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ON THE ω -LIMIT SETS OF PRODUCT MAPS

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ABSTRACT. Let $\omega(\cdot)$ denote the union of all ω -limit sets of a given map. As the main result of this paper we prove that, for given continuous interval maps f_1, \dots, f_m , the union of all ω -limit sets of the product map $f_1 \times \dots \times f_m$ and the cartesian product of the sets $\omega(f_1), \dots, \omega(f_m)$ coincide.

This result enriches the theory of multidimensional permutation product maps, i.e., maps of the form

$$F(x_1, \dots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \dots, f_{\sigma(m)}(x_{\sigma(m)})),$$

where σ is a permutation of the set of indices $\{1, \dots, m\}$. Especially, for any such map F , we prove that the set $\omega(F)$ is closed and we also show that $\omega(F)$ cannot be a proper subset of the center of the map F . These results solve open questions mentioned, e.g., in [F. Balibrea, J. S. Cánovas, A. Linero, *New results on topological dynamics of antitriangular maps*, Appl. Gen. Topol.].

1. INTRODUCTION AND MAIN RESULTS

Let X be a compact metric space and let $C(X)$ be the set of continuous maps from X into itself. Put $I := [0, 1]$ and $I^m := I \times \dots \times I$ (m times). For a given $\varphi \in C(X)$ and $x \in X$ we consider an *orbit* $\text{Orb}_\varphi(x) := \{\varphi^n(x)\}_{n \geq 0}$, and we define the ω -*limit set* $\omega_\varphi(x)$ of the point x with respect to the map φ as the set of limit points of $\text{Orb}_\varphi(x)$. Finally,

$$\omega(\varphi) = \bigcup_{x \in X} \omega_\varphi(x)$$

is the ω -*limit set* of the map φ .

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Consider now $f_1, \dots, f_m \in C(X)$ and the equality

$$(1.1) \quad \omega(f_1 \times \dots \times f_m) = \omega(f_1) \times \dots \times \omega(f_m).$$

Clearly in some particular cases, for instance in the case of m identity maps, this equation is satisfied. On the other side the equality (1.1) need not be true in general, even when considering the product of two relatively simple restrictions of interval maps: see Section 6. Nevertheless, in the case of a finite product of interval maps, the equality holds as we present in the following theorem being the main result of this paper.

Theorem A. *Let $f_1, \dots, f_m \in C(I)$. Then*

$$\omega(f_1 \times \dots \times f_m) = \omega(f_1) \times \dots \times \omega(f_m)$$

and, consequently, $\omega(f_1 \times \dots \times f_m)$ is closed.

In fact the authors were originally motivated by another open problem (see [3]) of the theory of permutation product maps. Here a map $F : I^m \rightarrow I^m$ is a *permutation product map* if it is of the form $F(x_1, \dots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \dots, f_{\sigma(m)}(x_{\sigma(m)}))$ for some $f_1, \dots, f_m \in C(I)$, where σ is a permutation of the set of indices $\{1, \dots, m\}$. When σ is the identity we obtain a *product map*. We denote the set of (m -dimensional) permutation product maps by $C_P(I^m)$. For example, in the two-dimensional case, permutation maps are either product maps $F(x, y) = (f(x), g(y))$, or maps of the type $F(x, y) = (g(y), f(x))$, sometimes called *antitriangular* maps, for instance, see [3]. The original motivation of this paper was to study the validity of the equality

$$(1.2) \quad \omega(F) = \omega(g \circ f) \times \omega(f \circ g).$$

for an antitriangular map F . Since, obviously, $F^2 = (g \circ f) \times (f \circ g)$, a close relation between (1.1) and (1.2) is given by the well-known fact $\omega(F) = \omega(F^n)$ for every positive integer n , see [5] (in particular $\omega(F) = \omega(F^2)$). Consequently, it is deduced from Theorem A that (1.2) holds.

Corollary B. *Let $f, g \in C(I)$ and let $F(x, y) = (g(y), f(x))$. Then*

$$\omega(F) = \omega(g \circ f) \times \omega(f \circ g)$$

and, consequently, $\omega(F)$ is closed.

