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Some Consequences of Herbrand and McNaughton Theorems in Fuzzy Logic

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This paper deals with the formal theory of first-order fuzzy logic in narrow sense (FLn), namely its form with evaluated syntax since in our oppinion, it might serve as a pivotal fuzzy logic. It has been presented in many papers (see, e.g. [6]) and in details is explained in the book [10]. Note that from the point of metamathematics presented in the book [2], it is a specific logical system (called there Rational Pavelka logic) which extends Łukasiewicz logic. The truth values are therefore taken from the interval $[0, 1]$ and the system of basic logical connectives is the Łukasiewicz one. Its specific feature is introduction of symbols for all truth values (called *logical constants*) into the language. Though we work with logical constants for all the truth values, its has been demonstrated (for example in [9]) that only rational logical constants are necessary and the rest can be used as a significant technical, though dispensable, means.

Our program is to formulate the most important results of classical logic also in FLn since the former is a special case of the latter. Namely, we focus on the formulation of the analogue of the classical Herbrand theorem.Recall that the generalization of the classical completeness theorem holds in fuzzy logic. It states, roughly speaking, that the provability and truth degrees of a formula in a fuzzy theory (a theory given by a fuzzy set of axioms) are equal. The generalization of the Hilbert–Ackermann consistency theorem has been proved in [7]. On the basis of it, the fuzzy analogue of the Herbrand theorem has been proved in [8]. In this paper, we present few properties of fuzzy theories and prove a stronger version of the Herbrand theorem. Furthermore, we recall the result of McNaughton [4] stating that all functions representable by logical formulas are piecewise linear and, conversely, each piecewise linear function with the integer coefficients can be represented by a logical formula. Our reasoning is based on the proof presented in [12], which is constructive and thus, it allows to represent each piecewise linear function by its normal form — a formula analogous to the perfect disjunctive normal form.

1 Preliminaries

We will recall only few basic notions of FLn. The reader may find the precise definitions and full proofs of theorems (if missing) in the cited papers.

In general, the set of truth values is supposed to form a complete residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle. \tag{1}$$

There are good reasons to assume, moreover, that (1) is the Łukasiewicz algebra

$$\mathcal{L}_{\mathrm{L}} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle \tag{2}$$

where

$$a \rightarrow b = 1 \wedge (1 - a + b), \qquad \text{(Łukasiewicz implication)}$$
$$a \otimes b = 0 \vee (a + b - 1). \qquad \text{(Łukasiewicz conjunction).}$$

Furthermore, we define the following operations

$$\begin{aligned} \neg a &= a \to 0 = 1 - a, && \text{(negation)} \\ a \oplus b &= \neg(\neg a \otimes \neg b) = 1 \wedge (a + b), && \text{(Łukasiewicz disjunction)} \\ a^n &= \underbrace{a \otimes \cdots \otimes a}_{n\text{-times}}, && \text{(n-fold conjunction)} \\ na &= \underbrace{a \oplus \cdots \oplus a}_{n\text{-times}}. && \text{(n-fold disjunction)} \end{aligned}$$

The *language* J of FLn consists of variables, constants, n-ary functional and predicate symbols, binary predicate symbol $=$ (equality sign), logical constants $\mathbf{a}$, $a \in [0, 1]$, the binary connective $\Rightarrow$ and symbol for the general quantifier $\forall$.

Terms and formulas are defined as usual with the exception that all logical constants are atomic formulas. The common abbreviations of formulas $\neg A$ (negation), $A \vee B$ (disjunction), $A \wedge B$ (conjunction), $A \mathbin{\&} B$ (Łukasiewicz conjunction), $A \Leftrightarrow B$ (equivalence), $(\exists x)A$, A^k are introduced (see [5, 6, 10]). Moreover, we will use also the abbreviation $A \nabla B$ defined by

$$A \nabla B := \neg(\neg A \mathbin{\&} \neg B)$$

and call it *Lukasiewicz disjunction*. By F_J, we denote the set of all well formed formulas of the language J.

As explained in the cited works, syntax of fuzzy logic is evaluated by *syntactic evaluations* taken from L. An *evaluated formula* is a couple

$$a/A$$

where $A \in F_J$ and $a \in L$. By $A(x_1, \ldots, x_n)$ we understand a formula the free variables of which appear among $x_1, \ldots, x_n$.

A *theory* T in the language J of FLn (called a *fuzzy theory*) is a triple

$$T = \langle \mathrm{Ax}^{\mathrm{L}}, \mathrm{Ax}^{\mathrm{S}}, \{r_{MP}, r_G\}\rangle$$

where $\mathrm{Ax}^{\mathrm{L}}, \mathrm{Ax}^{\mathrm{S}} \underset{\sim}{\subset} F_J$ are *fuzzy sets* (or, equivalently, sets of evaluated formulas) of logical and special axioms, respectively and r_{MP}, r_G are the inference rules of modus ponens and generalization, respectively. By $J(T)$ we denote the language of fuzzy theory. A *fuzzy predicate calculus* is the fuzzy theory with $\mathrm{Ax}^{\mathrm{S}} = \emptyset$. By F_J, we denote the set of all the well formed formulas for the given language J of FLn.

A *structure* for the language $J(T)$ is denoted by

$$\mathcal{D} = \langle D, f_D, \ldots, P_D, \ldots, u \ldots\rangle$$

where $f_D, \ldots$ are crisp n-ary functions assigned to the respective functional symbols, $P_D \underset{\sim}{\subset} D^n, \ldots$ are fuzzy relations assigned to the predicate symbols and $u, \ldots \in D$ are elements assigned to the constants of $J(T)$. If T is a fuzzy theory and $(\mathrm{Ax}^{\mathrm{L}} \cup \mathrm{Ax}^{\mathrm{S}})(A) \le \mathcal{D}(A)$ for all axioms of T then $\mathcal{D}$ is a *model* of T, $\mathcal{D} \models T$. Obviously, each structure is a model of the fuzzy predicate calculus.

In our approach to syntax, we work with evaluated proofs. If w is a proof in theory T then we write $\mathrm{Val}_T(w)$ for its value. If A is a formula and w its proof then we will write w_A to stress this fact. Recall that

$$C^{syn}(\mathrm{Ax}^{\mathrm{L}} \cup \mathrm{Ax}^{\mathrm{S}})(A) = \bigvee\{\mathrm{Val}(w_A) \mid w_A \text{ is a proof of } A\} \tag{3}$$

denotes the operation of syntactical consequence and

$$C^{sem}(\mathrm{Ax}^{\mathrm{L}} \cup \mathrm{Ax}^{\mathrm{S}})(A) = \bigwedge\{\mathcal{D}(A) \mid \mathcal{D} \models T\} \tag{4}$$

that of the semantical one.

If $C^{syn}(\mathrm{Ax}^{\mathrm{L}} \cup \mathrm{Ax}^{\mathrm{S}})(A) = a$ then A is a *theorem* in degree a of the theory T and we write

$$T \vdash_a A.$$

If $C^{sem}(\mathrm{Ax}^{\mathrm{L}} \cup \mathrm{Ax}^{\mathrm{S}})(A) = a$ then the formula A is *true* in degree a in the theory T and we write

$$T \models_a A.$$

We write $T \vdash A$, $T \models A$ instead of $T \vdash_1 A$, $T \models_1 A$, respectively and say that A is a theorem of (true in) theory T.

If T is a fuzzy theory and $\Gamma \underset{\sim}{\subseteq} F_{J(T)}$ a fuzzy set of formulas then $T' = T \cup \Gamma$ is a fuzzy theory whose special axioms are extended by formulas from Γ (as a union of fuzzy sets).

We say that a fuzzy theory T is *contradictory* (in the strong sense) iff there is a formula A and proofs w_A and $w_{\neg A}$ such that

$$\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) > 0. \tag{5}$$

Otherwise, we say that T is *consistent.*

The proof of the following theorem can be found in [6].

Theorem 1 *A fuzzy theory T is contradictory iff $T \vdash A$ holds for every formula $A \in F_{J(T)}$.*

In [7] we introduced the equality predicate $=$ satisfying the following (common) logical axioms:

(E1) $x = x$

(E2) There are $m_1, \dots, m_n$ such that

$$(x_1 = y_1)^{m_1} \Rightarrow \dots \Rightarrow (x_n = y_n)^{m_n} \Rightarrow (f(x_1, \dots, x_n) = f(y_1, \dots, y_n))$$

(E3) There are $m_1, \dots, m_n$ such that

$$(x_1 = y_1)^{m_1} \Rightarrow \dots \Rightarrow (x_n = y_n)^{m_n} \Rightarrow (p(x_1, \dots, x_n) \Rightarrow p(y_1, \dots, y_n))$$

for every n-ary functional symbol f and n-ary predicate symbol p.

