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# EQ-logics: non-commutative fuzzy logics based on fuzzy equality[☆]

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**Abstract**

In this paper, we develop a specific formal logic in which the basic connective is fuzzy equality and the implication is derived from the latter. Moreover, the fusion connective (strong conjunction) is non-commutative. We call this logic EQ-logic.

First, we formulate the basic EQ-logic which is rich enough to enjoy the completeness property. Furthermore, we introduce two extensions which seem to us interesting. The first one is IEQ-logic which is EQ-logic with double negation. The second one adds prelinearity that enables us to prove a stronger variant of the completeness property. Finally, we extend the latter logic by three more axioms including the residuation one (importation–exportation law) and prove that the resulting logic is equivalent with MTL-logic. Formal proofs in this paper proceed mostly in an equational style.

*Keywords:* EQ-algebra; residuated lattice; mathematical fuzzy logic; fuzzy equality

## 1. Introduction

When tracing back the development of logic we can distinguish two basic directions: (a) the fundamental connective is implication and the basic inference rule is modus ponens, and (b) the fundamental connective is logical equivalence (taken as an equality between truth values) and the basic inference rules are equanimity (from $A$ and $A \equiv B$ infer $B$) and Leibniz (from $A \equiv B$ infer $C[p := A] \equiv C[p := B]$) ones. Direction (a) is the more popular than (b); the latter, however, gains gradually still more and more interest, too (cf., e.g., [8, 9, 17]). We think that there are at least two main reasons for that.

First, equality (equivalence) seems to be more essential connective than implication. This idea can be traced already to G. W. Leibniz[1] (cf. [1]) and F. P. Ramsey (cf. [16]). They even tried to construct foundations of mathematics using equality only. Unfortunately, this is impossible in classical propositional or predicate logic. Surprisingly, this is possible in higher-order logic (type theory) as has been shown by L. Henkin in [10].

The second reason comes out of the idea that formal proofs can be more effectively formed in an equational style. Various arguments in favor of such approach have been given in [8], namely that the equational style makes it possible to develop and present calculations in a rigorous manner, without complexity and detail overwhelming. Consequently, proofs in this style are relatively easy to construct. Logic thus becomes a tool, rather than simply an object of study.

A natural question thus raises, how is it in fuzzy logics? Recall that the structure of truth values of the core fuzzy logics (see [4]) is extension of the MTL-algebra. If we set equivalence to be interpreted by the biresiduation ($a \leftrightarrow b = (a \to b) \wedge (b \to a)$) then we face a methodological discrepancy coming out of the natural assumption that basic connectives should be interpreted by basic operations of the corresponding algebra.

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[1]For example, Leibniz proclaimed that "a fully satisfactory logical calculus must be an equational one".

Note that when starting from fuzzy equivalence, we can derive the implication as[2]

$$A \Rightarrow B := (A \wedge B) \equiv A.$$

This gives rise to the concept of an EQ-algebra ([12, 15]) which is a lower semilattice endowed by two operations of multiplication and fuzzy equality. Its axioms reflect the basic properties which we expect from the equality to fit the supporting structure, namely the ordered set.

A challenging question is immediately raised: do EQ-algebras differ from residuated lattices, or are only a reformulation being term-equivalent to the latter; and if they differ, do they play any role in fuzzy logic where the position of residuated lattices seems to be unshakable? It turned out that EQ-algebras generalize residuated lattices, especially because they relax the tie between multiplication and residuation (i.e. between conjunction and implication in logic). Each residuated lattice gives rise to an EQ-algebra but not vice versa. At the same time, after checking properties of the implication in relation with the already developed fuzzy type theory based on EQ-algebras [14, 13], we conclude that it preserves all the essential ones (including, e.g., the exchange principle and the modus ponens). Let us stress, however, that there are EQ-algebras with the same implication but different fuzzy equalities, i.e. the fuzzy equality, in general, cannot be reconstructed from the implication (in special cases, of course, this is possible, for example, in all linearly ordered EQ-algebras).

In this paper, we develop a fuzzy-equality based logic which we call EQ-logic. The fusion connective **&** is, in general, non-commutative. First, we form a basic EQ-logic which is rich enough to enjoy the completeness property. Further, we introduce two extensions which seem to us interesting. The first one is IEQ-logic which is EQ-logic with the law of double negation. The second one adds prelinearity that enables us to prove a stronger variant of the completeness property. Finally, we extend the latter logic by three more axioms including the residuation one (importation–exportation law, cf. [11]) and prove that the resulting logic is equivalent with MTL (recall that the MTL-logic is the basis of all the core fuzzy logics; cf. [4]).

The paper is structured as follows: the next section contains overview of the concept of EQ-algebra and its main properties. In Section 3, we introduce the basic EQ-logic and show its main properties. In Section 4 we extend this logic to obtain further kinds of EQ-logics. It should be emphasized that formal proofs in this paper proceed mostly in the equational style.

## 2. EQ-algebras

In this section, we briefly overview the main concepts of EQ-algebras. We focus especially on the properties important for the development of EQ-logics.

**Definition 1**
*A non-commutative EQ-algebra $\mathcal{E}$ is an algebra of type (2, 2, 2, 0), i.e.*

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle,$$

*where for all $a, b, c, d \in E$:*

- *(E1) $\langle E, \wedge, \mathbf{1} \rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element $\mathbf{1}$). We put $a \leq b$ iff $a \wedge b = a$, as usual.*
- *(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid and $\otimes$ is isotone w.r.t. $\leq$.*
- *(E3) $a \sim a = \mathbf{1}$ (reflexivity axiom)*
- *(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution axiom)*
- *(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence axiom)*
- *(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity axiom)*

[2]This definition can be recognized already in the works of G. W. Lebniz — cf. [1].

*(E7)* $a \otimes b \leq a \sim b$ *(boundedness axiom)*

The operation $\wedge$ is called meet (infimum), $\otimes$ is called multiplication and $\sim$ is the fuzzy equality.

For all $a, b \in E$ we put

$$a \to b = (a \wedge b) \sim a \tag{1}$$

and call this operation an *implication*.

The following theorem confirms that $\sim$ has fundamental properties of the fuzzy equality, namely reflexivity, symmetry and transitivity, and that the implication is also transitive.

**Theorem 1**
*Let $\mathcal{E}$ be an EQ-algebra. The following holds for all $a, b, c \in E$:*

*(a) Symmetry:* $a \sim b = b \sim a$ ,

*(b) Transitivity:* $(a \sim b) \otimes (b \sim c) \leq a \sim c$ ,

*(c) Transitivity of implication:* $(a \to b) \otimes (b \to c) \leq a \to c$ .

*(d)* $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$.

The proof of (a)–(c) can be found in [15]. Let us notice that commutativity of $\otimes$ is not used to prove these claims. It is possible to generalize EQ-algebra even more when dropping associativity of $\otimes$. Formula (d) is the antitonicity property of implication in the first variable. This formula has been included in the original definition of EQ-algebras. In [6], it has been shown that it can be derived from the other ones and so, we can omit it from the list of axioms of EQ-algebras.

Let $\mathcal{E}$ contain also the bottom element $\mathbf{0}$. Then we put

$$\neg a = a \sim \mathbf{0}, \qquad a \in E \tag{2}$$

and call $\neg a$ a *negation* of $a \in E$.

**Definition 2**
*Let $\mathcal{E}$ be an EQ-algebra and $a, b, c, d \in E$. We say that $\mathcal{E}$ is:*

*(i)* separated *if for all $a \in E$,*

*(E8)* $a \sim b = \mathbf{1}$ *implies* $a = b$.

*(ii)* good *if*

*(E9)* $a \sim \mathbf{1} = a$.

*(iii)* residuated *if for all $a, b, c \in E$,*

*(E10)* $(a \otimes b) \wedge c = a \otimes b$ *iff* $a \wedge ((b \wedge c) \sim b) = a$.

*(iv)* involutive *(IEQ-algebra) if for all $a \in E$,*

*(E11)* $\neg\neg a = a$.

*(v)* prelinear *if for all $a, b \in E$,*

*(E12)* $\sup\{a \to b, b \to a\} = \mathbf{1}$.

*(vi)* lattice EQ-algebra *($\ell$EQ-algebra) if it is a lattice and*

*(E13)* $((a \vee b) \sim c) \otimes (d \sim a) \leq (d \vee b) \sim c$.

The following lemmas summarize some of the properties of EQ-algebras useful in the sequel. Note that their proofs do not require commutativity of $\otimes$ (see [6, 15]).