Moreover the same idea is easily generalizable to the case of general permutation product maps (see Corollary C), since for any $F \in C_P(I^m)$ there is l such that F^l is a product map. For instance, if

$$F(x_1, x_2, x_3, x_4, x_5) = (f_3(x_3), f_4(x_4), f_1(x_1), f_5(x_5), f_2(x_2)),$$

then

$$F^6(x_1, x_2, x_3, x_4, x_5) = (k_1(x_1), k_2(x_2), k_3(x_3), k_4(x_4), k_5(x_5)),$$

with $k_1 = (f_3 \circ f_1)^3$, $k_2 = (f_4 \circ f_5 \circ f_2)^2$, $k_3 = (f_1 \circ f_3)^3$, $k_4 = (f_5 \circ f_2 \circ f_4)^2$ and $k_5 = (f_2 \circ f_4 \circ f_5)^2$, and

$$\omega(F) = \omega(F^6) = \omega(k_1) \times \omega(k_2) \times \omega(k_3) \times \omega(k_4) \times \omega(k_5).$$

In general, for a given antitriangular map $F \in C_P(I^m)$, a permutation σ of order m is a decomposition of cyclic permutations σ_j of order o_j , $j = 1, \dots, p$, with $p \leq m$, $\sum_j o_j = m$, and σ_j acting on the indices j_1, \dots, j_{m_j} , with $\{i_s\}_{s=1}^{m_i} \cap \{j_t\}_{t=1}^{m_j} = \emptyset$, for any $i \neq j$, $i, j \in \{1, \dots, p\}$. Then $F(x_1, \dots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \dots, f_{\sigma(m)}(x_{\sigma(m)}))$ implies that $F^{\text{lcm}(o_j)}$ is a direct product of m interval maps, say $h_i \in C(I)$, $i = 1, 2, \dots, m$, and we can apply Theorem A. Here $\text{lcm}(o_j)$ denotes the least common multiple of all o_j , $j = 1, \dots, p$. Hence we have:

Corollary C. *If $F \in C_P(I^m)$ is given by*

$$F(x_1, \dots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \dots, f_{\sigma(m)}(x_{\sigma(m)})),$$

then

$$\omega(F) = \omega(h_1) \times \omega(h_2) \times \omega(h_3) \times \dots \times \omega(h_m),$$

where each h_j , $j = 1, \dots, m$, is an appropriate composition of maps from the family $\{f_1, \dots, f_m\}$, and, consequently, $\omega(F)$ is closed.

Thus, we have found a large class of two-dimensional maps whose ω -limit sets are closed. This property is not held for general two-dimensional maps on I^2 . For instance, Kolyada in [11] found such examples of so-called triangular maps $T(x, y) = (f(x), g_x(y))$ of the square I^2 ($f, g_x \in C(I)$ for all $x \in I$). Also, Hero in [9], using the technique of inverse limits, constructed an example of a plane homeomorphism for which the ω -limit set is not closed. In the case of interval or, in fact, graphs maps, $\omega(f)$ is always closed (see [16, 10, 17, 7]), but this is not generally true either for one-dimensional continua. An example of a continuous map on a dendrite whose ω -limit set is not closed can be found in [7].

We also prove (Corollary D) that the center of a permutation product map is contained in its ω -limit set. As expected, this property is not true for general two-dimensional maps or even for triangular maps of the square I^2 (see again [11]).

The key tool is the following. For any direct product of interval maps we have

$$C(f_1 \times \dots \times f_m) = C(f_1) \times \dots \times C(f_m)$$

as a consequence of the fact $C(g) = \overline{P(g)}$ if $g \in C(I)$ (see [8]), where $C(\cdot)$ and $P(\cdot)$ denote the center and the set of periodic points of a map, respectively. Since $C(f_j) \subseteq \omega(f_j)$, and $C(F^n) = C(F)$ for all $F \in C_P(I^m)$ and $n \geq 1$ (see [5]), Corollary C implies:

Corollary D. *Let $F \in C_P(I^m)$. Then $C(F) \subseteq \omega(F)$.*

The paper is organized as follows. In Section 3 and 4 we study some properties of ω -limit sets of one-dimensional maps. We use these results to prove Theorem A in Section 5. Section 6 provides a counterexample to Theorem A after replacing the interval I by a Cantor set M . But firstly we introduce some basic notions.

2. DEFINITIONS AND NOTATIONS

For any $\varphi \in C(X)$, an n -th iteration is defined inductively by $\varphi^n = \varphi \circ \varphi^{n-1}$, $n \in \mathbb{N}$, where φ^0 is the identity map on X .

For a subset $M \subseteq X$, we use \overline{M} to denote its closure. We say that $x \in X$ is a *periodic point* of a *period* $n \in \mathbb{N}$ if $\varphi^n(x) = x$ and $\varphi^i(x) \neq x$ for $0 < i < n$. If $n = 1$, x is called a *fixed point*. We use $P(\varphi)$ to denote the set of periodic points of φ . By $R(\varphi)$ we denote the set of *recurrent points* of φ , i.e., the set of points $x \in X$ satisfying $x \in \omega_\varphi(x)$. Finally, the center $C(\varphi)$ of the map φ is defined as the closure of the set of recurrent points.

