

UNIVERSITY OF OSTRAVA

Institute for Research and Applications of Fuzzy Modeling

---

# Pseudo-Riemann-Stieltjes integral

ŠTAJNER-PAPUGA Ivana, GRBIĆ Tatjana and DAŇKOVÁ  
Martina

Research report No. 137

2008

*Submitted/to appear:*

Information Sciences

*Supported by:*

MSM6198898701 of the MŠMT ČR

University of Ostrava  
Institute for Research and Applications of Fuzzy Modeling  
30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-59-7091401 fax: +420-59-6120478  
e-mail: stjajner@im.ns.ac.yu (corresponding author)

## Abstract

An extension of the classical Riemann-Stieltjes integral to the field of pseudo-analysis is being investigated through this paper. Core of the construction presented here consists of generalized pseudo-operations given by monotone generating function.

*Key words and phrases:* pseudo-operatins, generating function, Riemann-Stieltjes integral

## 1 Introduction

Motivation for the research presented here lies both in applicability of the classical Riemann-Stieltjes integral and capability of the pseudo-analysis, generalization of the classical analysis, to extend classical notions in order to extends the range of the possible applications. It is known that the classical Riemann-Stieltjes integral has applications in several areas of analysis as well as in probability theory, stochastic processes, physics, econometrics, biometrics and informetrics (see [3, 4, 5, 7, 12]). Therefore, the pseudo Riemann-Stieltjes integral introduced in [22] presents an attempt to obtain a generalization of this well known classical notion in the pseudo-analysis' framework ([1, 2, 8, 9, 11, 15, 18, 21, 23]) that might broaden the area of possible applications.

Although pseudo-analysis is mainly focused on the pseudo-operations within a semiring ([11, 15, 17, 18, 20]), the initial construction of the pseudo-Riemann-Stieltjes integral is given in a more general setting based on so called generalized generated pseudo-operations. Operations in question are of the form  $x \oplus y = g^{(-1)}(g(x) + g(y))$  and  $x \odot y = g^{(-1)}(g(x)g(y))$ , where  $g$  is a strictly monotone generating function and  $g^{(-1)}$  its pseudo-inverse. This approach allows us to consider not only the strict case which is in the core of  $g$ -calculus ([14, 16, 17, 18]), but the nilpotent case as well. However, of the special interest for this paper remains a monotone bijection as a generator which gives us the strict case.

This paper is organized as follows: section 2 consists of preliminary notions, such as pseudo-operations, generalized generated pseudo-operations and measure-like set function. Set function given in this section is based on generating function  $g$  and, under some additional conditions, is monotone set function ([18, 23]). Section 3 contains pseudo-Riemann-Stieltjes integral and some estimations that follow from properties of the generator  $g$ . The fourth section considers pseudo-Riemann-Stieltjes integral given by means of a monotone bijection as a generator. This section contains some basic properties and investigates connections of the given integral with the classical Lebesgue and Riemann integrals,  $g$ -integral ([17, 18]) and another Lebesgue type of integral known as pseudo-Lebesgue-Stieltjes integral ([13]). Also, problems of existence and convergence are addressed in this section. A short conclusion is given in the fifth section.

## 2 Preliminary notions

Since the core of the pseudo-analysis are pseudo-operations, i.e., generalizations of the classical operations, a short overview of this notion follows ([11, 15, 18]).

Let  $[a, b]$  be a closed subinterval of  $[-\infty, +\infty]$  (in some cases semiclosed subintervals will be considered) and let  $\preceq$  be a total order on  $[a, b]$ . A semiring is the structure  $([a, b], \oplus, \odot)$  where the following holds:

- $\oplus$  is *pseudo-addition*, i.e., a function  $\oplus : [a, b] \times [a, b] \rightarrow [a, b]$  which is commutative, non-decreasing (with respect to  $\preceq$ ), associative and with a zero element, denoted by  $\mathbf{0}$ ;
- $\odot$  is *pseudo-multiplication*, i.e., a function  $\odot : [a, b] \times [a, b] \rightarrow [a, b]$  which is commutative, positively non-decreasing ( $x \preceq y$  implies  $x \odot z \preceq y \odot z$ ,  $z \in [a, b]_+ = \{x : x \in [a, b], \mathbf{0} \preceq x\}$ ), associative and for which there exists a unit element denoted by  $\mathbf{1}$ ;
- $\mathbf{0} \odot x = \mathbf{0}$ ;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ .

We can distinguish three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and non-idempotent

pseudo-multiplication. Semirings with strict pseudo-operations defined by strictly monotone and continuous generator function  $g : [a, b] \rightarrow [0, +\infty]$ , i.e.  $g$ -semirings, form the second class, and semirings with both idempotent operations belong to the third class. More on this structure as well as on corresponding measures and integrals can be found in [11, 15, 17, 18, 20].

**Remark 1** For  $[a, b]$  being the unit interval, some possible choices for pseudo-operations are t-norms, t-conorms and uninorms (see [6, 9, 10]). For example, if the pseudo-addition insures that the total order  $\leq$  is the usual order and  $[a, b]_+ = [a, b]$ , then the induced pseudo-multiplication is a t-norm or a uninorm (or, for  $[a, b] \neq [0, 1]$ , extension of a t-norm or a uninorm to  $[a, b]$ ). For this choice of pseudo-operations problem of maintaining the structure of a semiring, namely distributivity of  $\odot$  over  $\oplus$ , was investigated in [9, 10].

Of the special interest for this paper are not only  $g$ -operations, i.e., operations given by the strictly monotone and continuous generator  $g : [a, b] \rightarrow [0, +\infty]$  in the following manner

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)g(y)) \quad (1)$$

(see [17, 18]), but the generalization of  $g$ -operations given by the following definition.

