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Institute for Research and Applications of Fuzzy Modeling

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Radim Bělohlávek

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University of Ostrava  
Institute for Research and Applications of Fuzzy Modeling  
Bráfova 7, 701 03 Ostrava 1, Czech Republic

tel.: +420-69-622 2808 fax: +420-69-22 28 28

e-mail: belohlav@osu.cz

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Radim Bělohlávek

*Institute for Res. and Appl. of Fuzzy Modeling, University of Ostrava  
Bráfova 7, CZ-701 03 Ostrava, Czech Republic  
e-mail: belohlav@osu.cz*

and

*Dept. Computer Science, Technical University of Ostrava,  
tř. 17. listopadu, CZ-708 33 Ostrava, Czech Republic*

**Abstract.** The concept of Galois connection between power sets is generalized from the point of view of fuzzy logic. Studied is the case where the structure of truth values forms a complete residuated lattice. It is proved that fuzzy Galois connections are in one-to-one correspondence with binary fuzzy relations. A representation of fuzzy Galois connections by (classical) Galois connections is provided.

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## 1 Introduction and preliminaries

This paper deals with Galois connections between power sets viewed from the perspective of fuzzy logic and fuzzy set theory. These theories develop on a formal level ZADEH's [14] ideas of graded approach to vagueness. Up to now there are several results fulfilling the program of investigating mathematical foundations of human-like reasoning [3, 6, 8].

A remarkable role in mathematics and in general in human reasoning is played by *Galois connections* [1, 9]. We will be concerned with Galois connections between power sets of two sets  $X$  and  $Y$  (shortly: Galois connection between  $X$  and  $Y$ ), i.e. a pair  $\langle \uparrow, \downarrow \rangle$  of mappings  $\uparrow : 2^X \rightarrow 2^Y$ ,  $\downarrow : 2^Y \rightarrow 2^X$ , satisfying (a)  $A_1 \subseteq A_2$  implies  $A_2^\uparrow \subseteq A_1^\uparrow$ , (b)  $B_1 \subseteq B_2$  implies  $B_2^\downarrow \subseteq B_1^\downarrow$ , (c)  $A \subseteq (A^\uparrow)^\downarrow$ , (d)  $B \subseteq (B^\downarrow)^\uparrow$ , for all  $A, A_1, A_2 \in 2^X$ ,  $B, B_1, B_2 \in 2^Y$ . Several examples of Galois connections in mathematics can be found e.g. in [1, pp. 123–124]. In general, a Galois connection is met whenever  $X$  is a set (of objects),  $Y$  is a set (of attributes of objects),  $A^\uparrow$  (for  $A \in 2^X$ ) is the set of all (attributes)  $y \in Y$  which are related to (shared by) all (objects)  $x \in A$ , and  $B^\downarrow$  (for  $B \in 2^Y$ ) is the set of all (objects)  $x \in X$  which are related to (have) all (properties)  $y \in B$ . Denoting the relation (to have an attribute) in question between  $X$  and  $Y$  by  $I$ , we have an example of Galois connection given by

$$A^\uparrow = \{y \in Y \mid \langle x, y \rangle \in I \text{ for all } x \in A\},$$

$$B^\downarrow = \{x \in X \mid \langle x, y \rangle \in I \text{ for all } y \in B\}.$$

In this context, the properties (a) and (b) of Galois connections capture the very natural rules “the more objects, the less common attributes”, and vice-versa. It has been proved by ORE [9] that this case of Galois connections is representative: each Galois connection between  $X$  and  $Y$

is of the above form (i.e. induced by some  $I \subseteq X \times Y$ ). Interestingly, Galois connections between power sets are the cornerstone for the theory of concept lattices (i.e. hierarchical structures of concepts in the sense of Port-Royal school), see e.g. the seminal paper [13]. Our main concern is to generalize the concept of Galois connection from the point of view of fuzzy logic and fuzzy set theory. In Section 2 we introduce the concept of fuzzy Galois connection, show how fuzzy Galois connections are induced by binary fuzzy relations and prove the generalization of ORE's theorem. Representation of fuzzy Galois connections by (classical) Galois connections is provided in Section 3.

Recall that a *fuzzy set* [14] in a universe set  $X$  is any function  $A : X \rightarrow L$ , where  $L$  is a suitable set (of truth values). The value  $A(x)$  is called the membership degree of  $x$  in  $A$  and it is interpreted as the truth value of “ $x$  is element of  $A$ ”. Similarly, a fuzzy relation between  $X$  and  $Y$  is any function  $I : X \times Y \rightarrow L$ . By  $\{a/x\}$  (where  $a \in L$ ,  $x \in X$ ) it is meant a fuzzy set given by  $\{a/x\}(x) = a$  and  $\{a/x\}(x') = 0$  for  $x' \in X$ ,  $x' \neq x$ . The crucial step is the choice of an appropriate structure on  $L$ . A very general one is that of a residuated lattice. A *residuated lattice* [7, 6, p. 42] is an algebra  $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$  where

- (i)  $\langle L, \wedge, \vee, 0, 1 \rangle$  is a lattice with the least element 0 and the greatest element 1,
- (ii)  $\langle L, \otimes, 1 \rangle$  is a commutative monoid, i.e.  $\otimes$  is associative, commutative, and the identity  $x \otimes 1 = x$  holds,
- (iii)  $\otimes$  and  $\rightarrow$  satisfy the adjointness property, i.e.

$$x \leq y \rightarrow z \quad \text{iff} \quad x \otimes y \leq z$$

holds for each  $x, y, z \in L$  ( $\leq$  denotes the lattice ordering).