Clearly, whenever $x \in R(\varphi)$ and $U \subseteq X$ is any open neighborhood x , one can find an infinite increasing sequence of return times $\{m_i\}$ such that $\varphi^{m_i}(x) \in U$ for $i = 1, 2, \dots$. Consequently, recurrent points can be classified depending on the properties of the sequence of return times. For instance, if, for any neighborhood U of x , $\{m_i\}$ is relatively dense in \mathbb{N} , that is, the distances $|m_{i+1} - m_i|$ are bounded, then x is *uniformly recurrent*. If, for any U , there is an integer m such that $m_i = m \cdot i$, the point x is *regularly recurrent*. Denote by $x \in UR(\varphi)$, $x \in RR(\varphi)$ a uniformly recurrent and regularly recurrent point, respectively. Obviously, any regularly recurrent point is also uniformly recurrent.

A set $M \subseteq X$ is φ -invariant (or only invariant) if $\varphi(M) \subseteq M$. Further, M is *minimal* with respect to φ if it is nonempty, closed and φ -invariant and if no proper subset of M has these three properties. It is well known that there is a connection between minimal sets and uniformly recurrent points: any point of a given minimal set is uniformly recurrent and, conversely, the ω -limit set of any uniformly recurrent point is minimal.

In what follows we use extensively the notion of periodic interval (now $X = I$ and $f \in C(I)$). In the literature this notion is defined in a

number of related, but not exactly equivalent ways, so it is convenient to distinguish among them. We say that a compact interval $J \subseteq I$ is *weakly periodic* (respectively, *periodic*, *strongly periodic*) of *period* $r \in \mathbb{N}$ if $f^r(J) = J$ and r is minimal with respect to this property (respectively, $f^r(J) = J$ and the intervals $\{f^i(J)\}_{i=0}^{r-1}$ have pairwise disjoint interiors, $f^r(J) = J$ and the intervals $\{f^i(J)\}_{i=0}^{r-1}$ are pairwise disjoint). In any case $\text{Orb}_f(J)$, the *orbit* of J , denotes the union $J \cup f(J) \cup \dots \cup f^{r-1}(J)$.

Remark 1. If J is weakly periodic of period r , then either J is strongly periodic of period r , or r is even and $J \cup f^{r/2}(J)$ is strongly periodic of period $r/2$. This result has been proved in [4], and for the sake of completeness, we include here its proof.

It clearly suffices to show that if $1 \leq k < r$ satisfies $f^k(J) \cap J \neq \emptyset$ then $k = r/2$. Suppose not. Then we can assume that $k < r/2$, which in view of the minimality of r implies (after rewriting $g = f^k$) that the intervals J , $g(J)$ and $g^2(J)$ are pairwise different. As neither $g(J)$ can be strictly contained in J nor it can strictly contain J (because $g^r(J) = J$) we can for example assume that $g(J)$ is to the right of J , that is, there are points of $g(J)$ to the right of J , and J is to the left of $g(J)$, that is, there are points of J to the left of $g(J)$.

We claim that $g^2(J)$ is then to the right of $g(J)$. Actually, if $g^2(J)$ is to the left of $g(J)$, then either $g^2(J)$ is to the right of J and so $g^r(J \cup g(J))$ is strictly contained in $J \cup g(J)$, or $g^2(J)$ is to the left of J and then $g^r(J \cup g(J))$ strictly contains $J \cup g(J)$; in both cases we arrive at a contradiction.

Repeating the argument we find that $g^{j+1}(J)$ is to the right of $g^j(J)$ for every j , which is impossible because $g^r(J) = J$. \square

Following [14], we say that an ω -limit set ω of $f \in C(I)$ has a *periodic decomposition* of period r if there exists a weakly periodic interval $J \subseteq I$ of period r such that $\omega \subseteq \text{Orb}_f(J)$, and such that for $0 \leq i < r$ the convex hulls $\text{conv}(\omega_i) := \text{conv}(f^i(J) \cap \omega)$ have at most endpoints in common.

For a given $f \in C(I)$, consider the system $\{\omega_f(x) : x \in I\}$ of its ω -limit sets partially ordered by inclusion. It is well known that each ω -limit set of f is a subset of some *maximal* ω -limit set (see [15]). Following the terminology of [15], we say that a maximal ω -limit set of f is of the *second kind* (or a *basic set*) if it contains periodic points and is infinite; otherwise we call it a maximal set of f of the *first kind*. We denote by $\mathcal{A}_1(f)$ (respectively, $\mathcal{A}_2(f)$) the system of *infinite* maximal ω -limit sets of the first kind (respectively, of the second kind). The sets from $\mathcal{A}_1(f)$ are also called *solenoidal* sets.

