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A Formal Theory of Generalized Intermediate Syllogisms☆

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Abstract

This paper is a continuation of the formal theory of *intermediate quantifiers* (expressions such as *most, few, almost all, a lot of, many, a great deal of, a large part of, a small part of*) introduced by Novák in [12]. The theory is a fuzzy-logic formalization of the concept introduced by Peterson in his book [17]. In this paper, we syntactically prove that 105 generalized Aristotle's syllogism introduced in Peterson's book are in our theory valid. At the same time, we also proved that various syllogisms listed there as invalid are invalid also in our theory. Therefore, we believe that our theory provides a reasonable mathematical model of the generalized syllogistics.

Key words: Generalized quantifiers; intermediate quantifiers; fuzzy type theory; evaluative linguistic expressions; Aristotle's syllogisms.

1. Introduction

This paper is a continuation of [12] where a formal theory of the, so called, *intermediate quantifiers* has been introduced. Recall that the latter are expressions such as *most, few, almost all, a lot of, many, a great deal of, a large part of, a small part of*, etc. This class of quantifiers has been deeply studied by Peterson in the book [17] from the point of view of their semantics. However, Peterson has not introduced any special formal logical system for them. Moreover, despite typically vague character of intermediate quantifiers, the proposed semantics is basically classical.

The theory proposed in [12] addresses both drawbacks. Namely, a formal theory of intermediate quantifiers has been introduced as a special theory of the fuzzy type theory (FTT) — the higher order fuzzy logic [9, 14]. The main idea consists in the assumption that intermediate quantifiers are just classical quantifiers $\forall$ or $\exists$ whose universe of quantification is modified using an evaluative linguistic expression (expressions such as "very small", "roughly big", "more or less medium", etc.). The meaning of the latter, however, is imprecise and so, the meaning of intermediate quantifiers is imprecise as well and can be modeled using fuzzy logic. Thus, intermediate quantifiers are special formulas consisting of two parts:

(i) Characterization of the size of a given fuzzy set using specific measure and some evaluative linguistic expression,

(ii) ordinary quantification (general or existential) of the resulting formula.

The formulas are constructed in a certain extension of a special formal theory T^{Ev} of FTT which describes semantics of *trichotomous evaluative linguistic expressions* (see [11]).

Recall that the main merits of our theory is relative simplicity (they are only shorts for special formulas of the already established theory), sufficient generality to encompass a wide class of generalized quantifiers, a unified definition of all of them, and the possibility to study their properties in syntax only so that we are free to introduce a variety of possible interpretations.

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One of essential contributions of the Peterson's theory is a list of generalized Aristotle's syllogisms. Namely, he introduced and informally proved validity of 105 of them. In [12], we proved that 24 of them are valid also in our theory. In this paper, we continue in proving the remaining ones so that, finally, we proved validity of all the 105 generalized syllogism. At the same time, we also proved that various syllogisms listed there as invalid are invalid also in our theory. Therefore, we believe that our theory provides a reasonable mathematical model of the generalized syllogistics. Let us also emphasize that all our proofs are syntactical and so, our theory is very general.

In view of the classical theory of generalized quantifiers, our quantifiers are of type $\langle 1, 1\rangle$ (cf. [6, 16, 18]) which are isomorphism-invariant (cf. [5, 3]). Fulfilling further general properties will be the topic of some further paper.

The structure of the paper is the following: Section 2 provides overview of the three principal components of our approach: fuzzy type theory (FTT), formal theory of trichotomous evaluative linguistic expressions, and a formal syntactical theory of fuzzy sets inside FTT. In Section 3 we review the formal theory of intermediate quantifiers and list some of their basic properties used later. Section 4 contains all 105 generalized Aristotle's syllogism together with formal proofs of their validity. Section 5 contains discussion of our theory and demonstration of its behavior on a simple model.

2. Preliminaries

In this section, we briefly overview the main points of fuzzy type theory and the formal theory of evaluative linguistic expressions. We will work within Łukasiewiczz FTT (cf. [13]).

2.1. Syntax of FTT

Types and formulas.. The basic syntactical objects of FTT are classical — see [1], namely the concepts of type and formula. The atomic types are ϵ (elements) and o (truth values). General types are denoted by Greek letters $\alpha, \beta, \ldots$. The set of all types is denoted by *Types*.

The *language* of FTT denoted by J consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ ($\alpha \in \mathit{Types}$) and auxiliary symbols λ and brackets. We will consider the following concrete special constants: $\mathbf{E}_{(o\alpha)\alpha}$ for every $\alpha \in \mathit{Types}$ and $\mathbf{D}_{(oo)}$.

Formulas[1] are formed from variables, constants (each of specific type), and the symbol λ. Thus, each formula A is assigned a type (we write A_α). A set of formulas of type α is denoted by Form_α. The set of all formulas is $\mathit{Form} = \bigcup_{\alpha \in \mathit{Types}} \mathit{Form}_\alpha$.

Recall that if $B \in \mathit{Form}_{\beta\alpha}$ and $A \in \mathit{Form}_\alpha$ then $(BA) \in \mathit{Form}_\beta$. Similarly, if $A \in \mathit{Form}_\beta$ and $x_\alpha \in J$, $\alpha \in \mathit{Types}$, is a variable then $\lambda x_\alpha\, A \in \mathit{Form}_{\beta\alpha}$. The set of all formulas is $\mathit{Form} = \bigcup_{\alpha \in \mathit{Types}} \mathit{Form}_\alpha$.

The following special formulas are defined:

(i) *Equivalence:* $\equiv \;:=\; \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha, \qquad \alpha \in \mathit{Types}.$

As usual, we will write $x_o \equiv y_o$ instead of $(\equiv y_o)x_o$ and similarly for the other formulas defined below. Note that if A_α, B_α are formulas then $(A_\alpha \equiv B_\alpha)$ is a formula of type o; if $\alpha = o$ then $\equiv$ is the logical equivalence.

(ii) *Implication:* $\Rightarrow \;:=\; \lambda x_o \lambda y_o\, (x_o \wedge y_o) \equiv x_o$

(iii) *Negation:* $\neg := \lambda x_o (x_o \equiv \bot)$.

(iv) *Strong conjunction* $\mathbin{\&} := \lambda x_o (\lambda y_o (\neg(x_o \Rightarrow \neg y_o)))$.

(v) *Conjunction:* $\wedge \;:=\; \lambda x_o (\lambda y_o (x_o \mathbin{\&} (x_o \Rightarrow y_o)))$.

(vi) *Disjunction:* $\vee \;:=\; \lambda x_o (\lambda y_o (x_o \Rightarrow y_o) \Rightarrow y_o)$.

(vii) *Delta connective:* $\boldsymbol{\Delta} \;:=\; \lambda x_o \mathbf{D}_{oo} x_o$.

[1] In the up-to-date type theory, "formulas" are quite often called "lambda-terms". We prefer the former in this paper because FTT is logic and so, the term "formula" is more natural.

Furthermore, the n-times strong conjunction of A_o is denoted by A_o^n and n-times strong disjunction (denoted by nA_o). The general ($\forall$) and existential ($\exists$) quantifiers are defined as special formulas (see [9]). The following priority of the logical connectives will sometimes be used: (1) $\neg, \boldsymbol{\Delta}$; (2) $\boldsymbol{\&}, \boldsymbol{\nabla}, \wedge, \vee$; (3) $\equiv$; (4) $\Rightarrow$.

Axioms and inference rules.. The Łukasiewiczz FTT has the following logical axioms:

(FT1) $\boldsymbol{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(FT2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$

(FT2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$

(FT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

(FT4) $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \wedge (y_o \Rightarrow x_o))$

(FT5) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon)$

(FT6) $(A_o \equiv \top) \equiv A_o$

(FT7) $x_o \Rightarrow (y_o \Rightarrow x_o)$

(FT8) $(x_o \Rightarrow y_o) \Rightarrow ((y_o \Rightarrow z_o) \Rightarrow (x_o \Rightarrow z_o))$

(FT9) $(\neg y_o \Rightarrow \neg x_o) \equiv (x_o \Rightarrow y_o)$

(FT10) $(x_o \vee y_o) \equiv (y_o \vee x_o)$

(FT11) $(g_{oo}(\boldsymbol{\Delta} x_o) \wedge g_{oo}(\neg \boldsymbol{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\boldsymbol{\Delta} y_o)$

(FT12) $\boldsymbol{\Delta}(x_o \wedge y_o) \equiv \boldsymbol{\Delta} x_o \wedge \boldsymbol{\Delta} y_o$

(FT13) $\boldsymbol{\Delta}(x_o \vee y_o) \Rightarrow \boldsymbol{\Delta} x_o \vee \boldsymbol{\Delta} y_o$

(FT14) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$

(FT15) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon$

(for the details see [9, 14]). The inference rules are the following:

(R) Infer B' from $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$ which comes from B by replacing one occurrence of A_α by A'_α (provided that A_α in B is not an occurrence of a variable immediately preceded by λ).

(N) Infer $\boldsymbol{\Delta} A_o$ from A_o.

The inference rules of modus ponens and generalization are derived rules in FTT. The concepts of *provability* and *proof* are defined in the same way as in classical logic. A *theory* T over FTT is a set of formulas of type o ($T \subset Form_o$). By $J(T)$ we denote the language of the theory T. By $T \vdash A_o$ we mean that A_o is provable in T.

Theorem 1 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \boldsymbol{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

2.2. Semantics of FTT

The truth values form an MV-algebra (see [2, 15]) extended by the delta operation, i.e. the residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta \rangle, \tag{1}$$

where $\mathbf{0}$ is the least and $\mathbf{1}$ is the greatest element. The $L = \langle L, \otimes, \mathbf{1} \rangle$ is a commutative monoid, the $\rightarrow$ is residuation fulfilling the *adjunction property*

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c,$$

and $a \vee b = (a \rightarrow b) \rightarrow b$ holds for all $a, b, c \in L$. The Δ is a unary operation fulfiling 6 special axioms (cf. [4, 9]).

A special case of the algebra of truth values is the standard Łukasiewicz$_\Delta$ algebra

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1, \Delta \rangle \tag{2}$$

where

$$\begin{aligned} \wedge &= \text{minimum}, & \vee &= \text{maximum}, \\ a \otimes b &= 0 \vee (a + b - 1), & a \rightarrow b &= 1 \wedge (1 - a + b), \\ \neg a &= a \rightarrow 0 = 1 - a, & \Delta(a) &= \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let J be a language of FTT and $(M_\alpha)_{\alpha \in Types}$ be a basic frame. The *frame* is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \tag{3}$$

so that the following holds:

(i) The $\mathcal{L}_\Delta$ is a structure of truth values (i.e., an MV-algebra). We put $M_o = L$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{L}_\Delta$.

(ii) $=_\alpha$ is a fuzzy equality on M_α and $=_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$.

A function p such that $p(x_\alpha) \in M_\alpha$, $\alpha \in Types$, is an assignment of elements from $\mathcal{M}$ to variables. The set of all assignments over $\mathcal{M}$ will be denoted by $Asg(\mathcal{M})$. An interpretation $\mathcal{M}_p$ is a function that assign every formula A_α, $\alpha \in Types$ and every assignment p a corresponding element of type α. A *general model* is a frame $\mathcal{M}$ such that for every formula A_α, $\alpha \in Types$ and every assignment $p \in Asg(\mathcal{M})$, the interpretation $\mathcal{M}_p$ gives

$$\mathcal{M}_p(A_\alpha) \in M_\alpha.$$

In the sequel we will write $\mathcal{M}(A_\alpha)$ only. We say that a frame $\mathcal{M}$ is a *model* of a theory T if all axioms are true in the degree $\mathbf{1}$ in $\mathcal{M}$. If A_o is true in the degree $\mathbf{1}$ in all models of T then we write $T \models A_o$.

The following completeness theorem can be proved (the proof is analogous to the proof of completeness given in [9, 14]).

Theorem 2 (completeness)
(a) A theory T is consistent iff it has a general model $\mathcal{M}$.

(b) For every theory T and a formula A_o

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

The following properties will be needed in the sequel.

Theorem 3 (propositional properties)
Let $A, B, C \in Form_o$. The the following is provable:

(a) $\vdash ((A \mathbin{\&} B) \Rightarrow C) \equiv (A \Rightarrow (B \Rightarrow C))$,

(b) $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$,

(c) $\vdash (A \mathbin{\&} B) \Rightarrow A, \quad \vdash (A \wedge B) \Rightarrow A$,

(d) $\vdash (A \mathbin{\&} B) \equiv (B \mathbin{\&} A)$,

(e) $\vdash (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,

(f) $\vdash (C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (B \wedge A)))$,

(g) $\vdash (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$,

(h) $\vdash (A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$,

(i) $\vdash ((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$,

(j) $\vdash (A \wedge B) \mathbin{\&} (A \wedge C) \Rightarrow (A \wedge (B \mathbin{\&} C))$,

(k) $\vdash (A \wedge (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow (A \wedge C))$,

(l) $\vdash \boldsymbol{\Delta} A \Rightarrow A$,

(m) $\vdash (A \mathbin{\&} (B \Rightarrow C)) \Rightarrow ((A \mathbin{\&} B) \Rightarrow (A \mathbin{\&} C))$.