Residuated lattices have been introduced by DILWORTH and WARD [12]. Note that several other names are used for residuated lattices, e.g. integral commutative residuated  $l$ -monoid [1, pp. 324–325], residuated abelian semigroup with a unit [2, pp. 211–214], or commutative complete lattice ordered semigroup with infinity [5, 4].

In each residuated lattice it holds  $x \leq y$  implies  $x \otimes z \leq y \otimes z$  (isotonicity), and  $x \leq y$  implies  $z \rightarrow x \leq z \rightarrow y$  (isotonicity in the second argument) and  $x \rightarrow z \geq y \rightarrow z$  (antitonicity in the first argument). The operation  $\otimes$  is thus a *t-norm* (see e.g. [6]),  $\rightarrow$  is called *residuum*. In the following we will deal with *complete residuated lattices*, i.e.  $\langle L, \wedge, \vee, 0, 1 \rangle$  is assumed to be a complete lattice. The following identities of complete residuated lattices will be needed (see e.g. [5, 11]):

$$a = 1 \rightarrow a \tag{1}$$

$$a \leq (a \rightarrow b) \rightarrow b \tag{2}$$

$$a \otimes (a \rightarrow b) \leq b \tag{3}$$

$$a \otimes \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I} (a \otimes b_i) \tag{4}$$

$$\left( \bigvee_{i \in I} a_i \right) \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b) \tag{5}$$

$$a \rightarrow \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \rightarrow b_i) \tag{6}$$

A semantically complete first-order many-valued logic with semantics defined over complete residuated lattices is described in [7]. Several special classes of residuated lattices serve as structures of truth values of logical calculi which are semantically complete w.r.t. these structures

(for details see e.g. [6, 8, 10, 11]). The semantics of conjunction and implication is modeled by the operations  $\otimes$  and  $\rightarrow$ , respectively. Supremum ( $\vee$ ) and infimum ( $\wedge$ ) are intended for modeling of the general and the existential quantifier, respectively.

The most studied and applied set of truth values is the real interval  $[0, 1]$ . The most important are the Lukasiewicz, Gödel, and product algebras (see [6] for their role and the definitions) defined by the following  $t$ -norms:  $a \otimes b = \max(a+b-1, 0)$  (Lukasiewicz),  $a \otimes b = \min(a, b)$  (Gödel), and  $a \otimes b = a \cdot b$  (product), with the corresponding residua given by  $a \rightarrow b = \min(1 - a + b, 1)$ ,  $a \rightarrow b = 1$  if  $a \leq b$  and  $= b$  else,  $a \rightarrow b = 1$  if  $a \leq b$  and  $= b/a$  else, respectively. Another important structure of truth values is given by  $L = \{a_0, a_1, \dots, a_n\}$ ,  $0 = a_0 < \dots < a_n = 1$ , with the  $t$ -norm  $a_k \otimes a_l = a_{\max(k+l-n, 0)}$  and the corresponding residuum  $a_k \rightarrow a_l = a_{\min(n-k+l, n)}$ . A special case of the latter algebra is the Boolean algebra  $\mathbf{2}$  of classical logic with the support  $2 = \{0, 1\}$ . Note that each of the preceding residuated lattices is complete.

Fuzzy sets (fuzzy relations) are also called  $\mathbf{L}$ -sets ( $\mathbf{L}$ -relations) if the structure  $\mathbf{L}$  is to be emphasized [4, 5]. In this perspective, classical sets (relations) are identified with  $\mathbf{2}$ -sets ( $\mathbf{2}$ -relations).  $\mathbf{2}$ -sets (and relations) are called *crisp*. The set of all  $\mathbf{L}$ -sets in a given universe  $X$  will be denoted by  $L^X$ .

## 2 Fuzzy Galois connections

In this section we study fuzzy Galois connections between the sets of all fuzzy sets in two given universes and their correspondence to binary fuzzy relations. We suppose  $\mathbf{L}$  to be a complete residuated lattice. First we have to internalize the concept of a Galois connection for the case of fuzzy logic. Given two fuzzy sets  $A_1, A_2 \in L^X$  we define the *subsethood degree* [4]  $Subs(A_1, A_2)$  of  $A_1$  in  $A_2$  by  $Subs(A_1, A_2) = \bigwedge_{x \in X} (A_1(x) \rightarrow A_2(x))$ . Note that  $Subs(A_1, A_2)$  is naturally interpreted as the truth value of “for all  $x \in X$  it holds that if  $x$  belongs to  $A_1$  then  $x$  belongs to  $A_2$ ” and that for  $\mathbf{L} = \mathbf{2}$ ,  $Subs$  coincides with the usual subsethood relation. As usual, we write  $A_1 \subseteq A_2$  for  $Subs(A_1, A_2) = 1$ .