Finally, we say that an interval $J \subset I$ is *wandering* if all its iterates are pairwise disjoint and the orbits of its points do not converge to any periodic orbit.

3. ON SOLENOIDAL SETS

In this section we study ω -limit sets of the first kind. The starting point of this analysis is given by the fact that they exhibit a “periodic” behavior.

Proposition 1. [15] *Let $f \in C(I)$ and let $\omega = \omega_f(x)$ be an infinite maximal ω -limit set of f . Then ω is of the first kind if and only if ω has periodic decompositions of arbitrarily high periods.*

From Proposition 1, Remark 1 and well-known results on solenoidal sets mentioned e.g. in [5] it follows:

Proposition 2. *For any solenoidal set $\omega \subset I$ there are a strictly increasing sequence of positive integers $\{r_m\}_{m=1}^{\infty}$ and a decreasing sequence of strongly periodic intervals $\{J_m\}$ of periods $\{r_m\}$ such that*

$$\omega \subseteq K = \bigcap_{m=1}^{\infty} \bigcup_{i=0}^{r_m-1} f^i(J_m).$$

Moreover, ω is a perfect set and:

- (i) *Any one-point connected component of K consists of a regularly recurrent point belonging to ω .*
- (ii) *If J is a nondegenerate component of K , then at least one of its endpoints belongs to ω and none of its interior points belongs to ω . Moreover, J is a wandering interval.*

In the next two lemmas we fix a point u belonging to a solenoidal set $\omega = \omega_f(x)$. Without loss of generality we assume that a subsequence of $\{f^n(x)\}$ approaches u from the left, i.e. for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$(3.1) \quad f^n(x) \in [u - \varepsilon, u].$$

Let $\{r_m\}$ and $\{J_m\}$ be the corresponding sequences of numbers and strongly periodic intervals and define K as in Proposition 2. We can assume that

$$(3.2) \quad u \in J_m$$

for every m . In what follows, $\text{Int } M$ denotes the interior of M as a subset of \mathbb{R} , that is, $\text{Int } I = (0, 1)$.

Lemma 1. *For every $\varepsilon > 0$ there is an r -periodic interval J such that $u \in J$ and $\text{Int } J \subseteq \bigcup_{i=0}^{\infty} f^{ri}([u - \varepsilon, u])$.*

Proof. Since the sequence $\{f^n(x)\}$ approaches u from the left, there is $s \in \mathbb{N}$ such that $f^s([u - \varepsilon, u]) \cap [u - \varepsilon, u] \neq \emptyset$. Notice that $f^{si}([u - \varepsilon, u]) \cap f^{s(i+1)}([u - \varepsilon, u]) \neq \emptyset$ for all integers i and each $f^{si}([u - \varepsilon, u])$ is a connected set. Consequently, $M := \bigcup_{i=0}^{\infty} f^{si}([u - \varepsilon, u])$ is an interval.

Moreover, $\{f^{si}(M)\}_{i=0}^{\infty}$ is a decreasing sequence of intervals and we claim that its intersection $S := \bigcap_{i=0}^{\infty} f^{si}(M)$ is a nondegenerate interval. From $\{f^{si}(u)\} \subseteq M$ we take a subsequence $\{f^{sn_k}(u)\}_k$ convergent to $w \in S$. Since $u \in M$, we have $f^{sn_k}(u) \in f^{sn_k}(M)$ for all k . If S is not an interval, then $\bigcap_{i \geq 0} \overline{f^{si}(M)}$ consists of exactly one point w . On the other hand, it is clear that $f^s(w) = w$, so w is a periodic point, which contradicts that $w \in \omega_f(u) \subset \omega$ with ω of the first kind. This ends the claim.

Let $J' = \overline{S}$. Clearly $f^s(J') = J'$. Let r' be the minimal positive integer satisfying $f^{r'}(J') = J'$. Now, according to Remark 1, we have that either J' is r' -periodic (then we define $r = r'$ and $J = J'$), or r' is even, $\text{Int } J' \cap \text{Int } f^{r'/2}(J') \neq \emptyset$, and $J' \cup f^{r'/2}(J')$ is $r'/2$ -periodic (then we define $r = r'/2$ and $J = J' \cup f^{r'/2}(J')$). Notice that in both cases $r|s$.

The statement $\text{Int } J \subseteq \bigcup_{i=0}^{\infty} f^{ri}([u - \varepsilon, u])$ is clear from the construction, so it only rests to show $u \in J$.