Theorem 4 (predicate properties)
Let $A, B \in Form_o$ and $\alpha \in Types$. Then the following is provable:

(a) $\vdash (\forall x_\alpha)(A \Rightarrow B) \Rightarrow ((\forall x_\alpha)A \Rightarrow (\forall x_\alpha)B)$,

(b) $\vdash (\forall x_\alpha)(A \Rightarrow B) \Rightarrow ((\exists x_\alpha)A \Rightarrow (\exists x_\alpha)B)$,

(c) $\vdash (\exists x_\alpha)(A \mathbin{\&} B) \equiv ((\exists x_\alpha)A \mathbin{\&} B)$, x_α *is not free in* B,

(d) $\vdash (\forall x_\alpha)(A \Rightarrow B) \Rightarrow ((\exists x_\alpha)\boldsymbol{\Delta} A \Rightarrow B)$,

(e) $\vdash (\exists x_\alpha)\boldsymbol{\Delta} A \Rightarrow (\exists x_\alpha)A$.

(f) $\vdash (\forall x_\alpha)B \Rightarrow B_{x_\alpha}[A_\alpha]$,

(g) $\vdash B_{x_\alpha}[A_\alpha] \Rightarrow (\exists x_\alpha)B$,

(provided that A_α is substitutable to B for all free occurrences of x_α in (f), (g)).

Lemma 1
Let T be a theory and $\mathbf{u}_\alpha$, $\alpha \in Types$ be a new special constant such that $\mathbf{u}_\alpha \notin J(T)$. Then

$$T \vdash (\exists x_\alpha)\boldsymbol{\Delta} A \quad \textit{iff} \quad T \cup A_{x_\alpha}[\mathbf{u}_\alpha] \quad \textit{is a conservative extension of } T.$$

2.3. Trichotomous evaluative linguistic expressions

Syntactical characterization. Trichotomous evaluative linguistic expressions are expressions of natural language, for example, *small, medium, big, about fourteen, very short, more or less deep, quite roughly strong*. In this paper, we will consider only simple expressions with the following syntactical structure:

$$\langle\text{linguistic hedge}\rangle\langle\text{TE-adjective}\rangle \tag{4}$$

where ⟨TE-adjective⟩ is an evaluative adjective which includes also the class of gradable adjectives. A typical feature of TE-adjectives is that they form pairs of antonyms (e.g., small–big) completed by the middle member (medium). Canonical TE-adjectives are *small, medium, big*. In a concrete situation, of course, they can be replaced by more proper adjectives such as *short, medium short, long*, etc.

The ⟨linguistic hedge⟩ is an intensifying adverb making the meaning of the evaluative expressions either more, or less specific. We distinguish the following:

$$\langle\text{linguistic hedge}\rangle := \textsf{empty hedge} \mid \langle\text{narrowing hedge}\rangle \mid \langle\text{widening hedge}\rangle \mid \langle\text{specifying hedge}\rangle$$

Typical examples are *extremely, significantly, very* (narrowing), *more or less, roughly, quite roughly, very roughly* (widening) or *rather* (specifying). Note a special empty hedge which makes it possible to deal with evaluative expressions, e.g., "large" and "very large" in a unified way.

We will also consider *negative evaluative expressions*

$$\textsf{not}\ (\textsf{empty hedge}\langle\text{TE-adjective}\rangle). \tag{5}$$

Formalization. The meaning of evaluative expressions is formalized within a formal logical theory T^{Ev} which is a special theory of Ł-FTT. The meaning of evaluative linguistic expression is constructed as special formula representing *intension* whose interpretation in a model is a function from the set of possible worlds (in our theory, we prefer to speak about *contexts*) into a set of fuzzy sets. For each possible world, intension determines the corresponding extension, which is a fuzzy set in some universe constructed using a specific *horizon* which can be shifted along the latter. All the details the formal theory T^{Ev} including its special axioms and the motivation can be found in [11, 10].

The language of T^{Ev} contains besides the standard constants $\top, \bot$ (truth falsity) also a constant $\dagger$ which represents a middle truth value (in the standard Łukasiewicz algebra, it is interpreted by 0.5). Further constant is $\sim$ for an additional fuzzy equality on the set of truth values L. The theory T^{Ev} has 11 special axioms which characterize properties of both constants, properties of contexts (see below) and properties of special formulas which represent linguistic hedges.

By context in T^{Ev}, we understand a formula $w_{\alpha o}$ whose interpretation is a function $w : L \longrightarrow M_\alpha$. Hence, the context determines in M_α a triple of elements $\langle v_L, v_S, v_R\rangle$ where $v_L, v_S, v_R \in M_\alpha$ and $v_L = \mathcal{M}_p(w\bot)$, $v_S = \mathcal{M}_p(w\dagger)$, $v_R = \mathcal{M}_p(w\top)$.

For the theory of intermediate quantifiers, we may consider only abstract expressions such as "very small" which contain no specification of " what is indeed small"[2]. Consequently, they have only one (abstract) context and so, their intension actually coincides with their extension.

The fuzzy equality $\sim$ makes it possible to introduce three horizons:

$$\begin{aligned} LH_{oo} &:= \lambda z_o \cdot \bot \sim z_o, \\ MH_{oo} &:= \lambda z_o \cdot \dagger \sim z_o, \\ RH_{oo} &:= \lambda z_o \cdot \top \sim z_o. \end{aligned}$$

The left horizon LH is a function assigning to each z_o a truth degree of the fuzzy equality with $\bot$; similarly right RH and middle MH horizons.

A *hedge* is represented by a formula $\boldsymbol{\nu} \in \mathit{Form}_{oo}$ whose interpretation is a function ν on truth values that is monotone, sends some truth value to the top and some other truth value to the bottom, and there is an inner truth value b so that $\nu(a) \le a$ for all $a \le b$ and $a \le \nu(a)$ for all $b \le a$. Hedges can be partially ordered: $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2$ expresses that hedge $\boldsymbol{\nu}_1$ is narrower than $\boldsymbol{\nu}_2$ ($\boldsymbol{\nu}_2$ is wider than $\boldsymbol{\nu}_1$). We refer the reader to [11] for the more detailed explanation and technical details.

We introduce the following special hedges: $\{\mathit{Ex}, \mathit{Si}, \mathit{Ve}, \mathit{ML}, \mathit{Ro}, \mathit{QR}, \mathit{VR}\}$ (*extremely, significantly, very, more or less, roughly, quite roughly, very roughly*, respectively) which are ordered as follows:

$$\mathit{Ex} \preceq \mathit{Si} \preceq \mathit{Ve} \preceq \bar{\boldsymbol{\nu}} \preceq \mathit{ML} \preceq \mathit{Ro} \preceq \mathit{QR} \preceq \mathit{VR} \tag{6}$$

The hedges $\mathit{Ex}, \mathit{Si}, \mathit{Ve}$ have narrowing effect with respect to the empty hedge and $\mathit{ML}, \mathit{Ro}, \mathit{QR}, \mathit{VR}$ have widening effect; the $\bar{\boldsymbol{\nu}}$ is empty hedge.

The following formulas represent intensions of simple evaluative expressions (4):

[2]For example, "very small animal" suggests considering various sizes of animals depending on the context. Expressions of the form '$\mathcal{A}$ is ⟨noun⟩' where $\mathcal{A}$ is an evaluative expression, are called *evaluative (linguistic) predications.*

(i) $Sm := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(LH\, z)$,

(ii) $Me := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(MH\, z)$,

(iii) $Bi := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(RH\, z)$.

Note that the structure of these formulas represents construction of the corresponding extensions, whose interpretation in a model is schematically depicted in Figure 1. To simplify the explanation, we will often use a general metavariable *Ev* standing for intensions (i)–(iii) above.

A specific role in this theory is played by the formulas $Sm\,\boldsymbol{\Delta}$, $Me\,\boldsymbol{\Delta}$, $Bi\boldsymbol{\Delta}$ where the connective $\boldsymbol{\Delta}$ has been used as a specific hedge that can be taken as a linguistic hedge "utmost" (or, alternatively a "limit"). This makes it possible to include in our theory also classical quantifiers without necessity to introduce them as special case different from the rest of the theory.

A *canonical model* of T^{Ev} is based on a frame

$$\mathcal{M}^0 = \langle (M_\gamma, =_\gamma)_{\gamma\in Types}, \mathcal{L}_\Delta\rangle$$

where $\mathcal{L}_\Delta$ is the standard Łukasiewicz$_\Delta$-algebra and $M_\epsilon = \mathbb{R}$ (set of real numbers). Interpretation of the constant $\mathcal{M}^0(\dagger) = 0.5$.

Interpretation of special formulas of T^{Ev} in the canonical model, together with construction of extensions of evaluative expressions, is schematically depicted in Figure 1. According to our theory, it is

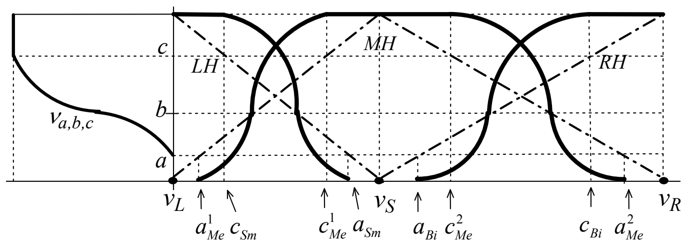

Figure 1: Scheme of the construction of extensions of evaluative expressions ($\nu_{a,b,c}$ is a function interpreting hedge $\boldsymbol{\nu}$ in $\mathcal{M}^0$; it is turned 90° counterclockwise).

easy to see that the kernel of the fuzzy set is shortened or prolonged if a hedge with a narrowing effect (such as *very*) or that with a widening effect (such as *roughly*) is present (linguistic arguments for such behavior are given in [7]).

The following properties of evaluative expressions will be used in the sequel:

Theorem 5
Let $\boldsymbol{\nu}$ be a hedge. Then

(a) $T^{\mathrm{Ev}} \vdash (\forall z)((Sm\,\boldsymbol{\nu})z \Rightarrow \neg(Bi\,\boldsymbol{\nu})z)$,

(b) $T^{\mathrm{Ev}} \vdash (\forall z)((Bi\,\boldsymbol{\nu})z \Rightarrow \neg(Sm\,\boldsymbol{\nu})z)$.

For the proof see [11].

3. Intermediate Quantifiers and Generalized Syllogisms

In this section we will show that all the 105 forms of the generalized syllogisms analyzed in [17] are formally valid in our theory of intermediate quantifiers. First, we start with basic definitions.

3.1. The theory of intermediate quantifiers

Recall the basic idea stating that these quantifiers are classical general or existential quantifiers but the universe of quantification is modified and the modification can be imprecise. We introduce a theory T^{IQ} which is a special theory of Ł-FTT extending the theory T^{Ev} of evaluative linguistic expressions introduced in the previous section.

We start with the definition of a measure on fuzzy sets. This is syntactically represented by a special formula. Its interpretation is a function $M_\alpha \longrightarrow L$.

Definition 1
Let $R \in Form_{o(o\alpha)(o\alpha)}$ be a formula. Put

$$\mu := \lambda z_{o\alpha}\, \lambda x_{o\alpha}\, (R z_{o\alpha}) x_{o\alpha}. \tag{7}$$

We say that the formula $\mu \in Form_{o(o\alpha)(o\alpha)}$ represents a measure on fuzzy sets *in the universe of type $\alpha \in Types$ if it has the following properties:*

(M1) $\boldsymbol{\Delta}(x_{o\alpha} \equiv z_{o\alpha}) \equiv ((\mu z_{o\alpha}) x_{o\alpha} \equiv \top)$,

(M2) $\boldsymbol{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \boldsymbol{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \boldsymbol{\Delta}(x_{o\alpha} \subseteq y_{o\alpha}) \Rightarrow ((\mu z_{o\alpha}) x_{o\alpha} \Rightarrow (\mu z_{o\alpha}) y_{o\alpha})$

(M3) $\boldsymbol{\Delta}(z_{o\alpha} \not\equiv \emptyset_{o\alpha}) \,\&\, \boldsymbol{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})(z_{o\alpha} - x_{o\alpha}) \equiv \neg(\mu z_{o\alpha}) x_{o\alpha})$.

Definition 2
Let $\mathcal{S} \subseteq Types$ be a distinguished set of types and $\{R \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$ be a set of new constants. The theory of intermediate quantifiers *T^{IQ} w.r.t. $\mathcal{S}$ is a formal theory of Ł-FTT with the language $J^{\mathrm{Ev}} \cup \{R_{o(o\alpha)(o\alpha)} \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$ which is extension of T^{Ev} such that $\mu \in Form_{o(o\alpha)(o\alpha)}$, $\alpha \in \mathcal{S}$, defined in (7) represents a measure on fuzzy sets in the sense of Definition 1.*

The reason for considering only some types $\mathcal{S}$ are possible difficulties with interpretation of the measure μ over sets of very large cardinalities which can thus be avoided. This means that our theory is not fully general. We do not see it as a limitation, though, because one can hardly imagine the meaning of "most X's" over a set of inaccessible cardinality.

Intermediate quantifiers have been formally defined by Novák in [12]. The following definition is a slight modification of the original definition by considering strong conjunction instead of the ordinary one.

Definition 3
Let T^{IQ} be a theory of intermediate quantifiers in the sense of Definition 2 and $Ev \in Form_{oo}$ be intension of some evaluative expression. Furthermore, let $A, B \in Form_{o\alpha}$ be formulas and $z \in Form_{o\alpha}$ and $x \in Form_\alpha$ variables where $\alpha \in \mathcal{S}$. Then a type $\langle 1, 1\rangle$ intermediate generalized quantifier interpreting the sentence

"$\langle$Quantifier$\rangle$ *B's are A*"

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(B, A) := (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B) z)), \tag{8}$$

$$(Q^{\exists}_{Ev}\, x)(B, A) := (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\exists x)(zx \wedge Ax)) \wedge Ev((\mu B) z)). \tag{9}$$

If *presupposition* is needed then we must use the following form of the intermediate quantifiers.