A special kind of fuzzy equality is the sharp one interpreted by

$$\mathcal{D}(t = s) = \begin{cases} 1 & \text{if } \mathcal{D}(t) = \mathcal{D}(s), \\ 0 & \text{otherwise} \end{cases}$$

in every model $\mathcal{D}$.

In the sequel, we will use the symbol $*$ to denote first, the extended order relation

$$a >^* b \quad \text{iff} \quad \text{either } b < 1 \text{ and } a > b, \quad \text{or} \quad a = 1 \text{ and } b = 1.$$

and second, the operaton of truncation

$$x^* := \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

for any real number x.

2 Some Properties of Fuzzy Theories

Let us first recall some important theorems used in the sequel, which are proved in the cited literature.

Theorem 2 (deduction) *Let A be a closed formula and $T' = T \cup \{^1/A\}$. Then to every B there is n such that*

$$T \vdash_a A^n \Rightarrow B \quad \textit{iff} \quad T' \vdash_a B.$$

Theorem 3 (reduction for the consistency) *Let $\Gamma \underset{\sim}{\subseteq} F_J$ be a fuzzy set of closed formulas. A theory $T' = T \cup \Gamma$ is contradictory iff there are $m_1, \dots, m_n$ and $A_1, \dots, A_n \in \mathrm{Supp}(\Gamma)$ such that*

$$T \vdash_c \neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}$$

where $a_i = \Gamma(A_i)$, $i = 1, \dots, n$ and $c >^* \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$.

Corollary 1 *Let A be a closed formula A theory $T' = T \cup \{{}^{\neg a}/\neg A\}$ is contradictory iff $T \vdash_b mA$ for some m and $b >^* ma$.*

Lemma 1 *If a fuzzy theory $T' = T \cup \{{}^{a}/A\}$ is contradictory then for every formula B there is m such that $T \vdash A^m \Rightarrow B$.*

The following lemmas demonstrate some further properties of fuzzy theories that will be used in the next section. The use of the completeness theorem will be explicitly stressed whenever its use is necessary.

Lemma 2 *Let $T \vdash_a A$ and $T \vdash_b B$. Then $T \vdash_c A \nabla B$ where $c \geq a \oplus b$.*

PROOF: This follows from the completeness theorem and the properties of infimum. □

By induction we immediately obtain the following corollary.

Corollary 2 *If $T \vdash_a A$ then $T \vdash_b mA$ for every m where $b \geq ma$.*

Lemma 3 *If $T \vdash_a A \Rightarrow B$ then $T \vdash_b (\exists x)A \Rightarrow (\exists x)B$ where $a \leq b$.*

PROOF: To every proof $w_{A \Rightarrow B}$ we construct a proof of $(\exists x)A \Rightarrow (\exists x)B$ using the substitution axiom $\vdash B \Rightarrow (\exists x)B$, transitivity, rule of generalization and a provable formula $\vdash (\forall x)(A \Rightarrow B) \Leftrightarrow ((\exists x)A \Rightarrow B)$ having the same value as $w_{A \Rightarrow B}$. □

The following simple but useful lemma has been presented already in [6].

Lemma 4 *Let T_1, T_2 be fuzzy theories and A a formula. If for every $a, c \in L$ $T_1 \vdash_a A$ implies $T_2 \vdash_b A$ where $a \leq b$, and at the same time $T_2 \vdash_c A$ implies $T_1 \vdash_d A$ where $c \leq d$ then*

$$T_1 \vdash_a A \quad \textit{iff} \quad T_2 \vdash_a A.$$

Lemma 5 *Let T be a consistent theory and $T \vdash_a A$ where A is a closed formula. Let $T' = T \cup \{{}^{b}/\neg A\}$. Then T' is consistent iff $b \leq \neg a$.*

PROOF: If $b > \neg a$ then $b \otimes a > 0$. At the same time $T' \vdash_{b'} \neg A$ where $b' \geq b$. Thence, $T' \vdash_d A \,\&\, \neg A$ for $d > 0$. Consequently, T' is contradictory.

Conversely, let T' be contradictory. We have to show that $b > \neg a$. It is sufficient to suppose that $b = \neg a$ since then, for $b < \neg a$, the lemma will immediately follow from the properties of extension of fuzzy theories.

We will use the completeness theorem and suppose two cases. Let there be a model $\mathcal{D} \models T$ such that $\mathcal{D}(A) = a$. Then $\mathcal{D}(\neg A) = \neg a$, i.e. $\mathcal{D} \models T'$ and thus, T' is consistent.

Let $1 > \mathcal{D}(A) > a$ hold for each model $\mathcal{D} \models T$. Since by the assumption, T' is contradictory then by Lemma 1, given $B := \mathbf{0}$ there is m_0 such that $T' \vdash (\neg A)^{m_0} \Rightarrow \mathbf{0}$, which is equivalent to $T' \vdash m_0 A$. Choose $\mathcal{D}$ and m_0 such that $\mathcal{D}(m_0 A) = m_0 \mathcal{D}(A) = 1$. Then for each $\mathcal{D}'$ such that $a < \mathcal{D}'(A) < \mathcal{D}(A)$ we have $m_0 \mathcal{D}'(A) < 1$ which, by completeness, implies that $T' \vdash_c m_0 A$ for some $c < 1$ and consequently, T' is not contradictory. We conclude that $b > \neg a$. □

The following is immediate.

Corollary 3 *T' is contradictory iff $b > \neg a$.*

Lemma 5 is the main motivation for the following concept.

Definition 1 *Let T be a consistent theory and A a formula. Then the element $\neg a$ is a consistency threshold for A in T if $T' = T \cup \{{}^{b}/\neg A\}$ is contradictory for all $b > \neg a$ and consistent otherwise.*

An immediate consequence of Lemma 5 is the following.

Lemma 6 *Let T be a consistent theory. For every formula A, $T \vdash_a A$ iff $\neg a$ is a consistency threshold for A in T.*

We close this section by a simple lemma demonstrating that closed and open axioms lead to the same consistency behaviour of fuzzy theories.

Lemma 7 *Let A be a formula and A' be its closure. Then $T' = T \cup \{{}^{a}/A'\}$ is contradictory iff $T'' = T \cup \{{}^{a}/A\}$ is contradictory.*

PROOF: Let T' be contradictory and $w_{C\,\&\,\neg C}$ be a proof containing a special axiom a/A' such that

$$\mathrm{Val}(w_{C\,\&\,\neg C}) > 0.$$

If we extend it by the proof

$$a/A, a/(\forall x)A, \ldots, a/A'$$

then a/A took the role of special axiom instead of a/A' and thus, we obtain a proof of $C\,\&\,\neg C$ in T'' with the value greater than 0. Consequently, T'' is contradictory.

The proof of the converse is similar. We extend the proof of $C \wedge \neg C$ in T'' by

$$a/A', 1/A' \Rightarrow A'', \ldots, a/A \ldots$$

where a/A' took the role of the new special axiom and $A' \Rightarrow A''$ is an instance of the substitution axiom. We obtained a proof of the contradiction in T' with a non-zero value. □

3 Herbrand Theorem in Fuzzy Logic

In this section, we prove a slightly modified, stronger version of the Herbrand theorem in FLn, originally proved in [8].

Recall that a formula A is a *fuzzy quasitautology* in the degree a if

$$\models_a B_1 \,\&\, \cdots \,\&\, B_k \Rightarrow A$$

where B_i, $i = 1, \ldots, k$ are closed instances of the equality axioms. Formally, we will write

$$\models_a^Q A.$$

The following is a generalization of the Hilbert–Ackermann consistency theorem.

Theorem 4 (consistency) *Open theory T is contradictory iff there are positive integers $p_1, \ldots, p_n$ and special axioms $A_1, \ldots, A_n$ of the theory T such that*

$$\models_b^Q \neg \bar{A}_1^{p_1} \nabla \cdots \nabla \neg \bar{A}_n^{p_n}$$

where $\bar{A}_i$ are instances of the special axioms and $b >^* \neg(a_1^{p_1} \otimes \cdots \otimes a_n^{p_n})$ where $a_i = \mathrm{Ax}^{\mathrm{S}}(A_i)$, $i = 1, \ldots, n$.

Recall that the matrix of a formula $A := (Q_1 x_1) \cdots (Q_n x_n) B(x_1, \ldots, x_n)$ in prenex form is the formula $B(x_1, \ldots, x_n)$ where Q_i are the quantifiers $\exists$ or $\forall$. The *Herbrand existential formula A_H* is *constructed from A* by substitution of new functional symbols in the same way as in classical logic.

The following is the first generalization of the classical Herbrand theorem to FLn with evaluated syntax.