**Definition 1** A fuzzy Galois connection ( $\mathbf{L}$ -Galois connection) between the sets  $X$  and  $Y$  is a pair  $\langle \uparrow, \downarrow \rangle$  of mappings  $\uparrow : L^X \rightarrow L^Y$ ,  $\downarrow : L^Y \rightarrow L^X$ , satisfying

$$Subs(A_1, A_2) \leq Subs(A_2^\uparrow, A_1^\uparrow) \quad (7)$$

$$Subs(B_1, B_2) \leq Subs(B_2^\downarrow, B_1^\downarrow) \quad (8)$$

$$A \subseteq (A^\uparrow)^\downarrow \quad (9)$$

$$B \subseteq (B^\downarrow)^\uparrow. \quad (10)$$

for every  $A, A_1, A_2 \in L^X$ ,  $B, B_1, B_2 \in L^Y$ .

Note that crisp Galois connections are just  $\mathbf{L}$ -Galois connections for  $\mathbf{L} = \mathbf{2}$ . In the following we write  $A^{\uparrow\downarrow}$  for  $(A^\uparrow)^\downarrow$  etc., and similarly,  $B^{\downarrow\uparrow}$  for  $(B^\downarrow)^\uparrow$  etc. The next theorem provides us with a simple characterization of  $\mathbf{L}$ -Galois connections.

**Theorem 2** A pair  $\langle \uparrow, \downarrow \rangle$  forms an  $\mathbf{L}$ -Galois connection between  $X$  and  $Y$  iff

$$Subs(A, B^\downarrow) = Subs(B, A^\uparrow) \quad (11)$$

for all  $A \in L^X$ ,  $B \in L^Y$ .

*Proof.* Let  $\langle \uparrow, \downarrow \rangle$  be an  $\mathbf{L}$ -Galois connection. From  $B \subseteq B^{\downarrow\uparrow}$  we get  $\text{Subs}(B^{\downarrow\uparrow}, A^\uparrow) \leq \text{Subs}(B, A^\uparrow)$ , hence by (7)

$$\text{Subs}(A, B^\downarrow) \leq \text{Subs}(B^{\downarrow\uparrow}, A^\uparrow) \leq \text{Subs}(B, A^\uparrow).$$

Repeating the arguments we get  $\text{Subs}(B, A^\uparrow) \leq \text{Subs}(A, B^\downarrow)$ . Therefore (11) holds.

Conversely, let (11) hold. From  $\text{Subs}(A^\uparrow, A^\uparrow) = 1$  we get  $\text{Subs}(A, A^{\uparrow\downarrow}) = 1$ , i.e.  $A \subseteq A^{\uparrow\downarrow}$  proving (9). (10) may be proved symmetrically. From  $A_2 \subseteq A_2^{\uparrow\downarrow}$  it follows by (11) that

$$\text{Subs}(A_1, A_2) \leq \text{Subs}(A_1, A_2^{\uparrow\downarrow}) = \text{Subs}(A_2^\uparrow, A_1^\uparrow)$$

proving (7). The proof of (8) is symmetric.  $\square$

We have noted that crisp Galois connections are in one-to-one correspondence with binary crisp relations. Our aim is to generalize this result for the case of  $\mathbf{L}$ -Galois connections. First, we show how a  $\mathbf{L}$ -Galois connection may be obtained from a binary  $\mathbf{L}$ -relation. To this end, suppose there is an  $\mathbf{L}$ -relation  $I$  between the sets  $X$  and  $Y$ , i.e.  $I \in L^{X \times Y}$ . We introduce operators  $\uparrow : L^X \rightarrow L^Y$  and  $\downarrow : L^Y \rightarrow L^X$ . Recall that for the crisp case and  $A \in 2^X$  we have

$$y \in A^\uparrow \quad \text{iff} \quad \text{for all } x \in X : \text{if } x \in A \text{ then } \langle x, y \rangle \in I.$$

Rewriting in the language of algebras of fuzzy logic yields for  $A \in L^X$

$$A^\uparrow(y) = \bigwedge_{x \in X} (A(x) \rightarrow I(x, y)) \quad \text{for } y \in Y \quad (12)$$

which will be taken for the definition of  $\uparrow : L^X \rightarrow L^Y$  for any complete residuated lattice  $\mathbf{L}$ . Similarly, for  $\downarrow$  we put for  $B \in L^Y$

$$B^\downarrow(x) = \bigwedge_{y \in Y} (B(y) \rightarrow I(x, y)) \quad \text{for } x \in X. \quad (13)$$

$I$  may represent a vague relationship between a set of objects and a set of attributes. In this case,  $A^\uparrow$  and  $B^\downarrow$  have the same linguistic interpretation as described in Section 1 but the semantics is modeled using fuzzy logic. The mappings  $\uparrow$  and  $\downarrow$  defined by (12) and (13) will be called *induced by  $I$* . To stress this fact we will also write  $\uparrow_I$  and  $\downarrow_I$  instead of  $\uparrow$  and  $\downarrow$ , respectively. The fundamental properties of  $\uparrow$  and  $\downarrow$  are described by the following lemma which generalizes the result of BIRKHOFF [1, p. 122].