Definition 4
Let T^{IQ} be the theory of intermediate quantifiers in the sense of Definition 2 and $Ev \in Form_{oo}$ be intension of some evaluative expression. Then an intermediate generalized quantifier with presupposition is represented by the following formula:

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\exists x) zx \,\&\, (\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B) z)).$$

Note that only non-empty subsets of B are considered in this definition.

3.2. Generalized Aristotle's syllogisms

Before the presentation of intermediate generalized syllogisms we will introduce definitions of several specific intermediate quantifiers based on the analysis provided by Peterson in his book [17].

Definition 5
Let T^{IQ} be a theory of intermediate quantifiers. Let $A, B \in Form_{o\alpha}$ be formulas and $z \in Form_{o\alpha}$ and $x \in Form_\alpha$ variables where $\alpha \in \mathcal{S}$. Then the following intermediate quantifiers can be introduced:

$$\textbf{A:}\ \textit{All } B \textit{ are } A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\forall x)(Bx \Rightarrow Ax),$$

$$\textbf{E:}\ \textit{No } B \textit{ are } A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),$$

$$\textbf{P:}\ \textit{Almost all } B \textit{ are } A := Q^{\forall}_{Bi\ Ex}(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$

$$\textbf{B:}\ \textit{Few } B \textit{ are } A\ (:= \textit{Almost all } B \textit{ are not } A) := Q^{\forall}_{Bi\ Ex}(B, \neg A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$

$$\textbf{T:}\ \textit{Most } B \textit{ are } A := Q^{\forall}_{Bi\ Ve}(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$

$$\textbf{D:}\ \textit{Most } B \textit{ are not } A := Q^{\forall}_{Bi\ Ve}(B, \neg A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$

$$\textbf{K:}\ \textit{Many } B \textit{ are } A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$

$$\textbf{G:}\ \textit{Many } B \textit{ are not } A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, \neg A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$

$$\textbf{I:}\ \textit{Some } B \textit{ are } A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\exists x)(Bx \wedge Ax),$$

$$\textbf{O:}\ \textit{Some } B \textit{ are not } A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax).$$

The following theorem demonstrates that important role in our theory is played by monotonicity.

Theorem 6 (Valid implications)
Let **A**, ..., **G** *be the basic intermediate quantifiers defined above. Then the following sets of implications are provable in T^{IQ}:*

(a) $T^{IQ} \vdash \mathbf{A} \Rightarrow \mathbf{P}$, $\quad T^{IQ} \vdash \mathbf{P} \Rightarrow \mathbf{T}$, $\quad T^{IQ} \vdash \mathbf{T} \Rightarrow \mathbf{K}$.

(b) $T^{IQ} \vdash \mathbf{E} \Rightarrow \mathbf{B}$, $\quad T^{IQ} \vdash \mathbf{B} \Rightarrow \mathbf{D}$, $\quad T^{IQ} \vdash \mathbf{D} \Rightarrow \mathbf{G}$.

For the proof see [12].

Recall the classical definition: A *syllogism* or *logical appeal* is a kind of logical argument in which the *conclusion* is inferred from two *premises*. The first one we will called *major premise* the second one is called *minor premise*.

The syllogisms will be written as triples of formulas $\langle P_1, P_2, C\rangle$. To obtain the *intermediate syllogisms* means that we take any traditional syllogism (valid or not) and replace one or more of its formulas by formulas containing intermediate quantifiers.

Definition 6
We say that the syllogism $\langle P_1, P_2, C\rangle$ is strongly valid *if $T^{IQ} \vdash P_1\ \&\ P_2 \Rightarrow C$, or equivalently, if $T^{IQ} \vdash P_1 \Rightarrow (P_2 \Rightarrow C)$. We say that $\langle P_1, P_2, C\rangle$ is* weakly valid *if $T^{IQ} \cup \{P_1, P_2\} \vdash C$.*

Since in the rest of this paper we deal just with the theory T^{IQ}, we will write simply $\vdash \dots$ instead of $T^{\mathrm{IQ}} \vdash \dots$.

3.3. Classification of syllogisms

From classical theory of syllogisms we know that the latter are divided into four figures. Suppose that Q_1, Q_2, Q_3 are intermediate quantifiers and $X, Y, M \in Form_{o\alpha}$ are formulas representing properties. Then the following figures can be considered:

Figure I	**Figure II**	**Figure III**	**Figure IV**
$\dfrac{\begin{array}{c} Q_1\ M \text{ is } Y \\ Q_2\ X \text{ is } M \end{array}}{Q_3\ X \text{ is } Y}$	$\dfrac{\begin{array}{c} Q_1\ Y \text{ is } M \\ Q_2\ X \text{ is } M \end{array}}{Q_3\ X \text{ is } Y}$	$\dfrac{\begin{array}{c} Q_1\ M \text{ is } Y \\ Q_2\ M \text{ is } X \end{array}}{Q_3\ X \text{ is } Y}$	$\dfrac{\begin{array}{c} Q_1\ Y \text{ is } M \\ Q_2\ M \text{ is } X \end{array}}{Q_3\ X \text{ is } Y}$

Below, we introduce 105 valid forms of intermediate generalized syllogisms. First we start with the table presenting the traditional Aristotle's syllogisms.

Figure I	**Figure II**	**Figure III**	**Figure IV**
AAA	*EAE*	*AAI*	*AAI*
EAE	*AEE*	*IAI*	*AEE*
AII	*EIO*	*AII*	*IAI*
EIO	*AOO*	*EAO*	*EAO*
AAI	*EAO*	*OAO*	*EIO*
EAO	*AEO*	*EIO*	*AEO*

We continue with syllogisms which contain the intermediate quantifier *most*.

Figure I	**Figure II**	**Figure III**	**Figure IV**
AAT	*AED*	*ATI*	*AED*
ATT	*ADD*	*ETO*	*ETO*
ATI	*ADO*	*TAI*	*TAI*
EAD	*EAD*	*DAO*	
ETD	*ETD*		
ETO	*ATO*		

The following are syllogisms with intermediate quantifiers *most* and *many*.

Figure I	**Figure II**	**Figure III**	**Figure IV**
AAK	*AEG*	*AKI*	*AEG*
ATK	*ADG*	*EKO*	*EKO*
AKI	*AGO*	*KAI*	*KAI*
AKK	*AGG*	*GAO*	
EAG	*EAG*	***TTI***	
ETG	*ETG*	***DTO***	
EKO	*EKO*		
EKG	*EKG*		

We finish with intermediate syllogisms containing the intermediate quantifiers *almost all,most, many* and *few*.

Figure I	**Figure II**	**Figure III**	**Figure IV**
AAP	*AEB*	*PAI*	*AEB*
APP	*ABB*	*EPO*	*PAI*
APT	*ABD*	*BAO*	*EPO*
APK	*ABG*	*API*	
API	*ABO*	***PPI***	
EAB	*EAB*	***TPI***	
EPB	*EPB*	***KPI***	
EPD	*EPD*	***PTI***	
EPG	*EPG*	***PKI***	
EPO	*EPO*	***BPO***	
		DPO	
		GPO	
		BTO	
		BKO	

By bold type we denote twelve non-trivial intermediate syllogisms which are generalization of the corresponding intermediate syllogisms presented in Peterson's book.

4. Valid intermediate generalized syllogisms

In whole this section, $X, Y, M \in Form_{o\alpha}$ are formulas representing properties and $x \in Form_{\alpha}$ is a variable of type α.

4.1. Figure I.

Theorem 7
The following syllogisms are strongly valid.

$$\textbf{AAA-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{All } X \text{ are } Y} \qquad \textbf{AAT-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{Most } X \text{ are } Y}$$

$$\textbf{AAK-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{Many } X \text{ are } Y} \qquad \textbf{AAP-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{Almost all } X \text{ are } Y}$$

PROOF: Using Definition 5 we can formally write syllogism **AAA-I** as

$$\frac{\begin{array}{c}(\forall x)(Mx \Rightarrow Yx)\\ (\forall x)(Xx \Rightarrow Mx)\end{array}}{(\forall x)(Xx \Rightarrow Yx).}$$

Then the strong validity follows from Theorem 3(e):

$$\vdash (Mx \Rightarrow Yx) \Rightarrow ((Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow Yx))$$

By the rule of generalization and by Theorem 4(a) we have

$$\vdash (\forall x)(Mx \Rightarrow Yx) \Rightarrow ((\forall x)(Xx \Rightarrow Mx) \Rightarrow (\forall x)(Xx \Rightarrow Yx))$$

which is just strong validity of **AAA-I**.

Analogously, we obtain from syllogism **AAA-I** strong validity of the other three syllogisms **AAT-I, AAK-I**, and **AAP-I**. □

Theorem 8
All the syllogisms below are strongly valid.

$$\textbf{EAE-I:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{No } X \text{ are } Y} \qquad \textbf{EAB-I:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{Few } X \text{ are } Y}$$

$$\textbf{EAD-I:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{Most } X \text{ are not } Y} \qquad \textbf{EAG-I:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{All } X \text{ are } M\end{array}}{\text{Many } X \text{ are not } Y}$$

PROOF: The first syllogism can be formally written as

$$\frac{\begin{array}{l}(\forall x)(Mx \Rightarrow \neg Yx)\\ (\forall x)(Xx \Rightarrow Mx)\end{array}}{(\forall x)(Xx \Rightarrow \neg Yx).}$$

Then the strong validity of **EAE-I** follows immediately from the provable formula

$$\vdash (Mx \Rightarrow \neg Yx)\,\&\,(Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow \neg Yx);$$

then we proceed analogously as in the proof of **AAA-I**.

From **EAE-I** we can analogously prove also strong validity of the other three syllogisms **EAB-I, EAD-I**, and **EAG-I**. □

Now we introduce two classical syllogisms (on the left is linguistic form, on the right the corresponding formal expression).

Theorem 9
The following syllogisms are strongly valid.

$$\textbf{EIO-I:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{Some } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow \neg Yx)\\ (\exists x)(Xx \wedge Mx)\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

$$\textbf{AII-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Some } X \text{ are } M\end{array}}{\text{Some } X \text{ are } Y} \qquad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\exists x)(Xx \wedge Mx)\end{array}}{(\exists x)(Xx \wedge Yx)}$$

PROOF: From $\vdash (Mx \Rightarrow \neg Yx) \Rightarrow ((Xx \wedge Mx) \Rightarrow (Xx \wedge \neg Yx))$, rule of generalization and by Theorem 4(a),(b) we obtain strong validity of the classical syllogism **EIO-I**. Analogously we can also prove strong validity of **AII-I**. □

Theorem 10
All the syllogisms below are weakly valid.

$$\textbf{ATT-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Most } X \text{ are } M\end{array}}{\text{Most } X \text{ are } Y} \qquad \textbf{AKK-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Many } X \text{ are } M\end{array}}{\text{Many } X \text{ are } Y}$$

$$\textbf{APP-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Almost all } X \text{ are } M\end{array}}{\text{Almost all } X \text{ are } Y} \qquad \textbf{APT-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Almost all } X \text{ are } M\end{array}}{\text{Most } X \text{ are } Y}$$

$$\textbf{ATK-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Most } X \text{ are } M\end{array}}{\text{Many } X \text{ are } Y} \qquad \textbf{APK-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Almost all } X \text{ are } M\end{array}}{\text{Many } X \text{ are } Y}$$

PROOF: Analogously as above we can formally write the first syllogism as follows:

$$\textbf{ATT-I:}\quad \begin{array}{l} (\forall x)(Mx \Rightarrow Yx) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(X)z)) \\ \hline (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge (Bi\ Ve)(\mu(X)z)). \end{array}$$

Let us denote by $Ez := (Bi\ Ve)(\mu(X)z)$. By the properties of quantifiers and the properties of FTT we can prove

$$\vdash (\forall x)(Mx \Rightarrow Yx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow (\forall x)(zx \Rightarrow Yx)).$$

By the major premise and the properties of FTT we obtain

$$\vdash (\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \Rightarrow (\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx))$$

and thus,

$$\vdash ((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge Ez) \Rightarrow \\ (\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge Ez).$$

Finally, by the rule of generalization with respect to $(\forall z)$ and by Theorem 4(b) we obtain

$$\vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge Ez) \Rightarrow \\ (\exists z)(\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge Ez).$$

If we replace Ez by $(Bi\ Ve)(\mu(X)z)$ then by the minor premise we obtain just the conclusion of **ATT-I** which means weak validity of it.

If we put $Ez := (Bi\ Ex)(\mu(X)z)$, we obtain weak validity of **APP-I**; by putting $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})(\mu(X)z)$ we obtain weak validity of **AKK-I**. Using Theorem 6(a), we can prove from the syllogism **APP-I** weak validity of **APT-I** and **APK-I**. Analogously, we obtain weak validity **ATK-I** from **ATT-I**. □

Theorem 11
The following syllogisms are weakly valid.

$$\textbf{EPB-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Almost all } X \text{ are } M \\ \hline \text{Few } X \text{ are } Y \end{array} \qquad \textbf{EPD-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Almost all } X \text{ are } M \\ \hline \text{Most } X \text{ are not } Y \end{array}$$

$$\textbf{EPG-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Almost all } X \text{ are } M \\ \hline \text{Many } X \text{ are not } Y \end{array} \qquad \textbf{ETD-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Most } X \text{ are } M \\ \hline \text{Most } X \text{ are not } Y \end{array}$$

$$\textbf{ETG-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Most } X \text{ are } M \\ \hline \text{Many } X \text{ are not } Y \end{array} \qquad \textbf{EKG-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Many } X \text{ are } M \\ \hline \text{Most } X \text{ are not } Y \end{array}$$

PROOF: The syllogism **EPB-I** can be formally written as follows:

$$\textbf{EPB-I:}\quad \begin{array}{l} (\forall x)(Mx \Rightarrow \neg Yx) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ex)(\mu(X)z)) \\ \hline (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\ Ex)(\mu(X)z)). \end{array}$$

Obviously as above we start with the provable formula $\vdash (Mx \Rightarrow \neg Yx)\,\&(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx)$. Using the same properties as in the proof above we obtain the weak validity of **EPB-I**.

