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L-fuzzy quantifiers of the type ⟨1⟩ determined by fuzzy measures[☆]

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Abstract

The aim of this paper is, first, to introduce two new types of fuzzy integrals, namely, ⊗-fuzzy integral and →-fuzzy integral. The first integral is based on a fuzzy measure of **L**-fuzzy sets and the second one on a complementary fuzzy measure of **L**-fuzzy set, where **L** is a complete residuated lattice. Some of their properties and a relation to the fuzzy (Sugeno) integral are investigated. Second, using these integrals, two classes of monadic **L**-fuzzy quantifiers of the type ⟨1⟩ are defined, namely, **L**-fuzzy quantifiers of the type ⟨1⟩ determined by fuzzy measures and **L**-fuzzy quantifiers of the type ⟨1⟩ determined by complementary fuzzy measures. Several semantic properties of these **L**-fuzzy quantifiers are studied.

Key words: fuzzy measure, fuzzy integral, fuzzy logic, monadic **L**-fuzzy quantifier

1. Introduction

This paper continues the study of semantic properties of monadic **L**-fuzzy quantifiers originated by Glöckner [3] and elaborated in [10] by studying one specific but important class of them, namely, fuzzy quantifiers of the type ⟨1⟩ determined by fuzzy measures.

Quantifiers of the type ⟨1⟩ are denotations of important noun phrases of natural language, e.g. "something" in "Something is broken.", "everyone" in "Everyone likes Bob.", "nobody" in "Nobody knows everything.", etc. Moreover, classical logical quantifiers "for all" and "there exists" also belong to this type. It is claimed (e.g. in [24]) that from the point of view of natural language semantics, quantifiers of the type ⟨1, 1⟩ (e.g. "every" in "Every book has leaves.", "most" in "Most birds fly.") are more basic and more important. However, it is usual and advantageous to start with the type ⟨1⟩ quantifiers, because they are

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simpler and there are important relationships between them and quantifiers of the type $\langle 1, 1\rangle$.

Generalized quantifiers evolved, from pioneering works of Mostowski [19], Lindström [14], Barwise and Cooper [1], into quite large research field with deep results. For overview as well as new results see the recent monograph [24]. Quantifiers of type $\langle 1\rangle$ (like "everyone", "something", etc.) are usually modeled, given a universe M, as a mapping $Q_M : \mathcal{P}_M \longrightarrow \{true, false\}$ (or, equivalently, as subsets of power set $\mathcal{P}_M$). It is possible to introduce many properties of (models of) quantifiers, characterizing their behavior from various points of view, for example, permutation invariance (PI), isomorphism invariance (ISOM), extension (EXT), and others.

When we think about the definition and properties of generalized quantifiers (like e.g. *many*, *a few* and others), we feel that their truth values should not change abruptly if we gradually change cardinalities of corresponding sets of objects. Consider for example sentence "Many people read books." If the number of people who read books increase by 1, it would be very strange if truth value of this sentence changes from false to true. Therefore, it was inevitable that researchers started to consider more than two truth values in this context, and so-called *fuzzy quantifiers* emerged, starting from a generalization of the definition from the previous paragraph, where instead of $\{true, false\}$ we consider some other structure of truth values, notably the interval $[0, 1]$ of real numbers.

Research in the field of fuzzy quantifiers started with works of Zadeh [31], Thiele [28], Ralescu [25] and others, see also [15, 16, 20]. Important contribution was made by Hájek in [9], where he considers especially quantifier "many" using relative frequencies. He also points out interconnections between generalized quantifiers and modalities. A comprehensive study of fuzzy quantifiers was undertaken by Glöckner [3] (see also [4]). In the recent paper [22], Novák studies so-called intermediate quantifiers, mainly from syntactic point of view in the frame of fuzzy type theory [21]. An attempt to model linguistic quantifiers by fuzzy (Sugeno) integral was presented by Ying in [30].

Aside from the theoretical importance of generalized quantifiers of the type $\langle 1\rangle$, we were also motivated by the applications of fuzzy logic e.g. in time series analysis, decision making and other fields where it can be advantageous to be able to use other than classical quantifiers "for all" and "exists". For example, in time series analysis and modeling, we can use quantifiers like "many" for filtering off of outliers, i.e. values that are numerically distant from (the course of) the data.

The semantic interpretation of many generalized quantifiers is often connected to measurement of "size" of sets in concern. Consider e.g. the quantifier "many". The truth value of proposition "many books have a red cover" clearly depends on the "size" of the set of red books. Therefore, it is natural to consider measures (and integrals) of (fuzzy) sets as natural tools for the modeling of important classes of monotonically non-decreasing and monotonically non-increasing generalized quantifiers.

Fuzzy measures and integrals ([29], see also [5, 12]) are important tools allowing to compare classical or fuzzy sets with respect to their size. Standardly, fuzzy measures are set functions defined on some algebra of sets which are monotone with respect to inclusion and they assign zero to the empty set. In this contribution, fuzzy measures are defined on algebras of fuzzy sets (measure spaces) and, generally, they attain values from a complete residuated lattice **L**. Two types of fuzzy integral, namely, $\otimes$-fuzzy integral and $\rightarrow$-fuzzy integral, are then defined on an arbitrary fuzzy measure space and complementary fuzzy measure space, respectively. Integrals of $\otimes$ type will be used as models of quantifiers like *all* and *some*, while integrals of $\rightarrow$ type as models of *no* and *not all*, etc. However, if the structure of truth values **L** is a complete MV-algebra, then it is possible to define the $\rightarrow$-fuzzy integral from the $\otimes$-fuzzy integral using negation (see Theorem 3.18). Contrary to usual definitions of fuzzy integrals, these integrals can be used to integrate all (fuzzy) sets, that means, also (fuzzy) sets which are not standardly measurable with respect to the used fuzzy measure space. This enables us to introduce fuzzy quantifiers over spaces of all (fuzzy) sets and not only over spaces of all measurable (fuzzy) sets (cf. [30]). However, it is surprising that we are able to show that the well-known Sugeno integral [27] is, under certain conditions, a special case of our fuzzy integral (see Theorem 3.11, p. 17).

Our structure of truth values is a complete residuated lattice. We prefer to work in this structure (and not in, e.g., interval $[0, 1]$ of real numbers) for several reasons. First, it is advantageous to work in more general structure, if all its important properties we need are fulfilled in this structure. Then all results are valid also for particular examples, including the interval $[0, 1]$ mentioned above. Second, our structure of truth values is not in general linearly ordered. Therefore, our results are valid also for structures of truth values with incomparable elements, which can be useful in some applications. Third, working in abstract algebraic system is often more convenient than working in one particular structure.

The paper is structured as follows. Section 2 contains some basic facts about the structure of truth values and **L**-fuzzy sets. In Section 3, we first introduce notions of *fuzzy measurable* and *fuzzy measure spaces*, and also of *complementary fuzzy measure spaces*. Then, isomorphism between fuzzy measure spaces is defined. In the next part of this section, the $\otimes$-fuzzy integral is defined and its basic properties are proved. We also show that the definition of this integral can be simplified if **L** is a complete MV-algebra. Then, connections to the Sugeno integral are discussed and a theorem is proved saying that if **L** is a complete Heyting algebra, then the Sugeno integral and the $\otimes$-fuzzy integral coincide. Finally, $\rightarrow$-fuzzy integral on a complementary fuzzy measure space is defined and it is shown, besides other results, that if **L** is a complete MV-algebra, the $\rightarrow$-fuzzy integral is definable from the $\otimes$-fuzzy integral.

Next Section 4 summarizes properties of **L**-fuzzy quantifiers of the type $\langle 1 \rangle$, using definition from [10]. In Section 5, we define **L**-fuzzy quantifiers of the type $\langle 1 \rangle$ determined by fuzzy measures and show that these quantifiers are bounded by $\exists$ and $\forall$. Then we study conditions under which these quantifiers belong to classes PI, ISOM, EXT, etc. Further, quantifiers determined by complementary fuzzy measures (hence by $\rightarrow$-fuzzy integral) are defined and analogous properties are studied. Finally, in Section 6, we point out some conclusions and sketch directions of the further research.

2. Preliminaries

2.1. Structures of truth values

In this paper, we suppose that the structure of truth values is a *complete residuated lattice* (see e.g. [2]), i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \bot, \top \rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \bot, \top \rangle$ is a complete lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, respectively, $\langle L, \otimes, \top \rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{1}$$

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. The operations $\otimes$ and $\rightarrow$ are usually called multiplication and residuum, respectively. Since the operations $\wedge$ and $\otimes$ have a lot of common properties, which may be used for various alternative constructions, we will denote them, in general, by the symbol $\odot$. Thus, if we deal with the operation $\odot$ then we will consider either the operation $\wedge$ or the operation $\otimes$, whereas none of them is specified. A residuated lattice is *divisible*, if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$, and satisfies the *law of double negation*, if $(a \rightarrow \bot) \rightarrow \bot = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. For other information about residuated lattices we refer to [2, 23].

Example 2.1. It is easy to prove (see e.g. [9]) that the algebra

$$\mathbf{L}_{\mathrm{T}} = \langle [0,1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where T is a left continuous t-norm and $a \rightarrow_T b = \bigvee \{c \in [0,1] \mid T(a,c) \leq b\}$ defines the residuum, is a complete residuated lattice. Moreover, if T is the Łukasiewicz t-norm, i.e., $T(a,b) = \max(a+b-1, 0)$ for all $a, b \in [0,1]$, then $\mathbf{L}_{\mathrm{T}}$ is a complete MV-algebra called a *Łukasiewicz algebra (on* $[0,1]$*)*. The Łukasiewicz algebra will be denoted by $\mathbf{L}_{\mathrm{L}}$.

Example 2.2. Let $a, b \in [0, \infty]$ be such that $a < b$. One checks easily that $\mathbf{L}_{[a,b]} = \langle [a,b], \min, \max, \rightarrow, a, b \rangle$, where

$$c \rightarrow d = \begin{cases} d, & \text{if } d < c, \\ b, & \text{otherwise,} \end{cases} \tag{2}$$

is a complete residuated lattice. Note that $\mathbf{L}_{[a,b]}$ is a special example of a more general residuated lattice called a *Heyting algebra*.[1]

[1] A Heyting algebra is a residuated lattice with $\otimes = \wedge$.

Let us recall some basic properties of complete residuated lattices and complete MV-algebras that will be useful in the following text.

Theorem 2.1 ([2, 23]). *Let* $\mathbf{L}$ *be a complete residuated lattice,* $a \in L$ *and* $\{b_i \mid i \in I\}$ *be a set of elements from* L *over an index set* I*. Then we have*

(a) $a \otimes \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \otimes b_i)$,

(b) $a \to \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \to b_i)$,

(c) $(\bigvee_{i\in I} b_i) \to a = \bigwedge_{i\in I}(b_i \to a)$,

(d) $a \otimes \bigwedge_{i\in I} b_i \le \bigwedge_{i\in I}(a \otimes b_i)$,

(e) $\bigvee_{i\in I}(a \to b_i) \le a \to \bigvee_{i\in I} b_i$,

(f) $\bigvee_{i\in I}(b_i \to a) \le \bigwedge_{i\in I} b_i \to a$.

If $\mathbf{L}$ *is an MV-algebra, then the inequalities (d)-(f) may be replaced by the equalities and we have*

(g) $a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i)$,

(h) $a \vee \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \vee b_i)$.

Let us define the following additional operations for all $a, b \in L$ and sets $\{a_i \mid i \in I\}$ of elements from L over a countable (possibly empty) index set I:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a) \qquad (\textit{biresiduum})$$

$$\neg a = a \to \bot \qquad (\textit{negation})$$

$$\bigotimes_{i\in I} a_i = \begin{cases} \top, & I = \emptyset, \\ \bigwedge_{K\in \mathrm{Fin}(I)} \bigotimes_{i\in K} a_i, & \text{otherwise}, \end{cases} \qquad (\textit{countable multiplication})$$

where $\mathrm{Fin}(I)$ denotes the set of all finite subsets of I. If $|I| = n$ is a finite index set and $a_i = a$ for any $i \in I$, then we will write $a^n = \bigotimes_{i\in I} a_i$.

2.2. $\mathbf{L}$*-fuzzy sets*

Let $\mathbf{L} = \langle L, \wedge, \vee, \to, \otimes, \bot, \top\rangle$ be a complete residuated lattice and M be a universe of discourse (possibly empty). A mapping $A : M \to L$ is called an $\mathbf{L}$*-fuzzy set on* M. A value $A(m)$ is called a *membership degree of* m *in the* $\mathbf{L}$*-fuzzy set* A. The set of all $\mathbf{L}$-fuzzy sets on M is denoted by $\mathcal{F}_{\mathbf{L}}(M)$. Obviously, if $M = \emptyset$, then the empty mapping $\emptyset$ is the unique $\mathbf{L}$-fuzzy set on $\emptyset$ and thus $\mathcal{F}(\emptyset) = \{\emptyset\}$. An $\mathbf{L}$-fuzzy set A on M is called *crisp*, if there is a subset X of M such that $A = 1_X$, where 1_X denotes the characteristic function of X. Particularly, $1_\emptyset$ denotes the empty $\mathbf{L}$-fuzzy set on M, i.e., $1_\emptyset(m) = \bot$ for any $m \in M$. This convention will be also kept for $M = \emptyset$. The set of all crisp $\mathbf{L}$-fuzzy sets on M is denoted by $\mathcal{P}_{\mathbf{L}}(M)$. An $\mathbf{L}$-fuzzy set A is *constant*, if there is $c \in L$ such that $A(m) = c$ for any $m \in M$. For simplicity, a constant **L**-fuzzy set is denoted by the corresponding element of L, e.g., a, b, c.[2] Let us denote $\mathrm{Supp}(A) = \{m \mid m \in M \ \&\ A(m) > \bot\}$ and $\mathrm{core}(A) = \{m \mid m \in M \ \&\ A(m) = \top\}$ the *support* and *core* of an **L**-fuzzy set A, respectively. Obviously, $\mathrm{Supp}(1_X) = \mathrm{core}(1_X) = X$ for any crisp **L**-fuzzy set. An **L**-fuzzy set A is called *normal*, if $\mathrm{core}(A) \neq \emptyset$.