To begin with this observe that $u \in K$ and $\omega_f(u) \subseteq \omega$. Moreover, by the construction of J , every limit point of $\{f^{ri}(u)\}$ must belong to J . If $\{u\}$ is a one-point connected component of K , then u is regularly recurrent by Proposition 2. Thus, $\{f^{ri}(u)\}$ accumulates at u and $u \in J$.

If $\{u\}$ is not a one-point connected component of K , then, again by Proposition 2, it is an endpoint of a nondegenerate component $[u, v]$ of K , which is a wandering interval containing no points of ω in its interior. We emphasize that $u < v$ because we assume condition (3.1). Now, if $\{f^{ri}(u)\}$ accumulates at u , we obtain $u \in J$, so we can assume that $\{f^{ri}(u)\}$ accumulates at $v \in J$. Since the orbit of v is infinite (because it belongs to a wandering interval) and $f^s(J) = J$, some iterate of v belongs to $\text{Int } J$. Then there exists iterate of u in $\text{Int } J$ and also $f^k(x) \in \text{Int } J$ for some positive integer k . As a consequence of the periodicity of J we get $f^n(x) \in \text{Orb}(J)$ for any $n \geq k$. Since $u \in \omega_f(f^k(x))$ and $\text{Orb}(J)$ is a closed invariant set by f , we conclude that u belongs to some interval $f^t(J)$, $0 \leq t < r$. If $u \notin J$, then $t > 0$ and $f^t(J)$ is to the left of J . In this case, if $[a, b]$ is the (possibly

degenerate) interval connecting J and $f^t(J)$, then the f^r -invariance of J and $f^t(J)$ implies that $f^r(a) \leq a$, $f^r(b) \geq b$, hence $[a, b]$ contains some fixed point of f^r . This is impossible because $[u, v]$ is a wandering interval covering $[a, b]$. \square

Lemma 2. *For every $\varepsilon > 0$ there are $0 < \delta < \frac{\varepsilon}{2}$ and $k \in \mathbb{N}$ such that*

$$[u - \delta, u] \subseteq \bigcap_{n=0}^{\infty} f^{kn}([u - \varepsilon, u]).$$

Proof. Let r and J be as in Lemma 1. Let m be such that $r_m > 2r$. From the r -periodicity of J , r_m -periodicity of J_m and (3.2) we deduce that all three intervals J_m , $f^r(J_m)$, and $f^{2r}(J_m)$ intersect J . These intervals are pairwise disjoint, so one of them is contained in J (recall that $r_m > 2r$). From here, by Lemma 1, it is clear that $J_m \subseteq J$ and r divides r_m since $f^{r_m}(J_m) = J_m \subseteq J$ implies $f^{r_m}(J) = J$.

We claim that u cannot be the left endpoint of J_m . The reason is that, by Proposition 2, J_m contains infinitely many points from ω . Hence $\{f^n(x)\}$ eventually falls into the orbit of J_m so it cannot approach u from the left; we also use the strong periodicity of J_m . This contradicts assumption (3.1) and finishes the claim.

Let $0 < \delta < \frac{\varepsilon}{2}$ be such that $[u - \delta, u] \subseteq J_m$. We next show that this number δ is adequate to our purposes (for an appropriate positive integer k). To simplify the notation we next write $g = f^r$, when the terms ‘‘orbit’’, ‘‘fixed point’’, and so on, refer to g . We emphasize that the orbit of J_m is contained in J . Taking into account that $r|r_m$, $r < r_m$ and $r_m > 2$, we deduce that some interval Q of the orbit of J_m is mapped to the right by g , that is, $g(Q) > Q$, and another interval \tilde{Q} of the orbit of J_m is mapped to the left (that is, $g(\tilde{Q}) < \tilde{Q}$). Let $L < R$ be two such consecutive intervals, that is, $g(L) > L$, $g(R) < R$ and $\text{conv}(L \cup R) \cap \text{Orb}(J_m) = L \cup R$. Then there exists a fixed point p lying between L and R . In particular, $p \in \text{Int } J$.