**Definition 2** Let  $g$  be a strictly monotone real valued function defined on  $[a, b] \subseteq [-\infty, +\infty]$  such that  $0 \in \text{Ran}(g)$ . The generalized generated pseudo-addition  $\oplus$  and the generalized generated pseudo-multiplication  $\odot$  are given by

$$x \oplus y = g^{(-1)}(g(x) + g(y)), \quad (2)$$

$$x \odot y = g^{(-1)}(g(x)g(y)), \quad (3)$$

where  $g^{(-1)}$  is pseudo-inverse function for function  $g$ .

**Remark 3** Since  $g : [a, b] \rightarrow [-\infty, \infty]$ , the range of a generator can contain values  $-\infty$  and  $+\infty$  and, in that case, if necessary, individual conventions will be used for expressions which can not be well-defined by continuous extension.

In Definition 2, constrains normally put on the generating function  $g$  (while defining  $g$ -semiring) are weakened in such sense that now generating function does not need to be continuous, just strictly monotone. Also, the range now is not limited only to nonnegative values. Therefore, in order to define generalized  $g$ -operations, it is necessary to use the pseudo-inverse function of the generator  $g$  instead of the classical inverse function. This is done in order to obtain more general setting.

**Remark 4** For non-decreasing function  $f : [a, b] \rightarrow [a_1, b_1]$ , where  $[a, b]$  and  $[a_1, b_1]$  are closed subintervals of the extended real line  $[-\infty, +\infty]$ , the pseudo-inverse is  $f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) < y\}$ . If  $f$  is non-increasing function, its pseudo-inverse is  $f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) > y\}$ . More on this subject can be found in [9].

**Remark 5** It is obvious that if the generator  $g$  from the previous definition is a monotone bijection  $g : [a, b] \rightarrow [0, \infty]$ ,  $g$ -semiring ([18]) is obtained and operations are given by (1).

**Remark 6** Importance of operations generated by monotone generating functions can be easily illustrated through the theory of triangular norms ([9]). It is well known fact that a strictly decreasing function  $t : [0, 1] \rightarrow [0, \infty]$ , under some conditions, can lead to a  $t$ -norm of the form

$$T(x, y) = t^{(-1)}(t(x) + t(y)). \quad (4)$$

With some additional constrains put on generator  $t$ , (4) is a continuous Archimedean  $t$ -norm. Specially, for  $t(0) = \infty$ , i.e.,  $t$  being a bijection, (4) is a strict  $t$ -norm of the form  $T(x, y) = t^{(-1)}(t(x) + t(y))$ , and it is an example of pseudo-operation from  $g$ -semiring on  $[0, 1]$ . On the other hand, if  $t(0) < \infty$ , the obtained  $t$ -norm is nilpotent and it remains in the form (4). This nilpotent case can be of the special interest for future studies since it has not been considered in the  $g$ -calculus so far. More on triangular norms, both properties and applications, can be found in [9].

As in the case of  $g$ -semiring, monotonicity of generating function  $g$  is closely connected with the order  $\preceq$  on  $[a, b]$ , i.e.,  $x \preceq y \Leftrightarrow g(x) \leq g(y)$ . Additionally,  $x \prec y$  if and only if  $g(x) \leq g(y)$  and  $x \neq y$ .

It is obvious that operations (2) and (3) are commutative, however, they need not be associative. Some basic properties of (2) and (3) are following (see [22]):

(a) If  $g(x) + g(y), g(z)g(x), g(z)g(y) \in \text{Ran}(g)$ ,  $\odot$  is distributive over  $\oplus$ , i.e.,

$$z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y).$$

(b) Neutral element for  $\oplus$  is  $\mathbf{0} = g^{(-1)}(0)$ .

(c) If  $1 \in \text{Ran}(g)$ , the neutral element for  $\odot$  is  $\mathbf{1} = g^{(-1)}(1)$ .

(d)  $g^{(-1)}(0) \odot x = x \odot g^{(-1)}(0) = g^{(-1)}(0)$  for all  $x \in [a, b]$ .

(e)  $\oplus$  is a non-decreasing function, i.e., for  $x \preceq y$  we have  $x \oplus z \preceq y \oplus z$ .

(f)  $\odot$  is a positively non-decreasing function, i.e., for  $x \preceq y$  we have  $x \odot z \preceq y \odot z$ , while  $z \in [a, b]_+ = \{w \in [a, b] \mid w \succeq \mathbf{0}\}$ .

(g) In the general case, associativity does not hold for  $\oplus$ .

(h) In the general case, the cancellation law does not hold for  $\oplus$ .

**Remark 7** If both  $-\infty$  and  $+\infty$  are in  $\text{Ran}(g)$ , beside the usual continuous extensions, we have to suppose that the following hold

$$+\infty + (-\infty) = 0, \quad 0 \cdot \pm\infty = 0 \quad \text{and} \quad -\infty \cdot (+\infty) = -\infty$$

**Example 8** Let  $g : [-\infty, +\infty] \rightarrow [-\infty, +\infty]$  given by

$$g(x) = \begin{cases} e^x - 1, & x \in [-\infty, 0) \\ \ln(x + 1), & x \in [0, 2], \\ e^x, & x \in (2, +\infty] \end{cases}$$

be a generating function for pseudo-addition  $\oplus$ . Its pseudo-inverse is

$$g^{(-1)}(x) = \begin{cases} -\infty, & x \in [-\infty, -1], \\ \ln(x + 1), & x \in (-1, 0), \\ e^x - 1, & x \in [0, \ln 3], \\ 2, & x \in (\ln 3, e^2], \\ \ln x, & x \in (e^2, +\infty]. \end{cases}$$

This pseudo-inverse function is continuous and strictly increasing on  $\text{Ran}(g)$ .

Now, for this choice of generating function and corresponding pseudo-operation it can be easily shown that the following holds:

$$\left(\frac{3}{2} \oplus \frac{1}{2}\right) \oplus 3 = \ln(\ln 3 + e^3) \neq \ln\left(\ln \frac{15}{4} + e^3\right) = \frac{3}{2} \oplus \left(\frac{1}{2} \oplus 3\right),$$

therefore, associativity, in general, does not hold.