Let $\{A_i \mid i \in I\}$ be a non-empty family of **L**-fuzzy sets on M. Then the *union of* A_i is defined by

$$\left(\bigcup_{i\in I} A_i\right)(m) = \bigvee_{i\in I} A_i(m) \tag{3}$$

for any $m \in M$ and the *intersection of* A_i is defined by

$$\left(\bigcap_{i\in I} A_i\right)(m) = \bigwedge_{i\in I} A_i(m) \tag{4}$$

for any $m \in M$. Let A be an **L**-fuzzy set on M. The *complement* of A is an **L**-fuzzy set $\overline{A}$ on M defined by $\overline{A}(m) = \neg A(m)$ for any $m \in M$. Finally, an extension of the operations $\otimes$ and $\rightarrow$ on L to the operations on $\mathcal{F}_{\mathbf{L}}(M)$ is given by

$$(A \otimes B)(m) = A(m) \otimes B(m) \quad \text{and} \quad (A \rightarrow B)(m) = A(m) \rightarrow B(m) \tag{5}$$

for any $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $m \in M$, respectively. The following theorem shows the well-known relation between the operations of the union and intersection of sets which also holds for **L**-fuzzy sets, if we restrict ourselves to a special class of complete residuated lattices.

Theorem 2.2. *Let* **L** *be a complete residuated lattice satisfying the law of double negation and* $\{A_i \mid i \in I\}$ *be a non-empty family of* **L**-*fuzzy sets on* M. *Then*

$$\bigcup_{i\in I} A_i = \overline{\bigcap_{i\in I} \overline{A}_i} \quad \textit{and} \quad \bigcap_{i\in I} A_i = \overline{\bigcup_{i\in I} \overline{A}_i}. \tag{6}$$

Proof. Let $\{A_i \mid i \in I\}$ be a non-empty family of **L**-fuzzy sets on M. Since **L** satisfies the law of double negation, then $(\bigwedge_{i\in I} a_i) \rightarrow \bot = \bigvee_{i\in I}(a_i \rightarrow \bot)$ and $(\bigvee_{i\in I} a_i) \rightarrow \bot = \bigwedge_{i\in I}(a_i \rightarrow \bot)$ hold for any index set $\{a_i \mid i \in I\}$ (see e.g. [2]). Hence, we can write

$$\left(\overline{\bigcap_{i\in I} \overline{A}_i}\right)(m) = \left(\bigwedge_{i\in I}(A_i(m) \rightarrow \bot)\right) \rightarrow \bot =$$

$$\left(\left(\bigvee_{i\in I} A_i(m)\right) \rightarrow \bot\right) \rightarrow \bot = \bigvee_{i\in I} A_i(m) = \left(\bigcup_{i\in I} A_i\right)(m)$$

[2] We suppose that the meaning of this symbol will be unmistakable from the context, that is, it should be clear when an element of L is considered and when a constant **L**-fuzzy set is assumed.

for any $m \in M$. Analogously, we can prove the second equality. $\square$

We say that an $\mathbf{L}$-fuzzy set A is an $\mathbf{L}$-*fuzzy subset of* an $\mathbf{L}$-fuzzy set B and denote by $A \subseteq B$, if, for any $m \in M$, we have $A(m) \leq B(m)$. Let $f : M \to M'$ be a mapping. A mapping $f^{\to} : \mathcal{F}_{\mathbf{L}}(M) \to \mathcal{F}_{\mathbf{L}}(M')$ defines by $f^{\to}(A)(m) = \bigvee_{m' \in f^{-1}(m)} A(m')$ is called the *fuzzy extension* of the mapping f. Obviously, if f is a bijective mapping, then $f^{\to}(A)(f(m)) = A(m)$ for any $m \in M$. Note that if $M = M' = \emptyset$, then $\emptyset : \emptyset \to \emptyset$ is the unique bijective mapping here. This empty mapping determines the unique mapping $\emptyset^{\to} : \mathcal{F}_{\mathbf{L}}(\emptyset) \to \mathcal{F}_{\mathbf{L}}(\emptyset)$ assigning $1_{\emptyset}$ to $1_{\emptyset}$.

3. Fuzzy integrals

In this section, we will introduce a notion of fuzzy measure and complementary fuzzy measure of $\mathbf{L}$-fuzzy sets and two types of fuzzy integrals that will be used to define $\mathbf{L}$-fuzzy quantifiers. Note that, usually, a fuzzy measure is simply a monotone set function from some algebra of (classical) sets to non-negative real numbers or to $[0, 1]$, possibly with continuity conditions [29]. Our fuzzy measures are monotone functions from algebra of fuzzy sets to a complete residuated lattice. Sometimes, a fuzzy measure is understood as a function from an algebra of classical of fuzzy sets to some algebra of fuzzy sets. Then classical or fuzzy set is measured by, e.g., some fuzzy number [7, 8, 26], and we can, for example, say: *The measure of set A is "approximately 0.5".* We will not deal with this type of fuzzy measures in this paper. For more information about fuzzy measures and integrals, we refer to [5, 6, 29].

3.1. Fuzzy measures of $\mathbf{L}$*-fuzzy sets*

In the following, we will consider algebras of $\mathbf{L}$-fuzzy sets as a base for defining fuzzy measures of $\mathbf{L}$-fuzzy sets.

Definition 3.1 ([29]). Let M be a non-empty universe of discourse. A subset $\mathcal{M}$ of $\mathcal{F}_{\mathbf{L}}(M)$ is an *algebra of* $\mathbf{L}$*-fuzzy sets on* M, if the following conditions are satisfied

(i) $1_{\emptyset}, 1_M \in \mathcal{M}$,

(ii) if $A \in \mathcal{M}$, then $\overline{A} \in \mathcal{M}$,

(iii) if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

A pair $(M, \mathcal{M})$ is called a *fuzzy measurable space*, if $\mathcal{M}$ is an algebra of $\mathbf{L}$-fuzzy sets on M.

Example 3.1. The sets $\{\emptyset, M\}$, $\mathcal{P}_{\mathbf{L}}(M)$, σ-algebras on M, or $\mathcal{F}_{\mathbf{L}}(M)$ are algebras of $\mathbf{L}$-fuzzy sets on M.

Example 3.2. Let us say that an **L**-fuzzy set A on M is a *simple* **L**-*fuzzy set* on M, if there exists a family of sets $\{M_i \mid i = 1, \dots, n\}$ such that $\bigcup_{i=1}^{n} M_i = M$, $M_i \neq M_j$ for any $i \neq j$ and $A(m) = A(m')$ holds for each $m, m' \in M_i$, where $i = 1, \dots, n$. Obviously, the set of all simple **L**-fuzzy sets on M is an algebra of **L**-fuzzy sets on M.

Example 3.3. Let $M = [0, 1]$ and $\mathbf{L}_{Ł}$ be the Łukasiewicz algebra (see Example 2.1). Then the set of all continuous mappings $A : M \to L$ is an algebra of $\mathbf{L}_{Ł}$-fuzzy sets in M.[3]

Let us introduce the concept of fuzzy measure and complementary fuzzy measure as follows. The first definition is a modification of the definition of a normed measure with respect to truth values (see e.g. [5, 12]).

Definition 3.2. Let $(M, \mathcal{M})$ be a fuzzy measurable space. A mapping $\mu : \mathcal{M} \to L$ is called a *fuzzy measure* on $(M, \mathcal{M})$, if

(i) $\mu(1_\emptyset) = \bot$ and $\mu(1_M) = \top$,

(ii) if $A, B \in \mathcal{M}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

A triplet $(M, \mathcal{M}, \mu)$ is called the *fuzzy measure space*, if $(M, \mathcal{M})$ is a fuzzy measurable space and μ is a fuzzy measure on $(M, \mathcal{M})$.

Definition 3.3. Let $(M, \mathcal{M})$ be a fuzzy measurable space. A mapping $\nu : \mathcal{M} \to L$ is called a *complementary fuzzy measure* on $(M, \mathcal{M})$, if

(i) $\nu(1_\emptyset) = \top$ and $\nu(1_M) = \bot$,

(ii) if $A, B \in \mathcal{M}$ such that $A \subseteq B$, then $\nu(A) \geq \nu(B)$.

A triplet $(M, \mathcal{M}, \nu)$ is called a *complementary fuzzy measure space*, if $(M, \mathcal{M})$ is a fuzzy measurable space and ν is a complementary fuzzy measure on $(M, \mathcal{M})$.

Example 3.4. Let $(M, \mathcal{M})$ be the fuzzy measurable space of all continuous mappings from Example 3.3. It is easy to see that

$$\mu(A) = \int_0^1 A(m)\ dm,$$

where $\int_0^1 A(m)\ dm$ denotes the Riemann integral, defines a fuzzy measure on $(M, \mathcal{M})$.

[3] Note that the set of all continuous mappings need not be an algebra of **L**-fuzzy sets for other residuated lattices determined by left T-norms, because the negation is not a continuous mapping in general.

Example 3.5. Let $\mathbf{L}$ be a complete residuated lattice with the support $[0,1]$ and $\mathbb{N}$ be the set of natural numbers with 0. For any non-empty countable (finite or denumerable) universe M, injective mapping $f : M \to \mathbb{N}$, $n \in \mathbb{N}$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, introduce

$$A_{f,n}(m) = \begin{cases} A(m), & \text{if } f(m) \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Further, for any injective mapping $f : M \to \mathbb{N}$ and $n \in \mathbb{N}$, define $\mu_{f,n} : \mathcal{F}_{\mathbf{L}}(M) \to [0,1]$ as follows

$$\mu_{f,n}(A) = \frac{\sum_{m \in \mathrm{Supp}(A_{f,n})} A_{f,n}(m)}{|\mathrm{Supp}(1_{M_{f,n}})|} \tag{8}$$

and, finally, define $\underline{\mu}_f, \overline{\mu}_f : \mathcal{F}_{\mathbf{L}}(M) \to [0,1]$ as follows

$$\underline{\mu}_f = \liminf_{n \to \infty} \mu_{f,n}(A), \tag{9}$$

$$\overline{\mu}_f = \limsup_{n \to \infty} \mu_{f,n}(A). \tag{10}$$

It is easy to see that $\mu_{f,n}$, $\underline{\mu}_f$ and $\overline{\mu}_f$ are fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by an injective mapping f.[4] If, for example, $M = \mathbb{N}$ and $f = \mathrm{id}$, then $\underline{\mu}_f(A) = \overline{\mu}_f(A) = 0$ for any $\mathbf{L}$-fuzzy set on a finite universe. For the set of all even or odd numbers, both fuzzy measures give $\frac{1}{2}$ and, for the set of all prime number, we obtain 0.

If M is finite, then $\underline{\mu}_f = \underline{\mu}_g = \overline{\mu}_f = \overline{\mu}_g$ for any injective mappings $f, g : M \to \mathbb{N}$ and

$$\underline{\mu}_f(A) = \overline{\mu}_f(A) = \frac{\sum_{m \in M} A(m)}{|M|}. \tag{11}$$

Hence, it is easy to see that $\mu_f(A) = \mu_f(h^{\to}(A))$ holds for any non-empty finite universe M, $A \in \mathcal{F}_{\mathbf{L}}(M)$, injective mapping $f : M \to \mathbb{N}$ and bijective mapping $h : M \to M$. Unfortunately, this equality fails for denumerable universes in general. In fact, consider $M = \mathbb{N}$, $f = \mathrm{id}$ and a bijective mapping $h : \mathbb{N} \to \mathbb{N}$ such that the image of all even numbers is the set of prime numbers. Then both measures give $\frac{1}{2}$ for the set of all even numbers, however, 0 for the set of all prime numbers. As a simple consequence of this inequality we obtain the inequality between $\underline{\mu}_f$ and $\underline{\mu}_g$ (and also between $\overline{\mu}_f$ and $\overline{\mu}_g$) which can arise for some examples of injective mappings f and g, if the denumerable universes are permitted.

[4] Note that $\underline{\mu}_f$ and $\overline{\mu}_f$ could be understood as a generalization of lower and upper weighted densities (well known in number theory) which are examples of so-called lower and upper asymptotic fuzzy measures (see [17]).

Example 3.6. Let μ_f be one of the fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by f defined in (9) and (10). If $h : [0,1] \to [0,1]$ is a non-decreasing mapping with $h(0) = 0$ and $h(1) = 1$, then $h \circ \mu_f$ is a fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by μ_f and h. If $h : [0,1] \to [0,1]$ is a non-increasing mapping with $h(0) = 1$ and $h(1) = 0$, then $h \circ \mu_f$ is a complementary fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by μ_f and h.

Theorem 3.1. *Let $(M, \mathcal{M})$ be a fuzzy measurable space. If μ (ν) is a fuzzy measure (a complementary fuzzy measure) on $(M, \mathcal{M})$, then $\nu'(A) = \neg\mu(A)$ ($\mu'(A) = \neg\nu(A)$) defines a complementary fuzzy measure (a fuzzy measure) on $(M, \mathcal{M})$.*

Proof. This is a straightforward consequence of properties of the residuum.□

Definition 3.4. Let $(M, \mathcal{M})$ be a fuzzy measurable space and $X \in \mathcal{F}_{\mathbf{L}}(M)$. We say that X is $\mathcal{M}$-*measurable*, if $X \in \mathcal{M}$.

Let $(M, \mathcal{M})$ be a fuzzy measurable space and $X \in \mathcal{F}_{\mathbf{L}}(M)$. Denote $\mathcal{M}_X$ the set of all $\mathcal{M}$-measurable sets which are contained in X, i.e.,

$$\mathcal{M}_X = \{A \mid A \in \mathcal{M} \text{ and } A \subseteq X\}. \tag{12}$$

Note that $1_\emptyset \in \mathcal{M}_X$ for each $X \in \mathcal{F}_{\mathbf{L}}(M)$ and if X is a $\mathcal{M}$-measurable set, then also $X \in \mathcal{M}_X$. If $X = M$, then we will write only $\mathcal{M}$ instead of $\mathcal{M}_M$.

Theorem 3.2. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. A mapping $\mu^* : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\mu^*(X) = \bigvee_{A \in \mathcal{M}_X} \mu(A) \tag{13}$$

is a fuzzy measure on the fuzzy measurable space $(M, \mathcal{F}_{\mathbf{L}}(M))$. We say that μ^* is the inner fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by μ.