**Lemma 3** *For each  $\mathbf{L}$ -relation  $I \in L^{X \times Y}$ , the pair  $\langle \uparrow, \downarrow \rangle$  of mappings defined by (12) and (13) forms an  $\mathbf{L}$ -Galois connection between  $X$  and  $Y$ .*

*Proof.* We prove (7) and (9). The conditions (8) and (10) may be proved analogously. Prove (7). We first show that

$$\text{Subs}(A_1, A_2) \otimes (A_2(x) \rightarrow I(x, y)) \leq (A_1(x) \rightarrow I(x, y)) \quad (14)$$

holds for any  $y \in Y$ . Thus let  $y \in Y$ . Since  $\text{Subs}(A_1, A_2) = \bigwedge_{x' \in X} A_1(x') \rightarrow A_2(x')$  we have  $\text{Subs}(A_1, A_2) \leq A_1(x) \rightarrow A_2(x)$  for any  $x \in X$ . By this fact and by (3) we have

$$\text{Subs}(A_1, A_2) \otimes A_1(x) \leq A_1(x) \otimes (A_1(x) \rightarrow A_2(x)) \leq A_2(x). \quad (15)$$

Furthermore, by (2),

$$A_2(x) \leq (A_2(x) \rightarrow I(x, y)) \rightarrow I(x, y) .$$

We therefore have

$$\text{Subs}(A_1, A_2) \otimes A_1(x) \leq (A_2(x) \rightarrow I(x, y)) \rightarrow I(x, y)$$

which is equivalent (applying the adjointness property twice) to (14). Hence, (14) holds. Using the fact that  $a_i \leq b_i$ ,  $i \in I$ , implies  $\bigwedge_{i \in I} a_i \leq \bigwedge_{i \in I} b_i$  and (4) we further conclude

$$\begin{aligned} & \text{Subs}(A_1, A_2) \otimes \bigwedge_{x \in X} (A_2(x) \rightarrow I(x, y)) \leq \\ & \leq \bigwedge_{x \in X} (\text{Subs}(A_1, A_2) \otimes (A_2(x) \rightarrow I(x, y))) \leq \bigwedge_{x \in X} (A_1(x) \rightarrow I(x, y)) . \end{aligned}$$

Applying adjointness property we obtain

$$\begin{aligned} & \text{Subs}(A_1, A_2) \leq \\ & \leq \bigwedge_{x \in X} (A_2(x) \rightarrow I(x, y)) \rightarrow \bigwedge_{x \in X} (A_1(x) \rightarrow I(x, y)) = \\ & = A_2^\uparrow(y) \rightarrow A_1^\uparrow(y) . \end{aligned}$$

Since  $y$  is arbitrary, we have

$$\text{Subs}(A_1, A_2) \leq \bigwedge_{y \in Y} (A_2^\uparrow(y) \rightarrow A_1^\uparrow(y)) = \text{Subs}(A_2^\uparrow, A_1^\uparrow)$$

proving (7). Prove (9). Let  $A \in L^X$ ,  $x \in X$ . We first prove that

$$A(x) \leq ( \bigwedge_{x' \in X} (A(x') \rightarrow I(x', y)) ) \rightarrow I(x, y) \tag{16}$$

holds for each  $y \in Y$ . By the adjointness property, (16) holds iff

$$A(x) \otimes ( \bigwedge_{x' \in X} (A(x') \rightarrow I(x', y)) ) \leq I(x, y)$$

which is due to commutativity of  $\otimes$  and the adjointness property equivalent to

$$( \bigwedge_{x' \in X} (A(x') \rightarrow I(x', y)) ) \leq A(x) \rightarrow I(x, y)$$

which holds evidently, hence (16) holds. From (16) and from the properties of infimum it follows

$$A(x) \leq \bigwedge_{y \in Y} ( \bigwedge_{x' \in X} (A(x') \rightarrow I(x', y)) ) \rightarrow I(x, y) = A^{\uparrow\downarrow}(x) ,$$

i.e.,  $A \subseteq A^{\uparrow\downarrow}$  proving (9). □

We are going to prove that each **L**-Galois connection is in the above sense induced by some **L**-relation. For this purpose we need the following lemma.