Also, it can be shown that  $\frac{3}{2} \oplus \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{3}$ , i.e., the cancellation law does not hold either.

Necessity of assumption  $g(x) + g(y), g(z)g(x), g(z)g(y) \in \text{Ran}(g)$  for distributivity of  $\odot$  over  $\oplus$  can be illustrated by the following:

$$\ln 10 \odot (1 \oplus 1) = \ln(\ln 3^{10}) \neq 2 = (\ln 10 \odot 1) \oplus (\ln 10 \odot 1).$$

Since the generalized pseudo-addition  $\oplus$  is not necessarily an associative operation, further on the following notation will be used:

$$\bigoplus_{i=1}^n \alpha_i = (\dots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \dots) \oplus \alpha_n,$$

where  $\alpha_i \in [a, b]$   $i \in \{1, 2, \dots, n\}$ .

Another necessary notion is the notion of metric given by generating function  $g$ . Let  $d : [a, b]^2 \rightarrow [0, +\infty]$  be a function of the form

$$d(x, y) = |g(x) - g(y)|, \quad (5)$$

where  $x, y \in [a, b]$  and  $g$  is a generating function for  $\oplus$ . Obviously,  $d$  fulfills all conditions for being a metric.

**Remark 9** Since values  $-\infty$  and  $+\infty$  can belong to  $\text{Ran}(g)$ , convention  $\infty - \infty = 0$  is necessary for well-defined metric.

## 2.1 $g_\phi$ -set-function

The essential notion for this research is also the notion of measure-type set function introduced by means of generating function  $g$  and defined on family of subintervals of the real line in the following manner.

Let  $\mathcal{C}$  be a family of semiclosed subintervals  $(c, d]$  of  $\mathbb{R}$  where  $c \leq d$ , then  $\mathcal{C}$  is semiring of sets, i.e.,  $\emptyset \in \mathcal{C}$ , if  $A, B \in \mathcal{C}$  then  $A \cap B \in \mathcal{C}$  and for all  $A, B \in \mathcal{C}$  there exists  $C_1, \dots, C_n$  from  $\mathcal{C}$  such that  $C_i \cap C_j = \emptyset$  for  $i \neq j$  and  $A \setminus B = \cup_{k=1}^n C_k$ .

**Definition 10** Let  $g$  be a generating function from Definition 2 and let  $\phi$  be a bounded function defined on the real line. A mapping  $m : \mathcal{C} \rightarrow [a, b]$  is called  $g_\phi$ -set-function if

$$m((c, d]) = g^{(-1)}(\phi(d) - \phi(c)). \quad (6)$$

Some basic properties of the  $g_\phi$ -set-function are ([22]):

- (a)  $m(\emptyset) = g^{(-1)}(0)$ ,
- (b) if functions  $g$  and  $\phi$  are of the same monotonicity,  $m$  is a monotone set function.

It is possible to show pseudo-(sub/super)additivity for some classes of generating functions.

**Proposition 11** Let  $\mathcal{P} = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$  be an  $n$ -partition of some interval  $(c, d] \in \mathcal{C}$  such that  $c = x_0 \leq x_1 \leq \dots \leq x_n = d$ .

- i) If  $g : [a, b] \rightarrow [-\infty, \infty]$  is either strictly increasing right-continuous or strictly decreasing left-continuous function such that  $+\infty \in \text{Ran}(g)$ , then  $m$  is pseudo-subadditive on  $\mathcal{P}$ , i.e.,

$$m((c, d]) = m\left(\bigcup_{i=0}^{n-1} (x_i, x_{i+1}]\right) \leq \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]). \quad (7)$$

- ii) If  $g : [a, b] \rightarrow [-\infty, \infty]$  is either strictly increasing left-continuous or strictly decreasing right-continuous function such that  $-\infty \in \text{Ran}(g)$ , then  $m$  is pseudo-superadditive on  $\mathcal{P}$ , i.e.,

$$m((c, d]) = m\left(\bigcup_{i=0}^{n-1} (x_i, x_{i+1}]\right) \geq \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]). \quad (8)$$

- iii) If  $g : [a, b] \rightarrow [-\infty, \infty]$  is a monotone bijection, then  $m$  is pseudo-additive on  $\mathcal{P}$ , i.e.,

$$m((c, d]) = m\left(\bigcup_{i=0}^{n-1} (x_i, x_{i+1}]\right) = \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]). \quad (9)$$

*Proof.* For  $g$  strictly increasing right-continuous or strictly decreasing left-continuous generating function that fulfills condition  $g(b) = +\infty$  or  $g(a) = +\infty$ , respectively, holds  $g \circ g^{(-1)}(x) \geq x$ , for all  $x \in [-\infty, +\infty]$ . Now, for some  $n$ -partition  $\mathcal{P}$ , this implies

$$\begin{aligned} \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]) &\succeq g^{(-1)} \left( \sum_{i=0}^{n-1} g(m((x_i, x_{i+1}])) \right) \\ &\succeq g^{(-1)} \left( \sum_{i=0}^{n-1} \phi(x_{i+1}) - \phi(x_i) \right) = m((c, d]). \end{aligned}$$

Proof for (ii) is similar and based on  $g \circ g^{(-1)}(x) \leq x$  that holds for all  $x \in [-\infty, +\infty]$ . In (iii) pseudo-inverse coincides with inverse, which gives us (9).  $\square$

**Remark 12** If we do not allow negative values for the generator, e.g., if generator is monotone bijection that maps  $[a, b]$  to  $[0, +\infty]$  as in  $g$ -calculus, assumption of monotonicity for  $\phi$  is needed.