Proof. Obviously, $\mu^*(A) = \mu(A)$ for any $A \in \mathcal{M}$. Hence, $\mu^*(1_\emptyset) = \bot$ and $\mu^*(1_M) = \top$. Since $\mathcal{M}_X \subseteq \mathcal{M}_Y$ for any $X \subseteq Y$, then $\mu^*(X) \le \mu^*(Y)$ and μ^* is a fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$. □

Example 3.7. Let $(M, \mathcal{P}_{\mathbf{L}}(M), \mu)$ be an arbitrary fuzzy measurable space (recall that $\mathcal{P}_{\mathbf{L}}(M)$ is the power set of M). Then the inner fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ is defined by

$$\mu^*(A) = \begin{cases} \mu(A'), & \text{if } 1_{\mathrm{core}(A)} = A', \\ \bot, & \text{otherwise.} \end{cases} \tag{14}$$

Thus all **L**-fuzzy sets that are not normal have the inner fuzzy measure equal to $\bot$.

Example 3.8. Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz algebra and $(M, \mathcal{M}, \mu)$ be the fuzzy measure space of continuous $\mathbf{L}_{\text{Ł}}$-fuzzy sets from Example 3.4. Then, for example, we obtain $\mu^*(1_{[a,b]}) = b - a$, however, $1_{[a,b]} \notin \mathcal{M}$ in general.

Theorem 3.3. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. A mapping $\nu^* : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\nu^*(X) = \bigwedge_{A \in \mathcal{M}_X} \nu(A) \tag{15}$$

is a complementary fuzzy measure on the fuzzy measurable space $(M, \mathcal{F}_{\mathbf{L}}(M))$. We say that ν is the inner complementary fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by ν.

Proof. This is analogous to the proof of Theorem 3.2. □

In the following part we will define an isomorphism between fuzzy measure spaces and then between complementary fuzzy measure spaces.

Definition 3.5. Let $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{M} \to \mathcal{M}'$ is an *isomorphism between* $(M, \mathcal{M})$ *and* $(M', \mathcal{M}')$, if

(i) g is a bijective mapping with $g(1_\emptyset) = 1_\emptyset$,

(ii) $g(A \cup B) = g(A) \cup g(B)$ and $g(\overline{A}) = \overline{g(A)}$ hold for any $A, B \in \mathcal{M}$,

(iii) there exists a bijective mapping $f : M \to M'$ with $A(m) = g(A)(f(m))$ for any $A \in \mathcal{M}$ and $m \in M$.

Theorem 3.4. *Let $(M, \mathcal{M})$, $(M', \mathcal{M}')$ be fuzzy measurable spaces and $g : \mathcal{M} \to \mathcal{M}'$ be a surjective mapping. Then g is an isomorphism between $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ if and only if there exists a bijective mapping $f : M \to M'$ such that $g = f^{\to}$.*

Proof. First, let $g : \mathcal{M} \to \mathcal{M}'$ be an isomorphism of spaces $(M, \mathcal{M})$ and $(M', \mathcal{M}')$. Then there exists a bijective mapping $f : M \to M'$ (according to (iii)) such that $A(m) = g(A)(f(m))$ holds for any $m \in M$. Since $f^{\to}(A)(f(m)) = A(m) = g(A)(f(m))$ for any $m \in M$, then clearly $f^{\to}(A) = g(A)$ for any $A \in \mathcal{M}$ and thus $g = f^{\to}$.

Conversely, let $g : \mathcal{M} \to \mathcal{M}'$ be a surjective mapping such that $g = f^{\to}$ for some bijective mapping $f : M \to M'$. Let us suppose that $g(A) = g(B)$ for some $A, B \in \mathcal{M}$. Since f is a bijective mapping of M onto M', then we have $g(A)(m) = f^{\to}(A)(m) = A(f^{-1}(m)) = B(f^{-1}(m)) = f^{\to}(B)(m) = g(B)(m)$ for any $m \in M$ and thus $A = B$. Hence, g is a bijective mapping. Further, $g(1_\emptyset)(m) = f^{\to}(1_\emptyset)(m) = 1_\emptyset(f^{-1}(m)) = \bot$ holds for all $m \in M'$. Hence, $g(1_\emptyset) = 1_\emptyset$ and (i) is proved. If $A, B \in \mathcal{M}$, then $g(A \cup B)(m) = f^{\to}(A \cup B)(m) = (A \cup B)(f^{-1}(m)) = A(f^{-1}(m)) \vee B(f^{-1}(m)) = f^{\to}(A)(m) \vee f^{\to}(B)(m) = (g(A) \cup g(B))(m)$ holds for all $m \in \mathcal{M}'$. Analogously, it could be shown that

$g(\overline{A}) = \overline{g(A)}$ and thus (ii) is proved. Since f is a bijective mapping with $g(A)(f(m)) = f^{\rightarrow}(A)(f(m)) = A(m)$, then (*iii*) is also true and the proof is finished. □

Definition 3.6. Let $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{M} \to \mathcal{M}'$ is an *isomorphism between* $(M, \mathcal{M}, \mu)$ *and* $(M', \mathcal{M}', \mu')$ (or *between* $(M, \mathcal{M}, \nu)$ *and* $(M', \mathcal{M}', \nu')$), if

(i) g is an isomorphism between $(M, \mathcal{M})$ and $(M', \mathcal{M}')$,

(ii) $\mu(A) = \mu'(g(A))$ (or $\nu(A) = \nu'(g(A))$) for any $A \in \mathcal{M}$.

If g is an isomorphism between fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ or between complementary fuzzy measure spaces $(M, \mathcal{M}, \nu)$ and $(M', \mathcal{M}', \nu')$, then we write $g(M, \mathcal{M}, \mu) = (M', \mathcal{M}', \mu')$ or $g(M, \mathcal{M}, \nu) = (M', \mathcal{M}', \nu')$, respectively.

Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. If $f : M \to M'$ is a bijective mapping, then $(M', f^{\rightarrow}(\mathcal{M}), \mu_{f^{\rightarrow}})$, where

$$\mu_{f^{\rightarrow}}(f^{\rightarrow}(A)) = \mu(A) \tag{16}$$

holds for any $A \in \mathcal{M}$, is a fuzzy measure space isomorphic with $(M, \mathcal{M}, \mu)$. A simple consequence of Theorem 3.4 is the fact that each fuzzy measure space $(M', \mathcal{M}', \mu')$ isomorphic with $(M, \mathcal{M}, \mu)$ has the form $(M', f^{\rightarrow}(\mathcal{M}), \mu_{f^{\rightarrow}})$ for a suitable bijective mapping $f : M \to M'$. Analogously, to each isomorphic complementary fuzzy measure space $(M', \mathcal{M}', \nu')$ with $(M, \mathcal{M}, \nu)$ there is a bijective mapping $f : M \to M'$ such that $(M', f^{\rightarrow}(\mathcal{M}), \nu_{f^{\rightarrow}}) = (M', \mathcal{M}', \nu')$.

Let $[(M, \mathcal{M}, \mu)]$ or $[(M, \mathcal{M}, \nu)]$ denote the class of all fuzzy measure spaces and all complementary fuzzy measure spaces defined on M that are isomorphic with $(M, \mathcal{M}, \mu)$ or with $(M, \mathcal{M}, \nu)$, respectively. Obviously, we can write

$$[(M, \mathcal{M}, \mu)] = \{(M, f^{\rightarrow}(\mathcal{M}), \mu_{f^{\rightarrow}}) \mid f : M \to M \text{ is a bijective mapping}\},$$
$$[(M, \mathcal{M}, \nu)] = \{(M, f^{\rightarrow}(\mathcal{M}), \nu_{f^{\rightarrow}}) \mid f : M \to M \text{ is a bijective mapping}\}.$$

Note that there are fuzzy measure spaces such that they are isomorphic with themselves for any bijective mapping $f : M \to M$ and thus we obtain $[(M, \mathcal{M}, \mu)] = \{(M, \mathcal{M}, \mu)\}$ or $[(M, \mathcal{M}, \nu)] = \{(M, \mathcal{M}, \nu)\}$. A simple example of such fuzzy measure space is a fuzzy measure space $(M, \mathcal{M}, \overline{\mu}_f)$ on a finite universe M from Example 3.5.

3.2. ⊗-fuzzy integral

In this part, we will introduce a type of fuzzy integral that can be defined on an arbitrary fuzzy measure space $(M, \mathcal{M}, \mu)$. The form of this integral is motivated by our need to describe a class of models of **L**-fuzzy quantifiers of the type $\langle 1 \rangle$. Later we will show that this class of models is bounded by the models of determiners *all* and *some*. Note that the models of *all* and *some* are the same as the interpretations of quantifiers $\forall$ and $\exists$ in fuzzy logic, respectively, see e.g. [4, 10, 24, 30].

Definition 3.7. Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and X be a $\mathcal{M}$-measurable $\mathbf{L}$-fuzzy set. The $\otimes$-*fuzzy integral of* A *on* X is given by

$$\int_X^{\otimes} A \, d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)). \tag{17}$$

If $X = 1_M$, then we write $\int^{\otimes} A \, d\mu$.

To explain our approach to fuzzy integral, let us look at Figure 1, where an $\mathbf{L}_{[0,1]}$-fuzzy set A with the support $[0, x]$ is displayed (recall that $\otimes = \wedge$ in $\mathbf{L}_{[0,1]}$, see Example 2.2). The idea how to find $\int^{\otimes} A \, d\mu$ is based on some restrictions of its support $[0, x]$ which are expressed by $\mathbf{L}$-fuzzy sets from a defined algebra $\mathcal{M}$ of $\mathbf{L}$-fuzzy sets. If we take an $\mathbf{L}$-fuzzy set Y from $\mathcal{M}$, then a minimal membership value of A over the support of Y which is decreased according to the measure of Y is searched for. In our case, this is the value $A(m_0) \otimes \mu(Y)$. Here, the value $\mu(Y)$ may be interpreted as a degree of a size of the restricting $\mathbf{L}_{[0,1]}$-fuzzy set Y, where, naturally, smaller values determined by the measure are expected for more restricting $\mathbf{L}_{[0,1]}$-fuzzy sets. Obviously, from the point of view of fuzzy logic the quantifier "for all" is applied to these values (the bold curve in Figure 1). Now, the $\otimes$-fuzzy integral of A is obtained as the greatest value over all suitable restrictions of the original support $[0, x]$ described by $\mathbf{L}$-fuzzy sets from $\mathcal{M}$. From the point of view of fuzzy logic the quantifier "exists" is applied on these values. One can see a similarity to the Sugeno integral [27] (see also [6] or[29]), where the integral is computed as the supremum of the measures of a-cuts of A decreased using a. In the end of this section, we will show that the Sugeno integral is a special case of our proposed integral, see Theorem 3.11.

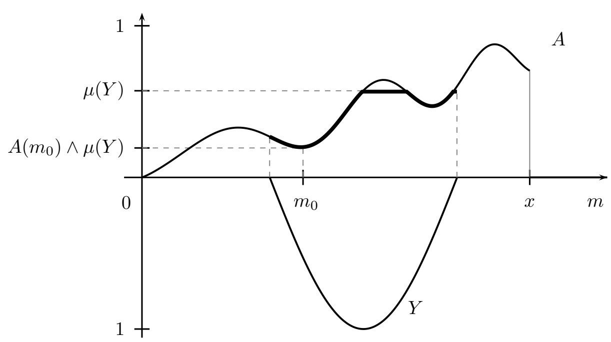

Figure 1: Computing the infimum part in the $\otimes$-fuzzy integral for an $\mathbf{L}_{[0,1]}$-fuzzy set Y.

Remark 3.9. It is easy to see that $\int_{1_\emptyset}^{\otimes} A \, d\mu = \bigvee \emptyset = \bot$ for any $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $\int_X^{\otimes} A \, d\mu \leq \int_Y^{\otimes} A \, d\mu$, whenever $X \subseteq Y$. Since $\int_{1_M}^{\otimes} A \, d\mu \neq \top$ in general, $\mu_A(X) = \int_X^{\otimes} A \, d\mu$ does not define a fuzzy measure on $(M, \mathcal{M})$ in the sense of Definition 3.2.

Remark 3.10. One can also define a $\wedge$-fuzzy integral of A on X in such a way that $\otimes$ is replaced by $\wedge$ in (17). Since $\otimes$ and $\wedge$ have many common properties, both types of fuzzy integral will have similar properties. Nevertheless, we prefer the $\otimes$-fuzzy integral in this paper, because it is closely related (due to the adjointness property) to $\rightarrow$-fuzzy integral that will be introduced in the following subsection.

Theorem 3.5. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. Then $\mu' : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\mu'(A) = \int^{\otimes} A \; d\mu \tag{18}$$

is a fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$.

Proof. We can write

$$\int^{\otimes} 1_{\emptyset} \; d\mu = \bigvee_{Y \in \mathcal{M} \setminus \{1_{\emptyset}\}} \; \bigwedge_{m \in \mathrm{Supp}(Y)} (1_{\emptyset}(m) \otimes \mu(Y)) = \bigvee_{Y \in \mathcal{M} \setminus \{1_{\emptyset}\}} \bot = \bot,$$

and

$$\int^{\otimes} 1_M \; d\mu = \bigvee_{Y \in \mathcal{M} \setminus \{1_{\emptyset}\}} \; \bigwedge_{m \in \mathrm{Supp}(Y)} (1_M(m) \otimes \mu(Y)) = \bigwedge_{m \in M} 1_M(m) \otimes \mu(1_M) = \top.$$

If $A \subseteq B$, i.e., $A(m) \leq B(m)$ for any $m \in M$, then using the isotonicity of $\otimes$ we can write

$$\int^{\otimes} A \; d\mu = \bigvee_{Y \in \mathcal{M} \setminus \{1_{\emptyset}\}} \; \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) \leq$$

$$\bigvee_{Y \in \mathcal{M} \setminus \{1_{\emptyset}\}} \; \bigwedge_{m \in \mathrm{Supp}(Y)} (B(m) \otimes \mu(Y)) = \int^{\otimes} B \; d\mu,$$

hence μ' is a fuzzy measure on the fuzzy measurable space $(M, \mathcal{F}_{\mathbf{L}}(M))$. □

Theorem 3.6. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. Then*

(i) $\int_X^{\otimes} (A \cap B) \; d\mu \leq \int_X^{\otimes} A \; d\mu \wedge \int_X^{\otimes} B \; d\mu$,

(ii) $\int_X^{\otimes} (A \cup B) \; d\mu \geq \int_X^{\otimes} A \; d\mu \vee \int_X^{\otimes} B \; d\mu$,

(iii) $\int_X^{\otimes} (c \otimes A) \; d\mu \geq c \otimes \int_X^{\otimes} A \; d\mu$,

(iv) $\int_X^{\otimes} (c \rightarrow A) \; d\mu \leq c \rightarrow \int_X^{\otimes} A \; d\mu$,

hold for any $X \in \mathcal{M}$, $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $c \in L$.