**Lemma 1**
*Let $\mathcal{E}$ be an EQ-algebra. For all $a, b \in E$ such that $a \leq b$ it holds that*

*(a)* $a \to b = \mathbf{1}\,,$

*(b)* $a \sim b = b \to a\,,$

*(c)* $c \to a \leq c \to b$ *and* $b \to c \leq a \to c\,.$

**Lemma 2**
*Let $\mathcal{E}$ be an good EQ-algebra. For all $a, b, c \in E$ it holds that*

*(a)* $a \otimes b \leq a$, $a \otimes b \leq a \wedge b$, $c \otimes (a \wedge b) \leq (c \otimes a) \wedge (c \otimes b)\,,$

*(b)* $a \sim b \leq a \to b$, $a \to a = \mathbf{1}\,,$

*(c)* $(a \to b) \otimes (b \to a) \leq a \sim b\,,$

*(d)* $a = b$ *implies* $a \sim b = \mathbf{1}\,,$

*(e)* $a = \mathbf{1} \to a$ *and* $a \to \mathbf{1} = \mathbf{1}\,,$

*(f)* $a \otimes (a \sim b) \leq b\,,$

*(g)* $b \leq a \to b\,.$

Obviously, if $\mathcal{E}$ is separated then $a \to b = \mathbf{1}$ implies $a \leq b$. Thus, it follows from Lemma 2(d) that EQ-algebra is separated iff for all $a, b \in E$ it holds that

$$a = b \quad \text{iff} \quad a \sim b = \mathbf{1}\,. \tag{3}$$

It is easy to show that *good EQ-algebras are separated.*

**Lemma 3**
*Let $\mathcal{E}$ be an IEQ-algebra. Then*

*(a)* $\mathcal{E}$ *is a good $\ell$EQ-algebra with join defined by*

$$a \vee b = \neg(\neg a \wedge \neg b).$$

*(b)* $a \sim b = \neg a \sim \neg b.$

*(c)* $a \otimes \neg a = \mathbf{0}.$

PROOF: The proof can be found in [15]. □

Similarly as in the case of residuated lattices, we say that an EQ-algebra is *representable* if it is subdirectly embeddable into a product of linearly ordered EQ-algebras. The following theorems characterize the representable class of good EQ-algebras (see [5]).

**Theorem 2 ([5])**
*Let $\mathcal{E}$ be a good EQ-algebra. Then the following properties are equivalent:*

*(a)* $\mathcal{E}$ *is representable.*

*(b)* $\mathcal{E}$ *satisfies the formula*

$$(a \to b) \vee (d \to (d \otimes (c \to (b \to a) \otimes c))) = \mathbf{1} \tag{4}$$

*for all $a, b, c, d \in E$.*

**Theorem 3 ([5])**
*If a good EQ-algebra $\mathcal{E}$ satisfies (4) then it is prelinear.*

## 3. Basic EQ-logic

In this section we introduce a propositional EQ-logic which we will call *basic*. This logic seems to be the simplest logic definable on the basis of an EQ-algebra of truth values.

**Definition 3**

*(i) The language of EQ-logic consists of propositional variables $p, q, \dots,$ binary connectives $\wedge, \&, \equiv$ and a truth (logical) constant $\top$.*

*(ii) Each propositional variable is a formula, $\top$ is a formula and if $A, B$ are formulas, then $A \wedge B$ (conjunction), $A \& B$ (fusion), and $A \equiv B$ (equality or equivalence) are formulas. Implication is defined as the following short:*

$$A \Rightarrow B := (A \wedge B) \equiv A. \tag{5}$$

The set of all formulas for the given language $J$ is denoted by $F_J$.

Let $J$ be a language of EQ-logic and $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle$ be a good non-commutative EQ-algebra (i.e. with an associative multiplication). A *truth evaluation* $e : F_J \longrightarrow E$ is defined as follows: if $p \in F_J$ is a propositional variable then $e(p) \in E$. Furthermore,

$$\begin{aligned} e(\top) &= \mathbf{1}, \\ e(A \wedge B) &= e(A) \wedge e(B), \\ e(A \& B) &= e(A) \otimes e(B), \\ e(A \equiv B) &= e(A) \sim e(B) \end{aligned}$$

for all formulas $A, B \in F_J$. A formula $A \in F_J$ is a tautology if $e(A) = \mathbf{1}$ for each truth evaluation $e : F_J \longrightarrow E$.

Notice that semantics of EQ-logic is defined using good, non-commutative, associative EQ-algebras. Despite the non-commutativity of $\otimes$, our logic has only one implication because it is derived from equivalence (i.e. its interpretation is not residual operation adjoint with multiplication). We are convinced that this is an advantage.

### *3.1. Logical axioms and inference rules*

The following formulas are axioms of the EQ-logic:

(EQ1) $(A \equiv \top) \equiv A$

(EQ2) $A \wedge B \equiv B \wedge A$

(EQ3) $(A \bigcirc B) \bigcirc C \equiv A \bigcirc (B \bigcirc C), \quad \bigcirc \in \{\wedge, \&\}$

(EQ4) $A \wedge A \equiv A$

(EQ5) $A \wedge \top \equiv A$

(EQ6) $A \& \top \equiv A$

(EQ7) $\top \& A \equiv A$

(EQ8a) $((A \wedge B) \& C) \Rightarrow (B \& C)$

(EQ8b) $(C \& (A \wedge B)) \Rightarrow (C \& B)$

(EQ9) $((A \wedge B) \equiv C) \& (D \equiv A) \Rightarrow (C \equiv (D \wedge B))$

(EQ10) $(A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$

(EQ11) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$

**Remark 1**
*The above axioms have been written using* $\Rightarrow$ *for better readability because formulas with this connective are more usual than those with* $\equiv$*. However, as we will see below, we apply equational-style proofs in the sense of [17], which are sequences of formulas of the form* $A_1 \equiv A_2, \ldots, A_{n-1} \equiv A_n$ *such that each of the individual theorems* $A_i \equiv A_{i+1}$ *has an independent individual proof. Therefore, a fussy presentation would require to rewrite all the axioms using* $\equiv$ *and the definition (5) only.*

By $C[p := A]$ we denote a formula resulting from $C$ (in the proofs below also called as a "C-part") by replacing all occurrences of a propositional variable $p$ in $C$ by the formula $A$. Then we can introduce the following inference rules of EQ-logic:

$$(EA) \quad \frac{A, A \equiv B}{B} \qquad (Leib) \quad \frac{A \equiv B}{C[p := A] \equiv C[p := B]}$$

The rule (EA) is the *equanimity rule* and (Leib) is the *Leibniz rule* for formulas (cf. [17] and elsewhere). The notion of provability is classical. A formal theory is any subset $T \subseteq F_J$. As usual, we suppose that $T$ is defined by a set of special axioms.

*3.2. Main properties*

As already indicated, we prefer equational-style of proofs to Hilbert style which seem to be more natural for EQ-logics. Note, that equational-style of proof is a sequence of formulas of the form

$$A_1 \equiv A_2, A_2 \equiv A_3, \ldots, A_{n-1} \equiv A_n, A_n \equiv A_{n+1}.$$

Each of the formulas $A_i \equiv A_{i+1}$ must be either a logical axiom, or an assumption, or it must be proved earlier, or derived using the Leibniz rule (this is often used in the proofs below). The proof consisting of a series of applications of the Leibniz inference rule is linked implicitly by the transitivity. Each step of the proof is completed by an informative annotation which explains how we arrived at the formula $A_i \equiv A_{i+1}$.

An equational proof says that its first formula is equivalent to the last one (or vice-versa). Thus, the equational proof needs not be built up to the final formula as in the case of Hilbert-style proof; whenever convenient, it can start with it and end up with some known formula. Moreover, none of the inference rules needs be mentioned explicitly in an equational proof, which reduces the amount of writing when presenting the proof. Consequently, the proofs are more concise and thus, easy to read and remember (for more details see [9] or [17]).

The following lemmas illustrate main properties of the basic EQ-logic. Their proofs are equational. However, to make the explanation concise and not overburdened with too many technical details, we will shorten the proofs whereever possible, or outline the main ideas only.

**Lemma 4**
*(a)* $\vdash A \equiv A$,

*(b)* $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)$.