Even more, if  $\phi$  is nonincreasing,  $g_\phi$ -set-function remains well-defined. Additionally, we can extend pseudo-inverse to the negative values (since, by definition from [9], for  $g : [a, b] \rightarrow [0, \infty]$  holds  $g^{(-1)} : [0, +\infty] \rightarrow [a, b]$ ) by assigning them  $\mathbf{0}$  which, in this case is  $a$  or  $b$ , depending on the generator's monotonicity. If  $g : [a, b] \rightarrow [0, \infty]$  is monotone bijection, this artificial extension also has to be applied on its inverse. In other words, pseudo-inverse of function  $g : [a, b] \rightarrow [-\infty, \infty]$  such that  $\text{Ran}(g) \subseteq [0, \infty]$  will be used in (6).

**Example 13** Let us consider generating function from Example 8 and function  $\phi$  given by

$$\phi(x) = \begin{cases} e^x, & x \in (-\infty, 0], \\ x^3 + 1, & x \in (0, 2], \\ 9, & x \in (2, \infty). \end{cases}$$

Now, since  $g \circ g^{(-1)}(x) < x$  for all  $x \in (\ln 3, e^2]$ , strict inequality in (8) can be obtained in some cases:

$$m((0, 1.5]) \oplus m((1.5, 3]) = g^{(-1)}(\ln 9) = 2 \prec \ln 8 = g^{(-1)}(8) = m((0, 3]).$$

If we consider strictly increasing right-continuous generator given by

$$g(x) = \begin{cases} e^x - 1, & x \in [-\infty, 0) \\ \ln(x + 1), & x \in [0, 2), \\ e^x, & x \in [2, +\infty], \end{cases}$$

and since  $g \circ g^{(-1)}(x) > x$  holds for all  $x \in [\ln 3, e^2]$ , strict inequality in (7) can be obtained, e.g.,

$$m((0, 1.5]) \oplus m((1.5, 3]) = g^{(-1)}(2e^2) = 2 + \ln 2 \succ \ln 8 = g^{(-1)}(8) = m((0, 3]).$$

Also, in both cases is possible to obtain the equality, e.g.,

$$m((0, 2]) \oplus m((2, 3]) = g^{(-1)}(8) = m((0, 3]).$$

### 3 Pseudo-Riemann-Stieltjes integral

The main aim of this section is to provide some estimations for pseudo-analysis' counterpart of the Riemann-Stieltjes integral (see [7]). As mentioned before, construction is done in more general setting.

Let us recall definitions of the Riemann-Stieltjes pseudo-sum and the pseudo-Riemann-Stieltjes integral ([22]).

Let  $\phi$  be a bounded function defined on the real line and let  $m$  be a  $g_\phi$ -set function given by (6). If  $\mathcal{P} = \{(\omega_i, (x_{i-1}, x_i])\}_{i=1}^n$  is a tagged partition of  $[c, d]$ , i.e.,  $c = x_0 \leq x_1 \leq \dots \leq x_n = d$  and  $\omega_i \in (x_{i-1}, x_i]$ , the Riemann-Stieltjes pseudo-sum of  $f$  with respect to  $\phi$  for a tagged partition  $\mathcal{P}$  is

$$\bigoplus_{\mathcal{P}} f = \bigoplus_{i=1}^n f(\omega_i) \odot m((x_{i-1}, x_i]),$$

where  $f : [c, d] \rightarrow [a, b]$ .

**Definition 14** Function  $f : [c, d] \rightarrow [a, b]$  is pseudo-Riemann-Stieltjes integrable with respect to  $\phi$  on  $[c, d]$  whenever there is a real number  $PI \in [a, b]$  satisfying the following condition: for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$d \left( \bigoplus_{\mathcal{P}} f, PI \right) < \varepsilon,$$

for all tagged partitions  $\mathcal{P}$  of  $[c, d]$  that fulfills  $\max\{x_i - x_{i-1} \mid 1 \leq i \leq n\} < \delta$ .

It is easy to check that the number  $PI$  from the previous definition, if exists, is uniquely determined. This number  $PI$  is pseudo Riemann-Stieltjes integral of  $f$  on  $[c, d]$ , and it will be denoted by  $(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi$ .

**Remark 15** Specially, for  $g(x) = x$  Definition 14 will give us the classical Riemann-Stieltjes integral  $(RS) \int_c^d f d\phi$  ([7]).

The following estimations are given for two classes of generating functions.

**Theorem 16** i) Let  $g : [a, b] \rightarrow [-\infty, \infty]$  be a strictly increasing left-continuous such that  $g(a) = -\infty$ , or strictly decreasing right-continuous such that  $g(b) = -\infty$ , and let  $f : [c, d] \rightarrow [a, b]$  be a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$  with respect to  $\phi$ . Then

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \preceq g^{(-1)} \left( (RS) \int_c^d g \circ f d\phi \right), \quad (10)$$

if integral on the righthand side of (10) exists.

ii) Let  $g : [a, b] \rightarrow [0, \infty]$  be either strictly increasing right-continuous such that  $g(b) = +\infty$ , or strictly decreasing left-continuous such that  $g(a) = +\infty$ , and let  $f : [c, d] \rightarrow [a, b]$  be a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$  with respect to  $\phi$ . Then

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \succeq g^{(-1)} \left( (RS) \int_c^d g \circ f d\phi \right), \quad (11)$$

if integral on the righthand side of (11) exists.

*Proof.* Proof for i) is based on  $g \circ g^{(-1)}(x) \leq x$ , which holds for that class of generators, and is analogous to ii). Therefore we will give only proof for ii).