Proof. It is easy to see that the statements are true for $X = 1_\emptyset$. Let us suppose that $X \neq 1_\emptyset$. Analogously as in the proof of the isotonicity of μ^* in Theorem 3.5, one checks easily that $\int_X^\otimes A\ d\mu \leq \int_X^\otimes B\ d\mu$, whenever $A \subseteq B$. Simple consequences of this isotonicity are the statements (i) and (ii). Let $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $c \in L$. Then we have

$$\int_X^\otimes (c \otimes A)\ d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (c \otimes A(m) \otimes \mu(Y)) \geq$$
$$\bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} c \otimes \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) =$$
$$c \otimes \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) = c \otimes \int_X^\otimes A\ d\mu,$$

since (a) and (c) of Theorem 2.1 hold. Finally, we can write

$$\int_X^\otimes (c \to A)\ d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} ((c \to A(m)) \otimes \mu(Y)) \leq$$
$$\bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (c \to (A(m) \otimes \mu(Y))) =$$
$$\bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} (c \to \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y))) \leq$$
$$c \to \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) = c \to \int_X^\otimes A\ d\mu,$$

where we use that $(a \to b) \otimes c \leq a \to (b \otimes c)$ holds in each residuated lattice. □

Theorem 3.7. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space and $c \in L$. Then we have*

(i) $\int^\otimes (c \otimes 1_X)\ d\mu = c \otimes \mu^*(1_X)$ *for any* $X \subseteq M$*,*

(ii) $\int^\otimes (c \otimes 1_X)\ d\mu = c \otimes \mu(1_X)$ *for any* $X \subseteq M$ *such that* $1_X \in \mathcal{M}$*,*

(iii) $\int^\otimes 1_X\ d\mu = \mu(1_X)$ *for any* $X \subseteq M$ *such that* $1_X \in \mathcal{M}$*,*

(iv) $\int^\otimes c\ d\mu = c$*.*

Proof. Let $X \subseteq M$ be an arbitrary set. Then we have

$$\int^\otimes (c \otimes 1_X)\ d\mu = \bigvee_{Y \in \mathcal{M} \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} ((c \otimes 1_X(m)) \otimes \mu(Y)) =$$
$$\bigvee_{Y \in \mathcal{M}_X} (c \otimes \mu(Y)) = c \otimes \bigvee_{Y \in \mathcal{M}_X} \mu(Y) = c \otimes \mu^*(1_X),$$

where (a) of Theorem 2.1 is used, and (i) is proved. Since $\mu^*(A) = \mu(A)$ for any $A \in \mathcal{M}$, then (ii) is true. The statements (iii) and (iv) are simple consequences of (ii).

Theorem 3.8. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. If $X \in \mathcal{M}$ is such that $1_{\mathrm{Supp}(Y)} \in \mathcal{M}_X$ for any $Y \in \mathcal{M}_X$, then, for any $A \in \mathcal{F}_{\mathbf{L}}(M)$, we have*

$$\int_X^{\otimes} A \; d\mu = \bigvee_{1_Y \in \mathcal{P}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in Y} (A(m) \otimes \mu(1_Y)), \tag{19}$$

where $\mathcal{P}_X = \{1_{\mathrm{Supp}(Z)} \mid Z \in \mathcal{M}_X\}$.

Proof. This is a straightforward consequence of the fact that $1_{\mathrm{Supp}(Y)} \in \mathcal{M}_X$ for any $Y \in \mathcal{M}_X$ and $\mu(1_{\mathrm{Supp}(Y)}) \geq \mu(Y)$. □

Theorem 3.9. *Let $\mathbf{L}$ be a complete MV-algebra, $(M, \mathcal{M}, \mu)$ be a fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $X \in \mathcal{M}$. Then*

$$\int_X^{\otimes} A \; d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \Big(\mu(Y) \otimes \bigwedge_{m \in \mathrm{Supp}(Y)} A(m)\Big). \tag{20}$$

Moreover, we have

$$\int_X^{\otimes} (c \otimes A) \; d\mu = c \otimes \int_X^{\otimes} A \; d\mu \tag{21}$$

for any $c \in L$.

Proof. Formula (20) follows from the equality in (d) of Theorem 2.1 holding in each MV-algebra. The equality (21) is trivial for $X = 1_\emptyset$. Let us suppose that $X \neq 1_\emptyset$. Then we have

$$\int_X^{\otimes} (c \otimes A) \; d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (c \otimes A(m) \otimes \mu(Y)) =$$

$$c \otimes \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) = c \otimes \int_X^{\otimes} A \; d\mu,$$

where (a) and the equality in (d) of Theorem 2.1 are applied. □

Theorem 3.10. *Let g be an isomorphism between fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ and $X \in \mathcal{M}$. Then we have*

$$\int_X^{\otimes} A \; d\mu = \int_{g(X)}^{\otimes} g(A) \; d\mu' \tag{22}$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$.

Proof. Obviously, if $X \in \mathcal{M}$, then $g(X) \in \mathcal{M}'$ and $g(\mathcal{M}_X \setminus \{1_\emptyset\}) = \mathcal{M}'_{g(X)} \setminus \{1_\emptyset\}$. If $f : M \to M'$ is the bijective mapping such that $g = f^{\to}$, then we have

$$\int_X^{\otimes} A\, d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) =$$
$$\bigvee_{g(Y) \in \mathcal{M}'_{g(X)} \setminus \{1_\emptyset\}} \bigwedge_{f(m) \in \mathrm{Supp}(g(Y))} (g(A)(f(m)) \otimes \mu'(g(Y))) =$$
$$\bigvee_{Z \in \mathcal{M}'_{g(X)} \setminus \{1_\emptyset\}} \bigwedge_{n \in \mathrm{Supp}(Z)} (g(A)(n) \otimes \mu'(Z)) = \int_{g(X)}^{\otimes} g(A)\, d\mu'$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$. □

As we have mentioned above, our approach to fuzzy quantifiers seems to be similar to the Sugeno approach. Let us show that the Sugeno integral is a special case of our proposed integral. Since the Sugeno integral is defined under different presumptions[5], we will use a slight modification of its definition with respect to the fuzzy measurable spaces over complete residuated lattices.

Let $\mathbf{L}$ be a complete residuated lattice and $(M, \mathcal{M})$ be a fuzzy measurable space such that $A \cap B \in \mathcal{M}$ for any $A, B \in \mathcal{M}$.[6] Denote $A_a = \{m \mid m \in M \ \&\ A(m) \geq a\}$. We say that an $\mathbf{L}$-fuzzy set A is *$\mathcal{M}$-Sugeno measurable*, if $1_{A_a} \in \mathcal{M}$ for any $a \in L$. The *Sugeno integral* is given, for any fuzzy measure space $(M, \mathcal{M}, \mu)$ with $A \cap B \in \mathcal{M}$ for any $A, B \in \mathcal{M}$, for any $\mathcal{M}$-Sugeno measurable $\mathbf{L}$-fuzzy set A and for any $X \in \mathcal{M}$, by

$$\int_X A\, d\mu = \bigvee_{a \in L} (a \wedge \mu(1_{A_a} \cap X)). \tag{23}$$

As we can see the Sugeno integral uses the operation of infimum. This leads us to restrict ourselves to complete Heyting algebras, where $\otimes = \wedge$.

Theorem 3.11. *Let $\mathbf{L}$ be a complete Heyting algebra, $(M, \mathcal{M}, \mu)$ be a fuzzy measure space with $A \cap B \in \mathcal{M}$ for any $A, B \in \mathcal{M}$, A be a $\mathcal{M}$-Sugeno measurable $\mathbf{L}$-fuzzy set and $X \in \mathcal{M}$. Then $\int_X A\, d\mu = \int_X^{\otimes} A\, d\mu$.*

Proof. Recall that $\otimes = \wedge$ in Heyting algebras. Obviously, the equality is trivial for $X = 1_\emptyset$. Let us suppose that $X \neq 1_\emptyset$ and A be a $\mathcal{M}$-Sugeno measurable

[5]Notably, in the standard definitions, the ranges of considered functions are subsets of non-negative real numbers (see [6, 13, 27]), but, in our case, the ranges of functions are subsets of the support of a given residuated lattice.

[6]Note that, according to Theorem 2.2, each complete residuated lattice satisfying the law of double negation has this property. Nevertheless, there are fuzzy measurable spaces which keep this property, but $\mathbf{L}$ does not satisfy the law of double negation. A simple example is a fuzzy measurable space $(M, \mathcal{M})$ such that $\mathcal{M} \subseteq \mathcal{P}_{\mathbf{L}}(M)$ and $\mathbf{L}$ is an arbitrary complete residuated lattice (e.g. $L_{[0,\infty]}$ from Example 2.2).

L-fuzzy set. First, let $Y \in \mathcal{M}_X \setminus \{1_\emptyset\}$ and put $a = \bigwedge_{m\in \mathrm{Supp}(Y)} A(m)$. Obviously, $Y \subseteq 1_{\mathrm{Supp}(Y)}$ and if $m \in \mathrm{Supp}(Y)$, then $A(m) \geq a$ and thus $m \in A_a$. Hence, we easily obtain $Y = Y \cap X \subseteq 1_{A_a} \cap X$ which implies $\mu(Y) \leq \mu(1_{A_a} \cap X)$. Then we can write

$$\bigwedge_{m\in \mathrm{Supp}(Y)} (A(m) \wedge \mu(Y)) = a \wedge \mu(Y) \leq a \wedge \mu(1_{A_a} \cap X) \leq \int_X A \, d\mu,$$

and hence

$$\int_X^{\otimes} A \, d\mu \leq \int_X A \, d\mu.$$

Conversely, let $a \in L$ be any element. If $1_{A_a} \cap X = 1_\emptyset$, then $a \wedge \mu(1_{A_a} \cap X) = \bot \leq \int_X^{\otimes} A \, d\mu$. Let us suppose that $1_{A_a} \cap X \neq 1_\emptyset$. One checks easily that $\mathrm{Supp}(1_{A_a} \cap X) \subseteq \mathrm{Supp}(1_{A_a})$ which implies $a \leq \bigwedge_{m\in \mathrm{Supp}(1_{A_a})} A(m) \leq \bigwedge_{m\in \mathrm{Supp}(1_{A_a} \cap X)} A(m)$. Then we can write

$$\begin{aligned} a \wedge \mu(1_{A_a} \cap X) \leq & \Big(\bigwedge_{m\in \mathrm{Supp}(1_{A_a} \cap X)} A(m) \Big) \wedge \mu(1_{A_a} \cap X) = \\ & \bigwedge_{m\in \mathrm{Supp}(1_{A_a} \cap X)} (A(m) \wedge \mu(1_{A_a} \cap X)) \leq \int_X^{\otimes} A \, d\mu. \end{aligned}$$

Hence, we obtain

$$\int_X A \, d\mu \leq \int_X^{\otimes} A \, d\mu \tag{24}$$

and the proof is finished. □

Remark 3.11. It is easy to see (from the proof of the previous theorem) that an analogical result may be obtained for the fuzzy integrals defined by (23), where $\wedge$ is replaced by $\otimes$, if the following equality

$$a \otimes \bigwedge_{i\in I} b_i = \bigwedge_{i\in I} (a \otimes b_i) \tag{25}$$

holds for any $a \in L$ and any set $\{b_i \mid i \in I\}$ of elements of L in the given complete residuated lattice. Note that (25) is satisfied, for example, in the complete MV-algebras (see [2, 23]) or in the complete residuated lattices determined by the continuous t-norms (see [11]).

3.3. $\rightarrow$-fuzzy integral

In this part, we will introduce another type of fuzzy integral that can be defined on an arbitrary complementary fuzzy measure space $(M, \mathcal{M}, \nu)$. The form of this integral is motivated by our need to describe another class of models of **L**-fuzzy quantifiers of the type $\langle 1 \rangle$ which are kind of negations of the previous ones. Later we will show that this class of models is bounded by the models of determiners *no* and *not all*.

Definition 3.8. Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and X be a $\mathcal{M}$-measurable **L**-fuzzy set. The *$\rightarrow$-fuzzy integral of A on X* is given by

$$\int_X^{\rightarrow} A \; d\nu = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \; \bigvee_{m \in \mathrm{Supp}(Y)} (A(m) \rightarrow \nu(Y)). \tag{26}$$

If $X = 1_M$, then we write $\int^{\rightarrow} A \; d\nu$.

Remark 3.12. It is easy to see that $\int_{1_\emptyset}^{\rightarrow} A \; d\nu = \bigwedge \emptyset = \top$ for any $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $\int_X^{\rightarrow} A \; d\nu \leq \int_Y^{\rightarrow} A \; d\nu$, whenever $Y \subseteq X$. Since $\int_{1_M}^{\rightarrow} A \; d\nu \neq \bot$ in general, $\nu_A(X) = \int_X^{\rightarrow} A \; d\nu$ does not define a complementary fuzzy measure on $(M, \mathcal{M})$ in the sense of Definition 3.3.

Theorem 3.12. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. Then $\nu' : \mathcal{F}_{\mathbf{L}}(M) \rightarrow L$ defined by*

$$\nu'(A) = \int^{\rightarrow} A \; d\nu \tag{27}$$

is a complementary fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$.

Proof. This is analogous to the proof of Theorem 3.5, where the antitonicity of $\rightarrow$ in the first argument is applied.