Theorem 5 *Let T be fuzzy predicate calculus and $A \in F_{J(T)}$ a closed formula in prenex form and $a \geq 0$. Then*

$$T \vdash_b mA \quad \textit{iff} \quad \models_d^Q p_1 A_H^{(1)} \nabla \cdots \nabla p_n A_H^{(n)}$$

is a fuzzy quasitautology for some m and $p_1, \ldots, p_n$ where $b >^* ma$, $d >^* (p_1 + \cdots + p_n)a$, where $A_H^{(i)}$ are instances of the matrix of the formula A_H.

The following theorem is fundamental for the proof of the Herbrand theorem. Unlike [8], we use here the concept of the consistency threshold.

Theorem 6 *Let T be fuzzy predicate calculus with equality, A be a closed existential formula*

$$A := (\exists x_1) \cdots (\exists x_n) B$$

Then $T \vdash_a A$ iff to every $b < a$ there are positive integers $p_1, \ldots, p_n$ such that

$$\models_d^Q p_1 B_1 \nabla \cdots \nabla p_n B_n$$

is a fuzzy quasitautology where $B_1, \ldots, B_n$ are instances of the formula B and $d >^* (p_1 + \cdots + p_n)b$.

PROOF: First, note that

$$T \vdash \neg(\exists x)B \Leftrightarrow (\forall x)\neg B.$$

By Lemma 6, $T \vdash_a A$ iff $\neg a$ is a consistency threshold for A in T, i.e. iff

$$T' = T \cup \{{}^{\neg b}/(\forall x_1)\cdots(\forall x_n)\neg B\}$$

is contradictory for all $\neg b > \neg a$ and thus, for all $b < a$. However, by Lemma 7, this is equivalent to $T' = T \cup \{{}^{\neg b}/\neg B\}$ being contradictory.

Since T is consistent, T' is consistent for all $b \geq a$ by Lemma 5. Choose $b < a$. By Theorem 4 there is a quasitautology of instances of $\neg B$

$$\models^Q_d \neg(\neg B_1)^{p_1} \nabla \cdots \nabla \neg(\neg B_n)^{p_n}$$

where $d >^* \neg((\neg b)^{p_1} \otimes \cdots \otimes (\neg b)^{p_n})$. But this is equivalent with

$$\models^Q_d p_1 B_1 \nabla \cdots \nabla p_n B_n$$

where $d >^* \neg\neg(p_1 a \oplus \cdots \oplus p_n a) = (p_1 + \cdots + p_n)a$. □

Let us now introduce the special equality axioms

$$(\forall x)(B \Leftrightarrow C) \Rightarrow (\mathbf{r} = \mathbf{s})$$

where $\mathbf{r}$ and $\mathbf{s}$ are special constants for $(\forall x)B$ $(\forall x)C$, respectively. Analogous axioms are considered in classical logic in the proof of Herbrand theorem. We will denote by T_H the Henkin extension of the theory T, i.e. the extension by the *special axioms* of the form

$$A_x[\mathbf{r}] \Rightarrow (\forall x)A \tag{6}$$

in the degree 1. In every model of T_H there is an element $d \in D$ such that $d = \mathcal{D}(\mathbf{r})$ and $\mathcal{D}((\forall x)A) = \mathcal{D}(A_x[\mathbf{r}])$.

Furthermore, we denote by

$$T'_H = T_H \cup \{{}^1/(\forall x)(B \Leftrightarrow C) \Rightarrow (\mathbf{r} = \mathbf{s}) \mid B, C \in F_J\}$$

extension of T_H by the special equality axioms. It has been proved in [8] that T'_H is a conservative extension of T.

Now we are ready to prove the second version of the Herbrand theorem in FLn.

Theorem 7 *Let T be a fuzzy predicate calculus and $A \in F_{J(T)}$ a closed formula in prenex form. Then*

$$T \vdash_a A$$

iff to every $b < a$ there are $p_1, \ldots, p_n$ such that $\models^Q_d p_1 A^{(1)}_H \nabla \cdots \nabla p_n A^{(n)}_H$ is a fuzzy quasitautology, where $d >^* (p_1 + \cdots + p_n)b$ and $A^{(i)}_H$ are instances of the matrix of the Herbrand existential formula A_H constructed from A.

PROOF: First, we extend the language $J(T)$ by the new functional symbols from A_H and construct a fuzzy theory T'_H being a Henkin extension of T further extended by special equality axioms. This is a conservative extension of T. Then we prove that

$$T \vdash_a A \quad \text{iff} \quad T'_H \vdash_a A_H \tag{7}$$

where A_H is the Herbrand existential formula.

If $T \vdash_a A$ then $T'_H \vdash_{a'} A$ where $a \leq a'$. Using the substitution axiom of the form

$$\vdash (\forall a)B(y) \Rightarrow B_y[f(x_1, \ldots, x_n)]$$

and Lemma 3 we successively obtain $T'_H \vdash_b A_H$, $b \geq a$.

Conversely: to simplify the description of the proof, we assume, in the same way as in [11], that

$$A := (\exists x)(\forall y)(\exists z)(\forall w)B(x, y, z, w)$$

and

$$A_H := (\exists x)(\exists z)B(x, f(x), z, g(x, z)).$$

Furthermore, if u, v are closed terms then we denote by $\mathbf{r}_\forall(u)$ a special constant for

$$(\forall y)(\exists z)(\forall w)B(u, y, z, w)$$

and similarly, $\mathbf{r}_\forall(u, v)$ that for

$$(\forall w)B(u, \mathbf{r}_\forall(u), v, w).$$

First, we show that

$$T'_H \vdash B(u, \mathbf{r}_\forall(u), v, \mathbf{r}_\forall(u, v)) \Rightarrow A \tag{8}$$

for all terms u, v. Indeed, using the special Henkin axioms, as well as substitution, we have

$$\begin{aligned}
&T'_H \vdash B(u, \mathbf{r}_\forall(u), v, \mathbf{r}_\forall(u, v)) \Rightarrow (\forall w)B(u, \mathbf{r}_\forall(u), v, w),\\
&T'_H \vdash (\forall w)B(u, \mathbf{r}_\forall(u), v, w) \Rightarrow (\exists z)(\forall w)B(u, \mathbf{r}_\forall(u), z, w),\\
&T'_H \vdash (\exists z)(\forall w)B(u, \mathbf{r}_\forall(u), z, w) \Rightarrow (\forall y)(\exists z)(\forall w)B(u, y, z, w),\\
&T'_H \vdash (\forall y)(\exists z)(\forall w)B(u, y, z, w) \Rightarrow A.
\end{aligned}$$

Furthermore, we must show that

$$T'_H \vdash (\exists x)(\exists z)B(x, f(x), z, g(x, z)) \Rightarrow B(\mathbf{r}_A, \mathbf{r}_\forall(\mathbf{r}_A), \mathbf{r}_\exists, \mathbf{r}_\forall(\mathbf{r}_A, \mathbf{r}_\exists)) \tag{9}$$

where $\mathbf{r}_A$ is a special constant for A and $\mathbf{r}_\exists$ is a special constant for

$$(\exists z)(\forall w)B(\mathbf{r}_A, \mathbf{r}_\forall(\mathbf{r}_A), z, w).$$

This follows form the completeness theorem. Indeed, let $\mathcal{D} \models T'_H$. Then (9) is equivalent with

$$\bigvee_{d \in D} \bigvee_{d' \in D} \mathcal{D}(B[\mathbf{d}, f(\mathbf{d}), \mathbf{d}', g(\mathbf{d}, \mathbf{d}')]) \leq \mathcal{D}(B[\mathbf{d}_A, \mathbf{d}_\forall(\mathbf{d}_A), \mathbf{d}_\exists, \mathbf{d}_\forall(\mathbf{d}_A, \mathbf{d}_\exists)]) \tag{10}$$

where $\mathbf{d}_A$ is a constant for $d_A = \mathcal{D}(\mathbf{r}_A)$, and similarly the other constants represent $d_\forall(d_A) = \mathcal{D}(\mathbf{r}_\forall(\mathbf{r}_A))$, $d_\exists = \mathcal{D}(\mathbf{r}_\exists)$ and $d_\forall(d_A, d_\exists) = \mathcal{D}(\mathbf{r}_\forall(\mathbf{r}_A, \mathbf{r}_\exists))$. Since T'_H is a Henkin theory, we may suppose that the interpretation of the functional symbol f in $\mathcal{D}$ is such that $\mathcal{D}(f(\mathbf{d})) = d_\forall(d_A)$ and $\mathcal{D}(g(\mathbf{d}, \mathbf{d}')) = d_\forall(d_A, d_\exists)$. For the same reason the left-hand side of (10) is equal to

$$\mathcal{D}(B[\mathbf{d}_A, f(\mathbf{d}_A), \mathbf{d}_\exists, g(\mathbf{d}_A, \mathbf{d}_\exists)])$$

and, consequently, we have even equality in (10). Combining (9) and (8) we obtain

$$T'_H \vdash A_H \Rightarrow A. \tag{11}$$

Let $T'_H \vdash_c A_H$. Using (11) we conclude that $T'_H \vdash_d A$ where $c \leq d$ which due to Lemma 4 and the conservativeness of T'_H gives us $T \vdash_a A$.