PROOF: (a)

$A \wedge A \equiv A$ (EQ4)

$\Leftrightarrow \langle$(Leib) + (EQ4); "C-part": $\mathbf{p} \equiv A\rangle$

$A \equiv A$

(b)

$((A \wedge C) \equiv (A \wedge C)) \,\&\, ((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))$ (EQ9)

$\Leftrightarrow \langle$(Leib) + Lemma 6(a) + rule (T2);

"C-part": $\mathbf{p} \,\&\, ((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))\rangle$

$\top \,\&((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))$

$\Leftrightarrow \langle$(Leib) + (EQ7); "C-part": $\mathbf{p} \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))\rangle$

$((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))$

$\Leftrightarrow \langle$(Leib) + (EQ2) + (EQ3); "C-part": $((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv \mathbf{p})$

$((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge C) \wedge B))$

(i.e. $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)$)

□

**Lemma 5**
*The following are special derived rules:*

*(a)* $A \equiv \top \vdash A.$ *(rule (T1))*

*(b)* $A \vdash A \equiv \top.$ *(rule (T2))*

*(c)* $A \wedge D \equiv C, A \equiv B \vdash B \wedge D \equiv C.$ *(rule (C))*

*(d)* $(A \equiv D) \equiv C, A \equiv B \vdash (B \equiv D) \equiv C.$ *(rule (E))*

*(e)* $A \,\&\, D \equiv C, A \equiv B \vdash B \,\&\, D \equiv C.$ *(rule (F1))*

*(f)* $D \,\&\, A \equiv C, A \equiv B \vdash D \,\&\, B \equiv C.$ *(rule (F2))*

PROOF: (a) By the assumption and (EQ1) using equanimity rule.
(b)

$A \equiv \top$
$\Leftrightarrow \langle$axiom (EQ1)$\rangle$
$A$

(c)

$A \wedge D \equiv C$
$\Leftrightarrow \langle$(Leib) + assumption $A \equiv B$; "C-part": $\mathbf{p} \wedge D \equiv C\rangle$
$B \wedge D \equiv C$

(d), (e), (f) are proved in the same way as above. □

**Lemma 6**
*(a)* $A \equiv B \vdash B \equiv A,$

*(b)* $A \equiv B, B \equiv C \vdash A \equiv C,$

*(c)* $A, A \Rightarrow B \vdash B,$ *(Modus Ponens)*

*(d)* $\vdash (\top \Rightarrow A) \equiv A,$

*(e)* $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C,$

*(f)* $A, B \vdash A \,\&\, B,$

*(g)* $\vdash (A \equiv B) \equiv (B \equiv A),$

*(h)* $A \Rightarrow (B \equiv C), B \vdash A \Rightarrow C,$

(i) $\vdash (A \equiv D) \Rightarrow ((A \equiv B) \equiv (D \equiv B))$,

(j) $A \Rightarrow (B \equiv C), B \equiv D \vdash A \Rightarrow (D \equiv C)$,

(k) $A \Rightarrow (B \equiv C), C \equiv D \vdash A \Rightarrow (B \equiv D)$,

(l) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B)$,

(m) $\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D)$,

(n) $\vdash (A \Rightarrow B) \,\&\, (B \Rightarrow A) \Rightarrow (A \equiv B)$,

(o) $A \Rightarrow B, C \Rightarrow D \vdash (A \,\&\, C) \Rightarrow (B \,\&\, D)$,

(p) $A \equiv B, C \equiv D \vdash (A \,\&\, C) \equiv (B \,\&\, D)$,

(q) $\vdash ((A \equiv B) \equiv C) \,\&\, (A \equiv D) \Rightarrow ((D \equiv B) \equiv C)$,

(r) $\vdash B \Rightarrow (A \Rightarrow B)$.

PROOF:

(a)

$(A \,\&\, \top) \equiv A$ (EQ6)

$\Leftrightarrow \langle$(Leib) + assumption $A \equiv B$; "C-part": $\mathbf{p} \,\&\, \top \equiv A\rangle$

$(B \,\&\, \top) \equiv A$

$\Leftrightarrow \langle$(Leib) + (EQ6): $B \,\&\, \top \equiv B$; "C-part": $\mathbf{p} \equiv A\rangle$

$B \equiv A$

(b)

$A \equiv B$

$\Leftrightarrow \langle$(Leib) + assumption $B \equiv C$; "C-part": $A \equiv \mathbf{p}\rangle$

$A \equiv C$

(c)

$A$

$\Leftrightarrow \langle$(EQ1) + Lemma 6(a)$\rangle$

$A \equiv \top$

$\Leftrightarrow \langle$(Leib) + assumption $A \wedge B \equiv A$ + Lemma 6(a); "C-part": $\mathbf{p} \equiv \top\rangle$

$A \wedge B \equiv \top$

$\Leftrightarrow \langle$assumption $A$ + rule (T2) + rule (C)$\rangle$

$\top \wedge B \equiv \top$

$\Leftrightarrow \langle$(Leib) + axiom (EQ2): $\top \wedge B \equiv B \wedge \top$; "C-part": $\mathbf{p} \equiv \top\rangle$

$B \wedge \top \equiv \top$

$\Leftrightarrow \langle$(Leib) + axiom (EQ5): $B \wedge \top \equiv B$; "C-part": $\mathbf{p} \equiv \top\rangle$

$B \equiv \top$

$\Leftrightarrow \langle$ (EQ1) $\rangle$

$B$

(d)

$\top \wedge A \equiv \top$ (i.e. $\top \Rightarrow A$)

$\Leftrightarrow \langle$(Leib) + (EQ2) ; "C-part": $\mathbf{p} \equiv \top\rangle$

$A \wedge \top \equiv \top$

$\Leftrightarrow \langle$(Leib) + (EQ5); "C-part": $\mathbf{p} \equiv \top\rangle$

$A \equiv \top$

$\Leftrightarrow \langle$ (EQ1) $\rangle$

$A$

(e)

$A \wedge B \equiv A$ (assumption)

$\Leftrightarrow \langle$(Leib) + (EQ2); "C-part": $\mathbf{p} \equiv A\rangle$

$B \wedge A \equiv A$

$\Leftrightarrow \langle$(Leib) + assumption $B \wedge C \equiv B$ + Lemma 6(a); "C-part": $\mathbf{p} \wedge A \equiv A\rangle$

$(B \wedge C) \wedge A \equiv A$

$\Leftrightarrow \langle$(Leib) + axiom: $B \wedge C \equiv C \wedge B$; "C-part": $\mathbf{p} \wedge A \equiv A\rangle$

$(C \wedge B) \wedge A \equiv A$

$\Leftrightarrow \langle$(Leib) + axiom: $(C \wedge B) \wedge A \equiv A \wedge (C \wedge B)$ ; "C-part": $\mathbf{p} \equiv A\rangle$

$A \wedge (C \wedge B) \equiv A$

$\Leftrightarrow \langle$ axiom: $A \wedge (C \wedge B) \equiv A \Rightarrow (A \wedge C) \equiv A$ + Lemma 6(c) $\rangle$

$A \wedge C \equiv A$

(f)

$A$ (assumption)

$\Leftrightarrow \langle$(EQ6) + Lemma 6(a)$\rangle$

$A \,\&\, \top$

$\Leftrightarrow \langle$(Leib)+ assumption $B$ + rule (T2) + Lemma 6(a); "C-part": $A \,\&\, \mathbf{p}\rangle$

$A \,\&\, B$

(g)

$(A \equiv A) \,\&\, (B \equiv B)$ (2x Lemma 4(a) + Lemma 6(f))

$\Leftrightarrow \langle$(EQ10): $(A \equiv A) \,\&\, (B \equiv B) \Rightarrow (A \equiv B) \equiv (B \equiv A)$ + Lemma 6(c)$\rangle$

$(A \equiv B) \equiv (B \equiv A)$

(h)

$(A \wedge (B \equiv C)) \equiv A$ (assumption)

$\Leftrightarrow \langle$(Leib) + assumption B + rule (T2); "C-part": $A \wedge (\mathbf{p} \equiv C) \equiv A\rangle$

$A \wedge (\top \equiv C) \equiv A$

$\Leftrightarrow \langle$(Leib) + Lemma 6(g): $(\top \equiv C) \equiv (C \equiv \top)$ ; "C-part": $A \wedge \mathbf{p} \equiv A\rangle$

$A \wedge (C \equiv \top) \equiv A$

$\Leftrightarrow \langle$(Leib) + axiom: $(C \equiv \top) \equiv C$; "C-part": $A \wedge \mathbf{p} \equiv A\rangle$

$A \wedge C \equiv A$

(i)

$(A \equiv D) \,\&\, (B \equiv B) \Rightarrow (A \equiv B) \equiv (B \equiv D)$ (EQ10)

$\Leftrightarrow \langle$(Leib) + Lemma 4(a): $B \equiv B$ + rule (T2);
"C-part": $(A \equiv D) \,\&\, \mathbf{p} \Rightarrow ((A \equiv B) \equiv (B \equiv D))\rangle$

$(A \equiv D) \,\&\, \top \Rightarrow ((A \equiv B) \equiv (B \equiv D))$

$\Leftrightarrow \langle$(Leib) + axiom (EQ6): $(A \equiv D) \,\&\, \top \equiv (A \equiv D)$;
"C-part": $\mathbf{p} \Rightarrow ((A \equiv B) \equiv (B \equiv D))\rangle$