Theorem 3.13. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. Then*

(i) $\int_X^{\rightarrow} (A \cap B) \; d\nu \geq \int_X^{\rightarrow} A \; d\nu \vee \int_X^{\rightarrow} B \; d\nu$,

(ii) $\int_X^{\rightarrow} (A \cup B) \; d\nu \leq \int_X^{\rightarrow} A \; d\nu \wedge \int_X^{\rightarrow} B \; d\nu$,

(iii) $\int_X^{\rightarrow} (c \otimes A) \; d\nu \leq c \rightarrow \int_X^{\rightarrow} A \; d\nu$,

(iv) $\int_X^{\rightarrow} (c \rightarrow A) \; d\nu \geq c \otimes \int_X^{\rightarrow} A \; d\nu$,

hold for any $X \in \mathcal{M}$, $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $c \in L$.

Proof. Analogously to the proof of the antitonicity of μ^* in Theorem 3.12, one checks easily that $\int_X^{\rightarrow} A \; d\nu \geq \int_X^{\rightarrow} B \; d\nu$, whenever $A \subseteq B$. Simple consequences of this antitonicity are the statements (i) and (ii). Let $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $c \in L$. Then we have

$$\int_X^{\rightarrow} (c \otimes A) \; d\nu = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \; \bigvee_{m \in \mathrm{Supp}(Y)} ((c \otimes A(m)) \rightarrow \nu(Y)) =$$

$$\bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \; \bigvee_{m \in \mathrm{Supp}(Y)} (c \rightarrow (A(m) \rightarrow \nu(Y))) \leq$$

$$c \rightarrow \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \; \bigvee_{m \in \mathrm{Supp}(Y)} (A(m) \rightarrow \nu(Y)) = c \rightarrow \int_X^{\rightarrow} A \; d\nu,$$

where $(a \otimes b) \to c = a \to (b \to c)$ holding in each residuated lattice, (b) and (e) of Theorem 2.1 are used. Since $a \otimes (b \to c) \leq (a \to b) \to c$ holds in each residuated lattice, then using (a) and (d) of Theorem 2.1 we can write

$$\int_X^{\to} (c \to A)\, d\nu = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigvee_{m \in \mathrm{Supp}(Y)} ((c \to A(m)) \to \nu(Y)) \geq \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigvee_{m \in \mathrm{Supp}(Y)} (c \otimes (A(m) \to \nu(Y))) \geq c \otimes \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigvee_{m \in \mathrm{Supp}(Y)} (A(m) \to \nu(Y)) = c \otimes \int_X^{\to} A\, d\nu$$

and the proof is finished. □

Theorem 3.14. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space and $c \in L$. Then we have*

(i) $\int^{\to} (c \otimes 1_X)\, d\nu = c \to \nu^*(1_X)$ *for any* $X \subseteq M$*,*

(ii) $\int^{\to} (c \otimes 1_X)\, d\nu = c \to \nu(1_X)$ *for any* $X \subseteq M$ *such that* $1_X \in \mathcal{M}$*,*

(iii) $\int^{\to} 1_X\, d\nu = \nu(1_X)$ *for any* $X \subseteq M$ *such that* $1_X \in \mathcal{M}$*,*

(iv) $\int^{\to} c\, d\nu = \neg c$*.*

Proof. Let $X \subseteq M$ be arbitrary. Then using (b) of Theorem 2.1 we have

$$\int^{\to} (c \otimes 1_X)\, d\nu = \bigwedge_{Y \in \mathcal{M} \setminus \{1_\emptyset\}} \bigvee_{m \in \mathrm{Supp}(Y)} ((c \otimes 1_X(m)) \to \nu(Y)) = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} (c \to \nu(Y)) = c \to \big(\bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \nu(Y) \big) = c \to \nu^*(1_X),$$

since $\bigvee_{m \in \mathrm{Supp}(Y)} ((c \otimes 1_X(m)) \to \nu(Y)) = \top$ for any $Y \notin \mathcal{M}_X \setminus \{1_\emptyset\}$, and (i) is proved. The statement (ii) follows from (i) and the equality $\nu^*(A) = \nu(A)$ holding for any $A \in \mathcal{M}$ and (iii) and (iv) are simple consequences of (ii). □

Theorem 3.15. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. If $X \in \mathcal{M}$ is such that $1_{\mathrm{Supp}(A)} \in \mathcal{M}_X$ for any $A \in \mathcal{M}_X$, then, for any $A \in \mathcal{F}_{\mathbf{L}}(M)$, we have*

$$\int_X^{\to} A\, d\nu = \bigvee_{1_Y \in \mathcal{P}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in Y} (A(m) \to \nu(1_Y)), \tag{28}$$

where $\mathcal{P}_X = \{1_{\mathrm{Supp}(A)} \mid A \in \mathcal{M}_X\}$.

Proof. This is a straightforward consequence of the fact that $1_{\mathrm{Supp}(A)} \in \mathcal{M}_X$ for any $A \in \mathcal{M}_X$ and $\nu(1_{\mathrm{Supp}(A)}) \leq \nu(A)$. □

Theorem 3.16. *Let* $\mathbf{L}$ *be a complete MV-algebra,* $(M, \mathcal{M}, \mu)$ *be a complementary fuzzy measure space,* $A \in \mathcal{F}_{\mathbf{L}}(M)$ *and* $X \in \mathcal{M}$. *Then*

$$\int_X^{\rightarrow} A\ d\nu = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \big((\bigwedge_{m \in \mathrm{Supp}(Y)} A(m)) \rightarrow \nu(Y) \big). \tag{29}$$

Moreover, we have

$$\int_X^{\rightarrow} (c \otimes A)\ d\nu = c \rightarrow \int_X^{\rightarrow} A\ d\nu \tag{30}$$

for any $c \in L$.

Proof. The statement (29) is a simple consequence of the equality in (f) of Theorem 2.1 holding in each MV-algebra. The equality (30) is trivial for $X = 1_\emptyset$. Let us suppose that $X \neq 1_\emptyset$. Then we have

$$\int_X^{\rightarrow} (c \otimes A)\ d\nu = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \ \bigvee_{m \in \mathrm{Supp}(Y)} (c \otimes A(m)) \rightarrow \nu(Y) =$$
$$\bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \ \bigvee_{m \in \mathrm{Supp}(Y)} (c \rightarrow (A(m) \rightarrow \nu(Y))) =$$
$$c \rightarrow \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \ \bigvee_{m \in \mathrm{Supp}(Y)} (A(m) \rightarrow \nu(Y)) = c \rightarrow \int_X^{\rightarrow} A\ d\nu,$$

where $(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c)$ holding in each residuated lattice, (b) and the equality in (e) of Theorem 2.1 are used. □

Theorem 3.17. *Let* g *be an isomorphism between complementary fuzzy measure spaces* $(M, \mathcal{M}, \nu)$ *and* $(M', \mathcal{M}', \nu')$ *and* $X \in \mathcal{M}$. *Then we have*

$$\int_X^{\rightarrow} A\ d\nu = \int_{g(X)}^{\rightarrow} g(A)\ d\nu' \tag{31}$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$.

Proof. This is analogous to the proof of Theorem 3.10. □

The following statement shows that if we consider a complete MV-algebra, then we can restrict ourselves, for example, to $\otimes$-fuzzy integrals, since each $\rightarrow$-fuzzy integral is uniquely determined by the negation of a suitable $\otimes$-fuzzy integral.

Theorem 3.18. *Let* $\mathbf{L}$ *be a complete MV-algebra and* $(M, \mathcal{M})$ *be a fuzzy measurable space. Then*

$$\int_X^{\rightarrow} A\ d\nu' = \neg \int_X^{\otimes} A\ d\mu, \tag{32}$$
$$\int_X^{\otimes} A\ d\mu' = \neg \int_X^{\rightarrow} A\ d\nu \tag{33}$$

hold for any fuzzy measure μ *and complementary fuzzy measure* ν, *where* $\nu' = \neg\mu$ *and* $\mu' = \neg\nu$.

PROOF. We will prove just the equality (32). The equality (33) could be proved analogously. According to $(a \otimes b) \to c = a \to (b \to c)$, (c) and the equality in (f) of Theorem 2.1 holding in each MV-algebra, we can write

$$\neg \int_X^{\otimes} A\ d\mu = \left(\bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)) \right) \to \bot =$$
$$\bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} ((A(m) \otimes \mu(Y)) \to \bot) =$$
$$\bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \to (\mu(Y)) \to \bot)) = \int_X^{\to} A\ d\nu',$$

where, according to Theorem 3.1, $\nu' = \neg\mu$ is a complementary fuzzy measure on the fuzzy measurable space $(M, \mathcal{M})$. □

Let us show a simple example demonstrating that the equalities in the previous theorem need not be satisfied, if general residuated lattices are considered, and the usage of the →-fuzzy integral can be advantageous in some situations.

Example 3.13. Let $\mathbf{L}_{[0,1]}$ be the complete Heyting algebra (see Example 2.2). One checks easily that $\neg a = 0$, if $a \in (0, 1]$, and $\neg 0 = 1$. Consider $M = (0, \infty]$ and $X = (0, a)$, where $a \in [1, \infty)$. Obviously, $\mathcal{M} = \{1_\emptyset, 1_X, 1_{\overline{X}}, 1_M\}$ is an algebra of $\mathbf{L}_{[0,1]}$-fuzzy sets. Further, let us define a fuzzy measure on $(M, \mathcal{M})$ by $\mu(1_\emptyset) = \mu(1_X) = 0$ and $\mu(1_{\overline{X}}) = \mu(1_M) = 1$. The fuzzy measure μ may be interpreted as a measure of the size of $\mathbf{L}_{[0,1]}$-fuzzy sets from $\mathcal{M}$ with respect to 1_M and thus $1_{\overline{X}}$ and 1_M can be viewed as "big" $\mathbf{L}_{[0,1]}$-fuzzy sets contrary to $1_\emptyset$ and 1_X which are "small" $\mathbf{L}_{[0,1]}$-fuzzy sets. Finally, let us define an $\mathbf{L}_{[0,1]}$-fuzzy set $A : M \to [0, 1]$ as follows

$$A(m) = \begin{cases} \frac{1}{m}, & m \in \overline{X}, \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

According to Theorem 3.15, we can write

$$\int^{\otimes} A\ d\mu = \bigvee_{1_Y \in \{1_X, 1_{\overline{X}}, 1_M\}} \bigwedge_{m \in Y} (A(m) \wedge \mu(1_Y)) = \bigwedge_{m \in \overline{X}} \frac{1}{m} = 0.$$

An interpretation of the obtained value is that A is not a "big" $\mathbf{L}_{[0,1]}$-fuzzy set with respect to 1_M. Obviously, if an $\mathbf{L}_{[0,1]}$-fuzzy set is not "big", then it does not mean that it is "small". Hence, to investigate, whether an $\mathbf{L}_{[0,1]}$-fuzzy set is "small", it seems to be advantageous to use →-fuzzy integral, where the complementary fuzzy measure is defined by $\nu'(\emptyset) = \nu'(X) = 1$ and $\nu'(\overline{X}) = \nu'(M) = 0$ (i.e., $\nu' = \neg\mu$). Then we have

$$\int^{\to} A\ d\nu' = \bigvee_{1_Y \in \{1_X, 1_{\overline{X}}, 1_M\}} \bigwedge_{m \in Y} (A(m) \to \nu'(1_Y)) = \bigvee_{m \in \overline{X}} \neg \frac{1}{m} = \bigvee_{m \in \overline{X}} 0 = 0$$

and, hence, we obtain

$$\neg \int^{\otimes} A \, d\mu = \neg 0 = 1 \neq 0 = \int^{\rightarrow} A \, d\nu'.$$

The fact that $\int^{\rightarrow} A \, d\nu' = 0$ indicates that A is not a "small" $\mathbf{L}_{[0,1]}$-fuzzy set with respect to 1_M. Clearly, both results seem to be right, because A is "smaller" than $1_{\overline{X}}$, but "bigger" than 1_X with respect to the defined measurement.

4. L-fuzzy quantifiers of the type $\langle 1 \rangle$

In [10], we defined the monadic **L**-fuzzy quantifiers of the type $\langle 1^n, 1 \rangle$. Here, we restrict ourselves to their special subclass, namely, to the monadic **L**-fuzzy quantifiers of the type $\langle 1 \rangle$ that can be defined as follows.

Definition 4.1. Let **L** be a complete residuated lattice, M be a universe (possibly empty[7]). A mapping $Q_M : \mathcal{F}_{\mathbf{L}}(M) \rightarrow L$ is called a *monadic* **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ *limited to* M.

Definition 4.2. An *unlimited (finite, countable) monadic* **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ is a functional Q assigning to each (finite, countable) universe M a monadic **L**-fuzzy quantifier Q_M of the type $\langle 1 \rangle$ limited to M.

In the following text, we will occasionally omit some of the terms "unlimited", "monadic" and "of the type $\langle 1 \rangle$" and we will say only "**L**-fuzzy quantifier" or "unlimited **L**-fuzzy quantifier", if no confusion can arise. Let us demonstrate several examples of unlimited **L**-fuzzy quantifiers that are interpretations of well-known determiners in natural language (see [10]). We will use expressions all, some, not all and no as generic expressions which stand for natural language quantifiers of the type $\langle 1 \rangle$, e.g. "everything", "someone", "not everyone" and "nothing", respectively.

Example 4.1. Let **L** be a complete residuated lattice. Then

$$(\mathsf{all})_M(A) = \bigwedge_{m \in M} A(m),$$
$$(\mathsf{some})_M(A) = \bigvee_{m \in M} A(m),$$
$$(\mathsf{not\ all})_M(A) = \bigvee_{m \in M} \neg A(m),$$
$$(\mathsf{no})_M(A) = \bigwedge_{m \in M} \neg A(m),$$

[7] To define the behavior of generalized quantifiers for the empty universe is important in some situations. It happens, for example, when we study type $\langle 1, 1 \rangle$ quantifiers which are obtained from type $\langle 1 \rangle$ quantifiers by means of *relativization*. Then it is vital to have values of e.g. $(\mathsf{some})_\emptyset(1_\emptyset)$ defined, see discussion in [24], p. 137.

where M is an arbitrary universe and $A \in \mathcal{F}_{\mathbf{L}}(M)$, define unlimited **L**-fuzzy quantifiers of the type $\langle 1 \rangle$. Obviously, the definitions of all and some (interpretations of determiners *all* and *some*) are the same as the interpretations of $\forall$ and $\exists$ in fuzzy logic, respectively. The others are negations of the previous ones. Notice that $(\mathsf{all})_{\emptyset}(\emptyset) = (\mathsf{no})_{\emptyset}(\emptyset) = \top$ and $(\mathsf{some})_{\emptyset}(\emptyset) = (\mathsf{not\ all})_{\emptyset}(\emptyset) = \bot$.

