzy quantifiers (aimed especially at modeling of nondecreasing and nonincreasing quantifiers such as *many* and *few*) will prove itself to be useful and convenient in practical applications.

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