Let  $f$  be a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$  and  $g \circ f$  Riemann-Stieltjes integrable function on  $[c, d]$ , i.e., for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that for each partition  $\mathcal{P} = \{(\omega_i, (x_i, x_{i+1}))\}_{i=0}^{n-1}$  of  $[c, d]$ , where  $\max\{x_i - x_{i-1} \mid 1 \leq i \leq n\} < \delta$ , we have

$$d \left( \bigoplus_{\mathcal{P}} f, (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \right) < \varepsilon$$

and  $|\sum_{i=1}^n g \circ f(\omega_i) (\phi(x_i) - \phi(x_{i-1})) - (RS) \int_c^d g \circ f d\phi| < \varepsilon$ . Since for generators from this class holds

$g \circ g^{(-1)}(x) \geq x$ , we have  $g \left( \bigoplus_{\mathcal{P}} f \right) \geq \sum_{i=1}^n g \circ f(\omega_i) (\phi(x_i) - \phi(x_{i-1}))$ . Now,

$$\begin{aligned} & g \left( (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \right) - (RS) \int_c^d g \circ f d\phi \\ & > g \left( \bigoplus_{\mathcal{P}} f \right) - \varepsilon - \sum_{i=1}^n g \circ f(\omega_i) (\phi(x_i) - \phi(x_{i-1})) - \varepsilon \geq -2\varepsilon. \end{aligned}$$

This holds for all  $\varepsilon > 0$  and, after allowing  $\varepsilon \rightarrow 0$ , inequality (11) is obtained.  $\square$

For  $g : [a, b] \rightarrow [-\infty, \infty]$  being a monotone bijection, stronger connection between pseudo-Riemann-Stieltjes integral and Riemann-Stieltjes integral is given.

**Proposition 17** *If  $f$  is pseudo Riemann-Stieltjes integrable on  $[c, d]$  with respect to  $\phi$ , then  $g \circ f$  is a Riemann-Stieltjes integrable function on  $[c, d]$  with respect to  $\phi$  and*

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi = g^{-1} \left( (RS) \int_a^b g \circ f d\phi \right), \quad (12)$$

where  $g$  is a monotone bijection.

*Proof.* Follows directly from Definition 14 and properties of generator  $g$ . If  $f$  is pseudo-Riemann-Stieltjes integral with respect to bounded function  $\phi$ , then, by the definition, there exists a real number  $PI \in [a, b]$  and for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$d \left( \bigoplus_{\mathcal{P}} f, PI \right) < \varepsilon, \quad (13)$$

for all tagged partitions  $\mathcal{P}$  of  $[c, d]$  that fulfills  $\max\{x_i - x_{i-1} \mid 1 \leq i \leq n\} < \delta$ . Now, from (13), under the same conditions, follows

$$\left| \sum_{i=1}^n g \circ f(\omega_I) (\phi(x_i) - \phi(x_{i-1})) - g(PI) \right| < \varepsilon,$$

that give us Riemann-Stieltjes integrability for  $g \circ f$  with respect to  $\phi$  and equality (12).  $\square$

Also, if  $f$  is Riemann-Stieltjes integrable on  $[c, d]$  with respect to  $\phi$ , it can be proved that  $g^{-1} \circ f$  is a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$ .

**Proposition 18** *i) Let  $f : [c, d] \rightarrow (a, b)$  be a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$  with respect to a bounded nondecreasing function  $\phi$ . If generating function  $g$  is a strictly increasing left-continuous or strictly decreasing right-continuous with additional property  $g(a) = -\infty$ , or  $g(b) = -\infty$ , respectively, and  $g \circ f$  is Riemann-Stieltjes integrable, then there exists a number  $M \in [a, b]$  such that*

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \preceq g^{(-1)} (g(M) (\phi(d) - \phi(c))). \quad (14)$$

*ii) Let  $f : [c, d] \rightarrow (a, b)$  be a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$  with respect to a bounded nondecreasing function  $\phi$ . If generating function  $g$  is a strictly increasing right-continuous or strictly decreasing left-continuous with additional property  $g(b) = +\infty$ , or  $g(a) = +\infty$ , respectively, and  $g \circ f$  is Riemann-Stieltjes integrable, then there exists a number  $M \in [a, b]$  such that*

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \succeq g^{(-1)} (g(M) (\phi(d) - \phi(c))). \quad (15)$$

*iii) Let  $f : [c, d] \rightarrow (a, b)$  be a pseudo-Riemann-Stieltjes integrable function on  $[c, d]$  with respect to a bounded nondecreasing function  $\phi$ . If generating function  $g : [a, b] \rightarrow [-\infty, \infty]$  is a monotone bijection, then there exists a number  $M \in [a, b]$  such that*

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi = M \odot m((c, d]).$$

*Proof.* Proof is based on inequalities (10) and (11), equality (12) and the mean value theorem for the classical Riemann-Stieltjes integral ([7]).  $\square$

**Remark 19** If in claims i and ii of the previous proposition, additionally, value  $\phi(d) - \phi(c)$  belongs to  $Ran(g)$ , then (14) and (15) are of the form

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \preceq M \odot m((c, d])$$



and

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \succeq M \odot m((c, d]),$$

respectively.

Another important issue is integration of the pseudo-characteristic function, i.e., possibility of representing the pseudo-Riemann-Stieltjes integral itself as a  $g_\phi$ -set-function of some interval. Let  $[u, v] \subseteq [c, d]$ , let  $1 \in \text{Ran}(g)$ , and let  $\chi_{[u,v]}$  be the pseudo-characteristic function of  $[u, v]$ :

$$\chi_{[u,v]}(x) = \begin{cases} \mathbf{0}, & x \notin [u, v], \\ \mathbf{1}, & x \in [u, v]. \end{cases}$$

Now, Riemann-Stieltjes integrability of  $g \circ \chi_{[u,v]}$  and Proposition 17 imply following.

**Corollary 20** *If generating function  $g : [a, b] \rightarrow [-\infty, \infty]$  is a monotone bijection, then*

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} \chi_{[u,v]} d\phi = m((u, v]). \quad (16)$$

**Remark 21** Results given in this section are also applicable for nonnegative strictly monotone generators  $g : [a, b] \rightarrow [0, \infty]$  that can be a bijection, if we assume that  $\phi$  is not only bounded, but monotone as well. In case of nonincreasing function  $\phi$ , artificial extension of (pseudo) inverse from Remark 12 is needed.