Notice that, by Lemma 1, we have $\text{Int } J \subseteq \bigcup_{i=0}^{\infty} g^i([u - \varepsilon, u])$. Since one of the intervals of the orbit of J_m is contained in $\text{Int } J$, we get that the whole orbit of J_m is contained in $\bigcup_{i=0}^{\infty} g^i([u - \varepsilon, u])$. Since $p \in \text{Int } J$, we get $p \in g^{i'}([u - \varepsilon, u])$ for some i' and then $p \in g^i([u - \varepsilon, u])$ for every $i \geq i'$. Let $[c, d] = \text{conv}(\text{Orb}_g(J_m))$. Now two possibilities arise.

a) We firstly assume that two intervals from the orbit of J_m lying on the same side of p are mapped by g to different sides of p (remember that $r_m > 2r$). There is no loss of generality in assuming that these two intervals, call them R_1 and R_2 , are to the right of p . Let v be the right endpoint of $\text{conv}(R_1 \cup R_2)$. Find a point $w \in L$ and a number l such that $g^l(w) = c$. Similarly, there are a point $z \in R$ and a

number t such that $g^t(z) = d$. Then $g^l([w, p]) \supseteq [c, p] \supseteq [w, p]$ and $g^t([p, z]) \supseteq [p, d] \supseteq [p, z]$, thus $g^{sl}([w, p]) \supseteq [c, p]$ and $g^{st}([p, z]) \supseteq [p, d]$ for every $s \geq 1$. From this, $g^{lt}([w, z]) \supseteq [c, d]$. Moreover, using that L and R are neighboring, $p \in \text{conv}(L \cup R)$, and the assumption on R_1 and R_2 , we obtain $\text{conv}(L \cup R) \subseteq g(\text{conv}(R_1 \cup R_2)) \subseteq g([p, v])$.

Next, we show that $v \in g^j([u - \varepsilon, u])$ for some $j \geq i'$. To see this, by the periodicity of J_m , $v \in \text{Orb}(J_m)$ implies the existence of a point $z' \in \text{Orb}(J_m)$ such that $g^{i'}(z') = v$. Moreover, $z' \in \bigcup_{i=0}^{\infty} g^i([u - \varepsilon, u])$, so $z' \in g^i([u - \varepsilon, u])$ for some $i \geq 0$. Hence $v \in g^j([u - \varepsilon, u])$ for some $j \geq i'$.

Finally,

$$\begin{aligned} g^{j+1+lt}([u - \varepsilon, u]) &\supseteq g^{1+lt}([p, v]) \supseteq g^{lt}(\text{conv}(L \cap R)) \supseteq g^{lt}([w, z]) \\ &\supseteq [c, d] \supseteq J_m \supseteq [u - \delta, u], \end{aligned}$$

and similarly $g^i([u - \varepsilon, u]) \supseteq [c, d] \supseteq J_m \supseteq [u - \delta, u]$ for every $i \geq j + 1 + lt$. Now it suffices to take $k = r(j + 1 + lt)$ to finish this part of the proof.

b) Now we assume that all intervals to the left of p are mapped to the right of p , and conversely. Then the convex hulls of the intervals from the orbit of J_m to the same side of p are mapped by g^2 onto themselves. For instance assume that J_m (hence u) is to the left of p and find l' such that $g^{2l'}(u)$ belongs to an interval from the orbit of J_m to the left of L (this is possible because $r_m > 2r$). This number l' can be taken as large as necessary and, consequently, we can assume that $g^{2l'}([u - \varepsilon, u])$ contains p (and then also L). Find t such that $g^{2t}(L)$ is the interval from the orbit of J_m containing the endpoint c . Then

$$g^{2l'+2t}([u - \varepsilon, u]) \supseteq g^{2t}(\text{conv}(L \cap \{p\})) \supseteq [c, p] \supseteq J_m \supseteq [u - \delta, u],$$

and similarly $g^{2l}([u - \varepsilon, u]) \supseteq [c, p] \supseteq J_m \supseteq [u - \delta, u]$ for every $l \geq l' + t$, since $g^2([c, p]) \supseteq [c, p]$. Thus, if j is an even number large enough, the statement of the lemma is valid for $k = rj$. \square

We are ready to prove the main result of this section.

Proposition 3. *Let $f \in C(I)$ and let $x \in I$. Assume that $\omega_f(x)$ is solenoidal. Then for every $u \in \omega_f(x)$ there is a sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers with the following property: Let $\{s_j\}$ be a sequence of positive integers and put $n_j = \sum_{i=1}^j s_i k_i$ for every j . Then there is $z \in I$ such that $\{f^{n_j}(z)\}$ converges to u .*

Proof. Take any $\varepsilon_0 > 0$ and construct a sequence $\{\varepsilon_j\}_{j=0}^{\infty}$ such that, for every $j \geq 1$, if $\varepsilon = \varepsilon_{j-1}$, then ε_j and k_j are the numbers δ and k from Lemma 2, respectively. Then $\varepsilon_j \rightarrow 0$.