$(A \equiv D) \Rightarrow ((A \equiv B) \equiv (B \equiv D))$

$\Leftrightarrow \langle$(Leib) + Lemma 6(g): $(B \equiv D) \equiv (D \equiv B)$;
"C-part": $(A \equiv D) \Rightarrow ((A \equiv B) \equiv \mathbf{p})\rangle$

$(A \equiv D) \Rightarrow ((A \equiv B) \equiv (B \equiv D))$

(j)

$(A \wedge (B \equiv C)) \equiv A$ (assumption)

$\Leftrightarrow \langle$(Leib) + assumption $B \equiv D$; "C-part": $(A \wedge (\mathbf{p} \equiv C)) \equiv A\rangle$

$(A \wedge (D \equiv C)) \equiv A$

(k) Can be proven in a similar way as above.
(l) Using (EQ9), we find

$$\vdash (((A \wedge A) \equiv A) \,\&\, (B \equiv A)) \wedge (A \equiv (B \wedge A)) \equiv ((A \wedge A) \equiv A) \,\&\, (B \equiv A).$$

Applying the Leibniz rule, (EQ4) with rule (T2) and only then (EQ7), we obtain

$$\vdash (B \equiv A) \wedge (A \equiv (B \wedge A)) \equiv (B \equiv A).$$

Finally use Lemma 6(g) and (EQ2) with the Leibniz rule. The definition of $\Rightarrow$ then yields (l).

(m) Follows immediately from (EQ10) using the Leibniz rule and Lemma 6(g) in the form $\vdash (D \equiv B) \equiv (B \equiv D)$.

(n) Using Lemma 6(m) in the form

$$\vdash ((A \wedge B) \equiv A) \,\&\, ((B \wedge A) \equiv B) \Rightarrow ((A \wedge B) \equiv (B \wedge A)) \equiv (A \equiv B)$$

and (EQ2) as assumptions in Lemma 6(h) we obtain

$$\vdash ((A \wedge B) \equiv A) \,\&\, ((B \wedge A) \equiv B) \Rightarrow (A \equiv B)$$

and thus

$$\vdash (A \Rightarrow B) \,\&\, (B \Rightarrow A) \Rightarrow (A \equiv B).$$

(o) First, we use (EQ8a) $\vdash ((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C)$ and the Leibniz rule with the assumption $\vdash (A \wedge B) \equiv A$ to obtain $\vdash (A \,\&\, C) \Rightarrow (B \,\&\, C)$. Then we use (EQ8b) $\vdash (B \,\&(C \wedge D)) \Rightarrow (B \,\&\, D)$ and the Leibniz rule again with the second assumption $\vdash (C \wedge D) \equiv C$ to obtain $\vdash (B \,\&\, C) \Rightarrow (B \,\&\, D)$. Finally, we use these results as the assumptions in Lemma 6(e) to obtain $\vdash (A \,\&\, C) \Rightarrow (B \,\&\, D)$.

(p) From the assumptions, Lemma 6(l) and (c) we obtain $\vdash A \Rightarrow B$ and $\vdash C \Rightarrow D$ and thus, from Lemma 6(o) we find $\vdash (A \,\&\, C) \Rightarrow (B \,\&\, D)$. In the same way and, moreover, using Lemma 6(g) we obtain $\vdash B \Rightarrow A$ and $\vdash D \Rightarrow C$ and thus, from Lemma 6(o) we get $\vdash (B \,\&\, D) \Rightarrow (A \,\&\, C)$. Finally, form the previous formulas and Lemma 6(f), (n) and (c) we have $\vdash (A \,\&\, C) \equiv (B \,\&\, D)$.

(q) Using the assumptions $\vdash ((A \equiv B) \equiv C) \Rightarrow ((A \equiv B) \equiv C)$ (EQ4) and $\vdash (A \equiv D) \Rightarrow ((A \equiv B) \equiv (D \equiv B))$ (Lemma 6(i)) in Lemma 6(o) we obtain

$$\vdash (((A \equiv B) \equiv C) \,\&(A \equiv D)) \Rightarrow ((((A \equiv B) \equiv C) \,\&((A \equiv B) \equiv (D \equiv B))),$$

which together with axiom (EQ10) in the form

$$\vdash (((A \equiv B) \equiv C) \,\&((A \equiv B) \equiv (D \equiv B))) \Rightarrow (((A \equiv B) \equiv (A \equiv B)) \equiv ((D \equiv B) \equiv C))$$

yields (by Lemma 6(e)) the formula

$$\vdash (((A \equiv B) \equiv C) \,\&(A \equiv D)) \Rightarrow (((A \equiv B) \equiv (A \equiv B)) \equiv ((D \equiv B) \equiv C)).$$

Now we use Lemma 6(h) with the assumption of Lemma 4(a) and formula above to get

$$\vdash ((A \equiv B) \equiv C) \,\&(A \equiv D) \Rightarrow ((D \equiv B) \equiv C).$$

(r) Follows from Lemma 4(b) in the form

$$\vdash (\top \Rightarrow B) \Rightarrow ((\top \wedge A) \Rightarrow B)$$

and from multiple using the Leibniz rule with the obvious assumptions. □

**Lemma 7**

*(a)* $\vdash (A \equiv B) \,\&(B \equiv C) \Rightarrow (A \equiv C)$,

*(b)* $\vdash (A \,\&(A \equiv B)) \Rightarrow B$,

*(c)* $\vdash (A \Rightarrow B) \,\&(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$,

*(d)* $\vdash (A \,\&(A \Rightarrow B)) \Rightarrow B$,

*(e)* $\vdash (A \,\&\, B) \Rightarrow A$,

*(f)* $\vdash (A \,\&\, B) \Rightarrow B$,

*(g)* $\vdash B \Rightarrow (B \equiv \top)$,

*(h)* $A \Rightarrow (B \Rightarrow C) \vdash (A \,\&\, B) \Rightarrow C$,

*(i)* $\vdash A \,\&\, B \Rightarrow A \equiv B$,

*(j)* $\vdash (A \wedge B) \Rightarrow A$,

*(k)* $\vdash (C \Rightarrow A) \,\&(C \Rightarrow B) \Rightarrow (C \Rightarrow (A \wedge B))$,

*(l)* $\vdash (A \,\&\, B) \Rightarrow (A \wedge B)$,

*(m)* $\vdash (A \equiv B) \Rightarrow ((A \Rightarrow B) \wedge (B \Rightarrow A))$,

*(n)* $\vdash A \Rightarrow ((A \equiv B) \equiv B)$,

*(o)* $\vdash A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$.

PROOF: (a) Follows from (EQ9) in the form

$$\vdash ((B \wedge \top) \equiv A) \,\&\, (C \equiv B) \Rightarrow (A \equiv (C \wedge \top))$$

and by clear modifications using the Leibniz rule.

(b) Using Lemma 7(a) in the form

$$\vdash (\top \equiv A) \,\&\, (A \equiv B) \Rightarrow (\top \equiv B),$$

and double Leibniz rule with (EQ1) as assumptions.

(c) We start with (EQ9) in the form

$$\vdash ((B \wedge A) \equiv A) \,\&\, ((B \wedge C) \equiv B) \Rightarrow (A \equiv ((B \wedge C) \wedge A))$$

and by easy sequence of Leibniz rule we get formula

$$\vdash (A \Rightarrow B) \,\&\, (B \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C)),$$

which together with (EQ11) and Lemma 6(e) yields (b).

(d) Immediately from Lemma 7(c), $\vdash (\top \Rightarrow A) \,\&\, (A \Rightarrow B) \Rightarrow (\top \Rightarrow B)$ and the Leibniz rule used twice with Lemma 6(d) as the assumption.

(e) Follows from (EQ8b) in a form $\vdash (A \,\&\, (B \wedge \top)) \Rightarrow (A \,\&\, \top)$, using double Leibniz rule with (EQ5) and (EQ6).

(f) In the same way as above using (EQ8a).