By Theorem 6(a) we obtain from **EPB-I** weak validity of the other syllogisms **EPD-I** and **EPG-I**. Obviously, we obtain weak validity of **ETD-I** and hence, by Theorem 6(b), we get weak validity of **ETG-I**. Similarly, we get weak validity of the syllogism **EKG-I**. □

4.1.1. Figure I — syllogisms with presupposition:

In this subsection, we introduce syllogisms which require presupposition. First, we start with two classical syllogisms which require presupposition in the minor premise.

Theorem 12
The following syllogisms are strongly valid.

$$\textbf{EAO-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{All } X \text{ are } M \\ \hline \text{Some } X \text{ are not } Y \end{array} \qquad \begin{array}{l} (\forall x)(Mx \Rightarrow \neg Yx) \\ (\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx \\ \hline (\exists x)(Xx \wedge \neg Yx) \end{array}$$

$$\textbf{AAI-I:}\quad \begin{array}{l} \text{All } M \text{ are } Y \\ \text{All } X \text{ are } M \\ \hline \text{Some } X \text{ are } Y \end{array} \qquad \begin{array}{l} (\forall x)(Mx \Rightarrow Yx) \\ (\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx \\ \hline (\exists x)(Xx \wedge Yx) \end{array}$$

PROOF: From

$$\vdash (Mx \Rightarrow \neg Yx) \,\&\, (Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow \neg Yx)$$

we obtain

$$\vdash (Mx \Rightarrow \neg Yx) \,\&\, (Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow (Xx \wedge \neg Yx)).$$

Thus, by Theorem 4(a),(b) we get

$$\vdash (\forall x)(Mx \Rightarrow \neg Yx) \,\&\, (\forall x)(Xx \Rightarrow Mx) \Rightarrow ((\exists x)Xx \Rightarrow (\exists x)(Xx \wedge \neg Yx))$$

which is equivalent with

$$\vdash ((\forall x)(Mx \Rightarrow \neg Yx) \,\&\, (\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)).$$

This means that the syllogism **EAO-I** is strongly valid.

Analogously we can prove the second classical syllogism **AAI-I**. □

The following syllogisms also require the presupposition, i.e. only non-empty subsets of X are considered.

Theorem 13
All the syllogisms below are weakly valid.

$$\textbf{ETO-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Most } X \text{ are } M \\ \hline \text{Some } X \text{ are not } Y \end{array} \qquad \textbf{EKO-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Many } X \text{ are } M \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

$$\textbf{EPO-I:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{Almost all } X \text{ are } M \\ \hline \text{Some } X \text{ are not } Y. \end{array}$$

PROOF: Using Definition 5 we can write

$$\textbf{ETO-I:}\quad \begin{array}{l} (\forall x)(Mx \Rightarrow \neg Yx) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(X)z)) \\ \hline (\exists x)(Xx \wedge \neg Yx). \end{array}$$

Put $Ez := (Bi\ Ve)(\mu(X)z)$. From

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx)$$

and thus by the properties of FTT we get

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Xx) \,\&\, (zx \Rightarrow Mx) \Rightarrow ((zx \Rightarrow Xx) \,\&\, (zx \Rightarrow \neg Yx)).$$

By the rule of generalization, Theorem 4(a) and using the properties of FTT we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow \\ (\forall x)((zx \Rightarrow Xx) \mathbin{\&} (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx))). \quad (10)$$

Using the major premise and the substitution axiom (Theorem 4(f)) we get

$$T^{\mathrm{IQ}} \vdash (zx \Rightarrow Xx) \Rightarrow ((zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx))).$$

Using Theorem 3(l) we obtain

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(zx \Rightarrow Xx) \Rightarrow ((zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx))).$$

Then by the rule of generalization and by Theorem 4(a) we have

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(\forall x)(zx \Rightarrow Xx) \Rightarrow (\forall x)((zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx)))$$

and hence by Theorem 4(a),(b) we get

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(z \subseteq X) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow ((\exists x)zx \Rightarrow (\exists x)(Xx \wedge \neg Yx))).$$

This is equivalent with

$$T^{\mathrm{IQ}} \vdash (\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow Mx) \mathbin{\&} (\exists x)zx) \Rightarrow (\exists x)(Xx \wedge \neg Yx).$$

Now using Theorem 3(c) we conclude

$$T^{\mathrm{IQ}} \vdash ((\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow Mx) \mathbin{\&} (\exists x)zx) \wedge Ez) \Rightarrow (\exists x)(Xx \wedge \neg Yx).$$

Finally by the rule of generalization with respect to $\forall z$ and quantifiers properties we obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow Mx) \mathbin{\&} (\exists x)zx) \wedge Ez) \Rightarrow (\exists x)(Xx \wedge \neg Yx).$$

If we replace Ez by $(Bi\ Ve)(\mu(X)z)$ then by minor premise we obtain just conclusion of our syllogism **ETO-I** and thus weak validity of our syllogism.

If in the proof above we replace Ez by $\neg\, Sm(\bar{\boldsymbol{\nu}})(\mu(X)z)$ then we get the weak validity of **EKO-I**. Analogously by putting $(Bi\ Ex)(\mu(X)z)$ we get **EPO-I**. □

Theorem 14
The following syllogisms are weakly valid.

$$\textbf{ATI-I:}\ \begin{array}{l} \text{All } M \text{ are } Y \\ \text{Most } X \text{ are } M \\ \hline \text{Some } X \text{ are } Y \end{array} \qquad \textbf{AKI-I:}\ \begin{array}{l} \text{All } M \text{ are } Y \\ \text{Many } X \text{ are } M \\ \hline \text{Some } X \text{ are } Y \end{array}$$

$$\textbf{API-I:}\ \begin{array}{l} \text{All } M \text{ are } Y \\ \text{Almost all } X \text{ are } M \\ \hline \text{Some } X \text{ are } Y \end{array}$$

PROOF: We have

$$\textbf{ATI-I:}\ \begin{array}{l} (\forall x)(Mx \Rightarrow Yx) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(X)z)) \\ \hline (\exists x)(Xx \wedge Yx). \end{array}$$

The proof is constructed similarly as that of Theorem 13 but the initial formula is

$$\vdash (Mx \Rightarrow Yx) \mathbin{\&} (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Yx).$$

Analogously we may prove other two syllogisms **AKI-I** and **API-I**. □

4.2. Figure II.

The second figure will be opened with two classical syllogisms.

Theorem 15
Both the classical syllogisms below are strongly valid.

$$\textbf{EIO-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{Some } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\exists x)(Xx \wedge Mx)\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

$$\textbf{AOO-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{Some } X \text{ are not } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow Mx)\\ (\exists x)(Xx \wedge \neg Mx)\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

PROOF: By contraposition $\vdash (Yx \Rightarrow \neg Mx) \Rightarrow (Mx \Rightarrow \neg Yx)$ and by $\vdash (Mx \Rightarrow \neg Yx) \Rightarrow ((Xx \wedge Mx) \Rightarrow (Xx \wedge \neg Yx))$ we get

$$\vdash (Yx \Rightarrow \neg Mx) \Rightarrow ((Xx \wedge Mx) \Rightarrow (Xx \wedge \neg Yx)).$$

Then by Theorem 4(a),(b) we get

$$\vdash (\forall x)(Yx \Rightarrow \neg Mx) \Rightarrow ((\exists x)(Xx \wedge Mx) \Rightarrow (\exists x)(Xx \wedge \neg Yx))$$

which is just strong validity of the syllogism **EIO-II**. The second syllogism **AOO-II** is obtained similarly. □

Theorem 16
The following syllogisms are strongly valid.

$$\textbf{AEE-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } X \text{ are } M\end{array}}{\text{No } X \text{ are } Y} \qquad \textbf{EAE-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{All } X \text{ are } M\end{array}}{\text{No } X \text{ are } Y}$$

$$\textbf{AEB-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } X \text{ are } M\end{array}}{\text{Few } X \text{ are } Y} \qquad \textbf{AED-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } X \text{ are } M\end{array}}{\text{Most } X \text{ are not } Y}$$

$$\textbf{AEG-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } X \text{ are } M\end{array}}{\text{Many } X \text{ are not } Y} \qquad \textbf{EAB-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{All } X \text{ are } M\end{array}}{\text{Few } X \text{ are } Y}$$

$$\textbf{EAD-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{All } X \text{ are } M\end{array}}{\text{Most } X \text{ are not } Y} \qquad \textbf{EAG-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{All } X \text{ are } M\end{array}}{\text{Many } X \text{ are not } Y}$$

PROOF: The classical syllogism can be written as follows:

$$\frac{\begin{array}{l}(\forall x)(Yx \Rightarrow Mx)\\ (\forall x)(Xx \Rightarrow \neg Mx)\end{array}}{(\forall x)(Xx \Rightarrow \neg Yx).}$$

The strong validity of it is obtained by contraposition from

$$\vdash (Yx \Rightarrow Mx)\,\&\,(Xx \Rightarrow \neg Mx) \Rightarrow (Xx \Rightarrow \neg Yx)$$

and using Theorem 4(a).

From **AEE-II** by Theorem 3(b) we get the strong validity of the syllogism **EAE-II**. From **AEE-II** by Theorem 6(b) we may prove the other three syllogisms **AEB-II, AED-II** and **AEG-II**. Finally, from the strong validity of **EAE-II** by Theorem 6(b) we can prove the strong validity of **EAB-II, EAD-II** and **EAG-II**. $\square$

Theorem 17
All the syllogisms below are weakly valid.

$$\textbf{ABB-II:}\ \begin{array}{l}\text{All } Y \text{ are } M\\ \text{Few } X \text{ are } M\\ \hline \text{Few } X \text{ are } Y\end{array}\qquad \textbf{ADD-II:}\ \begin{array}{l}\text{All } Y \text{ are } M\\ \text{Most } X \text{ are not } M\\ \hline \text{Most } X \text{ are not } Y\end{array}$$

$$\textbf{AGG-II:}\ \begin{array}{l}\text{All } Y \text{ are } M\\ \text{Many } X \text{ are not } M\\ \hline \text{Many } X \text{ are not } Y\end{array}\qquad \textbf{ABD-II:}\ \begin{array}{l}\text{All } Y \text{ are } M\\ \text{Few } X \text{ are } M\\ \hline \text{Most } X \text{ are not } Y\end{array}$$

$$\textbf{ABG-II:}\ \begin{array}{l}\text{All } Y \text{ are } M\\ \text{Few } X \text{ are } M\\ \hline \text{Many } X \text{ are not } Y\end{array}\qquad \textbf{ADG-II:}\ \begin{array}{l}\text{All } Y \text{ are } M\\ \text{Most } X \text{ are not } M\\ \hline \text{Many } X \text{ are not } Y\end{array}$$

PROOF: The first syllogism **ABB-II** can be written using Definition 5 as follows:

$$\begin{array}{l}(\forall x)(Yx \Rightarrow Mx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Mx)) \wedge (Bi\,Ex)(\mu(X)z))\\ \hline (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\,Ex)(\mu(X)z)).\end{array}$$

Let us denote $Ez := (Bi\,Ex)(\mu(X)z)$. By contraposition and using Theorem 4(a) we can prove that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow Mx) \Rightarrow ((\forall x)(zx \Rightarrow \neg Mx) \Rightarrow (\forall x)(zx \Rightarrow \neg Yx)).$$

By the major premise and using the same steps as in the proof of **ATT-I** we obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Mx)) \wedge Ez) \Rightarrow (\exists z)(\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx) \wedge Ez). \quad (11)$$

If we replace Ez by $(Bi\,Ex)(\mu(X)z)$ then by the minor premise we obtain just weak validity of the syllogism **ABB-II**.

If in the prove above we put $Ez := (Bi\,Ve)(\mu(X)z)$ we obtain **ADD-II** and finally by putting $Ez := \neg\,Sm(\bar{\boldsymbol{\nu}})(\mu(X)z)$ we get the weak validity of **AGG-II**. From the syllogism **ABB-II** by Theorem 6(b) we obtain the weak validity of the syllogisms **ABD-II** and thus **ABG-II**. Analogously, from **ADD-II** by Theorem 6(b) we get **ADG-II**. $\square$

Theorem 18
The following syllogisms are weakly valid.

$$\textbf{ETD-II:}\ \begin{array}{l}\text{No } Y \text{ are } M\\ \text{Most } X \text{ are } M\\ \hline \text{Most } X \text{ are not } Y\end{array}\qquad \textbf{EPB-II:}\ \begin{array}{l}\text{No } Y \text{ are } M\\ \text{Almost all } X \text{ are } M\\ \hline \text{Few } X \text{ are } Y\end{array}$$

$$\textbf{EKG-II:}\ \begin{array}{l}\text{No } Y \text{ are } M\\ \text{Many } X \text{ are } M\\ \hline \text{Many } X \text{ are not } Y\end{array}\qquad \textbf{ETG-II:}\ \begin{array}{l}\text{No } Y \text{ are } M\\ \text{Most } X \text{ are } M\\ \hline \text{Many } X \text{ are not } Y\end{array}$$

$$\textbf{EPD-II:}\ \begin{array}{l}\text{No } Y \text{ are } M\\ \text{Almost all } X \text{ are } M\\ \hline \text{Most } X \text{ are not } Y\end{array}\qquad \textbf{EPG-II:}\ \begin{array}{l}\text{No } Y \text{ are } M\\ \text{Almost all } X \text{ are } M\\ \hline \text{Many } X \text{ are not } Y\end{array}$$

PROOF: Analogously as above we have

$$\textbf{ETD-II:}\quad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(X)z))\end{array}}{(\exists z)((\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\ Ve)(\mu(X)z)).}$$

By Ez we denote $(Bi\ Ve)(\mu(X)z)$. Using contraposition and by the properties of quantifiers we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow (\forall x)(zx \Rightarrow \neg Yx)).$$

Thus using the same steps as above we get

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow Mx)) \wedge Ez) \Rightarrow \\ (\exists z)(\boldsymbol{\Delta}(z \subseteq X) \mathbin{\&} (\forall x)(zx \Rightarrow \neg Yx) \wedge Ez). \quad (12)$$

Replacing Ez by $(Bi\ Ve)(\mu(X)z)$ and by the major premise we obtain weak validity of **ETD-II**.