Example 4.2. Let **L** be a residuated lattice. Then

$$(\mathsf{John})_M(A) = \begin{cases} A(m_{\mathsf{John}}), & \text{if } m_{\mathsf{John}} \in M, \\ \bot, & \text{otherwise,} \end{cases}$$

$$(\mathsf{neither\ Bill\ nor\ Jack})_M(A) = \begin{cases} \neg(A(m_{\mathsf{Bill}}) \wedge A(m_{\mathsf{Jack}})), & \text{if } \{m_{\mathsf{Bill}}, m_{\mathsf{Jack}}\} \subseteq M, \\ \top, & \text{otherwise,} \end{cases}$$

where M is an arbitrary universe and $A \in \mathcal{F}_{\mathbf{L}}(M)$, define unlimited **L**-fuzzy quantifiers of the type $\langle 1 \rangle$. These quantifiers are examples of noun phrases, which, according to Montague [18] and others, can be advantageously understood as type $\langle 1 \rangle$ quantifiers.

Now, let us recall some well-known semantic properties that are usually investigated in the case of the **L**-fuzzy quantifiers of the type $\langle 1 \rangle$. For more information as well as examples we refer to [3, 10].

Definition 4.3. Let Q, P be **L**-fuzzy quantifiers. Then we say that Q *is less than or equal to* P and denote it by $Q \leq P$, if, for any non-empty universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$, we have

$$Q_M(A) \leq P_M(A). \tag{35}$$

Further, we say that Q *is equal to* P and denote it by $Q = P$, if $Q \leq P$ and $P \leq Q$.

Definition 4.4. Let Q, P be **L**-fuzzy quantifiers. We say that Q *is identical to* P and denote it by $Q \equiv P$, if for any (possibly empty) universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$, we have

$$Q_M(A) = P_M(A). \tag{36}$$

Remark 4.3. Note that the behavior of **L**-fuzzy quantifiers for the empty universe is often unpredictable (e.g., $(\mathsf{all})_M(A) \leq (\mathsf{some})_M(A)$ for all $M \neq \emptyset$, but $(\mathsf{some})_{\emptyset}(1_{\emptyset}) \leq (\mathsf{all})_{\emptyset}(1_{\emptyset})$), therefore, we require only non-empty universes for their comparison in the first definition. Moreover, this restriction seems to be insignificant from the practical point of view. The second definition of identity of **L**-fuzzy quantifiers gives useful denotation.

Definition 4.5. An **L**-fuzzy quantifier Q is *permutation-invariant*, if for arbitrary universe M, bijective mapping $f : M \to M$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$ we have

$$Q_M(A) = Q_M(f^{\to}(A)). \tag{37}$$

The set of all permutation-invariant **L**-fuzzy quantifiers is denoted by PI.

Definition 4.6. An **L**-fuzzy quantifier Q is *isomorphism-invariant*, if for arbitrary universe M, bijective mapping $f : M \to M'$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$ we have

$$Q_M(A) = Q_{M'}(f^{\to}(A)). \tag{38}$$

The set of all isomorphism-invariant **L**-fuzzy quantifiers is denoted by ISOM.

Definition 4.7. An **L**-fuzzy quantifier Q satisfies *extension*, if for arbitrary universes M, M' with $M \subseteq M'$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$ we have

$$Q_M(A) = Q_{M'}(A). \tag{39}$$

The set of all **L**-fuzzy quantifiers satisfying extension is denoted by EXT.

Definition 4.8. Let Q be an **L**-fuzzy quantifier. We say that Q is *monotonically non-decreasing*, if for arbitrary universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $A' \in \mathcal{F}_{\mathbf{L}}(M)$ with $A \subseteq A'$ we have

$$Q_M(A) \leq Q_M(A') \tag{40}$$

and Q is *monotonically non-increasing*, if for arbitrary universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $A' \in \mathcal{F}_{\mathbf{L}}(M)$ with $A' \subseteq A$ we have

$$Q_M(A) \leq Q_M(A'). \tag{41}$$

For our purpose we will consider the following stronger definition of **L**-similarity of fuzzy sets. Recall that a mapping $R : \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \to L$ is called an **L**-*fuzzy relation on* $\mathcal{F}_{\mathbf{L}}(M)$. Let $[A\ R\ B]$ denote the degree in which **L**-fuzzy sets A and B belongs to the **L**-fuzzy relation R, i.e., $[A\ R\ B] = R(A, B)$. Let us define an **L**-fuzzy relation $\equiv_M \colon \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \to L$ by $[A \equiv_M B] = \top$, if there is a bijective mapping f of M onto M such that $f^{\to}(A) = B$, and $[A \equiv_M B] = \bot$, otherwise.

Definition 4.9. An **L**-fuzzy relation $\approx_M \colon \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \to L$ is called an **L**-*permutation equivalence* on $\mathcal{F}_{\mathbf{L}}(M)$, if

$$[A \equiv_M B] \leq [A \approx_M B] \tag{42}$$

$$[A \approx_M B] = [B \approx_M A] \tag{43}$$

$$[A \approx_M B] \leq [\overline{A} \approx_M \overline{B}] \tag{44}$$

$$[A \approx_M B] \otimes [B \approx_M C] \leq [A \approx_M C] \tag{45}$$

hold for arbitrary $A, B, C \in \mathcal{F}_{\mathbf{L}}(M)$.

Obviously, (43) and (45) are the common axioms of symmetry and transitivity, respectively. Further, (42) could be understood as a generalization of reflexivity from the permutation isomorphism point of view. Let $A, B \subseteq M$ are **L**-fuzzy sets which are similar. Then one could wish that the complements of A and B are also similar (at least in the degree in which A and B are **L**-equivalent). This idea is expressed in (44).

Example 4.4 (see [10]). Let **L** be a complete residuated lattice, M be any countable universe and Perm(M) denote the set of all bijective mappings of M onto M. Then

$$[A \approx_M^{\otimes} B] = \bigvee_{f \in \mathrm{Perm}(M)} \bigotimes_{m \in M} (A(m) \leftrightarrow B(m)) \tag{46}$$

defines the **L**-permutation equivalence $\approx_M^{\otimes}$ on $\mathcal{F}_{\mathbf{L}}(M)$.

Example 4.5 (see [10]). Let **L** be a complete residuated lattice, M be any universe and Perm(M) denote the set of all bijective mappings of M onto M. Then

$$[A \approx_M^{\wedge} B] = \bigvee_{f \in \mathrm{Perm}(M)} \bigwedge_{m \in M} (A(m) \leftrightarrow B(m)) \tag{47}$$

defines the **L**-permutation equivalence $\approx_M^{\wedge}$ on $\mathcal{F}_{\mathbf{L}}(M)$.

Definition 4.10. Let $\approx$ be a class of **L**-permutation equivalences such that for each (finite, countable) universe M there is a unique $\approx_M$ from $\approx$ defined on $\mathcal{F}_{\mathbf{L}}(M)$. A (finite, countable) **L**-fuzzy quantifier Q of the type $\langle 1 \rangle$ is *extensional with respect to* $\approx$, if it holds that

$$[A \approx_M A'] \leq Q_M(A) \leftrightarrow Q_M(A') \tag{48}$$

for each (finite, countable) universe M and $A, A' \in \mathcal{F}_{\mathbf{L}}(M)$. The set of all extensional **L**-fuzzy quantifiers with respect to $\approx$ is denoted by EXTENS($\approx$).

5. L-fuzzy quantifiers of the type ⟨1⟩ determined by fuzzy measures

5.1. Quantifiers determined by fuzzy measures of **L***-fuzzy sets*

Let $\mathcal{S}(M)$ denote a set of fuzzy measure spaces defined on a universe M. For better readability, we will denote by

$$\int_{(M,\mathcal{M})}^{\otimes} A \, d\mu \tag{49}$$

the $\otimes$-fuzzy integral $\int^{\otimes} A \, d\mu$ defined over a fuzzy measure space $(M, \mathcal{M}, \mu)$. Now we can define **L**-fuzzy quantifiers limited to M using fuzzy measure spaces from a set $\mathcal{S}(M)$ as follows.

Definition 5.1. Let $\mathcal{S}(M)$ be a (possibly empty) set of fuzzy measure spaces defined on a non-empty universe M. An **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ *limited to* M *determined by the fuzzy measure spaces from* $\mathcal{S}(M)$ is a mapping $Q_{\mathcal{S}(M)} : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{(M,\mathcal{M},\mu) \in \mathcal{S}(M)} \int_{(M,\mathcal{M})}^{\otimes} A \, d\mu. \tag{50}$$

Remark 5.1. It is easy to see that if $\mathcal{S}(M) = \emptyset$ for some (possibly non-empty set) M, then we have $Q_{\mathcal{S}(M)}(A) = \bot$ for any $A \in \mathcal{F}_{\mathbf{L}}(M)$.

It is easy to see that $\mathcal{S}(\emptyset) = \emptyset$ (there is no fuzzy measure space with $M = \emptyset$). Hence, each unlimited **L**-fuzzy quantifier Q based only on the formula (50) has $Q_\emptyset(1_\emptyset) = Q_{\mathcal{S}(\emptyset)}(1_\emptyset) = \bot$. However, for example, we have $(\mathsf{all})_\emptyset(1_\emptyset) = \top$. This motivates us to exclude the determination of $Q_\emptyset(1_\emptyset)$ by (50) in the following definition of unlimited **L**-fuzzy quantifier.

Definition 5.2. Let $\mathcal{S}$ be a functional assigning to each universe M a set $\mathcal{S}(M)$ of fuzzy measure spaces defined on M. An *unlimited* **L**-*fuzzy quantifier of the type* $\langle 1\rangle$ *determined by fuzzy measures over* $\mathcal{S}$ is an unlimited **L**-fuzzy quantifier of the type $\langle 1\rangle$ assigning an **L**-fuzzy quantifier $Q_{\mathcal{S}(M)}$ determined by the fuzzy measure spaces from $\mathcal{S}(M)$ to each *non-empty* universe M.

Example 5.2. Let M be a non-empty universe and $\mathcal{S}_i(M) = \{(M, \mathcal{F}_{\mathbf{L}}(M), \mu_i)\}$, where, for $i = 1, 2$, we have

$$\mu_1(A) = \begin{cases} \bot, & \text{if } A = 1_\emptyset, \\ \top, & \text{otherwise} \end{cases} \tag{51}$$

and

$$\mu_2(A) = \begin{cases} \top, & \text{if } A = 1_M, \\ \bot, & \text{otherwise.} \end{cases} \tag{52}$$

If Q is determined by the fuzzy measures from $\mathcal{S}_1(M)$ for all $M \neq \emptyset$, then $Q = \mathsf{some}$. In fact, if $M \neq \emptyset$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, then

$$Q_M(A) = Q_{\mathcal{S}(M)_1}(A) = \int^{\otimes}_{(M,\mathcal{F}_{\mathbf{L}}(M))} A \, d\mu_1 = \bigvee_{m\in M} \mu_1(\{m\}) \otimes A(m) =$$
$$\bigvee_{m\in M} \top \otimes A(m) = \bigvee_{m\in M} A(m) = (\mathsf{some})_M(A).$$

According to Definition 4.3, we have $Q = \mathsf{some}$. One checks easily that Q defined as $Q_{\mathcal{S}_2(M)}$ for all $M \neq \emptyset$ is equal to all.

Example 5.3. Let $\mathbf{L}_{[0,1]}$ be the Heyting algebra from Example 2.2 and

$$\mathcal{S}(M) = \{(M, \mathcal{F}_{\mathbf{L}}(M), \mu)\}.$$

One checks easily, using Theorem 3.8, that

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{1_Y \in \mathcal{P}_{\mathbf{L}}(M)\setminus\{1_\emptyset\}} \Big(\mu(1_Y) \wedge \bigwedge_{m\in Y} A(m)\Big),$$

because $\otimes$ coincides with $\wedge$ in Heyting algebras. Note that, according to Theorem 3.11, the value $Q_{\mathcal{S}(M)}(A)$ may be equivalently computed using the Sugeno integral.

Let M be a non-empty countable universe and μ_f denote one of the fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ defined by (9) and (10) in Example 3.5. Putting

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{1_Y \in \mathcal{P}_{\mathbf{L}}(M) \setminus \{1_\emptyset\}} \left(\mu_f(1_Y) \wedge \bigwedge_{m \in Y} A(m) \right)$$

for any non-empty countable universe M and $Q_\emptyset(1_\emptyset) = 1$, we obtain an unlimited countable **L**-fuzzy quantifier of the type $\langle 1 \rangle$ which is an interpretation of the quantifier *many things*. Define

$$\mu_f^{1/2}(A) = \begin{cases} 1, & \text{if } \mu_f(A) \geq \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$. Then putting

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{1_Y \in \mathcal{P}_{\mathbf{L}}(M) \setminus \{1_\emptyset\}} \left(\mu_f^{1/2}(1_Y) \wedge \bigwedge_{m \in Y} A(m) \right)$$

for any non-empty countable universe M and $Q_\emptyset(1_\emptyset) = 1$, we obtain an unlimited countable **L**-fuzzy quantifier of the type $\langle 1 \rangle$ which is an interpretation of the quantifier *at least half things*. If we restrict ourselves to the class of all finite **L**-fuzzy quantifiers, then one checks easily (using the equality $\mu_f(A) = \mu_f(h^{\rightarrow}(A))$ from Example 3.5) that both defined quantifiers are PI and ISOM. Moreover, they are EXTENS($\approx^{\wedge}$) and EXTENS($\approx^{\otimes}$) (see Theorem 5.6 and 5.7).