(g)

$$B \wedge B \equiv B \qquad \text{(EQ4)}$$

$$\Leftrightarrow \langle \text{ (Leib)+ (EQ1): } (B \equiv \top) \equiv B + \text{Lemma 6(a); "C-part": } (B \wedge \mathbf{p}) \equiv B \rangle$$

$$(B \wedge (B \equiv \top)) \equiv B \qquad \text{(i.e. } B \Rightarrow (B \equiv \top))$$

(h) Using the assumption and Lemma 6(r), (o) we derive

$$\vdash A \,\&\, B \Rightarrow ((B \wedge C) \equiv B) \,\&\, (B \equiv \top),$$

from here and by Lemma 7(a) in the form

$$\vdash ((B \wedge C) \equiv B) \,\&\, (B \equiv \top) \Rightarrow ((B \wedge C) \equiv \top)$$

we get

$$\vdash A \,\&\, B \Rightarrow ((B \wedge C) \equiv \top).$$

by Lemma 6(c). Now, we derive in an obvious way

$$\vdash A \,\&\, B \Rightarrow (((\top \wedge C) \wedge B) \equiv \top).$$

By (EQ11) we have

$$\vdash (((\top \wedge C) \wedge B) \equiv \top) \Rightarrow ((\top \wedge C) \equiv \top)$$

a thus

$$\vdash A \,\&\, B \Rightarrow ((\top \wedge C) \equiv \top).$$

Finally we use Lemma 6(d), $\vdash ((\top \wedge C) \equiv \top) \equiv C$, 6(j), (c) and (e).

(i) Obviously follows from (EQ10):

$$\vdash (A \equiv \top) \,\&\, (B \equiv \top) \Rightarrow (A \equiv B) \equiv (\top \equiv \top)$$

(j) From Lemma 4(a): $\vdash (A \wedge B) \equiv (A \wedge B)$ using (EQ4) and rule (C) we get $\vdash ((A \wedge A) \wedge B) \equiv (A \wedge B)$. We finish this proof by using the Leibniz rule twice and assumptions (EQ3) and (EQ2).

(k) Follows from (EQ9) in the form

$$\vdash ((C \wedge A) \equiv C)\,\&((C \wedge B) \equiv C) \Rightarrow (C \equiv ((C \wedge B) \wedge A))$$

and easy application of the Leibniz rule.

(l) First of all we use Lemma 7(e), (f) and Lemma 6(f) and then we finish this proof using Lemma 7(k) and Lemma 6(c).

(m) Follows from double using of Lemma 6(l), next Lemma 6(f), Lemma 7(k) and Lemma 6(c).

(n) We start with (EQ10) in the form $\vdash (A \equiv \top)\,\&(B \equiv B) \Rightarrow (A \equiv B) \equiv (B \equiv \top)$. By easy using Leibniz rule and (EQ1), Lemma 4(a), rule (T2) and (EQ6) we complete the proof.

(o) Using Lemma 7(n) in the form $\vdash A \Rightarrow ((A \equiv (A \wedge B)) \equiv (A \wedge B))$, definition of implication, Lemma 6(l) and (e) we obtain $\vdash A \Rightarrow ((A \Rightarrow B) \Rightarrow (A \wedge B))$. (EQ11) and Lemma 6(e) complete the proof. □

It should be noted that the commutativity of **&** in general does not hold. On the other hand, however, its commutativity holds in the following weaker form:

$$A \equiv B \vdash A \,\&\, B \equiv B \,\&\, A. \tag{6}$$

This follows from the assumption, Lemma 6(a) and (p).

**Lemma 8**
*All axioms of the basic EQ-logic are tautologies.*

PROOF: (EQ1)–(EQ7) are obviously tautologies. To verify (EQ9)–(EQ11) observe (E4)–(E7), Lemma 1(a) and also Theorem 1(a) in the proof of (EQ10). To verify (EQ8a) and (EQ8b) we use the property $a \wedge b \leq b$, axiom (E2) and Lemma 1(a). □

**Lemma 9**
*The inference rules of basic EQ-logic are sound in the following sense: Let $e : F_J \longrightarrow E$ be a truth evaluation where $E$ is a support of a good non-commutative EQ-algebra:*

*(a) If $e(A) = \mathbf{1}$ and $e(A \equiv B) = \mathbf{1}$ then $e(B) = \mathbf{1}$.*

*(b) If $e(B \equiv C) = \mathbf{1}$ then $e(A[p := B] \equiv A[p := C]) = \mathbf{1}$ for any formula $A$.*

PROOF: (a) If $a = 1$ and $a \sim b = 1$ then necessarily $b = 1$.

(b) By induction on the complexity of the formula $A$. If $A$ is either $\top$ or $q$ (other than $p$) then $e(A[p := B] \equiv A[p := C]) = e(A \equiv A) = \mathbf{1}$. If on the other hand $A$ is $p$ then $e(A[p := B] \equiv A[p := C]) = e(B \equiv C) = \mathbf{1}$.

For induction step, we choose an arbitrary nonatomic $A$ and prove

$$e(A[p := B] \equiv A[p := C]) = \mathbf{1}$$

that is

$$e(A[p := B]) \sim e(A[p := C]) = \mathbf{1}$$

and thus, the using property (3) and the fact that every good algebra is separated we conclude

$$e(A[p := B]) = e(A[p := C]) \tag{7}$$

using the induction hypothesis (I.H.) that the claim $e(D[p := B]) = e(D[p := C])$ is true for all formulae less complex than $A$.

Let $A$ be $E \wedge F$ and I.H. applies to $E$ and $F$. Now, $A[p := B] \equiv A[p := C])$ implies $e(E[p := B] \wedge F[p := B] \equiv E[p := C] \wedge F[p := C])$ and thus, we get (7) as follows:

$$\begin{aligned} e(E[p := B] \wedge F[p := B]) &= e(E[p := B]) \wedge e(F[p := B]) \\ &= e(E[p := C]) \wedge e(F[p := C]) \text{ (by I.H.)} \\ &= e(E[p := C] \wedge F[p := C]) \end{aligned}$$

The cases where $A$ is $E \mathbin{\&} F$ or $E \equiv F$ are proved analogously. □

The following is the standard Lindenbaum-Tarski technique.

**Definition 4**
*Put*

$$A \approx B \quad \text{iff} \quad \vdash A \equiv B, \quad A, B \in F_J.$$

*It follows from Lemmas 4(a), (g) and 7(a) that $\approx$ is an equivalence on $F_J$. Let us denote by $[A]$ an equivalence class of $A$ and put $\bar{E} = \{[A] \mid A \in F_J\}$. Finally we define*

$$\begin{aligned} \mathbf{1} &= [\top], \\ [A] \wedge [B] &= [A \wedge B], \\ [A] \otimes [B] &= [A \mathbin{\&} B], \\ [A] \sim [B] &= [A \equiv B]. \end{aligned}$$

**Lemma 10**
*The algebra $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$ is a good non-commutative EQ-algebra.*

PROOF: The "goodness property" follows from (EQ1). For the properties of $\wedge$ see axioms (EQ2)–(EQ5). Note that we have

$$\begin{aligned} [A] \leq [B] \quad \text{iff} \quad [A] \wedge [B] = [A] \quad \text{iff} \quad &\vdash (A \wedge B) \equiv A \\ \text{iff} \quad &\vdash A \Rightarrow B \quad \text{iff} \quad \vdash (A \Rightarrow B) \equiv \top \quad \text{iff} \quad [A] \to [B] = [\top]. \end{aligned}$$

For the associativity and isotonicity of $\otimes$, and existence of the neutral element see axioms (EQ3) and (EQ6)–(EQ8b). Axiom (E3) follows from Lemma 4(a), (E4) from (EQ9), (E5) from Lemma 6(m), (E6) from (EQ11) and (E7) from Lemma 7(i). □

**Theorem 4 (Soundness)**
*The basic EQ-logic is sound.*

PROOF: This is a consequence of Lemmas 8 and 9. □

**Theorem 5 (Completeness)**
*The following is equivalent for every formula $A$:*

*(a)* $\vdash A$,

*(b)* $e(A) = \mathbf{1}$ *for every good non-commutative EQ-algebra $\mathcal{E}$ and a truth evaluation $e : F_J \longrightarrow E$.*

PROOF: The implication (a) to (b) is soundness.

(b) to (a): By Lemma 10 the algebra $\bar{\mathcal{E}}$ of equivalence classes of formulas is a good non-commutative EQ-algebra. Thus, if (b) holds then it holds also for $e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T1). □

## 4. Other EQ-logics

In this section will turn our attention to some more special propositional EQ-logics, namely involutive EQ-logic (IEQ-logic), prelinear logic and finally residuated EQ-logic, which is equivalent to MTL-logic.

### *4.1. Involutive EQ-logic*

This logic is characteristic by keeping the law of double negation and thus, the contraposition property[3]. Therefore, we modify the language $J$ of the basic EQ-logic by replacing the logical constant $\top$ by $\bot$ (falsum). Furthermore, we introduce the following shorts of formulas:

$$\top := \bot \equiv \bot, \tag{8}$$
$$\neg A := A \equiv \bot, \tag{9}$$
$$A \vee B := \neg(\neg A \wedge \neg B). \tag{10}$$

Formula (9) is the definition of *negation* and (10) the definition of *disjunction* in this logic.