Let $\{s_j\}$ be a sequence of positive integers and write $n_j = \sum_{i=1}^j s_i k_i$. From Lemma 2 we have $f^{s_i k_i}([u - \varepsilon_{i-1}, u]) \supseteq [u - \varepsilon_i, u]$ for every i .

Find a closed subinterval $I_0 \subseteq [u - \varepsilon_0, u]$ so that $f^{n_1}(I_0) = f^{s_1 k_1}(I_0) = [u - \varepsilon_1, u]$. Observe that

$$f^{n_2}(I_0) = f^{s_2 k_2 + s_1 k_1}(I_0) = f^{s_2 k_2}([u - \varepsilon_1, u]) \supseteq [u - \varepsilon_2, u].$$

Then, inside I_0 we can find a closed subinterval I_1 holding $f^{n_2}(I_1) = [u - \varepsilon_2, u]$ and such that $f^{n_1}(I_1) \subseteq [u - \varepsilon_1, u]$. Proceeding by induction, we are able to find a decreasing sequence of subintervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_j \supseteq I_{j+1} \supseteq \cdots$$

satisfying $f^{n_j}(I_{j-1}) = [u - \varepsilon_j, u]$ and $f^{n_i}(I_{j-1}) \subseteq [u - \varepsilon_i, u]$ for all $j \geq 1$ and $i < j$. Hence $\{I_j\}$ is a decreasing sequence of nonempty compact sets. Let $z \in \bigcap_j I_j \neq \emptyset$. Then $f^{n_j}(z) \rightarrow u$. □

4. ON BASIC SETS

Now we are going to analyze a useful property of ω -limit sets of the second kind. Recall that $\varphi \in C(X)$ is *topologically mixing* if for given two nonempty open sets $U, V \subseteq X$ there is $n_0 \in \mathbb{N}$ such that $\varphi^n(V) \cap U \neq \emptyset$ for all $n \geq n_0$. We say that a φ -invariant set $Y \subseteq X$ is *topologically mixing* if $\varphi|_Y$ is topologically mixing. Following Proposition 1, if $f \in C(I)$ and $\omega \in \mathcal{A}_2(f)$, then it admits a maximal finite decomposition into periodic portions.

Lemma 3. *Given $f \in C(I)$ and $x \in I$, assume that $\omega := \omega_f(x) \in \mathcal{A}_2(f)$. Let $\{\omega_i\}_{i=0}^{s-1}$ be a maximal periodic decomposition of ω . Then, for any $i = 0, \dots, s-1$, $f^s|_{\omega_i}$ is topologically mixing.*

Proof. Clearly, it suffices to show that one periodic portion (say ω_0) of ω is topologically mixing. From [6, Theorem 4.1] we know that $f^s|_{\omega_0}$ is almost conjugated to some mixing interval map $g : J \rightarrow J$. This means that there exists a monotone continuous surjective map $\varphi : \omega_0 \rightarrow J$ satisfying the following properties:

- (i) $g \circ \varphi = \varphi \circ f^s$,
- (ii) for each $z \in J$, $\varphi^{-1}(z)$ contains at most two points of the set ω_0 .

Now we are ready to show that $f^s|_{\omega_0}$ is topologically mixing. Let U, V be any nonempty open sets in ω_0 . After taking if necessary smaller sets we can assume that both $\varphi(U)$ and $\varphi(V)$ are open sets in J and, moreover, if $\varphi(x) = \varphi(y)$ for some points $x, y \in \omega_0$, then either $x, y \in U$, or $x, y \notin U$ (we use (ii)).

Since g is mixing, $g^i(\varphi(V)) \cap \varphi(U) \neq \emptyset$ for any integer i greater than some $n_0 \in \mathbb{N}$. Then $\varphi((f^s)^i(V)) \cap \varphi(U) \neq \emptyset$ for any $i \geq n_0$ by (i). But this also implies $(f^s)^i(V) \cap U \neq \emptyset$ for any $i \geq n_0$, because if $x \in (f^s)^i(V)$ and $y \in U$ are such that $\varphi(x) = \varphi(y)$, then $y \in U$ forces $x \in U$ as well. We have shown that $f^s|_{\omega_0}$ is topologically mixing. \square

5. PROOF OF THEOREM A

Let $f_1, \dots, f_m \in C(I)$. We have trivially

$$\omega(f_1 \times \dots \times f_m) \subseteq \omega(f_1) \times \dots \times \omega(f_m).$$

Hence it suffices to show that if $(x_1, \dots, x_m) \in I^m$, then

$$(5.1) \quad \omega_{f_1}(x_1) \times \dots \times \omega_{f_m}(x_m) \subseteq \omega(f_1 \times \dots \times f_m).$$