If we put $Ez := (Bi\ Ex)(\mu(X)z)$ we get **EPB-II** and finally if we put $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})(\mu(X)z)$ we have the weak validity of the syllogism **EKG-II**. From **ETD-II** by Theorem 6(b) we obtain the weak validity of **ETG-II**. Finally we conclude that from **EPB-II** by Theorem 6(b) we get the weak validity of **EPD-II** and thus **EPG-II**. □

4.2.1. Figure II — syllogisms with presupposition

Analogously as in Figure-I we closed this section with the syllogisms which require the presupposition. We start with two classical syllogisms. The proofs will be constructed in a similarly way as in Figure I.

Theorem 19
The classical syllogisms below are strongly valid.

$$\textbf{EAO-II:}\quad \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{All } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad\qquad \textbf{AEO-II:}\quad \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: Because the syllogism assumes presupposition it has the following form:

$$\textbf{EAO-II:}\quad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\forall x)(Xx \Rightarrow Mx) \mathbin{\&} (\exists x)Xx\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

We start with the provable formula

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \mathbin{\&} (Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow (Xx \wedge \neg Yx)).$$

Then by Theorem 4(a,b) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx) \mathbin{\&} (\forall x)(Xx \Rightarrow Mx) \Rightarrow ((\exists x)Xx \Rightarrow (\exists x)(Xx \wedge \neg Yx))$$

which is equivalent with

$$T^{\mathrm{IQ}} \vdash ((\forall x)(Yx \Rightarrow \neg Mx) \mathbin{\&} (\forall x)(Xx \Rightarrow Mx) \mathbin{\&} (\exists x)Xx) \Rightarrow (\exists x)(Xx \wedge \neg Yx).$$

This means that the syllogism **EAO-II** is strongly valid.

Analogously we obtain the strong validity of the second classical syllogism **AEO-II**. □

Theorem 20
All the syllogisms below are weakly valid.

$$\textbf{EKO-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{Many } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \textbf{ETO-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{Most } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\textbf{EPO-II:}\ \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{Almost all } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \textbf{ABO-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{Few } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\textbf{ADO-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{Most } X \text{ are not } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \textbf{AGO-II:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{Many } X \text{ are not } M\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: The syllogism **ETO-II** can be written as follows:

$$\frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\exists x)zx\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(X)z))\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

The proof is constructed analogously as the proof of **ETO-I** but the initial formula is

$$\vdash (Yx \Rightarrow \neg Mx)\,\&(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx).$$

Similarly as the proof of **ETO-I** we get weak validity of **ETO-II**

Analogously we may also prove the weak validity of other syllogisms **EKO-II** and **EPO-II**. The syllogisms **ABO-II, ADO-II** and **AGO-II**. will be constructed obviously but the starting formula is

$$\vdash (Yx \Rightarrow Mx)\,\&(zx \Rightarrow \neg Mx) \Rightarrow (zx \Rightarrow \neg Yx).$$

□

4.3. Figure III.

The third figure will be opened with four classical syllogisms.

Theorem 21
The following syllogisms are strongly valid.

$$\textbf{IAI-III:}\ \frac{\begin{array}{l}\text{Some } M \text{ are } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \textbf{OAO-III:}\ \frac{\begin{array}{l}\text{Some } M \text{ are not } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\textbf{AII-III:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Some } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \textbf{EIO-III:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{Some } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: The first syllogism can be written as follows: $\dfrac{\begin{array}{l}(\exists x)(Mx \wedge Yx)\\ (\forall x)(Mx \Rightarrow Xx)\end{array}}{(\exists x)(Xx \wedge Yx).}$ From

$$\vdash (Mx \Rightarrow Xx) \Rightarrow ((Mx \wedge Yx) \Rightarrow (Xx \wedge Yx))$$

by Theorem 4(a),(b) and Theorem 3(b) we get

$$\vdash (\exists x)(Mx \wedge Yx) \Rightarrow ((\forall x)(Mx \Rightarrow Xx) \Rightarrow (\exists x)(Xx \wedge Yx))$$

which is just strong validity of the syllogism **IAI-III**.

In the same way we can prove strong validity of the other three syllogisms **OAO-III, AII-III** and **EIO-III**. □

4.3.1. Figure III — syllogisms with presupposition:

We continue with the syllogisms which require presupposition (only non-empty subsets of M are considered). First we start with two classical syllogisms with presupposition which is included in the minor premise.

Theorem 22
Both the classical syllogisms below are strongly valid.

$$\textbf{AAI-III:}\ \begin{array}{l}\text{All } M \text{ are } Y\\ \text{All } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{EAO-III:}\ \begin{array}{l}\text{No } M \text{ are } Y\\ \text{All } M \text{ are } X\\ \hline \text{Some } X \text{ are not } Y\end{array}$$

PROOF: With the presupposition the first syllogism looks in the following

$$\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\forall x)(Mx \Rightarrow Xx)\,\&(\exists x)Mx\\ \hline (\exists x)(Xx \wedge Yx).\end{array}$$

The syllogism immediately follows from

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Yx)\,\&(Mx \Rightarrow Xx) \Rightarrow (Mx \Rightarrow (Xx \wedge Yx))$$

by the properties of quantifiers.

Analogously we can prove the strong validity of the second classical syllogism **EAO-III**. □

Theorem 23
All the syllogisms below are weakly valid.

$$\textbf{TAI-III:}\ \begin{array}{l}\text{Most } M \text{ are } Y\\ \text{All } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{PAI-III:}\ \begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{All } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array}$$

$$\textbf{KAI-III:}\ \begin{array}{l}\text{Many } M \text{ are } Y\\ \text{All } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{ATI-III:}\ \begin{array}{l}\text{All } M \text{ are } Y\\ \text{Most } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array}$$

$$\textbf{AKI-III:}\ \begin{array}{l}\text{All } M \text{ are } Y\\ \text{Many } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{API-III:}\ \begin{array}{l}\text{All } M \text{ are } Y\\ \text{Almost all } M \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array}$$

PROOF: Analogously as above we can write the first syllogism as follows:

$$\begin{array}{l}(\exists z)((\boldsymbol{\Delta}(z \subseteq M)\,\&(\exists x)zx\,\&(\forall x)(zx \Rightarrow Yx)) \wedge (Bi\ Ve)(\mu(M)z))\\ (\forall x)(Mx \Rightarrow Xx)\\ \hline (\exists x)(Xx \wedge Yx).\end{array}$$

Let us put $Ez := (Bi\ Ve)((\mu M)z)$. The proof will be analogous to the proof of **ETO-I** and so, we will continue in succinct way.

We start with the provable formula

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx)\,\&(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Xx)$$

and hence

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx) \Rightarrow ((zx \Rightarrow Mx)\,\&(zx \Rightarrow Yx) \Rightarrow (zx \Rightarrow Xx)\,\&(zx \Rightarrow Yx)).$$

Then analogously as above in **ETO-I** using quantifiers properties, substitution axiom and the minor premise we obtain

$$T^{\mathrm{IQ}} \vdash (zx \Rightarrow Mx) \Rightarrow ((zx \Rightarrow Yx) \Rightarrow (zx \Rightarrow (Xx \wedge Yx))).$$

Now using the same properties as in **ETO-I** we conclude that

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Yx)) \wedge Ez) \Rightarrow (\exists x)(Xx \wedge Yx).$$

If we replace Ez by $(Bi\ Ve)((\mu M)z)$ then using the major premise we obtain the conclusion of our syllogism which gives just weak validity of the syllogism **TAI-III**.

From the syllogism **TAI-III** we obtain by Theorem 3(b) weak validity of **ATI-III**. If in the proof above we put $Ez := (Bi\ Ex)((\mu M)z)$ then we obtain weak validity of **PAI-III** and, hence, **API-III**. If put $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu M)z)$ then we get weak validity of **KAI-III** and thus, weak validity of **AKI-III**. □

Theorem 24
The following syllogisms are weakly valid.

BAO-III:
$$\begin{array}{l} \text{Few } M \text{ are } Y \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

DAO-III:
$$\begin{array}{l} \text{Most } M \text{ are not } Y \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

GAO-III:
$$\begin{array}{l} \text{Many } M \text{ are not } Y \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

PROOF: The first syllogism with the presupposition will be written as follows:

$$\begin{array}{l} (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\ Ex)(\mu(M)z)) \\ (\forall x)(Mx \Rightarrow Xx) \\ \hline (\exists x)(Xx \wedge \neg Yx). \end{array}$$

The proof will be constructed as the proof above but the initial formula is

$$\vdash (Mx \Rightarrow Xx) \,\&\, (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Xx)$$

and, hence,

$$\vdash (Mx \Rightarrow Xx) \,\&\, (zx \Rightarrow Mx) \,\&\, (zx \Rightarrow \neg Yx) \Rightarrow (zx \Rightarrow Xx) \,\&\, (zx \Rightarrow \neg Yx).$$

Further steps are similar as above. Thus we may prove that the syllogism **BAO-III** is weakly valid.

Analogously we give weak validity of other two syllogisms **DAO-III** and **GAO-III**. □

Theorem 25
The following syllogisms are weakly valid.

ETO-III:
$$\begin{array}{l} \text{No } M \text{ are } Y \\ \text{Most } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

EPO-III:
$$\begin{array}{l} \text{No } M \text{ are } Y \\ \text{Almost all } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

EKO-III:
$$\begin{array}{l} \text{No } M \text{ are } Y \\ \text{Many } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

PROOF: The first syllogism **ETO-III** can be written as follows:

$$\frac{\begin{array}{l}(\exists x)(Mx \Rightarrow \neg Yx)\\(\exists z)((\boldsymbol{\Delta}(z \subseteq M)\,\&(\exists x)zx\,\&(\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ve)(\mu(M)z))\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

the proof is constructed analogously as the proof of **ETO-I**.

Obviously we give the weak validity of other two syllogisms **EPO-III** and **EKO-III**. □

This section is finished by twelve non-trivial intermediate syllogisms which are generalization of the corresponding syllogisms presented in the Peterson's book. All the syllogisms require presupposition.

Theorem 26
All the syllogisms below are strongly valid.

$$\textbf{TTI-III:}\ \frac{\begin{array}{l}\text{Most } M \text{ are } Y\\ \text{Most } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \textbf{PPI-III:}\ \frac{\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Almost all } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

$$\textbf{TPI-III:}\ \frac{\begin{array}{l}\text{Most } M \text{ are } Y\\ \text{Almost all } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \textbf{PTI-III:}\ \frac{\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Most } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

$$\textbf{PKI-III:}\ \frac{\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Many } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \textbf{KPI-III:}\ \frac{\begin{array}{l}\text{Many } M \text{ are } Y\\ \text{Almost all } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

PROOF: The first syllogism with presupposition is written as follows:

$$\frac{\begin{array}{l}(\exists z)((\boldsymbol{\Delta}(z \subseteq M)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge (Bi\ Ve)(\mu(M)z))\\(\exists z)((\boldsymbol{\Delta}(z \subseteq M)\,\&(\exists x)zx\,\&(\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ve)(\mu(M)z))\end{array}}{(\exists x)(Xx \wedge Yx).}$$

Let us put $Ez := (Bi\ Ve)(\mu(M)z)$. Then we start with the formula

$$\vdash (zx \Rightarrow Yx)\,\&(zx \Rightarrow Xx) \Rightarrow (zx \Rightarrow (Xx \wedge Yx)).$$

By the properties of quantifiers we get

$$\vdash (\forall x)(zx \Rightarrow Yx)\,\&(\forall x)(zx \Rightarrow Xx) \Rightarrow ((\exists x)zx \Rightarrow (\exists x)(Xx \wedge Yx)).$$

Using the properties of FTT we get

$$\vdash \boldsymbol{\Delta}(z \subseteq M)\,\&((\forall x)(zx \Rightarrow Yx)\,\&(\forall x)(zx \Rightarrow Xx)) \Rightarrow \\ \boldsymbol{\Delta}(z \subseteq M)\,\&((\exists x)zx \Rightarrow (\exists x)(Xx \wedge Yx)).$$

Thus by Theorem 3(c),(m) we obtain

$$\vdash (\boldsymbol{\Delta}(z \subseteq M)\,\&(\forall x)(zx \Rightarrow Yx)\,\&(\forall x)(zx \Rightarrow Xx)) \Rightarrow \\ ((\boldsymbol{\Delta}(z \subseteq M)\,\&(\exists x)zx) \Rightarrow (\exists x)(Xx \wedge Yx)).$$