Logical axioms of IEQ-logic are the same as in basic EQ-logic plus the following ones:

(EQ12) $(A \wedge \bot) \equiv \bot$ (ex falso quodlibet)

(EQ13) $\neg\neg A \equiv A$ (double negation)

Axiom (EQ12) characterizes the basic property of $\bot$ and we can also write it as $\bot \Rightarrow A$ .

The fact that we introduced the double negation property in this logic makes it richer. For example, the disjunction is naturally introduced by (10) and all its expected properties can be easily proved. Let us also remark that the contraposition and double negation are often used by people when reasoning.

**Lemma 11**

*(a)* $\vdash (A \equiv B) \equiv (\neg A \equiv \neg B)$,

*(b)* $\vdash ((A \vee B) \equiv C) \,\&\, (D \equiv A) \Rightarrow (D \vee B) \equiv C$.

PROOF: (a) Follows from twice using of Lemma 6(m) in the forms

$$\vdash (A \equiv B) \,\&\, (\bot \equiv \bot) \Rightarrow (A \equiv \bot) \equiv (B \equiv \bot)$$

and

$$\vdash ((A \equiv \bot) \equiv (B \equiv \bot)) \,\&\, (\bot \equiv \bot) \Rightarrow ((A \equiv \bot) \equiv \bot) \equiv ((B \equiv \bot) \equiv \bot),$$

few simple modifications and finally using Lemma 6(f), (n) and (c).

(b) Follows from axiom (EQ9) in the form

$$\vdash ((\neg A \wedge \neg B) \equiv \neg C) \,\&\, (\neg D \equiv \neg A) \Rightarrow (\neg C \equiv (\neg D \wedge \neg B))$$

using the Leibniz rule, Lemma 11(a), (EQ13) and Lemma 6(g). □

A contradiction is the formula $A \,\&\, \neg A$. Then we say that a theory $T$ is *contradictory* if $T \vdash A \,\&\, \neg A$ for some formula $A \in F_J$. The following theorem demonstrates that contradictory theories in EQ-logic behave classically.

**Theorem 6**

*A theory $T$ is contradictory iff $T \vdash A$ for all $A \in F_{J(T)}$.*

[3]Surprisingly, the corresponding algebraic operations are not equivalent in EQ-algebras, i.e. the double negation (involution) implies contraposition but not vice-versa.

PROOF: If $T$ is contradictory then $T \vdash A \mathbin{\&} \neg A$, i.e. $T \vdash A \mathbin{\&} (A \equiv \bot)$. Since $T \vdash (A \mathbin{\&} (A \equiv \bot)) \Rightarrow \bot$ (Lemma 7(b)), using (Lemma 6(c)) we obtain $T \vdash \bot$. Let $B \in F_{J(T)}$ is an arbitrary formula. Then from $T \vdash \bot$ using (EQ12) and (Lemma 6(c)) we get $T \vdash B$.

Conversely, assume $T \vdash A$ and $T \vdash \neg A$. Then by Lemma 6(f) it follows that $T \vdash A \mathbin{\&} \neg A$, thus the theory $T$ is contradictory. □

Semantics of IEQ-logic is formed by non-commutative IEQ-algebras.

**Theorem 7 (Completeness)**
*The following is equivalent for every formula A:*

*(a)* $\vdash A$,

*(b)* $e(A) = \mathbf{1}$ *for every IEQ-algebra* $\mathcal{E}$ *and a truth evaluation* $e : F_J \longrightarrow E$.

PROOF: The implication (a) to (b): It is easy to verify that axioms (EQ12)–(EQ13) are tautologies. The remaining part of proof follows from soundness of basic EQ-logic.

(b) to (a): Form algebra $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$ of equivalence classes of formulas. It is obvious $\bar{\mathcal{E}}$ is IEQ-algebra. Now, if (b) holds then it holds also for $e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T1). □

*4.2. Prelinear EQ-logic*

This logic seems to be closest to the residuated fuzzy logics in the sense that a stronger variant of the completeness theorem holds in it. The language of this logic is the same as that of basic EQ-logic extended by a short

$$A \vee B := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A). \tag{11}$$

The axioms are the same as in the basic EQ-logic plus the following one:

(EQ14) $(A \Rightarrow B) \vee (D \Rightarrow (D \mathbin{\&} (C \Rightarrow ((B \Rightarrow A) \mathbin{\&} C))))$.

Semantics of this logic is formed by good non-commutative EQ-algebras in which condition (4) is satisfied.

**Theorem 8 (Completeness)**
*For every formula* $A \in F_J$ *and for every theory* $T$ *the following is equivalent:*

*(a)* $T \vdash A$.

*(b)* $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every linearly ordered good non-commutative EQ-algebra* $\mathcal{E}$.

*(c)* $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every good non-commutative EQ-algebra* $\mathcal{E}$ *satisfying condition (4).*

PROOF: (a) $\Rightarrow$ (c): All axioms of $T$ are true in all models of T (axiom (EQ14) is a tautology because of the property (4)).

(c) $\Rightarrow$ (a): If (c) holds then it holds also for $e : F_J \longrightarrow \bar{E}$, where $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$ is a good non-commutative EQ-algebra satisfying (4) (the Lindenbaum algebra). If $e(A) = \mathbf{1}$ then $[A] = [\top]$, i.e. $T \vdash A \equiv \top$ and so, $T \vdash A$ by rule (T2).

(c) $\Rightarrow$ (b): According to Definition 4 we use the Lindenbaum-Tarski technique to construct the algebra $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$. This algebra satisfies (4) (see (EQ14)), hence it is representable by Theorem 2. Thus we also have (b) $\Rightarrow$ (c). □

*4.3. EQ(MTL)-logic*

In this subsection we demonstrate how EQ-logic can be extended to make it equivalent with MTL-logic. The language $J$ of EQ(MTL)-logic is the same as that of basic EQ-logic with the exception that $\top$ is replaced by $\bot$. The truth is defined by (8) and disjunction by (11). Its axioms are (EQ1)–(EQ11) plus the following:

(EQ15) $((A \mathbin{\&} B) \Rightarrow C) \equiv (A \Rightarrow (B \Rightarrow C))$

(EQ16) $(A \wedge \bot) \equiv \bot$

(EQ17) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$

In the following lemma, demonstrates that **&** is commutative.

**Lemma 12**

*(a)* $\vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$,

*(b)* $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$,

*(c)* $\vdash (A \mathbin{\&} B) \equiv (B \mathbin{\&} A)$.

PROOF: (a) follows from Lemma 7(c), (EQ15), Lemma 6(l) and (c).

(b) First we use Lemma 7(o), Lemma 12(a), Lemma 6(c) and we get $\vdash (((B \Rightarrow C) \Rightarrow C) \Rightarrow (A \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$. Then we use Lemma 12(a) in the form $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (((B \Rightarrow C) \Rightarrow C) \Rightarrow (A \Rightarrow C))$ and using Lemma 6(e) we obtain (b).

(c) The equivalence will obtained when proving the implications left-to right and its opposite. From Lemma 4(a) and (l) we obtain $\vdash A \mathbin{\&} B \Rightarrow A \mathbin{\&} B$. Then we use (EQ15), Lemma 6(l) and (c) to get $\vdash A \Rightarrow (B \Rightarrow A \mathbin{\&} B)$. Using Lemma 12(b), (c) and (EQ15), Lemma 6(g), (EA), Lemma 6(l) and (c) we finally get $\vdash (A \mathbin{\&} B) \Rightarrow (B \mathbin{\&} A)$. In the same way we can prove the converse implication. Both implications, Lemma 6(f), (n) and (c) complete the proof. □

**Theorem 9**

*EQ(MTL)-logic is equivalent with MTL-logic.*

PROOF: We will first show that axioms and rules of MTL-logic are provable from axioms and rules of EQ(MTL)-logic. Recall that implication is defined by (5).

Axiom (A1) follows from Lemma 7(c) using (EQ15) and Equanimity rule. We do not have to prove axiom (A2), because it is provable from the other ones (see [2]). (A3) follows from Lemma 12(c) (see also [3]). (A4) is Lemma 7(j). (A5) follows from (EQ2), Lemma 6(l) and Lemma 6(c), (A6) from Lemma 7(e), 7(d) and 6(f) and next Lemma 7(k) and 6(c), (A7a) and (A7b) from (EQ15), Lemma 6(l) and 6(c). Axiom (A8) is just (EQ17) and (A9) is (EQ16). Of course, the modus ponens (MP) is the derived rule in EQ(MTL)-logic.