Finally, let $T'_H \vdash_a A_H$. Using Theorem 6, to every $b < a$ there are positive integers $p_1, \ldots, p_n$ and a fuzzy quasitautology

$$\models_d^Q p_1 B_1 \nabla \cdots \nabla p_n B_n$$

where $d >^* (p_1 + \cdots p_n)b$ and $B_1, \ldots, B_n$ are instances of the formula B. □

4 McNaughton Theorem in Fuzzy Logic

As we have seen above, Herbrand theorem, in fact, transforms provability of first-order formulas into truth evaluation of certain propositional ones. In classical logic, this makes possible to automatize proofs. Also in fuzzy logic, a simplification and, at least partial automatization, of quite complicated provability can be expected. A great role should be played by McNaughton theorem since this theorem makes it possible to transform formulas into a unified normal form, in analogy with classical logic. Unfortunately, normal forms in fuzzy logic are much more complicated than the classical normal forms.

Our aim is to prove that all functions representable by formulas of FLn are piecewise linear with integer coefficients and, conversely, each piecewise linear function can be represented by a formula of FLn. Analogous problem has been solved by McNaughton in [4] for infinite valued propositional Łukasiewicz logic. In comparison with it, we give the *constructive proofs* which allows to assign each piecewise linear function its standard realization by a formula (a normal form). Since FLn is an extension of the infinite-valued Łukasiewicz logic, our proof is valid also for the latter.

As in classical logic, each formula $A = A(x_1, \dots, x_n)$ is in a natural way associated with a superposition of operations from $\mathcal{L}$ obtained by replacing each logical connective with the corresponding operation. This superposition $f_A : L^n \to L$ is a truth evaluation of A and it corresponds with the truth value assigned to A in a model $\mathcal{D}$. All truth evaluations (and only they) are superpositions of operations from $\mathcal{L}$.

Now let us define the class of *functions representable by formulas* as a class of all possible truth evaluations. Denote this class by H and the class of all n-ary functions representable by formulas by H_n. In particular, H_0 consists of constants. It is easy to see that H is the closure of the set of operations from $\mathcal{L}$ with respect to superposition. Note that in FLn, all logical operations (and, hence, all functions representable by formulas) can be represented as superpositions of the implication and constants.

In classical (or any other finite-valued) logic with Lukasiewicz-like system of connectives, any function $f : L^n \to L$ is a truth function of some formula $A(x_1, \dots, x_n)$. However, this is not the case when L is infinite. The problem stated in this paper is to describe the class of functions representable by formulas of FLn (and consequently, of Lukasiewicz logic with $L = [0, 1]$). In the next section, it will be proved that this class consists of all piecewise linear functions with integer coefficients.

Now let us formulate a definition of a piecewise linear function.

Definition 2 *A set $D \subseteq L^n$ is a (convex) polyhedron if it can be given by a system of linear inequalities*

$$D = \{\mathbf{x} \in L^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$$

where $\mathbf{A}$ is an integer $m \times n$-matrix, $\mathbf{b}$ is a real $m \times 1$-vector and $\leq$ is defined componentwise. So defined a polyhedron will be referred below as specified *by the pair $\langle \mathbf{A}, \mathbf{b} \rangle$.*

For example, the n-cube L^n is a polyhedron because it can be specified by the matrix

$$\begin{pmatrix} \mathbf{E}_n \\ -\mathbf{E}_n \end{pmatrix}$$

and the vector

$$\begin{pmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{pmatrix},$$

where $\mathbf{E}_n$ is the identity matrix, $\mathbf{1}_n$ and $\mathbf{0}_n$ are column vectors of ones and zeros respectively.

Definition 3 *A function $f : L^n \to L$ is a piecewise linear (with integer coefficients) if there exist a finite number of polyhedra $D_1, \dots, D_r$ such that $\bigcup_{i=1}^r D_i = L^n$ and $f|_{D_1}, \dots, f|_{D_r}$ are linear functions with integer coefficients, i.e.*

$$f(\mathbf{x}) = \mathbf{c}_i \mathbf{x} + d_i \text{ if } \mathbf{x} \in D_i, \qquad 1 \leq i \leq r$$

for some integer row vector $\mathbf{c}_i$ and a real number d_i. The polyhedra $D_1, \dots, D_r$ will be called the linearity domains *of f.*

Denote the class of all piecewise linear functions by K and the class of all n-ary piecewise linear functions by K_n. Both K_0 as well as H_0 consists of constants. Note that all piecewise linear functions are continuous.

4.1 Closeness of the class of piecewise linear functions

It will be proved in this section that $H \subseteq K$. This fact is based on the following two lemmas.

Lemma 8 *All operations from $\mathcal{L}$ are piecewise linear functions.*

PROOF: It can be done by a direct check. □

To establish the target inclusion, it suffices to prove that K is closed with respect to superposition. The following lemma will help to do it:

Lemma 9 *Let $f_1, \ldots, f_p \in K_n$. Then there exist non-empty polyhedra $D_1, \ldots, D_r$ such that $\bigcup_{i=1}^{r} D_i = L^n$ and $D_1, \ldots, D_r$ are linearity domains for each of $f_1, \ldots, f_p$.*

PROOF: Choose $D_{i1}, \ldots, D_{ip}$ be some linearity domains of $f_1, \ldots, f_p$ respectively and consider their intersection $D_i = D_{i1} \cap \ldots \cap D_{ip}$. Assume that $D_i \neq \emptyset$. Then D_i is a polyhedron specified by a pair $\langle \mathbf{A}_i, \mathbf{b}_i \rangle$, where

$$\mathbf{A_i} = \begin{pmatrix} \mathbf{A}_{i1} \\ \vdots \\ \mathbf{A}_{ip} \end{pmatrix}, \qquad \mathbf{b}_i = \begin{pmatrix} \mathbf{b}_{i1} \\ \vdots \\ \mathbf{b}_{ip} \end{pmatrix}$$

and the pairs $\langle \mathbf{A}_{i1}, \mathbf{b}_{i1} \rangle, \ldots, \langle \mathbf{A}_{ip}, \mathbf{b}_{ip} \rangle$ specify $D_{i1}, \ldots, D_{ip}$ respectively.

Since for any $j = 1, \ldots, p$, $D_i \subseteq D_{ij}$ and D_{ij} is a linearity domain of f_j, then D_i is a linearity domain for each of $f_1, \ldots, f_p$.

Finally, the union of all D_i corresponding to all possible sequences $(D_{i1}, \ldots, D_{ip})$ constitutes L^n:

$$\bigcup D_i = \bigcup_{\substack{D_{ij} \text{ is a linearity domain of } f_j \\ 1 \leq j \leq p}} (D_{i1} \cap \ldots \cap D_{ip}) = L^n.$$

Hence, all conditions of the lemma hold. □

Based on Lemma 9, the closeness of K can be established.

Theorem 8 *K is closed with respect to superposition.*

PROOF: Let $f_1, \ldots, f_p \in K_n$, $g \in K_p$, $n, p \geq 1$. Consider the superposition h of $f_1, \ldots, f_p$ and g:

$$h(x_1, \ldots, x_n) := g(f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)).$$

and prove that $h \in K_n$. To simplify the denotation, let us use a vector-function $\mathbf{f}$ of the form

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_p(\mathbf{x}))^\top.$$

In correspondence with Lemma 9, choose D_1 being a common linearity domain of $f_1, \ldots, f_p$ and D_2 being a linearity domain of g. Then there exist integer matrices C_1 $(p \times n)$ and $\mathbf{C}_2$ $(1 \times p)$, vector of reals $\mathbf{d}_1 (p \times 1)$ and a real number d_2 so that

$$\mathbf{f}(\mathbf{x}) = \mathbf{C}_1 \mathbf{x} + \mathbf{d}_1 \text{ if } \mathbf{x} = (x_1, \ldots, x_n)^\top \in D_1,$$
$$g(\mathbf{y}) = \mathbf{C}_2 \mathbf{y} + d_2 \text{ if } \mathbf{y} = (y_1, \ldots, y_p)^\top \in D_2.$$

Assume $D = D_1 \cap \mathbf{f}^{-1}(D_2)$. Then D is a polyhedron specified by a pair $\langle \mathbf{A}, \mathbf{b} \rangle$, where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \cdot \mathbf{C}_1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 - \mathbf{A}_2 \cdot \mathbf{d}_1 \end{pmatrix}$$

and pairs $\langle \mathbf{A}_1, \mathbf{b}_1 \rangle$ and $\langle \mathbf{A}_2, \mathbf{b}_2 \rangle$ specify D_1 and D_2 respectively. If $D \neq \emptyset$ and $\mathbf{x} \in D$ then

$$h(\mathbf{x}) = g(\mathbf{f}(\mathbf{x})) = g(\mathbf{C}_1 \mathbf{x} + \mathbf{d}_1) = \mathbf{C}_2(\mathbf{C}_1 \mathbf{x} + \mathbf{d}_1) + d_2 := \mathbf{c}\mathbf{x} + d,$$

where $\mathbf{c} = \mathbf{C}_2 \mathbf{C}_1$, $d = \mathbf{C}_2 \mathbf{d}_1 + d_2$.