By the properties of FTT we get

$$\vdash (\boldsymbol{\Delta}(z \subseteq M)\,\&(\forall x)(zx \Rightarrow Yx)) \Rightarrow \\ \{(\forall x)(zx \Rightarrow Xx) \Rightarrow ((\boldsymbol{\Delta}(z \subseteq M)\,\&(\exists x)zx) \Rightarrow (\exists x)(Xx \wedge Yx))\}$$

which gives

$$\vdash (\boldsymbol{\Delta}(z \subseteq M)\,\&(\forall x)(zx \Rightarrow Yx)) \Rightarrow \\ \{((\forall x)(zx \Rightarrow Xx)\,\&(\boldsymbol{\Delta}(z \subseteq M)\,\&(\exists x)zx)) \Rightarrow (\exists x)(Xx \wedge Yx)\}.$$

Furthermore,

$$\vdash \{Ez \wedge (\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx))\} \Rightarrow \\ \{Ez \wedge ((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)) \Rightarrow (\exists x)(Xx \wedge Yx))\}$$

which can be rewritten using the properties of FTT and by Theorem 3(k) into

$$\vdash \{(\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx)) \wedge Ez\} \Rightarrow \\ \{Ez \wedge (\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)) \Rightarrow (Ez \wedge (\exists x)(Xx \wedge Yx))\}.$$

Again by the properties of FTT we can write

$$\vdash \{((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx)) \wedge Ez) \mathbin{\&} \\ (Ez \wedge (\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)))\} \Rightarrow (Ez \wedge (\exists x)(Xx \wedge Yx))$$

which implies

$$\vdash \{((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx)) \wedge Ez) \mathbin{\&} \\ (Ez \wedge (\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)))\} \Rightarrow (\exists x)(Xx \wedge Yx).$$

Now by the properties of FTT, by the rule of generalization with respect to $(\forall z)$ and by quantifiers properties we obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)(\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx) \wedge Ez) \Rightarrow \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)) \wedge Ez) \Rightarrow (\exists x)(Xx \wedge Yx).$$

If we put $Ez := (Bi\ Ve)(\mu(M)z)$ then we obtain strong validity of our syllogism **TTI-III**.

If we put $Ez := (Bi\ Ex)(\mu(M)z)$ in the proof above we obtain the strong validity of **PPI-III**. From the strong validity of **TTI-III** and using Theorem 6(a) we get the strong validity of **PTI-III** and thus, of **TPI-III**. Analogously we may prove strong validity the other two syllogisms **PKI-III** and **KPI-III**. □

Theorem 27
The following syllogisms are strongly valid.

BPO-III:
Few M are Y
Almost all M are X

Some X are not Y

DPO-III:
Most M are not Y
Almost all M are X

Some X are not Y

GPO-III:
Many M are not Y
Almost all M are X

Some X are not Y

BTO-III:
Almost all M are Y
Most M are X

Some X are not Y

DTO-III:
Most M are not Y
Most M are X

Some X are not Y

BKO-III:
Few M are Y
Many M are X

Some X are not Y

PROOF: We prove the syllogism **GPO-III** which using Definition 5 can be written as

$$\begin{array}{l} (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow \neg Yx)) \wedge \neg\, Sm(\bar{\boldsymbol{\nu}})(\mu(M)z)) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ex)(\mu(M)z)) \\ \hline (\exists x)(Xx \wedge \neg Yx). \end{array}$$

From

$$\vdash (zx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Xx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx))$$

and hence by the properties of FTT and quantifiers we get

$$\vdash \{\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\forall x)(zx \Rightarrow \neg Yx) \,\&\, (\forall x)(zx \Rightarrow Xx)\} \Rightarrow \\ ((\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)).$$

This implies that

$$\vdash \{\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\forall x)(zx \Rightarrow \neg Yx)\} \Rightarrow \\ \{(\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Xx)) \Rightarrow (\exists x)(Xx \wedge \neg Yx)\}. \quad (13)$$

Now using Theorems 5(b) and 4(f) (substitution axiom) we obtain

$$\vdash ((Bi(\boldsymbol{\nu})(\mu(M)z) \Rightarrow \neg\, Sm(\boldsymbol{\nu})(\mu(M)z)).$$

From this, using (13) and by Theorem 3(g),(c) we get

$$\vdash (\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ex)(\mu(M)z) \Rightarrow \\ \{((\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\forall x)(zx \Rightarrow \neg Yx)) \wedge \neg\, Sm(\boldsymbol{\nu})(\mu(M)z)) \Rightarrow (\exists x)(Xx \wedge \neg Yx)\}.$$

Finally by the rule of generalization with respect to $(\forall z)$ and by the properties of quantifiers we obtain

$$\vdash (\exists z)\{(\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\forall x)(zx \Rightarrow \neg Yx)) \wedge \neg\, Sm(\boldsymbol{\nu})(\mu(M)z)\} \Rightarrow \\ (\exists z)\{(\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ex)(\mu(M)z)\} \Rightarrow \\ (\exists x)(Xx \wedge \neg Yx)$$

which is just strong validity of the syllogism **GPO-III**.

From the strong validity of **GPO-III** we get by Theorem 6(b) strong validity of the other syllogisms **DPO-III** and **BPO-III**. Analogously we can prove the strong validity of the syllogism **BKO-III**. In the same way as was constructed the proof of **TTI-III** we can prove also strong validity of the syllogisms **DTO-III**. Then using Theorem 6(b) we obtain strong validity of the last syllogism **BTO-III**. □

4.4. Figure IV.

Theorem 28

The classical syllogisms below are strongly valid.

$$\textbf{EIO-IV:}\ \begin{array}{l} \text{No } Y \text{ are } M \\ \text{Some } M \text{ are } X \\ \hline \text{Some } X \text{ are not} Y \end{array} \qquad \textbf{IAI-IV:}\ \begin{array}{l} \text{Some } Y \text{ are } M \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are } Y \end{array}$$

PROOF: By contraposition and the properties of FTT we prove

$$\vdash (Yx \Rightarrow \neg Mx) \Rightarrow ((Mx \wedge Xx) \Rightarrow (Xx \wedge \neg Yx)).$$

Then using Theorem 4(a),(b) we obtain strong validity of **EIO-IV**.

Analogously we can prove that **IAI-IV** is strongly valid. □

Theorem 29

The syllogisms below are strongly valid.

$$\textbf{AEE-IV:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{No } X \text{ are } Y} \qquad \textbf{AEB-IV:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{Few } X \text{ are } Y}$$

$$\textbf{AED-IV:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{Most } X \text{ are not } Y} \qquad \textbf{AEG-IV:}\ \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{Many } X \text{ are not } Y}$$

PROOF: The strong validity of the classical syllogism **AEE-IV** follows from the provable formula

$$\vdash (Yx \Rightarrow Mx) \mathbin{\&} (Mx \Rightarrow \neg Xx) \Rightarrow (Xx \Rightarrow \neg Yx) \tag{14}$$

and using quantifiers properties.

From **AEE-IV** by Theorem 6(b), we may prove also strong validity of the other three syllogisms **AEB-IV, AED-IV** and **AEG-IV**. □

4.4.1. Figure IV — syllogisms with presupposition

The last figure with presupposition will be divided into two groups. The first of them contains all the valid syllogisms with presupposition such that only non-empty subsets of Y are considered. The second one assumes that only non-empty subsets of M are considered.

Figure IV— only non-empty subsets of Y are considered.

Theorem 30
The following classical syllogism is strongly valid.

$$\textbf{\textit{AAI-IV:}}\ \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{All } M \textit{ are } X\end{array}}{\textit{Some } X \textit{ are } Y} \qquad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow Mx) \mathbin{\&} (\exists x)Yx\\ (\forall x)(Mx \Rightarrow Xx)\end{array}}{(\exists x)(Xx \wedge Yx).}$$

PROOF: By the properties of FTT and quantifiers we get

$$\vdash (\forall x)(Yx \Rightarrow Mx) \mathbin{\&} (\forall x)(Mx \Rightarrow Xx) \mathbin{\&} (\exists x)Yx \Rightarrow ((\exists x)(Xx \wedge Yx))$$

which is just strong validity of our syllogism. □

Theorem 31
The following syllogisms are weakly valid.

$$\textbf{TAI-IV:}\ \frac{\begin{array}{l}\text{Most } Y \text{ are } M\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \textbf{PAI-IV:}\ \frac{\begin{array}{l}\text{Almost all } Y \text{ are } M\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

$$\textbf{KAI-IV:}\ \frac{\begin{array}{l}\text{Many } Y \text{ are } M\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

PROOF: Using Definition 5, the syllogism **TAI-IV** can be written as

$$\frac{\begin{array}{l}(\exists z)((\mathbf{\Delta}(z \subseteq Y) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(Y)z))\\ (\forall x)(Mx \Rightarrow Xx)\end{array}}{(\exists x)(Xx \wedge Yx).}$$

We start with the provable formula

$$\vdash (zx \Rightarrow Mx) \mathbin{\&} (Mx \Rightarrow Xx) \Rightarrow (zx \Rightarrow Xx)$$

and thus

$$\vdash (Mx \Rightarrow Xx) \mathbin{\&} (zx \Rightarrow Yx) \mathbin{\&} (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Yx) \mathbin{\&} (zx \Rightarrow Xx).$$

Then we continue analogously as in the proof of **ETO-I**.

Analogously we obtain the weak validity of **PAI-IV** and **KAI-IV**. □

Figure IV— only non-empty subsets of M are considered.

Theorem 32
The following classical syllogisms are strongly valid.

$$\textbf{EAO-IV:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array} \qquad \begin{array}{l} (\forall x)(Yx \Rightarrow \neg Mx) \\ (\forall x)(Mx \Rightarrow Xx)(\exists x)Mx \\ \hline (\exists x)(Xx \wedge \neg Yx) \end{array}$$

$$\textbf{AEO-IV:}\quad \begin{array}{l} \text{All } Y \text{ are } M \\ \text{No } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array} \qquad \begin{array}{l} (\forall x)(Yx \Rightarrow Mx) \\ (\forall x)(Mx \Rightarrow \neg Xx) \mathbin{\&} (\exists x)Mx \\ \hline (\exists x)(Xx \wedge \neg Yx) \end{array}$$

PROOF: By contraposition, properties of FTT and by quantifiers properties we get that

$$\vdash (\forall x)(Yx \Rightarrow \neg Mx) \mathbin{\&} (\forall x)(Mx \Rightarrow Xx) \mathbin{\&} (\exists x)Mx \Rightarrow (\exists x)(Xx \wedge \neg Yx)$$

which is just strong validity of **EAO-IV**. By Theorem 3(b) we may prove the second classical syllogism **AEO-IV**. □

The section is closed by three intermediate generalized syllogisms with presupposition.

Theorem 33
All the syllogisms below are weakly valid.

$$\textbf{ETO-IV:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{Most } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array} \qquad \textbf{EPO-IV:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{Almost all } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

$$\textbf{EKO-IV:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{Many } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

PROOF: Analogously as above the first syllogism we can write in the following

$$\begin{array}{l} (\forall x)(Yx \Rightarrow \neg Mx) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ve)(\mu(M)z)) \\ \hline (\exists x)(Xx \wedge \neg Yx). \end{array}$$

The proof of this syllogism is obtained by contraposition and by properties of FTT from

$$\vdash (Yx \Rightarrow \neg Mx) \mathbin{\&} (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx)$$

and hence

$$\vdash (Yx \Rightarrow \neg Mx) \Rightarrow ((zx \Rightarrow Xx) \mathbin{\&} (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Xx) \mathbin{\&} (zx \Rightarrow \neg Yx)).$$

Then we continue using the same steps as in the proof of **ETO-I** and we conclude that **ETO-IV** is weakly valid.

Similarly we may prove weak validity of the other two syllogisms **EKO-IV** and **EPO-IV**. □

5. Interpretation

In this section we will show how can be our syllogisms interpreted in a model. We introduce four examples of syllogisms which are either weakly or strongly valid. In all examples, we will consider a simple model with a finite set M_ϵ of elements.

The frame of the constructed model is the following:

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

where $M_o = [0,1]$ is the support of the standard Łukasiewicz$_\Delta$ algebra. The fuzzy equality $=_o$ is the Łukasiewicz biresiduation $\leftrightarrow$. Furthermore, $M_\epsilon = \{u_1, \ldots, u_r\}$ is a finite set with fixed numbering of its elements and $=_\epsilon$ is defined by

$$[u_i =_\epsilon u_j] = (1 - \min\left(1, \frac{|i-j|}{s}\right))$$

for some fixed natural number $s \leq r$. This a separated fuzzy equality w.r.t. the Łukasiewicz conjunction $\otimes$. It can be verified that all the logical axioms of Ł-FTT are true in the degree 1 in $\mathcal{M}$ (all the considered functions are weakly extensional w.r.t. $\mathcal{M}(\equiv)$). Moreover, $\mathcal{M}$ is nontrivial because $1 - \frac{|i-j|}{s} \in (0,1)$ implies $\frac{|i-j|}{s} \in (0,1)$ and thus, taking the assignment p such that $p(x_\epsilon) = u_i$, $p(y_\epsilon) = u_j$ and considering $A_o := x_\epsilon \equiv y_\epsilon$, we obtain $\mathcal{M}_p(A_o \vee \neg A_o) \in (0,1)$.