Second, we will show, that axioms and inference rules of EQ(MTL)-logic are provable in MTL-logic. In the proof, we will use the properties of MTL-logic (see [7, Proposition 1]). We will refer to them by the numeration established in that paper, for example (EG9). For convenience, we have also summarized all these formulas into Appendix. Note, however, that not all properties from [7] are used below.

Recall that the role of equivalence is in MTL taken by the bi-implication $A \Leftrightarrow B := (A \Rightarrow B) \mathbin{\&} (B \Rightarrow A)$. Hence, the EQ-axioms proved below hold with respect to the latter and we take $A \equiv B := A \Leftrightarrow B$. To prove the required equivalences in the EQ-axioms, we prove the corresponding two implications and then put them together by (EG5) and twice use of (MP).

(EQ1):

(L.1) $\vdash (A \equiv \top) \Rightarrow (\top \Rightarrow A)$ ((EG25))

(L.2) $\vdash ((\top \Rightarrow A) \Rightarrow (\top \Rightarrow A)) \Rightarrow (\top \Rightarrow ((\top \Rightarrow A) \Rightarrow A))$ ((EG2))

(L.3) $\vdash (\top \Rightarrow A) \Rightarrow A$ ((EG3), L.2, (MP); (EG19), (MP))

(L.4) $\vdash (A \equiv \top) \Rightarrow A$ (L.1, L.3, (A1), 2x (MP))

For the proof of the converse implication we first need to prove the following property:

$$\vdash (A \,\&\, B) \Rightarrow (A \equiv B)$$

This immediately follows from twice use of (EG1), then (EG5) and twice (MP) and finally we use (EG7) and (MP) again, and the definition of equivalence. The second implication can be proved as follows:

(L.1) $\vdash A \Rightarrow (A \,\&\, \top)$ ((EG20), (A3), (A1), 2x (MP))

(L.2) $\vdash (A \,\&\, \top) \Rightarrow (A \equiv \top)$ (the property above)

(L.3) $\vdash A \Rightarrow (A \equiv \top)$ (L.1, L.2, (A1), 2x (MP))

(EQ2): This follows from twice use of (A5), (EG5) and twice (MP).

(EQ3): Associativity of $\wedge$ using (EG21), (EG5) and twice (MP) and associativity of **&** follows from (EG8), (EG5) and twice (MP).

(EQ4): It follows from (EG9), (EG12'), (EG5) and twice (MP).

(EQ5): We immediately get one implication from (EG9) and the second implication we get from (EG20), (A3), (EG9) applying (A1) and (MP).

(EQ6): One implication is just (A2). The converse implication follows from (EG20), (A3), (A1) and twice (MP).

(EQ7): One implication is just (EG20). The converse implication follows from (A3), (A2) and (A1) applying (MP).

(EQ8a):

(L.1) $\vdash (A \wedge B) \Rightarrow B$ ((EG9))

(L.2) $\vdash ((A \wedge B) \Rightarrow B) \Rightarrow (((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C))$ ((EG6))

(L.3) $\vdash ((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C)$ (L.1, L.2, (MP))

(EQ8b): This follows from (EQ8a) by using twice (A3), (A1) and (MP).

(EQ9): It is easy to see that using properties (EG5), (EG7) and (MP) this axiom follows from the following implications:

$$\vdash ((A \wedge B) \Rightarrow C) \,\&\, (D \Rightarrow A) \Rightarrow ((D \wedge B) \Rightarrow C),$$
$$\vdash (C \Rightarrow (A \wedge B)) \,\&\, (A \Rightarrow D) \Rightarrow (C \Rightarrow (D \wedge B)).$$

The first implication:

(L.1) $\vdash (D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow B$ ((A2), (EG9), (A1), 2x(MP))

(L.2) $\vdash ((D \wedge B) \Rightarrow D) \Rightarrow ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow (D \,\&\, (D \Rightarrow A)))$ ((EG6))

(L.3) $\vdash (D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow (D \,\&\, (D \Rightarrow A))$ ((EG9), L.2, (MP))

(L.4) $\vdash (D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow A$ (L.3, (EG4), (A1), 2x(MP))

(L.5) $\vdash ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow A) \,\&\, ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow B)$
(L.4, L.1, (EG5), 2x(MP))

(L.6) $\vdash ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow A) \wedge ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow B)$
(L.5, (EG9), (MP))

(L.7) $\vdash ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow A) \wedge ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow B) \Rightarrow$
$((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow (A \wedge B))$ ((EG12))

(L.8) $\vdash (D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow (A \wedge B)$ (L.6, L.7, (MP))

(L.9) $\vdash ((D \Rightarrow A) \,\&\, (D \wedge B)) \,\&\, ((A \wedge B) \Rightarrow C) \Rightarrow (A \wedge B) \,\&\, ((A \wedge B) \Rightarrow C)$ ((A3), L.8, (A1), 2x (MP); (EG6), (MP))

(L.10) $\vdash (((A \wedge B) \Rightarrow C) \,\&\, (D \Rightarrow A)) \,\&\, (D \wedge B) \Rightarrow C$ ((A3), L.9, (A1), 2x(MP); (EG4), (A1), 2x(MP))

(L.11) $\vdash ((A \wedge B) \Rightarrow C) \,\&\, (D \Rightarrow A) \Rightarrow ((D \wedge B) \Rightarrow C)$ (L.10, (A7b), (MP))

Now, we prove the second implication:

(L.1) $\vdash (A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow B$ ((A2), (EG9), (A1), 2x(MP))

(L.2) $\vdash ((A \wedge B) \Rightarrow A) \Rightarrow ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow (A \,\&\, (A \Rightarrow D)))$ ((EG6))

(L.3) $\vdash (A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow (A \,\&\, (A \Rightarrow D))$ ((EG9), L.2, (MP))

(L.4) $\vdash (A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow D$ (L.3, (EG4), (A1), 2x(MP))

(L.5) $\vdash ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow D) \,\&\, ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow B)$ ( L.4, L.1, (EG5), 2x(MP))

(L.6) $\vdash ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow D) \wedge ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow B)$ ( L.5, (EG9), (MP))

(L.7) $\vdash ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow D) \wedge ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow B) \Rightarrow ((A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow (D \wedge B))$ ((EG12))

(L.8) $\vdash (A \wedge B) \,\&\, (A \Rightarrow D) \Rightarrow (D \wedge B)$ (L.6, L.7, (MP))

(L.9) $\vdash (A \Rightarrow D) \Rightarrow ((A \wedge B) \Rightarrow (D \wedge B))$ ((A3), L.8, (A1), 2x(MP); (A7b), (MP))

(L.10) $\vdash ((A \wedge B) \Rightarrow (D \wedge B)) \Rightarrow ((C \Rightarrow (A \wedge B)) \Rightarrow (C \Rightarrow (D \wedge B)))$ ((EG11'))

(L.11) $\vdash (A \Rightarrow D) \Rightarrow ((C \Rightarrow (A \wedge B)) \Rightarrow (C \Rightarrow (D \wedge B)))$ (L.9, L.10, (A1), 2x (MP))

(L.12) $\vdash (C \Rightarrow (A \wedge B)) \,\&\, (A \Rightarrow D) \Rightarrow (C \Rightarrow (D \wedge B))$ (L.11, (A7a), (MP), (A3), (A1), 2x (MP))

(EQ10): Similarly as above, we prove the following two implications:

$$\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \equiv C) \Rightarrow (D \equiv B)),$$
$$\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((D \equiv B) \Rightarrow (A \equiv C)).$$

(L.1) $\vdash (B \equiv A) \,\&\, (A \equiv C) \Rightarrow (B \equiv C)$ ((EG24))

(L.2) $\vdash ((B \equiv A) \,\&\, (A \equiv C)) \,\&\, (C \equiv D) \Rightarrow (B \equiv C) \,\&\, (C \equiv D)$ (L.1, (EG6), (MP))

(L.3) $\vdash (B \equiv C) \,\&\, (C \equiv D) \Rightarrow (B \equiv D)$ ((EG24))

(L.4) $\vdash ((B \equiv A) \,\&\, (A \equiv C)) \,\&\, (C \equiv D) \Rightarrow (B \equiv D)$ (L.2, L.3, (A1), 2x(MP))

(L.5) $\vdash (B \equiv A) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow (B \equiv D)$ ((EG8), L.4, (A1), 2x(MP))

(L.6) $\vdash (A \equiv B) \Rightarrow (B \equiv A)$ ((EG24))

(L.7) $\vdash (A \equiv B) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow (B \equiv A) \,\&\, ((A \equiv C) \,\&\, (C \equiv D))$ (L.6, (EG6), (MP))

(L.8) $\vdash (A \equiv B) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow (B \equiv D)$ (L.7, L.5, (A1), 2x(MP))