Finally,

$$\bigcup_{(D_1, D_2)} \left(D_1 \cap f^{-1}(D_2)\right) = L^n \cap f^{-1}(L^p) = L^n.$$

Thus, h is a piecewise linear function. Hence, K is closed with respect to superposition. □

The following theorem is a direct corollary of Lemma 9 and Theorem 8

Theorem 9 *H is included in K.*

4.2 Realization of piecewise linear functions by formulas

It has been proved in the previous subsection that $H \subseteq K$. Here, the converse inclusion is established. It is worth to be noted that the proof is constructive, i.e. given a piecewise linear function f, its representation as a superposition of operations from $\mathcal{L}$ is explicitly described. It is done in two steps. First, all "truncated linear" functions of the form $(\mathbf{cx}+d)^*$ ("linear pieces" of piecewise linear function f) are built and then f is "glued" of its linear pieces. The mechanism of such "gluing" is provided by Lemma 10 below.

Let $n \geq 1$ be an integer. Let us define a class $UL(n)$ whose elements are sets $E \subseteq L^n$ equaled to unit level sets of functions representable by formulas with n variables. Thus, $E \in UL(n)$ iff there exists a function $g \in H_n$ so that

$$E = \{\mathbf{x} \in L^n : g(\mathbf{x}) = \mathbf{1}\}.$$

For each $E \in UL(n)$ define H_E as the class of n-ary functions $f : L^n \to L$ coinciding on E with some function representable by formulas. Thus, $f \in H_E$ if there exists function $h \in H_n$ such that $g(\mathbf{x}) = h(\mathbf{x})$ whenever $\mathbf{x} \in E$. Obviously, $H_{L^n} = H_n$.

Now we can formulate an important lemma which allows to build piecewise linear functions from their linear pieces. The rest of this paper is based on the following lemma.

Lemma 10 *(a) Let $E_1, E_2 \in UL(n)$ and $f \in H_{E_1} \cap H_{E_2}$. Then $f \in H_{E_1 \cup E_2}$ and consequently $H_{E_1} \cap H_{E_2} = H_{E_1 \cup E_2}$.*

(b) Let $E_1, \ldots, E_r \in UL(n)$ and $\bigcup_{i=1}^r E_i = L^n$ and for all i, $1 \leq i \leq r$, $f \in H_{E_i}$. Then $f \in H_n$.

PROOF: (a) Since $f \in H_{E_1} \cap H_{E_2}$ then by definition, there exist $h_1, h_2 \in H_n$ such that $f\,|_{E_1} = h_1\,|_{E_1}$, $f\,|_{E_2} = h_2\,|_{E_2}$. Again, since $E_1, E_2 \in UL(n)$ then there exist $g_1, g_2 \in H_n$, so that $g_1 \mid E_1 = 1$, $g_2 \mid E_2 = 1$.

Let us define a function $h^{p,q} : L^n \to L$ by the expression

$$h^{p,q}(\mathbf{x}) := (h_1(\mathbf{x}) \otimes (g_1(\mathbf{x}))^p) \vee (h_2(\mathbf{x}) \otimes (g_2(\mathbf{x}))^q), \tag{12}$$

where p and q are natural numbers.

Obviously, $h^{p,q} \in H_n$. Now we aim at proving that there exist natural numbers $\bar{p}, \bar{q}$, so that $f\,|_{E_1 \cup E_2} = h^{\bar{p},\bar{q}}\,|_{E_1 \cup E_2}$. Thus, formula (1) "glues" h_1 and h_2 into one function h and $h\,|_{E_1} = h_1\,|_{E_1}$, $h\,|_{E_2} = h_2\,|_{E_2}$.

As follows from Theorem 9, $g_1, g_2, h_1, h_2 \in K_n$. Due to Lemma 9, there exist polyhedra $D_1, \ldots, D_r$ ($\bigcup_{i=1}^r D_i = L^n$) specified by pairs $\langle \mathbf{A}_1, \mathbf{b}_1 \rangle, \ldots, \langle \mathbf{A}_r, \mathbf{b}_r \rangle$, which are linearity domains for each of g_1, g_2, h_1, h_2. It follows that, for example, $g_1(\mathbf{x}) = \mathbf{c}_i\mathbf{x} + d_i$ if $\mathbf{x} \in D_i$, $1 \leq i \leq r$. Consider new polyhedra $D'_1, \ldots, D'_r$ specified by

$$\mathbf{A}'_i = \begin{pmatrix} \mathbf{A}_i \\ -\mathbf{c}_i \end{pmatrix}, \qquad \mathbf{b}'_i = \begin{pmatrix} \mathbf{b}_i \\ d_i - 1 \end{pmatrix}, \qquad 1 \leq i \leq r.$$

It is easy to see that $D'_1, \ldots, D'_r$ are also linearity domains of g_1, g_2, h_1, h_2 and that $D'_i = D_i \cap E_1$, $1 \leq i \leq r$, whence $\bigcup_{i=1}^r D'_i = E_1$.

Let $D'_i \neq \emptyset$. We will show first that there exists a natural number q_i so that

$$h_2(\mathbf{x}) \otimes (g_2(\mathbf{x}))^{q_i} \leq h_1(\mathbf{x}), \qquad \mathbf{x} \in D'_i. \tag{13}$$

For this, estimate the difference between the right-hand and left-hand sides of (13):

$$\begin{aligned} h_1(\mathbf{x}) - h_2(\mathbf{x}) \otimes (g_2(\mathbf{x}))^{q_i} &= h_1(\mathbf{x}) - (h_2(\mathbf{x}) + q_i \cdot g_2(\mathbf{x}) - q_i)^* \geq \\ &\geq h_1(\mathbf{x}) - h_2(\mathbf{x}) + q_i\,(1 - g_2(\mathbf{x})) \end{aligned} \tag{14}$$

and denote the right-hand side of (14) by $\varphi(\mathbf{x})$. If $\mathbf{x} \in D'_i \cap E_2$ then $h_1(\mathbf{x}) = f(\mathbf{x}) = h_2(\mathbf{x})$ and $g_2(\mathbf{x}) = 1$, whence $\varphi(\mathbf{x}) = 0$. On the other hand, if $\mathbf{x} \in D'_i \setminus E_2$ then $1 - g_2(\mathbf{x}) > 0$, and q_i could be chosen so large that $\varphi(\mathbf{x}) \geq 0$.

Now, eliminate the dependence of q_i on $\mathbf{x}$. It is known that any linear function on a convex polyhedron attains its minimum value at one of its vertices. Let $\mathbf{x}_{i1}, \ldots, \mathbf{x}_{ip}$ be the vertices of D'_i and q_i be so large that $\varphi(\mathbf{x}_{ij}) \geq 0$ for any $j = 1, \ldots, p$. Due to the choice of D'_i, $\varphi(\mathbf{x})$ is a linear function within D'_i. Therefore, φ attains its minimum on D'_i at some of $\mathbf{x}_{ij}$ $(1 \leq j \leq p)$. Since $\varphi(\mathbf{x}_{ij}) \geq 0$, we have $\varphi(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in D'_i$, whence (13) follows.

Let $\bar{q} = \max_{1 \leq i \leq r} q_i$. Since $\bigcup_{i=1}^r D'_i = E_1$, then for all $\mathbf{x} \in E_1$ and any natural number p

$$\begin{aligned} h^{p,\bar{q}}(\mathbf{x}) &= (h_1(\mathbf{x}) \otimes (g_1(\mathbf{x}))^p) \vee \left(h_2(\mathbf{x}) \otimes (g_2(\mathbf{x}))^{\bar{q}}\right) = \\ &= (h_1(\mathbf{x}) \otimes 1) \vee \left(h_2(\mathbf{x}) \otimes (g_2(\mathbf{x}))^{\bar{q}}\right) = h_1(\mathbf{x}) \vee \left(h_2(\mathbf{x}) \otimes (g_2(\mathbf{x}))^{\bar{q}}\right) = \\ &= h_1(\mathbf{x}) = f(\mathbf{x}). \end{aligned}$$

It could be proved that there exist natural number $\bar{p}$, so that for any $\mathbf{x} \in E_2$ $h^{\bar{p},\bar{q}}(\mathbf{x}) = f(\mathbf{x})$. Thus, $f\mid_{E_1 \cup E_2} = h\mid_{E_1 \cup E_2}$. Hence, $f \in H_{E_1 \cup E_2}$.