**Lemma 4** Let  $\langle \uparrow, \downarrow \rangle$  be a  $\mathbf{L}$ -Galois connection between  $X$  and  $Y$ . Then

$$\{a/x\}^\uparrow(y) = a \rightarrow \{1/x\}^\uparrow(y) \quad (17)$$

$$\{a/y\}^\downarrow(x) = a \rightarrow \{1/y\}^\downarrow(x) \quad (18)$$

$$\{a/x\}^\uparrow(y) = \{a/y\}^\downarrow(x) \quad (19)$$

holds for all  $a \in L$ ,  $x \in X$ ,  $y \in Y$ .

*Proof.* First, we show

**Claim A.**  $a \rightarrow \{1/y\}^\downarrow(x) \leq \{a/x\}^\uparrow(y)$  and  $a \rightarrow \{1/x\}^\uparrow(y) \leq \{a/y\}^\downarrow(x)$ .

*Proof of Claim A.* From

$$\begin{aligned} Subs(\{a/x\}, \{\{1/y\}^\downarrow(x)/x\}) &= \{a/x\}(x) \rightarrow \{1/y\}^\downarrow(x) = \\ &= a \rightarrow \{1/y\}^\downarrow(x) \end{aligned} \quad (20)$$

we have by (7)

$$Subs(\{\{1/y\}^\downarrow(x)/x\}^\uparrow, \{a/x\}^\uparrow) \geq a \rightarrow \{1/y\}^\downarrow(x). \quad (21)$$

From  $\{\{1/y\}^\downarrow(x)/x\} \subseteq \{1/y\}^\downarrow$  we conclude by (7)

$$\{\{1/y\}^\downarrow(x)/x\}^\uparrow \supseteq \{1/y\}^\downarrow^\uparrow,$$

i.e.

$$\{\{1/y\}^\downarrow(x)/x\}^\uparrow(y) \geq \{1/y\}^\downarrow^\uparrow(y) = 1,$$

thus

$$\{\{1/y\}^\downarrow(x)/x\}^\uparrow(y) = 1. \quad (22)$$

We thus have

$$\begin{aligned} &= Subs(\{\{1/y\}^\downarrow(x)/x\}^\uparrow, \{a/x\}^\uparrow) = \\ &= \bigwedge_{y' \in M} (\{\{1/y\}^\downarrow(x)/x\}^\uparrow(y') \{a/x\}^\uparrow(y')) \leq \\ &\leq \{\{1/y\}^\downarrow(x)/x\}^\uparrow(y) \rightarrow \{a/x\}^\uparrow(y) = \\ &= 1 \rightarrow \{a/x\}^\uparrow(y) = \{a/x\}^\uparrow(y), \end{aligned}$$

i.e.  $a \rightarrow \{1/y\}^\downarrow(x) \leq \{a/x\}^\uparrow(y)$ . The second part,  $a \rightarrow \{1/x\}^\uparrow(y) \leq \{a/y\}^\downarrow(x)$ , can be obtained symmetrically. *Q.E.D.*

**Claim B.**  $\{1/x\}^\uparrow(y) = \{1/y\}^\downarrow(x)$ .

*Proof of Claim B.* For  $a = 1$  we get by Claim A

$$\{1/y\}^\downarrow(x) = 1 \rightarrow \{1/y\}^\downarrow(x) \leq \{1/x\}^\uparrow(y)$$

and

$$\{1/x\}^\uparrow(y) = 1 \rightarrow \{1/x\}^\uparrow(y) \leq \{1/y\}^\downarrow(x),$$

i.e.  $\{1/x\}^\uparrow(y) = \{1/y\}^\downarrow(x)$ . *Q.E.D.*

Prove (17). By Claim A and Claim B we get

$$a \rightarrow \{1/x\}^\uparrow(y) = a \rightarrow \{1/y\}^\downarrow(x) \leq \{a/x\}^\uparrow(y),$$

the first inequality. The second inequality

$$\{a/x\}^\uparrow(y) \leq a \rightarrow \{1/x\}^\uparrow(y) \quad (23)$$

is equivalent (applying twice the adjointness property) to

$$a \leq \{a/x\}^\uparrow(y) \rightarrow \{1/x\}^\uparrow(y). \quad (24)$$

From  $\text{Subs}(\{1/x\}, \{a/x\}) = a$  we get by (7)

$$\begin{aligned} a &\leq \text{Subs}(\{a/x\}^\uparrow, \{1/x\}^\uparrow) \leq \\ &\leq \{a/x\}^\uparrow(y) \rightarrow \{1/x\}^\uparrow(y), \end{aligned}$$

i.e. (24) holds. We have proved (17). (18) may be proved analogously.

By (17), (18) and Claim B we have

$$\{a/x\}^\uparrow(y) = a \rightarrow \{1/x\}^\uparrow(y) = a \rightarrow \{1/y\}^\downarrow(x) = \{a/y\}^\downarrow(x),$$

proving (19).  $\square$

The following lemma generalizes ORE's result [9, Theorem 10].