To make $\mathcal{M}$ a model of T^{Ev} and T^{IQ}, we define interpretation of $\sim$ by $\mathcal{M}(\sim) = \leftrightarrow^2$, $\mathcal{M}(\dagger) = 0.5$ and put $\mathcal{M}(\boldsymbol{\nu})$ equal to a function $\nu_{a,b,c}$ which is a simple partially quadratic function given in [11]. In Figure 2, extensions of several evaluative expressions used below are depicted. It can be verified that $\mathcal{M} \models T^{\mathrm{Ev}}$.

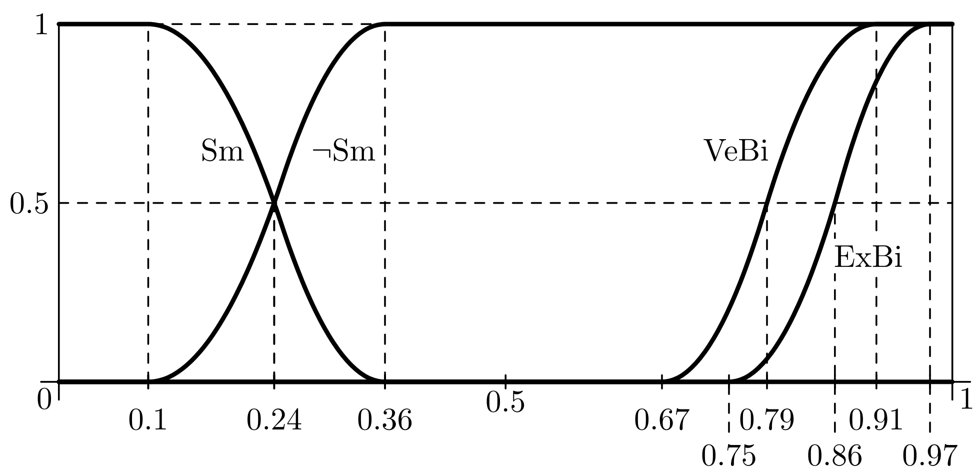

Figure 2: Shapes of the extensions of evaluative expressions in the context $[0,1]$ used in the examples below

The distinguished set $\mathcal{S} \subset Types$ is defined as follows: $\alpha \in \mathcal{S}$ iff α is a type not containing the type o of truth values. This means that all sets M_α for $\alpha \in \mathcal{S}$ are finite.

Let $A \underset{\sim}{\subset} M_\alpha$, $\alpha \in \mathcal{S}$ be a fuzzy set. We will put

$$|A| = \sum_{u \in \mathrm{Supp}(A)} A(u), \qquad u \in M_\alpha. \tag{15}$$

Furthermore, for fuzzy sets $A, B \underset{\sim}{\subset} M_\alpha$, $\alpha \in \mathcal{S}$ we define

$$F_R(B)(A) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = B, \\ \frac{|A|}{|B|} & \text{if } B \neq \emptyset \text{ and } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Interpretation of the constants $R \in Form_{o(o\alpha)(o\alpha)}$, $\alpha \in S$ is defined by $\mathcal{M}(R) = F_R$ where $F_R : \mathcal{F}(M_\alpha) \times \mathcal{F}(M_\alpha) \longrightarrow L$ is the function (16). It can be verified that axioms (M1)–(M3) are true in the degree 1 in $\mathcal{M}$. Thus, $\mathcal{M} \models T^{\mathrm{IQ}}$.

Below, we will demonstrate on concrete examples, how some of the syllogisms proved above behave on this model.

5.1. Example of weakly valid syllogism of Figure I.

Let us consider the following syllogism:

$$\textbf{ATT-I:}\quad \begin{array}{l} \text{All women are well dressed} \\ \text{Most people in the party are women} \\ \hline \text{Most people in the party are well dressed} \end{array}$$

Let M_ϵ be a set of people. Let $\mathrm{Wom}_{o\epsilon}$ be a formula representing "women" which is interpreted by $\mathcal{M}(\mathrm{Wom}_{o\epsilon}) = W \subseteq M_\epsilon$ where W is a classical set. Furthermore, let $\mathrm{Peop}_{o\epsilon}$ be a formula representing "people in the party" interpreted by $\mathcal{M}(\mathrm{Peop}_{o\epsilon}) = P \subseteq M_\epsilon$ where P is a classical set. Finally, let $\mathrm{Dress}_{o\epsilon}$ be a formula interpreted by $\mathcal{M}(\mathrm{Dress}_{o\epsilon}) = D \subseteq M_\epsilon$. Let us

Major premise: "All women are well dressed". From the assumption

$$\mathcal{M}((\forall x_\epsilon)(\mathrm{Wom}_{o\epsilon}(x_\epsilon) \Rightarrow \mathrm{Dress}_{o\epsilon}(x_\epsilon))) = \bigwedge_{m\in M_\epsilon} (\mathcal{M}(\mathrm{Wom}_{o\epsilon})(m) \to \mathcal{M}(\mathrm{Dress}_{o\epsilon})(m)) = 1$$

we conclude that $W \subseteq D$.

Minor premise: "Most people in the party are women". The assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Peop}_{o\epsilon})\,\&\,(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Wom}_{o\epsilon})) \wedge (Bi\ Ve)(\mu(\mathrm{Peop}_{o\epsilon})z_{o\epsilon}))) = 1 \quad (17)$$

leads to requirement to find the greatest subset $\mathcal{M}(z_{o\epsilon}) = W' \subseteq P$ such that:

$$\mathcal{M}(\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Peop}_{o\epsilon})) = 1, \quad (18)$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Wom}_{o\epsilon}) = 1, \quad (19)$$

$$\mathcal{M}((Bi\ Ve)(\mu(\mathrm{Peop}_{o\epsilon})z_{o\epsilon})) = 1. \quad (20)$$

One can verify that this holds if $W' = W$.

From (20) and the interpretation of evaluative expressions (see Figure 2) it follows that $\mathcal{M}(\mu(\mathrm{Peop}_{o\epsilon})z_{o\epsilon}) = F_R(P, W) \geq 0.91$. Thus, for example, if $|P| = 100$ then $|W| \geq 91$.

Conclusion: "Most people in the party are well dressed". The conclusion is the formula

$$Q^{\forall}_{Bi\ Ve}(\mathrm{Peop}_{o\epsilon}, \mathrm{Dress}_{o\epsilon}) := (\exists z_{o\epsilon})((\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Peop}_{o\epsilon})\,\&\,(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Dress}_{o\epsilon})) \wedge (Bi\ Ve)(\mu(\mathrm{Peop}_{o\epsilon})z_{o\epsilon})). \quad (21)$$

Because we are dealing with classical sets, we conclude that to find a truth value of (21) requires to find a set $\mathcal{M}(z_{o\epsilon}) = D'$, where $D' \subseteq P$ and $D' \subseteq D$, which maximizes the truth value

$$\mathcal{M}((Bi\ Ve)(\mu(\mathrm{Peop}_{o\epsilon})z_{o\epsilon})). \quad (22)$$

But from the first premise we know that $W \subseteq D$. From the fact that $F_R(P, W)$ provides the truth value 1 in (20) and from $D' \subseteq P$ we conclude that $W \subseteq D'$. Hence, $\mathcal{M}(\mu(\mathrm{Peop}_{o\epsilon})z_{o\epsilon}) = F_R(P, D')$ provides the truth value 1 in (22). Consequently,

$$\mathcal{M}(Q^{\forall}_{Bi\ Ve}(\mathrm{Peop}_{o\epsilon}, \mathrm{Dress}_{o\epsilon})) = 1$$

which verifies that this syllogism is weakly valid in our model.

For example, if $|P| = 100$ then the quantifier "most" means at least 91 people. By the discussed syllogism, if we know that all women are well dressed and most people in the party are women then we conclude that at least 91 people in the party are well dressed.

5.2. Example of weakly valid syllogism of Figure II.

$$\textbf{ETO-II:}\quad \frac{\begin{array}{l}\text{No lazy people pass exam}\\ \text{Most students pass exam}\end{array}}{\text{Some students are not lazy people}}$$

Suppose the same model and the definition of measure as above. Let M_ϵ be a set of people. Let $\mathrm{LP}_{o\epsilon}$ be a formula "lazy people" with the interpretation $\mathcal{M}(\mathrm{LP}_{o\epsilon}) = L \subseteq M_\epsilon$ where L is a classical set. Let $\mathrm{St}_{o\epsilon}$ be a formula "students" interpreted by $\mathcal{M}(\mathrm{St}_{o\epsilon}) = S \subseteq M_\epsilon$ where S is a classical set. Finally, let $\mathrm{Exam}_{o\epsilon}$ be a formula "students who pass exams" with the interpretation $\mathcal{M}(\mathrm{Exam}_{o\epsilon}) = E \subseteq M_\epsilon$ where E is a classical set.

Major premise: "No lazy people pass exam". From the assumption

$$\mathcal{M}((\forall x_\epsilon)(\mathrm{LP}_{o\epsilon}(x_\epsilon) \Rightarrow (\neg \mathrm{Exam}_{o\epsilon}(x_\epsilon)))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\mathrm{LP}_{o\epsilon}(m)) \to (1 - \mathcal{M}(\mathrm{Exam}_{o\epsilon}(m)))) = 1. \tag{23}$$

we conclude that $L \subseteq M_\epsilon - E$, i.e., $E \subseteq M_\epsilon - L$.

Minor premise: "Most students pass exam". The assumption

$$(\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{St}_{o\epsilon})\ \&(\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Exam}_{o\epsilon}))\ \wedge\ (Bi\ Ve)(\mu(\mathrm{St}_{o\epsilon}) z_{o\epsilon})) = 1 \tag{24}$$

means to find the greatest subset $\mathcal{M}(z_{o\epsilon}) = E' \subseteq S$ such that:

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{St}_{o\epsilon})) = 1 \tag{25}$$
$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Exam}_{o\epsilon})) = 1 \tag{26}$$
$$\mathcal{M}((Bi\ Ve)(\mu(\mathrm{St}_{o\epsilon}) z_{o\epsilon}) = 1. \tag{27}$$

This holds if $E' \subseteq S \cap E$. Furthermore, from (27) and the interpretation of evaluative expressions (Figure 2) we conclude that $\mathcal{M}(\mu(\mathrm{St}_{o\epsilon}) z_{o\epsilon}) = F_R(S, E') \geq 0.91$, which means that $|S \cap E| \geq 0.91|S|$. Thus, for example, if $|S| = 100$ then $|S \cap E| \geq 91$.

Conclusion: "Some students are not lazy people". The conclusion is the formula

$$Q^{\exists}_{Bi\boldsymbol{\Delta}}(\mathrm{St}_{o\epsilon}, \mathrm{LSt}_{o\epsilon}) := (\exists x_\epsilon)(\mathrm{St}_{o\epsilon}(x_\epsilon) \wedge (\neg \mathrm{LSt}_{o\epsilon}(x_\epsilon))). \tag{28}$$

The interpretation $\mathcal{M}(\mathrm{St}_{o\epsilon}(x_\epsilon) \wedge \neg \mathrm{LSt}_{o\epsilon}(x_\epsilon)) = S \cap (M_\epsilon - L)$. From both premises we have that $E' \subseteq (M_\epsilon - L)$ and $E' \subseteq S$ thus $E' \subseteq S \cap (M_\epsilon - L)$ which means that $S \cap (M_\epsilon - L) \neq \emptyset$ and we conclude that

$$\mathcal{M}(Q^{\exists}_{Bi\boldsymbol{\Delta}}(\mathrm{St}_{o\epsilon}, \mathrm{LSt}_{o\epsilon})) = 1$$

which means that the syllogism **ETO-II** is weakly valid in our model. In the example, we see even more — that at least 91 students of 100 are not lazy people.

5.3. Example of strongly valid syllogism of Figure III.

$$\textbf{PPI-III:}\quad \frac{\begin{array}{l}\text{Almost all old people are ill}\\ \text{Almost all old people have gray hair}\end{array}}{\text{Some people with gray hair are ill}}$$

Suppose the same frame and the measure as above. Let M_ϵ be a set of people. We consider four people with the following age: u_1 (40 years), u_2 (70 years), u_3 (82 years), u_4 (95 years). Now we define interpretation of the formulas from our syllogism as follows: Let $\mathrm{Old}_{o\epsilon}$ be a formula "old people " with the interpretation $\mathcal{M}(\mathrm{Old}_{o\epsilon}) = O \underset{\sim}{\subset} M_\epsilon$ defined by

$$O = \{{}^{0.3}/u_1, {}^{0.55}/u_2, {}^{0.8}/u_3, {}^{0.95}/u_4\}.$$

Let $\mathrm{Gr}_{o\epsilon}$ be a formula "people with gray hair" with the interpretation $\mathcal{M}(\mathrm{Gr}_{o\epsilon}) = G \underset{\sim}{\subset} M_\epsilon$ defined by

$$G = \{^{0.3}/u_1, ^{0.55}/u_2, ^{0.85}/u_3, ^{0.9}/u_4\}.$$

Finally, let $\mathrm{Ill}_{o\epsilon}$ be a formula "Ill people " with the interpretation $\mathcal{M}(\mathrm{Ill}_{o\epsilon}) = I \underset{\sim}{\subset} M_\epsilon$ defined by

$$I = \{^{0.2}/u_1, ^{0.4}/u_2, ^{0.75}/u_3, ^{0.95}/u_4\}.$$

Major premise: "Almost all old people are ill". Let the assumption be

$$\mathcal{M}((\exists z_{o\epsilon})((\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Ill}_{o\epsilon})) \wedge (Bi\ Ex)(\mu(\mathrm{Old}_{o\epsilon})z_{o\epsilon}))) = a \in (0,1]. \quad (29)$$