(b) Given (a), (b) can be obtained by induction. □ □

In the following lemma we will realize the first step of our procedure, i.e. we will show, how the "truncated linear" functions of the form $(\mathbf{cx}+d)^*$ can be realized. The proof of this lemma has many technical details and is omitted from the text. The interested reader can find it in [12].

Lemma 11 *Let $f : L^n \to L$ be a truncated linear function, i.e. $f(\mathbf{x}) = (\mathbf{cx}+d)^*$, where $\mathbf{c} = (c_1, \ldots, c_n)$ is vector of integer coefficients and $d \in \mathbf{R}$. Then $f \in H_n$.*

Now, let us prove the main result about representation of a piecewise linear function.

Theorem 10 *The class of functions representable by formulas of the propositional fuzzy logic coincides with the class of piecewise linear functions with integer coefficients.*

PROOF: Choose $f \in K_n$ and D being a linearity domain of f specified by pair $\langle \mathbf{A}, \mathbf{b} \rangle$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be rows of $\mathbf{A}$ and $\mathbf{b} = (b_1, \ldots, b_m)^\top$. Define a function $g : L^n \to L$ by

$$g(\mathbf{x}) := \neg (\mathbf{a}_1\mathbf{x} - b_1)^* \wedge \cdots \wedge \neg (\mathbf{a}_m\mathbf{x} - b_m)^* . \tag{15}$$

Due to Lemma (11), $g \in H$. Since $D = \{\mathbf{x} \in L^n : g(\mathbf{x}) = \mathbf{1}\}$, then $D \in UL(n)$. As a consequence of Lemma 11, $f \in H_D$ for each linearity domain D. Using Lemma 10(b), we obtain $f \in H_n$. Thus, the inclusion $K \subseteq H$ is proved. Together with Theorem (9), this yields $K = H$. □

5 Normal Form of Functions Representable by Formulas

Based on proofs of Lemmas (10), (11) and Theorem (10), one can, given a piecewise linear function, rewrite it as a superposition of operations from $\mathcal{L}$ in a standard way.

Theorem 11 *Let $f \in K_n$ be a piecewise linear function with linearity domains $D_1, \ldots, D_r$, so that $f(\mathbf{x}) = \mathbf{c}_i\mathbf{x} + d_i$ if $\mathbf{x} \in D_i$, $1 \le i \le r$ and $\bigcup_{i=1}^r D_i = L^n$. Let $D_1, \ldots, D_r$, are specified by pairs $\langle \mathbf{A}_1, \mathbf{b}_1 \rangle, \ldots, \langle \mathbf{A}_r, \mathbf{b}_r \rangle$ respectively, so that $\mathbf{a}_{i1}, \ldots, \mathbf{a}_{im_i}$ are rows of $\mathbf{A}_i$, and $b_{i1}, \ldots, b_{im_i}$ are elements of $\mathbf{b}_i$. Then $f(\mathbf{x})$ can be represented as follows:*

$$f(\mathbf{x}) = \bigvee_{i=1}^{r} \left((\mathbf{c}_i\mathbf{x} + d_i)^* \otimes \left(\bigwedge_{j=1}^{m_i} \neg (\mathbf{a}_{ij}\mathbf{x} - b_{ij})^* \right)^{p_i} \right), \tag{16}$$

where p_i, $1 \le i \le r$, are some (sufficiently large) natural numbers. Expression (16) will be called a normal form of a piecewise linear function f.

PROOF: Apply the "gluing" formula (12) $n-1$ times using (15) to represent linearity domains of f as unit level sets of functions from H. The result is (16). As we see, (16) rewrites $f(\mathbf{x})$ in terms of operations from $\mathcal{L}$ and functions $(\mathbf{c}_i\mathbf{x} + d_i)^*$. The latter can be expressed by suitable formulas, as a consequence of Lemma 11. This completes the construction of the normal form. □

As mentioned above, expression (16) is the result of a direct transformation of a piecewise linear function to a formula of the propositional calculus. Thus, each symbol in this expression is meaningfully connected with a function. Now we are interested in a more general formal representations. Admittedly, greater generality results in greater difficulty in interpreting the parameters.

Definition 4 *Polynomial functions P_m of rank $m \ge 1$ are inductively defined as follows:*

(i) For $m = 1$, $P_m(x_1, \ldots, x_n) = x_i$ for some $1 \le i \le n$.

(ii) Let $P_{m-1}(x_1, \ldots, x_n)$ be a polynomial function of rank $m-1$. Then the following are polynomial functions of rank k:

- $P_m(x_1, \ldots, x_n) = P_{m-1}(x_1, \ldots, x_n) \otimes x_i$,
- $P_m(x_1, \ldots, x_n) = P_{m-1}(x_1, \ldots, x_n) \oplus x_i$, $1 \le i \le n$,• $P_m(x_1, \ldots, x_n) = \neg P_{m-1}(x_1, \ldots, x_n)$.

The normal form (16) for piecewise linear function $f(x_1, \dots, x_n)$ with r linear parts can be rewritten by the following expression using polynomial functions:

$$f(x_1, \dots, x_n) = \bigvee_{i=1}^{r} \left(P_{m_i}(x_1, \dots, x_n, a_i) \otimes \left(\bigwedge_{j=1}^{l_i} \neg Q_{k_j}(x_1, \dots, x_n, b_j) \right)^{p_i} \right) \tag{17}$$

where $P_{m_i}(x_1, \dots, x_n, a_i)$ and $Q_{k_j}(x_1, \dots, x_n, b_j)$ are polynomials, l_i is the number of linear hyperplanes limiting the polyhedron which is the domain of i-th linear part and p_i is some natural number.

Finally, expression (17) can be rewritten in a slightly more general and thus, concise form using the distributivity of $\otimes$ with respect to $\wedge$ as follows:

$$f(x_1, \dots, x_n) = \bigvee_{i=1}^{r} \bigwedge_{j=1}^{s_i} R_{m_j}(x_1, \dots, x_n, a_{ij}, b_{ij}), \tag{18}$$

where $R_{m_j}(x_1, \dots, x_n, a_{ij}, b_{ij})$ are polynomials which are hardly interpretd in terms of the initial function. But precisely this expression reflects the general structure of the representation we are seeking for. In further investigations, we show the deep connection between this result and some metamathematical theorems in the formal theory of fuzzy logic.

6 Functions Representable by Formulas

As has been presented above, every formula of the propositional fuzzy logic corresponds to a truth function defined on $L = [0, 1]$. In the classical predicate logic, formulas represent relations and therefore, in the fuzzy predicate logic they represent fuzzy relations. On the basis of the concept of the truth evaluation, we give a concept of a *function representable by a formula* in the predicate fuzzy logic.

Let $A(x_1, \dots, x_n)$ be a quantifier-free formula and $P_1, \dots, P_m$ be instances of its atomic subformulas not being logical constants. Let us replace each instance P_i by a propositional variable p_i, $i = 1, \dots, m$. The result is a formula

$$\varphi_A(p_1, \dots, p_m) \tag{19}$$

of propositional fuzzy logic.

Definition 5 *Let $A(x_1, \dots, x_n)$ be a quantifier-free formula of the predicate FLn in the language J and $\mathcal{D}$ a structure for J with the support D. A formula A represents a membership function $r_A^{\mathcal{D}} : D^n \to L$ of a fuzzy relation $r_A^{\mathcal{D}}$ if $r_A^{\mathcal{D}}(d_1, \dots, d_n) = \mathcal{D}(A_{x_1, \dots, x_n}[\mathbf{d}_1, \dots, \mathbf{d}_n])$ where $\mathbf{d}_1, \dots \mathbf{d}_n$ are names of the respective elements $d_1, \dots, d_n \in D$. We will also say that the membership function of a fuzzy relation $r_A^{\mathcal{D}}$ is representable by the formula A.*

Let us note that all predicate and functional symbols in the formula A are uniquelly interpreted in the given structure $\mathcal{D}$.

It is not difficult to see that the following theorem is a direct consequence of the McNaughton theorem (our Theorem 10).

Theorem 12 *Let $A(x_1, \dots, x_n)$ be a quantifier-free formula which contains predicate symbols $P_1, \dots, P_m$. Then the membership function $r_A^{\mathcal{D}}$ can be expressed in the form*

$$r_A^{\mathcal{D}}(d_1, \dots, d_n) = l_A(P_{D,1}(d_1, \dots, d_n), \dots, P_{D,m}(d_1, \dots, d_n))$$

where $l_A : L^m \to L$ is a piecewise linear function representable by a formula $\varphi_A(p_1, \dots, p_m)$ in (19) and $P_{D,1}, \dots, P_{D,m}$ are fuzzy relations assigned to $P_1, \dots, P_m$ in $\mathcal{D}$, respectively.