**Lemma 5** *Let  $\langle \uparrow, \downarrow \rangle$  be a  $\mathbf{L}$ -Galois connection between  $X$  and  $Y$ . Then there is a  $\mathbf{L}$ -relation  $I \in L^{X \times Y}$  such that for the induced mappings  $\uparrow_I$  and  $\downarrow_I$  it holds  $\langle \uparrow_I, \downarrow_I \rangle = \langle \uparrow, \downarrow \rangle$ .*

*Proof.* Introduce  $I$  by

$$I(x, y) = \{1/x\}^\uparrow(y) = \{1/y\}^\downarrow(x)$$

which is correct due to Lemma 4. Observe that for  $A_i \in L^X$  it holds  $(\bigcup_{i \in I} A_i)^\uparrow = \bigcap_{i \in I} A_i^\uparrow$ . In fact, by (11), (5) and (6) we have for any  $C \in L^Y$

$$\begin{aligned} \text{Subs}(C, (\bigcup_{i \in I} A_i)^\uparrow) &= \text{Subs}((\bigcup_{i \in I} A_i), C^\downarrow) = \bigwedge_{x \in X} (\bigvee_{i \in I} A_i(x) \rightarrow C^\downarrow(x)) = \\ &= \bigwedge_{x \in X} \bigwedge_{i \in I} (A_i(x) \rightarrow C^\downarrow(x)) = \bigwedge_{i \in I} \bigwedge_{x \in X} (A_i(x) \rightarrow C^\downarrow(x)) = \\ &= \bigwedge_{i \in I} \text{Subs}(A_i, C^\downarrow) = \bigwedge_{i \in I} \text{Subs}(C, A_i^\uparrow) = \bigwedge_{i \in I} \bigwedge_{x \in X} (C(x) \rightarrow A_i^\uparrow(x)) = \\ &= \bigwedge_{x \in X} (C(x) \rightarrow \bigwedge_{i \in I} A_i^\uparrow(x)) = \text{Subs}(C, \bigcap_{i \in I} A_i^\uparrow). \end{aligned}$$

Take  $A \in L^X$ . We have

$$\begin{aligned} A^\uparrow(y) &= (\bigcup_{x \in X} \{A(x)/x\})^\uparrow(y) = \\ &= (\bigcap_{x \in X} \{A(x)/x\}^\uparrow)(y) = \bigwedge_{x \in X} \{A(x)/x\}^\uparrow(y) = \bigwedge_{x \in X} A(x) \rightarrow \{1/x\}^\uparrow(y) = \\ &= \bigwedge_{x \in X} A(x) \rightarrow I(x, y) = A^{\uparrow_I}(y), \end{aligned}$$

i.e.  $\uparrow = \uparrow_I$ . Using  $\{1/g\}^\uparrow(y) = \{1/m\}^\downarrow(x)$ ,  $\downarrow = \downarrow_I$  may be obtained symmetrically.  $\square$

As a direct consequence we get the following theorem which shows a one-to-one correspondence between  $\mathbf{L}$ -Galois connections and  $\mathbf{L}$ -relations.



**Theorem 6** For a binary  $\mathbf{L}$ -relation  $I \in L^{X \times Y}$  denote  $\langle \uparrow^I, \downarrow^I \rangle$  the mappings defined by (12) and (13). For an  $\mathbf{L}$ -Galois connection  $\langle \uparrow, \downarrow \rangle$  between  $X$  and  $Y$  denote  $I_{\langle \uparrow, \downarrow \rangle}$  the binary  $\mathbf{L}$ -relation from Lemma 5. Then  $\langle \uparrow^I, \downarrow^I \rangle$  is an  $\mathbf{L}$ -Galois connection and it holds

$$\langle \uparrow, \downarrow \rangle = \langle \uparrow_{I_{\langle \uparrow, \downarrow \rangle}}, \downarrow_{I_{\langle \uparrow, \downarrow \rangle}} \rangle \quad \text{and} \quad I = I_{\langle \uparrow^I, \downarrow^I \rangle}.$$

*Proof.* By Lemma 3 and Lemma 5 it suffices to prove  $I = I_{\langle \uparrow^I, \downarrow^I \rangle}$ . We have

$$\begin{aligned} I_{\langle \uparrow^I, \downarrow^I \rangle}(x, y) &= \{1/x\}^{\uparrow}(y) = \\ &= \bigwedge_{x' \in X} \{1/x\}(x') \rightarrow I(x', y) = 1 \rightarrow I(x, y) = I(x, y), \end{aligned}$$

proving the assertion.  $\square$

**Remark** As may be easily observed, any  $\mathbf{L}$ -Galois connection  $\langle \uparrow, \downarrow \rangle$  between  $X$  and  $Y$  forms a Galois connection between the complete lattices  $\langle L^X, \subseteq \rangle$  and  $\langle L^Y, \subseteq \rangle$  [1, 9]. This fact implies that the composite mappings  $\uparrow\downarrow : L^X \rightarrow L^X$  and  $\downarrow\uparrow : L^Y \rightarrow L^Y$  are closure mappings on the respective lattices and that the sets of all the closed elements of  $\uparrow\downarrow$  and  $\downarrow\uparrow$  (i.e.  $\mathbf{L}$ -sets  $A \in L^X$  and  $B \in L^Y$  such that  $A = A^{\uparrow\downarrow}$  and  $B = B^{\downarrow\uparrow}$ ) are dually isomorphic complete lattices w.r.t. the relation  $\subseteq$ . Furthermore,  $A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$  and  $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$  holds for any  $A \in L^X, B \in L^Y$ .