This leads to requirement to find the biggest fuzzy set $\mathcal{M}(z_{o\epsilon}) = X \underset{\sim}{\subset} M_\epsilon$ such that

$$\mathcal{M}(\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon})) = 1, \quad (30)$$
$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Ill}_{o\epsilon})) = b, \quad (31)$$
$$\mathcal{M}((Bi\ Ex)(\mu(\mathrm{Old}_{o\epsilon})z_{o\epsilon})) = c, \quad (32)$$

where $b \wedge c = a$. From (32) and Figure 2 it follows that

$$\mathcal{M}(\mu(\mathrm{Old}_{o\epsilon})z_{o\epsilon}) = F_R(O, X) > 0.75.$$

It can be verified that just the fuzzy set $O \underset{\sim}{\subset} M_\epsilon$ has the properties above and gives us the greatest degree in (29). Thus we conclude that for $\mathcal{M}(z_{o\epsilon}) = O \underset{\sim}{\subset} M_\epsilon$ we have that $c = 1$, $b = 0.85$ and hence

$$\mathcal{M}(Q^{\forall}_{Bi\ Ex}(\mathrm{Old}_{o\epsilon}, \mathrm{Ill}_{o\epsilon})) = a = 0.85. \quad (33)$$

Minor premise: "Almost all old people have gray hair". From the assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Gr}_{o\epsilon})) \wedge (Bi\ Ex)(\mu(\mathrm{Old}_{o\epsilon})z_{o\epsilon}))) = a' \in (0,1]. \quad (34)$$

Analogously as above this means to find the biggest fuzzy set $\mathcal{M}(z_{o\epsilon}) = Y \underset{\sim}{\subset} M_\epsilon$ such that

$$\mathcal{M}(\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon})) = 1 \quad (35)$$
$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Gr}_{o\epsilon})) = b' \quad (36)$$
$$\mathcal{M}((Bi\ Ex)(\mu(\mathrm{Old}_{o\epsilon})z_{o\epsilon})) = c' \quad (37)$$

where $b' \wedge c' = a'$. From (37) and Figure 2 it follows that

$$\mathcal{M}(\mu(\mathrm{Old}_{o\epsilon})z_{o\epsilon}) = F_R(O, Y) > 0.75.$$

Analogously as above, it can be verified that just the fuzzy set $O \underset{\sim}{\subset} M_\epsilon$ has the properties above and gives us the greatest degree in (34). Thus we obtain that $c' = 1$, $b' = 0.95$ and hence

$$\mathcal{M}(Q^{\forall}_{Bi\ Ex}(\mathrm{Old}_{o\epsilon}, \mathrm{Gr}_{o\epsilon})) = a' = 0.95. \quad (38)$$

Conclusion: "Some people with gray hair are ill". The conclusion is the formula

$$Q^{\exists}_{Bi\mathbf{\Delta}}(\mathrm{Gr}_{o\epsilon}, \mathrm{Ill}_{o\epsilon}) := (\exists x_\epsilon)(\mathrm{Gr}_{o\epsilon}(x_\epsilon) \wedge \mathrm{Ill}_{o\epsilon}(x_\epsilon)) \quad (39)$$

which is interpreted by

$$\mathcal{M}(Q^{\exists}_{Bi\mathbf{\Delta}}(\mathrm{Gr}_{o\epsilon}, \mathrm{Ill}_{o\epsilon})) = \bigvee_{m \in M_\epsilon} (\mathcal{M}(\mathrm{Gr}_{o\epsilon}(m)) \wedge \mathcal{M}(\mathrm{Ill}_{o\epsilon}(m))) = 0.9. \quad (40)$$

From (33),(38) and (40) we can see that $\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = 0.8 \leq \mathcal{M}(C) = 0.9$ which means that the syllogism above is strongly valid in our model.

We continue with two examples of invalid syllogisms which are generalization of invalid syllogisms introduced in Peterson's book. By *invalid* syllogism we mean that in the model which will be constructed, the premises are true in the degree 1 and the truth of the conclusion is smaller than 1.

5.4. First example of invalid syllogism

$$\textbf{TAT-III:}\quad \begin{array}{l} \text{Most bushes in the park are in blossom.} \\ \text{All bushes in the park are perennial.} \\ \hline \text{Most perennial in the park are in blossom.} \end{array}$$

Suppose the same frame as above. Let M_ϵ be a set of "vegetables in the park". Let $\mathrm{Bushe}_{o\epsilon}$ be a formula "bushes in the park" with the interpretation $\mathcal{M}(\mathrm{Bushe}_{o\epsilon}) = B \subseteq M_\epsilon$ where B is a classical set of 100 bushes. Furthermore, let $\mathrm{Bl}_{o\epsilon}$ be a formula "in blossom" with the interpretation $\mathcal{M}(\mathrm{Bl}_{o\epsilon}) = F \subseteq M_\epsilon$ where F is a classical set of 95 vegetables in blossom. Finally, let $\mathrm{Per}_{o\epsilon}$ be a formula "perennial" with the interpretation $\mathcal{M}(\mathrm{Per}_{o\epsilon}) = P \subseteq M_\epsilon$ where P is a classical set of 120 perennial. Let us

Major premise "Most bushes in the park are in blossom". The assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Bushe}_{o\epsilon})\,\&(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Bl}_{o\epsilon}x_\epsilon)) \wedge (Bi\ Ve)(\mu(\mathrm{Bushe}_{o\epsilon})z_{o\epsilon}))) = 1 \tag{41}$$

means to find the biggest subset $\mathcal{M}(z_{o\epsilon}) = F' \subseteq B$ such that:

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Bushe}_{o\epsilon})) = 1 \tag{42}$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Bl}_{o\epsilon})) = 1 \tag{43}$$

$$\mathcal{M}((Bi\ Ve)(\mu(\mathrm{Bushe}_{o\epsilon})z_{o\epsilon})) = 1. \tag{44}$$

It can be verified that this holds if $F' = F$.

From (44) and Figure 2 it follows that $\mathcal{M}(\mu(\mathrm{Bushe}_{o\epsilon})z_{o\epsilon}) = F_R(B, F) \geq 0.91$. This means that if $|B| = 100$, then $|F| \geq 91$

Minor premise "All bushes in the park are perennial". The assumption

$$\mathcal{M}((\forall x_\epsilon)(\mathrm{Bushe}_{o\epsilon}(x_\epsilon) \Rightarrow \mathrm{Per}_{o\epsilon}(x_\epsilon))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\mathrm{Bushe}_{o\epsilon}(m) \to \mathcal{M}(\mathrm{Per}_{o\epsilon}(m))) = 1 \tag{45}$$

means that $B \subseteq P$ and hence $F' \subseteq P$.

Conclusion "Most perennial in the park are in blossom". The conclusion is the following formula:

$$Q^{\forall}_{Bi\ Ve}(\mathrm{Per}_{o\epsilon}, \mathrm{Bl}_{o\epsilon}) := (\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Per}_{o\epsilon})\,\&(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Bl}_{o\epsilon}x_\epsilon)) \wedge (Bi\ Ve)(\mu(\mathrm{Per}_{o\epsilon})z_{o\epsilon})). \tag{46}$$

From the first premise for $\mathcal{M}(z_{o\epsilon}) = F'$ we have that

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Bl}_{o\epsilon})) = 1.$$

From the second one we obtain that

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Per}_{o\epsilon})) = 1 \tag{47}$$

From Figure 2 and from

$$\mathcal{M}(\mu(\mathrm{Per}_{o\epsilon})z_{o\epsilon}) = F_R(P, F') = 0.83$$

we obtain that

$$\mathcal{M}((Bi\ Ve)(\mu(\mathrm{Per}_{o\epsilon})z_{o\epsilon})) < 1.$$

Consequently, we conclude that

$$M(Q^{\forall}_{Bi\ Ve}(\mathrm{Per}_{o\epsilon}, \mathrm{Bl}_{o\epsilon})) < 1$$

which means that the syllogism **TAT-III** is invalid in our model .

5.5. Second example of invalid syllogism

$$\textbf{TAK-III:}\quad \frac{\begin{array}{l}\text{Most good dancers in the party are young people.}\\ \text{All good dancers in the party are very nice dressed.}\end{array}}{\text{Most very nice dressed dancers in the party are young people.}}$$

Suppose the same frame as above. Let M_ϵ be a set of "dancers in the party". We consider four dancers with the following age: d_1 (35 years), d_2 (45 years), d_3 (60 years), d_4 (70 years). Now we define the interpretation of the formulas from our syllogism in the following: Let $\text{Dance}_{o\epsilon}$ be a formula "good dancers in the party" with the interpretation $\mathcal{M}(\text{Dance}_{o\epsilon}) = D \underset{\sim}{\subseteq} M_\epsilon$ defined as follows

$$D = \{^{0.7}/d_1, ^{0.3}/d_2, ^{0.1}/d_3, ^{0.05}/d_4\}.$$

Furthermore, let $\text{Young}_{o\epsilon}$ be a formula "young people" with the interpretation $\mathcal{M}(\text{Young}_{o\epsilon}) = Y \underset{\sim}{\subseteq} M_\epsilon$ defined by

$$Y = \{^{0.9}/d_1, ^{0.8}/d_2, ^{0.75}/d_3, ^{0.6}/d_4\}.$$

Finally, let $\text{VeDr}_{o\epsilon}$ be a formula " very nice dressed" with the interpretation $\mathcal{M}(\text{VeDr}_{o\epsilon}) = V \underset{\sim}{\subseteq} M_\epsilon$ defined as follows

$$V = \{^{0.95}/d_1, ^{0.9}/d_2, ^{0.85}/d_3, ^{0.7}/d_4\}.$$

Major premise "Most good dancers in the party are young people". The assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \text{Dance}_{o\epsilon})\,\&\,(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \text{Young}_{o\epsilon}x_\epsilon)) \wedge (Bi\ Ve)(\mu(\text{Dance}_{o\epsilon})z_{o\epsilon}))) = 1 \tag{48}$$

means to find the biggest fuzzy subset $D' \underset{\sim}{\subseteq} M_\epsilon$ such that $\mathcal{M}(z_{o\epsilon}) = D'$ and the following holds:

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \text{Dance}_{o\epsilon})) = 1 \tag{49}$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \text{Young}_{o\epsilon})) = 1 \tag{50}$$

$$\mathcal{M}((Bi\ Ve)(\mu(\text{Dance}_{o\epsilon})z_{o\epsilon})) = 1. \tag{51}$$

It can be verified that this holds if $D' = D$.

From (51) and Figure 2 it follows that $\mathcal{M}(\mu(\text{Dance}_{o\epsilon})z_{o\epsilon}) = F_R(D, D') \geq 0.91$.

Minor premise "All good dancers in the party are very nice dressed". The assumption is

$$\mathcal{M}((\forall x_\epsilon)(\text{Dance}_{o\epsilon}(x_\epsilon) \Rightarrow \text{VeDr}_{o\epsilon}(x_\epsilon))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\text{Dance}_{o\epsilon}(m) \rightarrow \mathcal{M}(\text{VeDr}_{o\epsilon}(m))) = 1. \tag{52}$$

Conclusion "Most very nice dressed dancers in the party are young people". The conclusion is the following formula:

$$Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(\text{VeDr}_{o\epsilon}, \text{Young}_{o\epsilon}) := (\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \text{VeDr}_{o\epsilon})\,\&\,(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \text{Young}_{o\epsilon}x_\epsilon)) \wedge (\neg(Sm\,\bar{\boldsymbol{\nu}}))(\mu(\text{VeDr}_{o\epsilon})z_{o\epsilon})). \tag{53}$$

From the first premise for $\mathcal{M}(z_{o\epsilon}) = D' = D$ we have

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \text{Young}_{o\epsilon})) = 1.$$

From the second one we obtain that

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \text{VeDr}_{o\epsilon})) = 1 \tag{54}$$

because (52) is equivalent with (54). From Figure 2 and from

$$\mathcal{M}(\mu(\text{VeDr}_{o\epsilon})z_{o\epsilon}) = F_R(V, D') = 0.34$$

we obtain that

$$\mathcal{M}(\neg(Sm\,\bar{\boldsymbol{\nu}})(\mu(\mathrm{VeDr}_{o\epsilon})z_{o\epsilon})) < 1.$$

Consequently, we conclude that

$$M(Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(\mathrm{VeDr}_{o\epsilon}, \mathrm{Young}_{o\epsilon})) < 1$$

which means that the syllogism **TAK-III** is invalid in our model .

6. Conclusion

In this paper, we continued development of the formal theory of intermediate quantifiers whose basic idea is that they are just classical quantifiers but taken over modified universe where the modification is obtained by using an evaluative linguistic expression. The intermediate quantifier is thus represented by a specific formula.

Following the book of P. Peterson [17] where 105 syllogisms which generalized the classical Aristotle's ones were informally demonstrated to be valid, we showed that all of them are valid also in our theory. The validity is in most cases weak (the provability of the conclusion follows from the provability of both premises) but in some cases it is even strong which means that the implication $P_1 \,\&\, P_2 \Rightarrow C$ is provable and so, the truth values of C in any model is greater or equal to the truth value of $P_1 \,\&\, P_2$.

There are several interesting problems to be solved in the future: the structure of the quantifiers should be analyzed in detail (e.g., the relations of contraries and subcontraries — how they should be defined in fuzzy logic); further kinds of generalized quantifiers should be defined and the corresponding generalized syllogism should be formulated; how the concept of possible worlds could be incorporated (cf. [8, 11]) because, as could be seen in our examples above, we are dealing with properties whose extension depends on the possible world; what is the position of the intermediate quantifiers in the general theory (cf. [3, 5, 16]). We are also convinced that our theory has the great application potential.

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