The proof of this theorem is a technical excercise and therefore, we will illustrate it by an example. Let $A(x, y) := P(x, x) \Rightarrow P(x, y)$. Then

$$\varphi_A(p, q) := p \Rightarrow q$$
$$l_A(p, q) = (1 - p + q)^*.$$

Furthermore, let $\mathcal{D}$ be a structure with support D and let the predicate symbol P be assigned a fuzzy relation $P_D \underset{\sim}{\subset} D \times D$. Then the desired function $r_A^{\mathcal{D}}$ is represented by

$$r_A^{\mathcal{D}}(x, y) = (1 - P_D(x, x) + P_D(x, y))^*.$$

Theorem 13 *Let T be a fuzzy predicate calculus and $A(x_1, \dots, x_n)$ be a quantifier-free formula. Then $T \models_a A$ iff*

$$a = \min_{\langle p_1, \dots, p_m\rangle \in L^m} l_A(p_1, \dots, p_m)$$

where l_A is a piecewise linear function representable by a formula φ_A in (19).

PROOF: The inequality $\min_{\langle p_1, \dots, p_m\rangle \in L^m} l_A(p_1, \dots, p_m) \leq a$ immediately follows from the definition of the semantical consequence (4) and the fact that l_A is a piecewise linear function.

To prove the converse inequality, let us consider a point $\langle p_1^0, \dots, p_m^0\rangle \in L^m$ at which l_A is minimal. Our task is to find a structure $\mathcal{D}$ such that $\mathcal{D}(P_i(t_{i1}, \dots, t_{in_i})) = p_i^0$, $i = 1, \dots, m$ where $P_i(t_{i1}, \dots, t_{in_i})$ is an instance of an atomic subformula of A.

Let us fix some set D of elements such that their number is not less than the number of different terms occurring in A. We assign an element $\mathcal{D}(t_{i1}) \in D$ to each term t_{ij} so that different terms are assigned different elements. Note that during this assignment we can partially determine some functions on D assigned to the functional symbols occurring in the terms from A.

Take an instance $P_i(t_{i1}, \dots, t_{in_i})$ which is constructed on the basis of a predicate symbol Q (to be specific). The desired structure $\mathcal{D}$ is then obtained if we choose a membership function $Q_D : D^n \to L$ of a corresponding fuzzy relation in such a way that the equality

$$\mathcal{D}(P_i(t_{i1}, \dots, t_{in_i})) = Q_D(\mathcal{D}(t_{i1}), \dots, \mathcal{D}(t_{in_i})) = p_i^0$$

holds true. Since the instance P_i has been taken arbitrarily, the proof is completed. □

7 McNaughton Theorem in Fuzzy Theories

If A is a formula and l_A is a piecewise linear function representable by a formula φ_A in (19) then we will write

$$m_A = \min_{\langle p_1, \dots, p_m\rangle \in L^m} l_A(p_1, \dots, p_m),$$
$$M_A = \max_{\langle p_1, \dots, p_m\rangle \in L^m} l_A(p_1, \dots, p_m).$$

Corollary 4 *Let the conditions of Theorem 13 be fulfilled and A' be a closure of A. Then $T \models_a A'$ iff $a = m_A$.*

Corollary 5 *Let T be a fuzzy predicate calculus and A an arbitrary formula in prenex form. Then $T \models_a A$ implies $m_B \leq a$ where B is a matrix of A.*

All the results below use the completeness theorem.

Theorem 14 *Let T be a fuzzy predicate calculus and A be a closed existential formula in prenex form*

$$A := (\exists x_1) \cdots (\exists x_n) B.$$

If $T \vdash_a A$ then $m_B \leq a \leq M_B$.

PROOF:

Let $T \vdash_a A$. By the completeness theorem, also $T \models_a A$. The lower bound m_B now follows from Corollary 5. Furthermore,

$$a \leq \bigwedge \{ \bigvee_{d_1, \dots, d_n \in D} r_B^{\mathcal{D}}(d_1, \dots, d_n) \mid \mathcal{D} \text{ is a model of } T\}. \tag{20}$$

Let us fix a model $\mathcal{D}$. By Theorem 12,

$$r_B^{\mathcal{D}}(d_1, \dots, d_n) = l_B(P_{D,1}(d_1, \dots, d_n), \dots, P_{D,m}(d_1, \dots, d_n))$$

where l_B is a piecewise linear function representable by B. Then the desired inequality follows from (20) and the fact that l_B is piecewise linear on a closed set.

□

Corollary 6 *Let T be fuzzy predicate calculus and A an arbitrary formula in prenex form. Then $T \vdash_a A$ implies $m_B \leq a \leq M_B$ where B is a matrix of A.*

PROOF: We have $T \models_a A$ by the completeness theorem. Moreover,

$$T \models_c (\exists x_1) \cdots (\exists x_n) B.$$

Then $a \leq c$ due to the definition of quantifiers. Then using Theorem 13 and Corollary 4 we obtain

$$m_B \leq a \leq c \leq M_B.$$

□

We will now present several lemmas and theorems on the consistency of fuzzy theories.

Lemma 12 *Let T be fuzzy predicate calculus and A a quantifier-free formula. Then $T' = T \cup \{^a/A\}$ is contradictory iff $a > M_A$.*

PROOF: Let $a \leq M_A$. By the same procedure as in the proof of Theorem 13 we construct a structure $\mathcal{D}$ in which $\mathcal{D}(A) = M_A$. Then $\mathcal{D} \models T'$ and so T' is consistent.

Conversely, let $a > M_A$. Let T' be consistent and $\mathcal{D}$ be a structure, in which $\mathcal{D}(A) \geq a$. On the other hand, $T \models_{\neg M_A} \neg A$ and thus, $D(\neg A) \geq \neg M_A$. But then $\mathcal{D}(A \,\&\, \neg A) \geq a' \otimes \neg M_A > M_A \otimes \neg M_A = 0$. There is no such structure and therefore T' is contradictory. □

The following is immediate.

Corollary 7 *Let A be a quantifier-free formula and $a \leq M_A$. Then $T' = T \cup \{^a/A\}$ is consistent.*

Corollary 8 *Let T be a fuzzy predicate calculus and A an existential formula in the prenex form. Then $T' = T \cup \{^a/A\}$ is contradictory iff $a > M_B$ where B is a matrix of A.*

PROOF: Let $a \leq M_B$. Let $\mathcal{D}$ be the structure considered in the first part of the proof of Lemma 12, in which $\mathcal{D}(B) = M_B$. Because M_B is maximum of all truth evaluations of B, it follows from the definition of the existential quantifier that $\mathcal{D}$ is also a model of T' and so, T' is consistent.

Conversely, let $a > M_B$, T' be consistent and $\mathcal{D}$ be a structure, in which $\mathcal{D}(A) \geq a$. Since A is existential, we obtain $T \models_{\neg M_B} \neg B$ by Corollary 4 and thus, $D(\neg A) \geq \neg M_B$. But then $\mathcal{D}(A \,\&\, \neg A) \geq a' \otimes \neg M_B > M_B \otimes \neg M_B = 0$. There is no such structure and therefore T' is contradictory.

□

Putting together our results on Herbrand models and on McNaughton normal forms, we get the following.

Theorem 15 *Let T be a fuzzy predicate calculus and A a formula in the prenex form with the matrix B. Then $T' = T \cup \{^a/A\}$ is contradictory iff $a > M_{B_H}$ where B_H is a matrix of the corresponding Herbrand existential formula A_H constructed from A.*

PROOF: Let T'_H be a Henkin extension of T', further extended by the special equality axioms and A_H be a Herbrand existential formula. After inspection of the first part of the proof of Theorem 7 we conclude that $T' \vdash_b A$ iff $T'_H \vdash_b A_H$. Moreover, recall that T'_H is a conservative extension of T. The theorem then follows from Corollary 8. □

The following theorem is a weaker form of the Hilbert-Ackermann theorem [7] based on the McNaughton theorem.

Theorem 16 *Let $A_1, \ldots, A_n$ be special axioms of the open fuzzy theory T, $\mathrm{Ax}^{\mathrm{S}}(A_j) = a_j$ for all $j = 1, \ldots, n$. If*

$$T \models_c \neg A_1 \,\nabla \cdots \nabla\, \neg A_n \quad \textit{and} \quad \neg(m_{A_1} \otimes \cdots \otimes m_{A_k}) < c < 1$$

then T is contradictory.

PROOF: Let $\mathcal{D}$ be a structure. Then $\mathcal{D}(A_j) \geq m_{A_j}$ due to Theorem 13 and so

$$\mathcal{D}(A_1 \,\&\, \cdots \,\&\, A_n) \geq m_{A_1} \otimes \cdots \otimes m_{A_n}.$$

By the assumption, $\mathcal{D}(\neg(A_1 \,\&\, \cdots \,\&\, A_n)) \geq c >^* \neg(m_{A_1} \otimes \cdots \otimes m_{A_n})$ which implies that T is contradictory. □

The following theorem is a quite surprizing consequence of Theorem 10 which demonstrates that extension of fuzzy theories does not always follow the properties of classical logic.