## 4 Pseudo-Riemann-Stieltjes integral on the extension of the $g$ -semiring

As seen in Example 13, lack of continuity for generating function can lead to significant change in the nature of  $g_\phi$ -set function (from pseudo super-additivity to pseudo-sub-additivity) just by changing value of generator at one point. This fact, together with property (16) and applicability of  $g$ -calculus in different areas (see [18, 19, 21]), dictates further line of our investigation that is based on monotone bijections as generators. Therefore, further on let  $g : [a, b] \rightarrow [-\infty, \infty]$  be a strictly monotone bijection. Then  $\oplus$  and  $\odot$  are operations of the form (1), i.e., we are considering an extension of the  $g$ -semiring.

Some basic pseudo-linear properties of the pseudo-Riemann-Stieltjes based on (12) were given in [22]:

- (a) for some  $\alpha \in (a, b)$ ,  $\alpha \odot f$  is pseudo-Riemann-Stieltjes integrable on  $[c, d]$  with respect to  $\phi$ ,  $f$  is pseudo-Riemann-Stieltjes integrable on  $[c, d]$  with respect to  $g(\alpha)\phi$  and

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} \alpha \odot f d\phi = \alpha \odot (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi = (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d(g(\alpha)\phi),$$

- (b)  $f \oplus h$  is pseudo-Riemann-Stieltjes integrable on  $[c, d]$  with respect to  $\phi$ , and

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f \oplus h d\phi = (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \oplus (pRS) \int_{[c,d]}^{(\oplus, \odot)} h d\phi,$$

- (c)  $f$  is pseudo Riemann-Stieltjes integrable on  $[c, d]$  with respect to  $\phi + \varphi$  and

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d(\phi + \varphi) = (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi \oplus (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\varphi, \quad (17)$$

where  $\oplus$  and  $\odot$  are pseudo-operations from Definition 2 given by a monotone bijection  $g$  and  $f, h : [c, d] \rightarrow (a, b)$  are pseudo-Riemann-Stieltjes integrable functions on  $[c, d]$  with respect to  $\phi$  and  $\varphi$ .

## 4.1 Problem of existence

Further investigation of this problem concerns existence of the pseudo-Riemann-Stieltjes integral. The first issue is the Cauchy type of criterion for the pseudo-Riemann-Stieltjes integral given by the following theorem.

**Theorem 22** *Let  $g : [a, b] \rightarrow [-\infty, \infty]$  be a monotone bijection and  $\oplus$  and  $\odot$  operations of the form (1). The function  $f : [c, d] \rightarrow (a, b)$  is pseudo-Riemann-Stieltjes integrable with respect to  $\phi$  on  $[c, d]$  if and only if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$d \left( \bigoplus_{\mathcal{P}_1} f, \bigoplus_{\mathcal{P}_2} f \right) < \varepsilon$$

for all tagged partitions  $\mathcal{P}_1 = \{(u_i, (x_{i-1}, x_i])\}_{i=1}^n$  and  $\mathcal{P}_2 = \{(v_j, (y_{j-1}, y_j])\}_{j=1}^m$  of interval  $[c, d]$  that fulfill  $\max\{x_i - x_{i-1}\} < \delta$  and  $\max\{y_j - y_{j-1}\} < \delta$ .

*Proof.* Proof follows directly from (12) and Cauchy criterion for the classical Riemann-Stieltjes integral ([7]).  $\square$

**Corollary 23** *Let  $g : [a, b] \rightarrow [-\infty, \infty]$  be a strictly monotone bijection,  $\oplus$  and  $\odot$  operations of the form (1) and  $f : [c, d] \rightarrow (a, b)$  a pseudo-Riemann-Stieltjes integrable function with respect to  $\phi$  on  $[c, d]$ . Then,  $f$  is pseudo Riemann-Stieltjes integrable function with respect to  $\phi$  on each subinterval of  $[c, d]$ , and, for some  $u \in (c, d)$ , holds*

$$(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi = (pRS) \int_{[c,u]}^{(\oplus, \odot)} f d\phi \oplus (pRS) \int_{[u,d]}^{(\oplus, \odot)} f d\phi.$$

Having the previous theorem in mind, it can be shown that bounded variations are of vital importance for the pseudo-Riemann-Stieltjes integral.

**Theorem 24** *Let  $g : [a, b] \rightarrow [-\infty, \infty]$  be a monotone bijection and  $\oplus$  and  $\odot$  operations of the form (1). If  $f : [c, d] \rightarrow (a, b)$  is continuous and  $\phi$  is of bounded variation on  $[c, d]$ , then  $(pRS) \int_{[c,d]}^{(\oplus, \odot)} f d\phi$  exists.*

*Proof.* Proof of this theorem is similar to the proof of the classical case ([7]).

Since  $\phi$  is of bounded variation, it can be represented as a difference of two monotone functions. Therefore, without loss of generality, based on (17), it can be supposed that  $\phi$  is just monotone function, e.g., nondecreasing function.

The first step is to show that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$d \left( \bigoplus_{\mathcal{P}} f, \bigoplus_{\mathcal{Q}} f \right) < \varepsilon (\phi(d) - \phi(c)) \quad (18)$$

where  $\mathcal{P} = \{(\omega_i, (x_{i-1}, x_i])\}$  is some arbitrary tagged partition of interval  $[c, d]$  that fulfill  $\max\{x_i - x_{i-1}\} < \delta$ , and  $\mathcal{Q} = \{(q_i, (q_{i-1}, q_i])\}_{i=1}^n$ ,  $(q_{i-1}, q_i]$  are intervals of equal length and  $\cup_{i=1}^n [q_{i-1}, q_i] = [c, d]$ . Inequality (18) follows from a new tagged partition  $\mathcal{P}' = \{(u_j, (y_{j-1}, y_j])\}$  which is constructed from partitions  $\mathcal{P}$  and  $\mathcal{Q}$  as follows: tags, now denoted with  $u_j$ , remain the same as in  $\mathcal{P}$ , while the intervals  $(y_{j-1}, y_j]$  are subintervals of intervals from  $\mathcal{Q}$ . It should be stressed that tags  $u_j$  do not have to belong to the corresponding intervals  $(y_{j-1}, y_j]$  and that intervals  $(y_{j-1}, y_j]$  are chosen in such manner that  $\bigoplus_{\mathcal{P}} f = \bigoplus_{\mathcal{P}'} f$  holds.