### 3 Representation of fuzzy Galois connections by Galois connections

Our aim now is to show that that  $\mathbf{L}$ -Galois connections may be represented by special systems of  $\mathbf{2}$ -Galois connections. For an  $\mathbf{L}$ -set  $A \in L^X$  and for any  $a \in L$ , the  $a$ -cut of  $A$  is the  $\mathbf{2}$ -set  ${}^a A = \{x \in X \mid A(x) \geq a\}$ . The  $a$ -cut  ${}^a I$  of a binary  $\mathbf{L}$ -relation is defined analogously.

**Lemma 7** Let  $I \in L^{X \times Y}$  be an  $\mathbf{L}$ -relation,  $\langle \uparrow, \downarrow \rangle$  be the  $\mathbf{L}$ -Galois connection induced by  $I$ , and for  $a \in L$  let  $\langle \uparrow^a, \downarrow^a \rangle$  be the  $\mathbf{2}$ -Galois connection induced by the  $\mathbf{2}$ -relation  ${}^a I$ . Then for every  $\mathbf{2}$ -sets  $A \in 2^X, B \in 2^Y, a \in L$ , we have

$${}^a(A^{\uparrow}) = A^{\uparrow^a}, \quad {}^a(B^{\downarrow}) = B^{\downarrow^a}, \quad (25)$$

and for every  $\mathbf{L}$ -sets  $A \in L^X, B \in L^Y, a \in L$ , we have

$${}^a(A^{\uparrow}) = \bigcap_{b \in L} ({}^b A)^{\uparrow_{a \otimes b}}, \quad {}^a(B^{\downarrow}) = \bigcap_{b \in L} ({}^b B)^{\downarrow_{a \otimes b}}. \quad (26)$$

*Proof.* Prove (25). Let  $A \in 2^X$ . By definition,  $y \in {}^a(A^{\uparrow})$  iff  $\bigwedge_{x \in X} (A(x) \rightarrow I(x, y)) \geq a$ . Since  $A(x) \in \{0, 1\}$  for each  $x \in X$ , this is equivalent to

$$\bigwedge_{x \in A} (A(x) \rightarrow I(x, y)) = \bigwedge_{x \in A} (1 \rightarrow I(x, y)) = \bigwedge_{x \in A} I(x, y) \geq a$$

which holds iff for each  $x \in A$  it holds  $\langle x, y \rangle \in {}^a I$ . The last statement is equivalent to  $y \in A^{\uparrow^a}$  proving the first part of (25). The second part can be proved analogously.

Prove (26). Let  $y \in {}^a(A^\uparrow)$ , i.e.  $\bigwedge_{x \in X} (A(x) \rightarrow I(x, y)) \geq a$ . We have  $y \in ({}^bA)^{\uparrow_{a \otimes b}}$  iff for each  $x \in {}^bA$  it holds  $\langle x, y \rangle \in {}^{a \otimes b}I$ . Suppose  $x \in {}^bA$ , i.e.  $A(x) \geq b$ . From  $a \leq A(x) \rightarrow I(x, y)$  we have  $a \otimes b \leq a \otimes A(x) \leq I(x, y)$ , i.e.  $\langle x, y \rangle \in {}^{a \otimes b}I$ . Since  $x$  was chosen arbitrarily, we conclude  $y \in ({}^bA)^{\uparrow_{a \otimes b}}$ , i.e.  ${}^a(A^\uparrow) \subseteq \bigcap_{b \in L} ({}^bA)^{\uparrow_{a \otimes b}}$ . Conversely, let  $y \in ({}^bA)^{\uparrow_{a \otimes b}}$ , i.e.  $x \in {}^bA$  implies  $\langle x, y \rangle \in {}^{a \otimes b}I$ . We have to show  $A^\uparrow(y) = \bigwedge_{x \in X} (A(x) \rightarrow I(x, y)) \geq a$ . Let  $x \in X$  and put  $b = A(x)$ . From  $x \in {}^bA$  it follows by assumption  $\langle x, y \rangle \in {}^{a \otimes b}I$ , i.e.  $a \otimes b \leq I(x, y)$ . We therefore have

$$a \leq b \rightarrow a \otimes b = A(x) \rightarrow a \rightarrow b \leq A(x) \rightarrow I(x, y),$$

hence  $a \leq \bigwedge_{x \in X} (A(x) \rightarrow I(x, y)) = A^\uparrow(y)$ . We thus have also  $\bigcap_{b \in L} ({}^bA)^{\uparrow_{a \otimes b}} \subseteq {}^a(A^\uparrow)$ , proving the first part of (26). The second part may be obtained symmetrically.  $\square$