Theorem 17 *Let T be a fuzzy predicate calculus. Then there exist evaluated formulas $a_1/A_1, \ldots, a_n/A_n$, $0 < a_1 \otimes \cdots \otimes a_n < 1$ and $A_1, \ldots, A_n \in F_{J(T)}$ such that*

$$T' = T \cup \{a_1/A_1, \ldots, a_n/A_n\}$$

is contradictory and

$$T'' = T \cup \{a_1 \otimes \cdots \otimes a_n/A_1 \,\&\, \cdots \,\&\, A_n\}$$

is consistent.

PROOF: We will demonstrate the proof only for the case of $n = 2$. Its extension to arbitrary (finite) n is straightforward.

Let us consider two partially linear functions $l_{A_1}(x), l_{A_2}(x)$ on $[0,1]$ such that $l_{A_1}(x) \otimes l_{A_2}(x) = c > 0$ for all $x \in [0,1]$. Such functions can be constructed, for example by setting $l_{A_1}(x) = a_1 + x\varepsilon$ and $l_{A_2}(x) = a_2 - x\varepsilon$ where ε is some suitable number and $a_1 + a_2 = c > 0$. Then $l_{A_1}(x) \otimes l_{A_2}(x) = a_1 + x\varepsilon + a_2 - x\varepsilon = a_1 + a_2$.

By Theorems 10 and 13, $l_{A_1}(x), l_{A_2}(x)$ correspond to some quantifier-free formulas A_1, A_2, respectively. Furthermore, let $T \models_{b_1} \neg A_1$ where, by the assumption, $0 < b_1 < 1$. By Theorem 13, $b_1 = \min l_{\neg A_1} = \neg \max l_{A_1}$ and since l_{A_1} is partially linear, we have $\max l_{A_1} = l_{A_1}(p_0)$ for some $p_0 \in [0,1]$. We put $a_1 = l_{A_1}(p_0) + \varepsilon$ and $a_2 = l_{A_2}(p_0) - \varepsilon$. Then

$$a_1 \otimes a_1 = l_{A_1}(p_0) + \varepsilon + l_{A_2}(p_0) - \varepsilon - 1 = c.$$

Hence, in any model $\mathcal{D} \models T$ we have $\mathcal{D}(A_1 \,\&\, A_2) \geq c$, which implies $\mathcal{D} \models T''$, i.e. T'' is consistent.

On the other hand, $a_1 > \neg b_1$, i.e. $b_1 > \neg a_1$ and consequently,

$$b_1 \otimes a_1 > \neg a_1 \otimes a_1 = 0.$$

But this means that, by completeness, $T \vdash_{b_1} \neg A_1$ and so, $T' \vdash_d \neg A_1$ for some $d \geq b_1$ and $T' \vdash_e A_1$ for some $e \geq a_1$. We conclude that T' is contradictory. □

This lemma is trivial for $a_1 \otimes a_2 = 0$ and does not hold for $a_1 \otimes a_2 = 1$ since then $a_1 \otimes a_2 = a_1 \wedge a_2$ which implies that in such a case T' is contradictory iff T'' is contradictory.

Lemma 13 *(a) Let $T' = T \cup \{a_1/A_1, \ldots, a_n/A_n\}$. If $T \vdash_c \neg A_1 \nabla \cdots \nabla \neg A_n$ where $c >^* \neg a_1 \oplus \cdots \oplus \neg a_n$ then T' is contradictory.*

(b) Let T be a fuzzy predicate calculus. Then there is a contradictory theory $T' = T \cup \{a_1/A_1, \ldots, a_n/A_n\}$ such that

$$T \vdash_c \neg A_1 \nabla \cdots \nabla \neg A_n$$

where $c \leq \neg a_1 \oplus \cdots \oplus \neg a_n$.

PROOF: (a) To simplify the notation, we put $D := A_1 \,\&\, \cdots \,\&\, A_n$. Then we may rewrite the condition into

$$T \vdash_c \neg D, \qquad c >^* \neg(a_1 \otimes \cdots \otimes a_n).$$

Now we can write a formal proof

$$c'/\neg D\ \{\textit{some proof of } D\},\ a_1/A_1\ \{\textit{spec. axiom}\}, \ldots,\ a_n/A_n\ \{\textit{spec. axiom}\}$$
$$\ldots,\ a_1 \otimes \cdots \otimes a_n/D,\ c' \otimes (a_1 \otimes \cdots \otimes a_n)/\neg D \,\&\, D\ \{\textit{modus ponens}\},$$

which implies $T' \vdash_d \neg D \,\&\, D$ for some $d \geq c \otimes (a_1 \otimes \cdots \otimes a_n) > 0$. This means that T' is contradictory.

(b) By Theorem 17, we can construct a contradictory T' such that $T'' = T \cup \{a_1 \otimes \cdots \otimes a_n/A_1 \,\&\, \cdots \,\&\, A_n\}$ is consistent. Then by Lemma 5, $a_1 \otimes \cdots \otimes a_n \leq \neg c$, i.e.

$$c \leq \neg(a_1 \otimes \cdots \otimes a_n) = \neg a_1 \oplus \cdots \oplus \neg a_n.$$

□

The following corollary demonstrates that the powers $p_1, \ldots, p_n$ in Theorem 4 cannot be set equal to 1.

Corollary 9 *Let T be a fuzzy predicate calculus. Then there is a contradictory open theory T' such that there are special axioms $A_1, \dots, A_n$ and a quasitautology*

$$\models_b^Q \neg \bar{A}_1 \nabla \cdots \nabla \neg \bar{A}_n$$

where $\bar{A}_i$ are instances of the special axioms and $b \leq \neg(a_1 \otimes \cdots \otimes a_n)$ where $a_i = \mathrm{Ax}^{\mathrm{S}}(A_i)$, $i = 1, \dots, n$.

PROOF: By Lemma 13(b) and the proof of Theorem 17, there is a contradictory theory

$$T'' = T \cup \{{}^{a_1}/\bar{A}_1, \dots, {}^{a_q}/\bar{A}_q\}$$

such that

$$T \models_c \neg \bar{A}_1 \nabla \cdots \nabla \neg \bar{A}_q \tag{21}$$

where $c \leq \neg a_1 \oplus \cdots \oplus \neg a_n$ and $\bar{A}_i$ are closed quantifier-free formulas. Note that among them, there can be also closed instances ${}^1/B_1, \dots, {}^1/B_k$ of the equality axioms since then $\mathcal{D}(\neg B_i) = 0$ in every model $\mathcal{D}$ and so, they have no influence on the truth value of $\mathcal{D}(\neg \bar{A}_1 \nabla \cdots \nabla \neg \bar{A}_q)$. Consequently, we may take (21) as a quasitautology

$$\models_b^Q \neg \bar{A}_1 \nabla \cdots \nabla \neg \bar{A}_n \tag{22}$$

and consider $T'' = T \cup \{{}^{a_1}/\bar{A}_1, \dots, {}^{a_n}/\bar{A}_n\}$ where $B_1, \dots, B_k$ are not among $\bar{A}_1, \dots, \bar{A}_n$. We show that the open theory $T' = T \cup \{{}^{a_1}/A_1, \dots, {}^{a_n}/A_n\}$ where A_i are open formulas, of which $\bar{A}_i$ are instance, is contradictory as well.

Indeed, let $w_{\bar{C}\,\&\,\neg \bar{C}}$ be a proof of contradiction in T'' with the value

$$\mathrm{Val}(w_{\bar{C}\,\&\,\neg \bar{C}}) > 0.$$

When replacing all closed terms in $w_{\bar{C}\,\&\,\neg \bar{C}}$ which occur in $\bar{A}_1, \dots, \bar{A}_n$ by new variables (not occurring in formulas of $w_{\bar{C}\,\&\,\neg \bar{C}}$) we obtain a proof $w_{C\,\&\,\neg C}$ of some (possibly open) formula $C \,\&\, \neg C$ with the value $\mathrm{Val}(w_{C\,\&\,\neg C}) > 0$.

□

8 Conclusion

In this paper, we continue our program to develop fuzzy logic in narrow sense in the direction to cover most of the (basic) results of classical logic. We focus on fuzzy logic in narrow sense of Łukasiewicz style because, as is demonstrated on a lot of places, it is the most developed formal theory of FLn which seems to fit well the program of fuzzy logic outlined by L. A. Zadeh (cf. [5, 13]). Besides others, it meets well the requirement to provide syntactical reasoning on truth values and thus, it fulfils the program of being formal tool for modeling of the vagueness phenomenon. Note also, that it is a nontrivial generalization of classical logic.

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