Now, for some two partitions  $\mathcal{P}_1$  and  $\mathcal{P}_2$  that fulfill requirements of the theorem, holds

$$\begin{aligned} d \left( \bigoplus_{\mathcal{P}_1} f, \bigoplus_{\mathcal{P}_2} f \right) &\leq d \left( \bigoplus_{\mathcal{P}_1} f, \bigoplus_{\mathcal{Q}} f \right) + d \left( \bigoplus_{\mathcal{Q}} f, \bigoplus_{\mathcal{P}_2} f \right) \\ &< 2\varepsilon (\phi(d) - \phi(c)). \end{aligned}$$

Therefore, the claim follows from Theorem 22.

This claim also follows from (12) and properties of the classical Riemann-Stieltjes integral.  $\square$

## 4.2 Problem of convergence

The following problem of convergence for sequences of pseudo-Riemann-Stieltjes integrals is of great importance to the  $g$ -calculus and the general theory of integration.

**Theorem 25** *Let  $g : [a, b] \rightarrow [-\infty, \infty]$  be a strictly monotone bijection,  $\oplus$  and  $\odot$  operations of the form (1) and  $f : [c, d] \rightarrow (a, b)$  a pseudo integrable function with respect to functions  $\{\phi_n\}$  and  $\phi$  of bounded variations such that  $\lim_{n \rightarrow \infty} \phi_n = \phi$ . Then*

$$\lim_{n \rightarrow \infty} (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d \phi_n = (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d \phi, \quad (19)$$

where limit in (19) is considered with respect to the metric  $d$ .

*Proof.* Since function  $f$  is pseudo-Riemann-Stieltjes integrable with respect to functions  $\{\phi_n\}$  and  $\phi$ , Definition 14 insures that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$d \left( \bigoplus_{\mathcal{P}} f, (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d \phi \right) < \frac{\varepsilon}{4} \quad \text{and} \quad d \left( \bigoplus_{\mathcal{P}} f, (pRS) \int_{[c,d]}^{(\oplus, \odot)} f d \phi_n \right) < \frac{\varepsilon}{4}, \quad (20)$$

for all  $n \in \mathbb{N}$  and for all tagged partitions  $\mathcal{P}$  of  $[c, d]$  that fulfills  $\max\{x_i - x_{i-1} \mid 1 \leq i \leq n\} < \delta$ . Also, since  $\lim_{n \rightarrow \infty} \phi_n = \phi$ , properties of the classical Riemann-Stieltjes integral (see [7]) will give us

$$\left| \sum_{i=1}^n g \circ f(\omega_i) (\phi(x) - \phi_n(x)) \right| < \frac{\varepsilon}{4}. \quad (21)$$

Further on prefixes (pRS) and (RS) are omitted in order to obtain shorter notation. Now, (20) and (21) imply

$$\begin{aligned} & d \left( \int_{[c,d]}^{(\oplus, \odot)} f d \phi_n, \int_{[c,d]}^{(\oplus, \odot)} f d \phi \right) \\ &= \left| g \left( \int_{[c,d]}^{(\oplus, \odot)} f d \phi_n \right) - g \left( \int_{[c,d]}^{(\oplus, \odot)} f d \phi \right) \right| \\ &= \left| \int_c^d g \circ f d \phi_n \pm \sum_{i=1}^n g \circ f(\omega_i) (\phi_n(x_i) - \phi_n(x_{i-1})) \right. \\ &\quad \left. \pm \sum_{i=1}^n g \circ f(\omega_i) (\phi(x_i) - \phi(x_{i-1})) - \int_c^d g \circ f d \phi \right| \\ &\leq \left| \int_c^d g \circ f d \phi_n - \sum_{i=1}^n g \circ f(\omega_i) (\phi_n(x_i) - \phi_n(x_{i-1})) \right| \\ &\quad + \left| \int_c^d g \circ f d \phi - \sum_{i=1}^n g \circ f(\omega_i) (\phi(x_i) - \phi(x_{i-1})) \right| \\ &\quad + \left| \sum_{i=1}^n g \circ f(\omega_i) (\phi_n(x_i) - \phi_n(x_{i-1})) \right. \\ &\quad \left. - \sum_{i=1}^n g \circ f(\omega_i) (\phi(x_i) - \phi(x_{i-1})) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

which proves the claim.  $\square$

One of the possible applications of this result is in the field of random variables ([3, 4]), and is given by the following corollary.

**Corollary 26** *Let generator  $g$  be a strictly monotone bijection and let  $f : [c, d] \rightarrow [a, b]$  be a pseudo Riemann-Stieltjes integrable function with respect to  $\{F_n\}$  and  $F$ , where  $\{F_n\}$  and  $F$  are distribution functions of random variables  $\{\xi_n\}$  and  $\xi$ . If the sequence of random variables  $\{\xi_n\}$  convergence in distribution to  $\xi$ , then*

$$\lim_{n \rightarrow \infty} (pRS) \int_{[c, d]}^{(\oplus, \odot)} f d F_n = (pRS) \int_{[c, d]}^{(\oplus, \odot)} f d F, \quad (22)$$

where limit in (22) is considered with respect to the metric  $d$ .