Theorem 5.1. *For each unlimited* **L**-*fuzzy quantifier* Q *of the type* $\langle 1 \rangle$ *determined by fuzzy measures over* $\mathcal{S}$*, we have*

$$\mathsf{all} \leq Q \leq \mathsf{some}. \tag{53}$$

Proof. Let Q be an unlimited **L**-fuzzy quantifier determined by fuzzy measures over $\mathcal{S}$. Let M be any non-empty universe. Obviously, for each $(M, \mathcal{M}, \mu) \in \mathcal{S}(M)$, we have $\mu_2(A) \leq \mu(A) \leq \mu_1(A)$ for any $A \in \mathcal{F}_{\mathbf{L}}(M)$. According to Example 5.2, we easy obtain

$$(\mathsf{all})_M(A) \leq Q_{\mathcal{S}(M)}(A) \leq (\mathsf{some})_M(A)$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$ and the proof is finished. □

In the following part, we will investigate the semantic properties of **L**-fuzzy quantifiers determined by fuzzy measures. The following theorem states a sufficient condition for **L**-fuzzy quantifiers to be permutation invariant.

Theorem 5.2. *Let* Q *be an unlimited* **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ *determined by fuzzy measures over* $\mathcal{S}$ *such that for each non-empty universe* M *we have* $\mathcal{S}(M) = [(M, \mathcal{M}, \mu)]$. *Then* $Q \in \mathrm{PI}$.

Proof. The condition (37) is trivially satisfied for $M = \emptyset$. Let us suppose that $M \neq \emptyset$ and $\mathcal{S}(M) = [(M, \mathcal{M}, \mu)]$. According to the definition of $[(M, \mathcal{M}, \mu)]$, we have $f^{\rightarrow}(M, \mathcal{M}', \mu') \in \mathcal{S}(M)$ for any $(M, \mathcal{M}', \mu') \in \mathcal{S}(M)$ and any bijective mapping $f : M \to M$. According to Theorem 3.10 and the form of $\mathcal{S}(M)$, for each $A \in \mathcal{F}_{\mathbf{L}}(M)$ and each bijective mapping $f : M \to M$, we can write

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{(M,\mathcal{M}',\mu') \in \mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M}')} A \; d\mu' =$$
$$\bigvee_{(M,\mathcal{M}',\mu') \in \mathcal{S}(M)} \int^{\otimes}_{f^{\rightarrow}(M,\mathcal{M}')} f^{\rightarrow}(A) \; d\mu'_{f^{\rightarrow}} =$$
$$\bigvee_{(M,\mathcal{M}',\mu') \in \mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M}')} f^{\rightarrow}(A) \; d\mu' = Q_{\mathcal{S}(M)}(f^{\rightarrow}(A)).$$

and thus $Q \in$ PI. □

Note that the specification of a necessary condition for **L**-fuzzy quantifiers being permutation invariant seems to be immensely complicated and it is still an open problem. In the following theorem, let us denote fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ by $\mathbf{M}$ and $\mathbf{M}'$, respectively.

Theorem 5.3. *Let Q be an unlimited* **L***-fuzzy quantifier of the type $\langle 1 \rangle$ determined by fuzzy measures over $\mathcal{S}$ such that, for any universes M, M' with the same cardinality, we have*

(i) if $\mathbf{M} \in \mathcal{S}(M)$ and $f : M \to M'$ is a bijection, then $f^{\rightarrow}(\mathbf{M}) \in \mathcal{S}(M')$,

(ii) if $\mathbf{M} \in \mathcal{S}(M)$ and $\mathbf{M}' \in \mathcal{S}(M')$, then $\mathbf{M}$ and $\mathbf{M}'$ are isomorphic.

Then $Q \in$ ISOM.

Proof. Let Q be an unlimited **L**-fuzzy quantifier of the type $\langle 1 \rangle$ determined by fuzzy measures over $\mathcal{S}$ such that the conditions (i) and (ii) are satisfied for any equipotent universes. The condition (38) is trivially satisfied for $M = \emptyset$. Let M be a non-empty universe, $f : M \to M'$ be a bijective mapping and $A \in \mathcal{F}_{\mathbf{L}}(M)$. We have to show that

$$Q_{\mathcal{S}(M)}(M)(A) = Q_{\mathcal{S}(M')}(f^{\rightarrow}(A)).$$

According to the presumptions (i) and (ii) and Theorem 3.10, we can write

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{(M,\mathcal{M},\mu) \in \mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M})} A \; d\mu =$$
$$\bigvee_{(M,\mathcal{M},\mu) \in \mathcal{S}(M)} \int^{\otimes}_{f^{\rightarrow}(M,\mathcal{M})} f^{\rightarrow}(A) \; d\mu_{f^{\rightarrow}} =$$
$$\bigvee_{(M',\mathcal{M}',\mu') \in \mathcal{S}(M')} \int^{\otimes}_{(M',\mathcal{M}')} f^{\rightarrow}(A) \; d\mu' = Q_{\mathcal{S}(M')}(f^{\rightarrow}(A)).$$

Hence, $Q \in$ ISOM. □

Theorem 5.4. *Let Q be an unlimited* $\mathbf{L}$*-fuzzy quantifier of the type* $\langle 1 \rangle$ *determined by fuzzy measures over* $\mathcal{S}$*. Then Q is a non-decreasing* $\mathbf{L}$*-fuzzy quantifier.*

Proof. This is a straightforward consequence of the fact that $\otimes$-fuzzy integrals are non-decreasing mappings and $\otimes$ is isotonic in both arguments. □

Remark 5.4. The following theorem shows that some is the only type $\langle 1 \rangle$ quantifier with the extension (EXT) property which can be successfully modeled by our $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1 \rangle$ determined by fuzzy measures. However, the quantifier all and quantifiers which are interesting from the point of view of fuzzy logic, for example "at least half things", "many things", "most things" do not possess the extension property, and we can model them, see Examples 5.2 and 5.3. Nevertheless, quantifiers which refer to absolute cardinalities, e.g. "at least three things", possess the extension property, therefore they cannot be successfully modeled by our quantifiers.

Theorem 5.5. *Let Q be an unlimited* $\mathbf{L}$*-fuzzy quantifier of the type* $\langle 1 \rangle$ *determined by fuzzy measures over* $\mathcal{S}$*. Then* $Q \in \mathrm{EXT}$ *if and only if* $Q \equiv \mathsf{some}$*.*

Proof. Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1 \rangle$ determined by fuzzy measures over $\mathcal{S}$. First, let us suppose that $Q \equiv \mathsf{some}$. Recall that, according to Example 5.2, the $\mathbf{L}$-fuzzy quantifier some is an unlimited $\mathbf{L}$-fuzzy quantifier determined by fuzzy measures over $\mathcal{S}_1$, i.e., $(\mathsf{some})_M = Q_{\mathcal{S}_1(M)}$ for any non-empty universe M. Let M, M' be any universes with $M \subseteq M'$. If $M = \emptyset$, then trivially

$$Q_\emptyset(1_\emptyset) = \bot = Q_{\mathcal{S}(M')_1}(1_\emptyset)$$

holds for any universe M'. If $M \neq \emptyset$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, then

$$Q_{\mathcal{S}(M)_1}(A) = \bigvee_{m \in M} A(m) = \bigvee_{m \in M'} A(m) = Q_{\mathcal{S}(M')_1}(A).$$

Hence, $Q \in \mathrm{EXT}$. Conversely, let us suppose that $Q \in \mathrm{EXT}$. We have to show that $Q \equiv \mathsf{some}$. According to Theorem 3.5, we have $\int^{\otimes}_{(M,\mathcal{M})} 1_\emptyset \, d\mu = \bot$ for any non-empty universe M and $(M, \mathcal{M}, \mu) \in \mathcal{S}(M)$. Hence, $Q_{\mathcal{S}(M)}(1_\emptyset) = \bot$ for any non-empty universe M. Since Q satisfies extension, then also

$$Q_\emptyset(1_\emptyset) = Q_{\mathcal{S}(M)}(1_\emptyset) = \bot = (\mathsf{some})_\emptyset(1_\emptyset). \tag{54}$$

Further, let us suppose that $M = \{m\}$. According to the definition of fuzzy measure of $\mathbf{L}$-fuzzy sets, we have $\mu(\{1_{\{m\}}\}) = \top$ for each fuzzy measure μ on a fuzzy measurable space $(\{m\}, \mathcal{M})$. Due to (ii) of Theorem 3.7, we can write

$$Q_{\mathcal{S}(\{m\})}(1_{\{m\}}) = \int^{\otimes}_{(\{m\},\mathcal{M})} 1_{\{m\}} \, d\mu = \mu(1_{\{m\}}) \otimes 1_{\{m\}}(m) = \top \otimes \top = \top.$$

Since $Q \in \mathrm{EXT}$, then, from (i) of Theorem 3.7, we have

$$\top = Q_{\mathcal{S}(\{m\})}(1_{\{m\}}) = Q_{\mathcal{S}(M)}(1_{\{m\}}) =$$

$$\bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M})} 1_{\{m\}}\; d\mu = \bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \mu^*(1_{\{m\}})$$

for any M such that $\{m\} \subseteq M$. Hence, for any $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $m \in M$, we can write

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M})} A\; d\mu \geq$$

$$\bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M})} A(m)\otimes 1_{\{m\}}\; d\mu = \bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} (A(m)\otimes \mu^*(1_{\{m\}}) =$$

$$A(m) \otimes \bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \mu^*(1_{\{m\}}) = \top \otimes A(m) = A(m),$$

where the monotonicity of $\otimes$-fuzzy integrals and (i) of Theorem 3.7 are used. Hence, we have

$$Q_{\mathcal{S}(M)}(A) \geq \bigvee_{m\in M} A(m) = (\mathsf{some})_M(A).$$

for any non-empty universe M and thus $Q \geq \mathsf{some}$. According to Theorem 5.1, we have $Q \leq \mathsf{some}$ and thus $Q = \mathsf{some}$. Due to (54) and the previous equality, we obtain $Q \equiv \mathsf{some}$. □

Theorem 5.6. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$ such that $\mathcal{S}(M) = [(M,\mathcal{M},\mu)]$ for each non-empty universe M. Then $Q \in \mathrm{EXTENS}(\approx^{\wedge})$.*

Proof. Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$ such that $\mathcal{S}(M) = [(M,\mathcal{M},\mu)]$ for each non-empty universe M. First, if $M = \emptyset$, then trivially $[1_\emptyset \approx^{\wedge}_M 1_\emptyset] = Q_\emptyset(1_\emptyset) \leftrightarrow Q_\emptyset(1_\emptyset) = \top$. Further, let $M \neq \emptyset$, $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $f : M \to M$ be a bijective mapping. One checks easily that

$$\int^{\otimes}_{(M,\mathcal{M}')} A\; d\mu' = \bigvee_{X\in\mathcal{M}\setminus\{1_\emptyset\}} \bigwedge_{m\in\mathrm{Supp}(X)} (A(f(m)) \otimes \mu(X)) \tag{55}$$

holds for any isomorphic $(M,\mathcal{M},\mu)$ and $(M,\mathcal{M}',\mu')$, where $f^{\to}(M,\mathcal{M},\mu) =$

$(M, \mathcal{M}', \mu')$. Using (55), we can write, for any bijective mapping $f : M \to M$,

$$
\begin{gathered}
Q_{\mathcal{S}(M)}(A) \leftrightarrow Q_{\mathcal{S}(M)}(B) = \\
\bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M})} A\; d\mu \leftrightarrow \bigvee_{(M,\mathcal{M}',\mu')\in\mathcal{S}(M)} \int^{\otimes}_{(M,\mathcal{M}')} B\; d\mu' = \\
\Big(\bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \bigvee_{X\in\mathcal{M}\setminus\{1_\emptyset\}} \bigwedge_{m\in\mathrm{Supp}(X)} (\mu(X)\otimes A(m))\Big) \leftrightarrow \\
\Big(\bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \bigvee_{X\in\mathcal{M}\setminus\{1_\emptyset\}} \bigwedge_{m\in\mathrm{Supp}(X)} (\mu(X)\otimes B(f(m)))\Big) \geq \\
\bigwedge_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \bigwedge_{X\in\mathcal{M}\setminus\{1_\emptyset\}} \bigwedge_{m\in\mathrm{Supp}(X)} \big((\mu(X)\otimes A(m)) \leftrightarrow (\mu(X)\otimes B(f(m)))\big) \geq \\
\bigwedge_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \bigwedge_{X\in\mathcal{M}\setminus\{1_\emptyset\}} \bigwedge_{m\in\mathrm{Supp}(X)} \big((\mu(X)\leftrightarrow\mu(X))\otimes(A(m)\leftrightarrow B(f(m)))\big) = \\
\bigwedge_{m\in M} (A(m)\leftrightarrow B(f(m)),
\end{gathered}
$$

where $\bigwedge_{i\in I}(a_i \leftrightarrow b_i) \leq (\bigvee_{i\in I} a_i) \leftrightarrow (\bigvee_{i\in I} b_i)$, $\bigwedge_{i\in I}(a_i \leftrightarrow b_i) \leq (\bigwedge_{i\in I} a_i) \leftrightarrow (\bigwedge_{i\in I} b_i)$ and $(a \leftrightarrow c)\otimes(b\leftrightarrow d) \leq (a\otimes b)\leftrightarrow(c\otimes d)$ holding in each complete residuated lattice are applied (see e.g. [2, 23]). Hence, we obtain

$$
Q_{\mathcal{S}(M)}(A) \leftrightarrow Q_{\mathcal{S}(M)}(B) \geq \bigvee_{f\in\mathrm{Perm}(M)} \bigwedge_{m\in M} (A(m)\leftrightarrow B(f(m)) = [A \approx^{\wedge}_{M} B]
$$

and thus $Q \in \mathrm{EXTENS}(\approx^{\wedge})$. □

Theorem 5.7. *Let Q be an unlimited countable* **L***-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$ such that $\mathcal{S}(M) = [(M, \mathcal{M}, \mu)]$ for each non-empty countable universe M. Then $Q \in \mathrm{EXTENS}(\approx^{\otimes})$.*

Proof. This is a straightforward consequence of the previous theorem and $[A \approx^{\wedge}_{M} B] \geq [A \approx^{\otimes}_{M} B]$ holding for each $M \neq \emptyset$ and $A, B \in \mathcal{F}_{\mathbf{L}}(M)$. □

5.2. Quantifiers determined by complementary fuzzy measures of **L***-fuzzy sets*

Analogously to the previous subsection, let $\mathcal{T}(M)$ denote a set of complementary fuzzy measure spaces defined on M and

$$
\int^{\rightarrow}_{(M,\mathcal{M})} A\; d\nu \tag{56}
$$

denote the $\rightarrow$-fuzzy integral $\int^{\rightarrow} A\; d\nu$ defined over a complementary fuzzy measure space $(M, \mathcal{M}, \nu)$. Now we can define **L**-fuzzy quantifiers limited to M using complementary fuzzy measure spaces from a set $\mathcal{T}(M)$ as follows.