There is no loss of generality in assuming that all ω -limit sets to the left of (5.1) are maximal. Now two possibilities arise. First assume that some of these ω -limit sets, say $\omega_{f_1}(x_1)$, is of the second kind. By Lemma 3, $\omega_{f_1}(x_1)$ has a finite number (say s) of topologically mixing portions. Since

$$\omega(f_1 \times \dots \times f_m) = \omega(f_1^s \times \dots \times f_m^s)$$

and $\omega_{f_r}(x_r) = \bigcup_{i=0}^{s-1} \omega_{f_r^s}(f_r^i(x_r))$, $r = 1, \dots, m$, it is not restrictive to assume that $\omega_{f_1}(x_1)$ itself is topologically mixing. Then (5.1) is exactly Theorem 12 from [1].

Thus, since a direct product of mixing maps is mixing, it only rests to consider the case when every set $\omega_{f_r}(x_r)$, $r = 1, \dots, m$, is either finite (a periodic orbit) or solenoidal. In both situations, if the points $u_r \in \omega_{f_r}(x_r)$ are given, then there are sequences $\{k_{r,j}\}_{j=1}^{\infty}$ of positive integers with the property described in Proposition 3 for u and $\{k_j\}$ (if $\omega_{f_r}(x_r)$ is periodic of period s , then $k_{r,j} = sj$ does the job). This property allows us, after defining $n_j = \sum_{i=1}^j k_{1,i} k_{2,i} \dots k_{m,i}$, to find points z_1, \dots, z_m such that each sequence $\{f_r^{n_j}(z_r)\}$ converges to u_r , $r = 1, \dots, m$. Hence $(u_1, \dots, u_m) \in \omega_{f_1 \times \dots \times f_m}(z_1, \dots, z_m)$. This implies (5.1) and finishes the proof of Theorem A in general.

6. A COUNTEREXAMPLE

The following map was used in [3] as a counterexample to the equality $\text{UR}(f \times f) = \text{UR}(f) \times \text{UR}(f)$. We next show that it can be used to contradict a similar equality concerning ω -limit points - namely $\omega(f \times g) = \omega(f) \times \omega(g)$ (see the introduction for the complete explanation).

Example 1. *There are a Cantor set $M \subset I$ and a continuous map $f : M \rightarrow M$ such that*

$$\omega(F) \neq \omega(f) \times \omega(f)$$

for the product map $F = f \times f$.

It is well known (see [13]) that if the number $\alpha \in (0, 1)$ is appropriately chosen, then $h \in C(I)$ defined by $h(x) = \max\{\alpha, 1 - |2x - 1|\}$ is of type 2^∞ in the Sharkovsky ordering and has a unique h -minimal Cantor set $M \subset I$, which is a solenoidal set for h . More precisely, there is a decreasing sequence of strongly periodic intervals $\{J_m\}$ of periods $\{2^m\}$ such that $M \subseteq K = \bigcap_{m=1}^\infty \bigcup_{i=0}^{2^m-1} h^i(J_m)$. Moreover, the interval $[u, v]$ of constancy of h is one of the components of K , with both u and v belonging to M .

Let f denote the restriction of h to M . We have $\omega(f) = M$ due to the minimality of M . Then it suffices to show that $(u, v) \notin \omega(F)$.

Suppose that $(u, v) \in \omega_F(x, y)$ for some $(x, y) \in M \times M$. Then there is a sequence $n_i \rightarrow \infty$ such that $h^{n_i}(x) \rightarrow u$ and $h^{n_i}(y) \rightarrow v$.

Fix arbitrarily the number m . Then u and v belong to the same interval $h^k(J_m)$ of the h -orbit of J_m . Since both $h^{n_i}(x)$ and $h^{n_i}(y)$ belong to $h^k(J_m)$, and $x, y \in M$ (so, $x \in \omega_h(h^{n_i}(x))$ and $y \in \omega_h(h^{n_i}(y))$), we get that there is an interval $h^l(J_m)$ containing both x and y . Since m was arbitrarily chosen, we conclude that the interval having x and y as its endpoints is wandering for h . Hence $|h^n(x) - h^n(y)| \rightarrow 0$ as $n \rightarrow \infty$, which is impossible. \square

Notice that a slight variation of the above example provides an antitriangular map $G(x, y) = (g(y), f(x))$ from M^2 into itself for which $\omega(G) \neq \omega(g \circ f) \times \omega(f \circ g)$; just take f as above, put $g = \text{Id}$ and realize that $G^2 = F$. It could be an interesting problem to search for conditions ensuring the equality mentioned in Theorem A.

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