**Definition 8** A system  $\{\langle \uparrow^a, \downarrow^a \rangle \mid a \in L\}$  of **2-Galois connections** is called **L-nested** if (1) for each  $a, b \in L$ ,  $a \leq b$ ,  $A \in 2^X$ ,  $B \in 2^Y$ , it holds  $A^{\uparrow^a} \supseteq A^{\uparrow^b}$ ,  $B^{\downarrow^a} \supseteq B^{\downarrow^b}$ , and (2) for every  $g \in G$ ,  $m \in M$ , the set  $\{a \in L \mid m \in \{g\}^{\uparrow^a}\}$  has the greatest element.

**Lemma 9** For  $j = 1, 2$ , let  $\langle \uparrow^j, \downarrow^j \rangle$ , and  $I_j$  be **L-Galois connections** and the corresponding **L-relations** between  $X$  and  $Y$ , i.e.  $I_j = I_{\langle \uparrow^j, \downarrow^j \rangle}$  and  $\langle \uparrow^j, \downarrow^j \rangle = \langle \uparrow^{I_j}, \downarrow^{I_j} \rangle$ . Then it holds

$$I_1 \subseteq I_2 \text{ iff for each } A \in L^X, B \in L^Y \text{ it holds } A^{\uparrow^1} \subseteq A^{\uparrow^2}, B^{\downarrow^1} \subseteq B^{\downarrow^2}.$$

*Proof.* The direct implication follows from (12) and (13). Conversely, from  $A^{\uparrow^1} \subseteq A^{\uparrow^2}$  we conclude  $I_1(x, y) = 1 \rightarrow I_1(g, m) = (1/x)^{\uparrow^1}(y) \leq (1/x)^{\uparrow^2}(y) = 1 \rightarrow I_2(x, y) = I_2(x, y)$  which means  $I_1 \subseteq I_2$ .  $\square$

**Theorem 10** For an **L-Galois connection**  $\langle \uparrow, \downarrow \rangle$  between  $X$  and  $Y$  denote  $\mathcal{C}_{\langle \uparrow, \downarrow \rangle} = \{\langle \uparrow^a, \downarrow^a \rangle \mid a \in L\}$  where  $\uparrow^a : 2^X \rightarrow 2^Y$  and  $\downarrow^a : 2^Y \rightarrow 2^X$  are defined by  $A^{\uparrow^a} = {}^a(A^\uparrow)$  and  $B^{\downarrow^a} = {}^a(B^\downarrow)$  for  $A \in 2^X$ ,  $B \in 2^Y$ . For an **L-nested system**  $\{\langle \uparrow^a, \downarrow^a \rangle \mid a \in L\}$  of **2-Galois connections** between  $X$  and  $Y$  denote  $\langle \uparrow^c, \downarrow^c \rangle$  the pair of mappings  $\uparrow^c : L^X \rightarrow L^Y$  and  $\downarrow^c : L^Y \rightarrow L^X$  defined by

$$A^{\uparrow^c}(y) = \bigvee \{a \mid y \in \bigcap_{b \in L} ({}^bA)^{\uparrow_{a \otimes b}}\}, \quad B^{\downarrow^c}(x) = \bigvee \{a \mid x \in \bigcap_{b \in L} ({}^bB)^{\downarrow_{a \otimes b}}\}$$

for  $A \in L^X$ ,  $B \in L^Y$ . Then it holds

- (1)  $\mathcal{C}_{\langle \uparrow, \downarrow \rangle}$  is a nested system of **L-Galois connections** between  $X$  and  $Y$ ,
- (2)  $\langle \uparrow^c, \downarrow^c \rangle$  is a **L-Galois connection** between  $X$  and  $Y$ ,
- (3)  $\mathcal{C} = \mathcal{C}_{\langle \uparrow^c, \downarrow^c \rangle}$  and  $\langle \uparrow, \downarrow \rangle = \langle \uparrow^c_{\langle \uparrow, \downarrow \rangle}, \downarrow^c_{\langle \uparrow, \downarrow \rangle} \rangle$ .

*Proof.* (1) follows by Lemma 9 from the fact that  $a \leq b$  implies  ${}^aI \supseteq {}^bI$ . (2) follows by Lemma 7 by the fact that for any  $A \in L^X$  it holds  $A(x) = \bigvee \{a \mid x \in {}^aA\}$  (and similarly for  $B \in L^Y$ ). (3) is a consequence of (1), (2), and Lemma 7.  $\square$

**Remark** Theorem 10 (3) assures there is a one-to-one correspondence between the class of all **L-Galois connections** and the class of all **L-nested systems** of **2-Galois connections** between  $X$  and  $Y$ . The existence of the one-to-one correspondence is apparent from the one-to-one correspondence between **L-Galois connections** and binary **L-relations** and the representation of **L-relations** by systems of **2-relations**. However, Theorem 10 provides the direct way.

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