Definition 5.3. Let $\mathcal{T}(M)$ be a (possibly empty) set of fuzzy measure spaces defined on a non-empty universe M. An $\mathbf{L}$*-fuzzy quantifier of the type* $\langle 1 \rangle$ *limited to* M *determined by the complementary fuzzy measure spaces from* $\mathcal{T}(M)$ is a mapping $Q_{\mathcal{T}(M)} : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by

$$Q_{\mathcal{T}(M)}(A) = \bigwedge_{(M,\mathcal{M},\nu)\in\mathcal{T}(M)} \int^{\rightarrow}_{(M,\mathcal{M})} A\ d\nu. \tag{57}$$

Remark 5.5. If $\mathcal{T}(M) = \emptyset$, then obviously $Q_{\mathcal{T}(M)}(A) = \bigwedge \emptyset = \top$ holds for any $A \in \mathcal{F}_{\mathbf{L}}(X)$.

Definition 5.4. Let $\mathcal{T}$ be a functional assigning to each universe M a set $\mathcal{T}(M)$ of complementary fuzzy measure spaces defined on M. An *unlimited* $\mathbf{L}$*-fuzzy quantifier of the type* $\langle 1 \rangle$ *determined by complementary fuzzy measures over* $\mathcal{T}$ is an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1 \rangle$ assigning an $\mathbf{L}$-fuzzy quantifier $Q_{\mathcal{T}(M)}$ determined by the complementary fuzzy measure spaces from $\mathcal{T}(M)$ to each *non-empty* universe M.

Example 5.6. Let M be a non-empty universe and $\mathcal{T}_i(M) = \{(M, \mathcal{F}_{\mathbf{L}}(M), \nu_i)\}$, where, for $i = 1, 2$, we have

$$\nu_1(A) = \begin{cases} \top, & \text{if } A = 1_\emptyset, \\ \bot, & \text{otherwise} \end{cases} \tag{58}$$

and

$$\nu_2(A) = \begin{cases} \bot, & \text{if } A = 1_M, \\ \top, & \text{otherwise.} \end{cases} \tag{59}$$

If Q is an unlimited $\mathbf{L}$-fuzzy quantifier determined by complementary fuzzy measures from $\mathcal{T}_1(M)$ for all $M \neq \emptyset$, then $Q = \mathsf{no}$. In fact, if $M \neq \emptyset$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, then

$$Q_M(A) = Q_{\mathcal{T}(M)_1}(A) = \int^{\rightarrow}_{(M,\mathcal{F}_{\mathbf{L}}(M))} A\ d\nu_1 = \bigwedge_{m\in M} (A(m) \to \nu_1(\{m\})) =$$
$$\bigwedge_{m\in M} (A(m) \to \bot) = \bigwedge_{m\in M} \neg A(m) = (\mathsf{no})_M(A).$$

According to Definition 4.3, we have $Q = \mathsf{no}$. One checks easily that Q defined as $Q_{\mathcal{T}_2(M)}$ for all $M \neq \emptyset$ is equal to $\mathsf{not\ all}$.

Theorem 5.8. *For each unlimited* $\mathbf{L}$*-fuzzy quantifier* Q *of the type* $\langle 1 \rangle$ *determined by complementary fuzzy measures over* $\mathcal{T}$*, we have*

$$\mathsf{no} \leq Q \leq \mathsf{not\ all}. \tag{60}$$

Proof. Let Q be an unlimited **L**-fuzzy quantifier determined by complementary fuzzy measures over $\mathcal{T}$. Let M be any non-empty universe. Obviously, for each $(M,\mathcal{M},\nu)\in\mathcal{T}(M)$, we have $\nu_1(A)\leq\nu(A)\leq\nu_2(A)$ for any $A\in\mathcal{F}_{\mathbf{L}}(M)$. According to Example 5.6, we easily obtain

$$(\mathsf{no})_M(A)\leq Q_{\mathcal{T}(M)}(A)\leq(\mathsf{not\ all})_M(A)$$

for any $A\in\mathcal{F}_{\mathbf{L}}(M)$ and the proof is finished. □

Again in this part, we will investigate the semantic properties of **L**-fuzzy quantifiers determined by complementary fuzzy measures. The following theorem states a sufficient condition for **L**-fuzzy quantifiers to be permutation invariant.

Theorem 5.9. *Let Q be an unlimited **L**-fuzzy quantifier of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$ such that, for each non-empty universe M, we have $\mathcal{T}(M)=[(M,\mathcal{M},\nu)]$. Then $Q\in$ PI.*

Proof. The condition (37) is trivially satisfied for $M=\emptyset$. Let us suppose that $M\neq\emptyset$ and $\mathcal{T}(M)=[(M,\mathcal{M},\nu)]$. Analogously to the proof of Theorem 5.2 and using Theorem 3.17, we can write

$$Q_{\mathcal{T}(M)}(A)=\bigwedge_{(M,\mathcal{M}',\nu')\in\mathcal{T}(M)}\int^{\rightarrow}_{(M,\mathcal{M}')}A\;d\nu'=$$
$$\bigwedge_{(M,\mathcal{M}',\mu')\in\mathcal{T}(M)}\int^{\rightarrow}_{f^{\rightarrow}(M,\mathcal{M}')}f^{\rightarrow}(A)\;d\nu'_{f^{\rightarrow}}=$$
$$\bigwedge_{(M,\mathcal{M}',\nu')\in\mathcal{T}(M)}\int^{\rightarrow}_{(M,\mathcal{M}')}f^{\rightarrow}(A)\;d\nu'=Q_{\mathcal{T}(M)}(f^{\rightarrow}(A))$$

for any $M\neq\emptyset$, bijective mapping $f:M\rightarrow M$ and $A\in\mathcal{F}_{\mathbf{L}}(M)$ and thus $Q\in$ PI. □

We can see that the proof of the previous theorem has the same form as in the case **L**-fuzzy quantifiers determined by fuzzy measures and only supremum was replaced by infimum and vice-versa. In the following text such proofs will be omitted.

Theorem 5.10. *Let Q be an unlimited **L**-fuzzy quantifier of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$ such that, for any universes M, M' with the same cardinality, we have*

(i) if $\mathbf{M}\in\mathcal{T}(M)$ and $f:M\rightarrow M'$ is a bijection, then $f^{\rightarrow}(\mathbf{M})\in\mathcal{T}(M')$,

(ii) if $\mathbf{M}\in\mathcal{T}(M)$ and $\mathbf{M}'\in\mathcal{T}(M')$, then $\mathbf{M}$ and $\mathbf{M}'$ are isomorphic.

Then $Q\in$ ISOM.

Proof. This is analogous to the proof of Theorem 5.3. □

Theorem 5.11. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$. Then Q is a non-increasing $\mathbf{L}$-fuzzy quantifier.*

Proof. This is a straightforward consequence of the fact that each complementary fuzzy measure is a non-increasing mapping and the operation $\rightarrow$ is isotonic in the second argument. □

The next result shows that our fuzzy quantifiers of the type $\langle 1\rangle$ generated by complementary fuzzy measures are not suitable for modeling of non-increasing quantifiers which refer to absolute cardinalities, such as "at most three things", see Remark 5.4. However, quantifiers which are interesting from the point of view of fuzzy logic, like "at most half things" do not possess the extension property and, therefore, can be modeled by our quantifiers generated by complementary fuzzy measures.

Theorem 5.12. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$. Then $Q\in \mathrm{EXT}$ if and only if $Q=\mathsf{not\ all}$.*

Proof. This is analogous to the proof of Theorem 5.5. □

Theorem 5.13. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$ such that $\mathcal{T}(M)=[(M,\mathcal{M},\nu)]$ for each non-empty universe M. Then $Q\in \mathrm{EXTENS}(\approx^{\wedge})$.*

Proof. Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier Q of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$ such that $\mathcal{T}(M)=[(M,\mathcal{M},\nu)]$ for each non-empty universe M. First, if $M=\emptyset$, then trivially $[1_\emptyset\approx^{\wedge}_M 1_\emptyset]=Q_\emptyset(1_\emptyset)\leftrightarrow Q_\emptyset(1_\emptyset)=\top$.

Further, let $M\neq\emptyset$, $A,B\in\mathcal{F}_{\mathbf{L}}(M)$ and $f:M\to M$ be a bijective mapping. One checks easily that

$$\int^{\rightarrow}_{(M,\mathcal{M}')} A\ d\nu' = \bigwedge_{X\in\mathcal{M}\setminus\{1_\emptyset\}}\ \bigvee_{m\in\mathrm{Supp}(X)} (A(f(m))\rightarrow\nu(X)) \tag{61}$$

holds for any isomorphic $(M,\mathcal{M},\nu)$ and $(M,\mathcal{M}',\nu')$, where $f^{\rightarrow}(M,\mathcal{M},\nu)=(M,\mathcal{M}',\nu')$. Using (61), we can write

$$Q_{\mathcal{T}(M)}(A)\leftrightarrow Q_{\mathcal{T}(M)}(B)=$$

$$\bigwedge_{(M,\mathcal{M},\mu)\in\mathcal{T}(M)}\int^{\rightarrow}_{(M,\mathcal{M})} A\ d\nu \leftrightarrow \bigwedge_{(M,\mathcal{M}',\mu')\in\mathcal{T}(M)}\int^{\rightarrow}_{(M,\mathcal{M}')} B\ d\nu' =$$

$$\Big(\bigwedge_{(M,\mathcal{M},\nu)\in\mathcal{T}(M)}\ \bigwedge_{X\in\mathcal{M}\setminus\{1_\emptyset\}}\ \bigvee_{m\in\mathrm{Supp}(X)} (A(m)\rightarrow\nu(X))\Big)\leftrightarrow$$

$$\bigwedge_{(M,\mathcal{M},\mu)\in\mathcal{T}(M)}\ \bigwedge_{X\in\mathcal{M}\setminus\{1_\emptyset\}}\ \bigwedge_{m\in\mathrm{Supp}(X)} \big((A(m)\rightarrow\nu(X))\leftrightarrow(B(f(m))\rightarrow\nu(X))\big)\geq$$

$$\bigwedge_{(M,\mathcal{M},\nu)\in\mathcal{T}(M)} \bigwedge_{X\in\mathcal{M}\setminus\{1_\emptyset\}} \bigwedge_{m\in\mathrm{Supp}(X)} \big((A(m) \leftrightarrow B(f(m))) \otimes (\nu(X) \leftrightarrow \nu(X))\big) = \bigwedge_{m\in M} (A(m) \leftrightarrow B(f(m)),$$

where $\bigwedge_{i\in I}(a_i \leftrightarrow b_i) \leq (\bigvee_{i\in I} a_i) \leftrightarrow (\bigvee_{i\in I} b_i)$, $\bigwedge_{i\in I}(a_i \leftrightarrow b_i) \leq (\bigwedge_{i\in I} a_i) \leftrightarrow (\bigwedge_{i\in I} b_i)$ and $(a \leftrightarrow c) \otimes (b \leftrightarrow d) \leq (a \to b) \leftrightarrow (c \to d)$ holding in each complete residuated lattice are applied. Hence, we obtain

$$Q_{\mathcal{T}(M)}(A) \leftrightarrow Q_{\mathcal{T}(M)}(B) \geq \bigvee_{f\in\mathrm{Perm}(M)} \bigwedge_{m\in M} (A(m) \leftrightarrow B(f(m)) = [A \approx^{\wedge} B].$$

and thus $Q \in \mathrm{EXTENS}(\approx^{\wedge})$. □

Theorem 5.14. *Let Q be an unlimited countable **L**-fuzzy quantifier of the type $\langle 1\rangle$ determined by complementary fuzzy measures over $\mathcal{T}$ such that $\mathcal{T}(M) = [(M, \mathcal{M}, \nu)]$ for each non-empty countable universe M. Then $Q \in \mathrm{EXTENS}(\approx^{\otimes})$.*

Proof. This is a straightforward consequence of the previous theorem and $[A \approx^{\wedge}_M B] \geq [A \approx^{\otimes}_M B]$ holding for each $M \neq \emptyset$ and $A, B \in \mathcal{F}_{\mathbf{L}}(M)$. □

6. Conclusion

Our aim in this paper was to study a special class of **L**-fuzzy quantifiers of the type $\langle 1\rangle$, namely fuzzy quantifiers determined by fuzzy measures and in this way to continue research started in [10]. To achieve this goal, we introduced new types of fuzzy integral, namely $\otimes$-fuzzy integral and $\to$-fuzzy integral and showed their properties. Among other things, we showed that the $\otimes$-fuzzy integral is a generalization of the well-known Sugeno integral. Then, we were able to introduce **L**-fuzzy quantifiers of the type $\langle 1\rangle$ determined by fuzzy measures (and by complementary fuzzy measures), to provide some examples and to show properties of these quantifiers.

In the future we will concentrate on studying of **L**-fuzzy quantifiers of the type $\langle 1, 1\rangle$ (and possibly also of the type $\langle 1^n, 1\rangle$) generated by fuzzy measures. Quantifiers of the type $\langle 1, 1\rangle$ serve as models of very important class of natural language determiners (cf. e.g. [24]), for example "a few X are Y", "almost all X are Y", etc. Our definitions of $\otimes$-fuzzy integral and $\to$-fuzzy integral allow us to define these quantifiers, and we believe that they provide an important class of models with interesting properties.

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