(L.9) $\vdash (A \equiv B) \mathbin{\&} ((C \equiv D) \mathbin{\&} (A \equiv C)) \Rightarrow (A \equiv B) \mathbin{\&} ((A \equiv C) \mathbin{\&} (C \equiv D))$ ((A3), (EG6), (MP); 2x((A3), A1, 2x (MP)))

(L.10) $\vdash (A \equiv B) \mathbin{\&} ((C \equiv D) \mathbin{\&} (A \equiv C)) \Rightarrow (B \equiv D)$ (L.9, L.8, (A1), 2x(MP))

(L.11) $\vdash (A \equiv B) \mathbin{\&} (C \equiv D) \Rightarrow ((A \equiv C) \Rightarrow (B \equiv D))$ ((EG8), L.10, (A1), 2x(MP); (A7b), (MP))

(L.12) $\vdash ((A \equiv C) \Rightarrow (B \equiv D)) \Rightarrow ((A \equiv C) \Rightarrow (D \equiv B))$ ((EG24), (EG11'), (MP))

(L.13) $\vdash (A \equiv B) \mathbin{\&} (C \equiv D) \Rightarrow ((A \equiv C) \Rightarrow (D \equiv B))$ ((L.11, L.12, (A1), 2x(MP))

The second implication is analogous. Using (EG5), both already proved implications, (EG9), (EG12) and (MP) we get

$$\vdash (A \equiv B) \mathbin{\&} (C \equiv D) \Rightarrow (((A \equiv C) \Rightarrow (D \equiv B)) \wedge ((D \equiv B) \Rightarrow (A \equiv C))).$$

Finally, using (EG29), (EG24), (EG25), (A1) and (MP) we obtain axiom (EQ10).

(EQ11):

(L.1) $\vdash (B \wedge C) \Rightarrow B$ ((EG9))

(L.2) $\vdash ((B \wedge C) \Rightarrow B) \Rightarrow ((A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B))$ ((EG11'))

(L.3) $\vdash (A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$ (L.1, L.2, (MP))

(EQ15): This follows immediately from (A7a), (A7b), (EG5) using (MP).
(EQ16): This is just (A9).
(EQ17): This is just (A8).

Now we derive both rules of EQ(MTL)-logic. The equanimity rule immediately follows from (EG25) and the assumptions applying (MP). The Leibniz rule requires the proof by induction on the complexity of the formula $C$. Assume that $\vdash A \equiv B$. Then

1. If $C$ is either $\top$ or $q$ (other than $p$) then $C[p := A] \equiv C[p := B]$ is $C \equiv C$ which is (EG24).

2. If $C$ is $p$ then we immediately get the assumption.

3. Let $C$ be $p \mathbin{\&} D$. Then, from the assumption and (EG26) applying (MP) we have $\vdash (A \mathbin{\&} D) \equiv (B \mathbin{\&} D)$.

4. Let $C$ be $p \wedge D$. We prove two implications.

   (L.1) $\vdash (A \Rightarrow B) \Rightarrow ((A \wedge D) \Rightarrow B)$ (Lemma 4(b))

   (L.2) $\vdash (A \wedge D) \Rightarrow B$ (assumption, (EG25), (MP), L.1, (MP))

   (L.3) $\vdash ((A \wedge D) \Rightarrow B) \mathbin{\&} ((A \wedge D) \Rightarrow D)$ (L.2, (EG9), (EG5), 2x(MP))

   (L.4) $\vdash ((A \wedge D) \Rightarrow B) \wedge ((A \wedge D) \Rightarrow D)$ (L.3, (EG9), (MP))

   (L.5) $\vdash (A \wedge D) \Rightarrow (B \wedge D)$ (L.4, (EG12), (MP))

   In the same way we obtain the converse implication.

5. Let $C$ be $p \Rightarrow D$. Then from the assumption, (EG27) and using (MP) we get $\vdash (A \Rightarrow D) \equiv (B \Rightarrow D)$.

6. Let $C$ be $D \Rightarrow p$. Then from the assumption, (EG28) and using (MP) we find $\vdash (D \Rightarrow A) \equiv (D \Rightarrow B)$.

□

## 5. Conclusion

This paper is an attempt at developing many-valued (fuzzy) logics by starting with equivalence (fuzzy equality) instead of implication. The former connective thus becomes basic and so, natural style of formal proofs in EQ-logic is equational. The inference rules are equanimity and Lebniz ones while the rule of modus ponens is derived.

The set of truth values is formed by a non-commutative good EQ-algebra in which the fuzzy equality is one of the basic operations. Moreover, the algebra is not residuated. The implication operation (as well as the corresponding connective) is derived.

Unlike classical logic, which can be equivalently developed starting either by implication or by equivalence (cf. [17]), the EQ-logic is not equivalent with the residuated fuzzy logics. We have introduced four kinds of EQ-logics: the basic one and its extensions — IEQ-logic (involutive) in which the law of double negation is kept, prelinear in which the stronger completeness theorem holds and finally and EQ(MTL)-logic which is proved to be equivalent with the residuated MTL-fuzzy logic. The last kind of logic has been considered especially to underline the difference between EQ- and the residuated core logics (these are expansions of the MTL-one, cf. [4]). Note that because of the result in [2], and the fact that axioms (A1), (A4)–(A9) of the MTL-logic are provable in EQ(MTL), the latter is already commutative.

There are many open questions in this research. First of all, it is not clear whether is it possible to replace the above two inference rules by the modus ponens only (we guess that the answer is negative). Besides further study of the properties of EQ-logics, the predicate version should be developed. Surprisingly, there is already a higher-order EQ-logic developed in [14] and so, the relation among all these logics should be also studied.

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**Appendix**

The following axioms of MTL-logic are used in this article:

(A1) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

(A2) $(A \,\&\, B) \Rightarrow A$

(A3) $(A \,\&\, B) \Rightarrow (B \,\&\, A)$

(A4) $(A \wedge B) \Rightarrow A$

(A5) $(A \wedge B) \Rightarrow (B \wedge A)$

(A6) $(A \,\&(A \Rightarrow B)) \Rightarrow (A \wedge B)$

(A7a) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \,\&\, B) \Rightarrow C)$

(A7b) $((A \,\&\, B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

(A8) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$

(A9) $\bot \Rightarrow A$

The following properties of MTL-logic are used in this article.

(EG1) $\vdash A \Rightarrow (B \Rightarrow A)$

(EG2) $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$

(EG3) $\vdash A \Rightarrow A$

(EG4) $\vdash (A \,\&(A \wedge B)) \Rightarrow B$

(EG5) $\vdash A \Rightarrow (B \Rightarrow (A \,\&\, B))$

(EG6) $\vdash (A \Rightarrow B) \Rightarrow ((A \,\&\, C) \Rightarrow (B \,\&\, C))$

(EG7) $\vdash ((A \Rightarrow B) \,\&(C \Rightarrow D)) \Rightarrow ((A \,\&\, C) \Rightarrow (B \,\&\, D))$

(EG8) $\vdash ((A \,\&\, B) \,\&\, C) \Rightarrow (A \,\&(B \,\&\, C)), \vdash (A \,\&(B \,\&\, C)) \Rightarrow ((A \,\&\, B) \,\&\, C)$

(EG9) $\vdash (A \wedge B) \Rightarrow A, \vdash (A \wedge B) \Rightarrow B, \vdash (A \,\&\, B) \Rightarrow (A \wedge B)$

(EG11') $\vdash (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

(EG12) $\vdash ((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$

(EG12') $\vdash A \Rightarrow (A \wedge A)$

(EG19) $\vdash \top$

(EG20) $\vdash A \Rightarrow (\top \,\&\, A)$

(EG21) $\vdash ((A \wedge B) \wedge C) \Rightarrow (A \wedge (B \wedge C)), \vdash (A \wedge (B \wedge C)) \Rightarrow ((A \wedge B) \wedge C)$

(EG24) $\vdash A \equiv A, \vdash (A \equiv B) \Rightarrow (B \equiv A), \vdash (A \equiv B) \,\&\, (B \equiv C) \Rightarrow (A \equiv C)$

(EG25) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B), \vdash (A \equiv B) \Rightarrow (B \Rightarrow A)$

(EG26) $\vdash (A \equiv B) \Rightarrow ((A \,\&\, C) \equiv (B \,\&\, C))$

(EG27) $\vdash (A \equiv B) \Rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$

(EG28) $\vdash (A \equiv B) \Rightarrow ((C \Rightarrow A) \equiv (C \Rightarrow B))$

(EG29) $\vdash (A \equiv B) \equiv ((A \Rightarrow B) \wedge (B \Rightarrow A))$