*Proof.* Proof follows directly from Theorem 25 and properties of random variables ([3, 4]).  $\square$

**Remark 27** As in the previous section, results from this section can be applied to strictly monotone non-negative generators  $g : [a, b] \rightarrow [0, \infty]$ , that can be a bijection, with additional assumption of monotonicity for  $\phi$ . Again, nonincreasingness demands extension of (pseudo) inverse function and leads to trivial conclusions of the form  $\mathbf{0} = \mathbf{0}$ .

Also, assumption that  $\phi$  is of a bounded variation that appears in previous subsection can be omitted since monotonicity of  $\phi$  will ensure that.

### 4.3 Connections of pseudo-Riemann-Stieltjes integral with different types of integrals

This section contains an overview of connections of the pseudo Riemann-Stieltjes integral with different types of integrals (see [22]).

#### 4.3.1 Lebesgue integral

A useful relationship between the Lebesgue integral and the pseudo Riemann-Stieltjes integral is given by the following.

Let generator  $g : [a, b] \rightarrow [-\infty, \infty]$  be a strictly increasing bijection and  $f : [c, d] \rightarrow (a, b)$  a measurable function. Let  $\mu$  be a Lebesgue measure and  $\phi_f$  a distribution function of  $f$  given by  $\phi_f(x) = \mu(\{t \in [c, d] \mid f(t) > x\})$ . Then

$$(L) \int_c^d g \circ f d\mu = -g \left( (pRS) \int_{[a, b]}^{(\oplus, \odot)} x d\phi_f \right), \quad (23)$$

if the integrals exist.

Analog theorem for the classical case has been given in [7]. Proofs are similar.

#### 4.3.2 Riemann integral

Let generator  $g : [a, b] \rightarrow [-\infty, \infty]$  be a monotone bijection and  $f : [c, d] \rightarrow (a, b)$  a continuous function. If  $\phi$  is differentiable at  $[c, d]$  and  $\phi'$  is Riemann integrable on  $[c, d]$ , then

$$(R) \int_c^d (g \circ f)(x) \phi'(x) dx = g \left( (pRS) \int_c^d f d\phi \right). \quad (24)$$

#### 4.3.3 $g$ -integral

Based on (23) and (24) we can obtain the following connections with the extended  $g$ -integral generated by a monotone bijection  $g : [a, b] \rightarrow [-\infty, \infty]$ :

- if  $f : [c, d] \rightarrow (a, b)$  a measurable function and  $g^{-1}$  is an even function, then  $\int_{[c,d]} f \odot d\nu = (pRS) \int_{[a,b]}^{(\oplus, \odot)} x d\phi_f$ ;
- if  $f : [c, d] \rightarrow (a, b)$  a measurable function and  $g^{-1}$  is an odd function, then  $\int_{[c,d]} f \odot d\nu = -(pRS) \int_{[a,b]}^{(\oplus, \odot)} x d\phi_f$ ;
- if  $f : [c, d] \rightarrow (a, b)$  is a continuous function, then

$$\int_{[c,d]} (f \odot \phi'_g) \odot d\nu = (pRS) \int_c^d f d\phi,$$

where  $\phi'_g = g^{-1} \circ \phi'$  and extended  $g$ -integral of function  $f$  is denoted by  $\int_{[c,d]} f \odot d\nu$ . More on  $g$ -integral can be found in ([17, 18]).

#### 4.3.4 Pseudo Lebesgue-Stieltjes

Pseudo-integration focused on pseudo-probability space  $(\Omega, S, P)$ , strictly increasing continuous generating function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , pseudo operations of the form (1) and Lebesgue type of integral has been investigated in [13]. Integral constructed in that framework is pseudo Lebesgue-Stieltjes integral. Therefore, the connection given here is presented in pseudo-probability's surrounding.

Let  $(\Omega, S, P)$  be a pseudo-probability space,  $\xi : \Omega \rightarrow [a, b]$  a random variable,  $F_g = P(\{\omega \mid \xi(\omega) < x\})$  pseudo-distribution function of the random variable  $\xi$  and  $f : [a, b] \rightarrow [a, b]$  a measurable function, then

$$g \left( (pLS) \int_{[a,b]}^{\oplus} f dF_g \right) = -g \left( (pRS) \int_{[a,b]} x d\phi_{f \circ \xi} \right),$$

where the integral on the left is a pseudo Lebesgue-Stieltjes integral ([13]).

This connection is a consequence of the results from [13] and (23).

## 5 Conclusion

Results given in this paper present another step in the investigation of pseudo integrals of the Riemann type. Some further research of pseudo-Riemann-Stieltjes integral will be concentrated on possible applications in different mathematical theories, as well as in some practical problems, e.g. the construction of concentration measures for Lorenz curves and stochastic processes.

## Acknowledgements

The support of grants MNZŽSS 144012 and bilateral project SK-SRB-19 (for the first two authors), project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" of the Academy of Sciences and Arts of Vojvodina supported by Provincial Secretariat for Science and Technological Development of Vojvodina (for the first author) and grant MSM6198898701 of the MŠMT ČR (for the third author) is kindly announced.

## References

- [1] P. Benvenuti, R. Mesiar, Integrals with Respect to a General Fuzzy Measure, in: Fuzzy Measures and Integrals, Theory and Applications (M. Grabisch, T. Murofushi, M. Sugeno, eds.), pp. 205-232, Springer-Verlag Company 2000.

- [2] P. Benvenuti, R. Mesiar, D. Vivona, Monotone Set Functions-Based Integrals, in: Handbook of Measure Theory (E. Pap, ed.), Volume II, pp. 1329-1379, Elsevier, North-Holland 2002.
- [3] P. Billingsley, Probability measures, John Wiley and Sons, Inc., New York, 1968.
- [4] P. Billingsley, Probability and Measures, 3rd edition, John Wiley and Sons, Inc., New York, 1995.
- [5] L. Egghe, Construction of Concentration Measures for General Lorenz Curves Using Reimann-Stieltjes Integral, Mathematical and Computer Modelling 35 (2002) 1149-1163.
- [6] J. C. Fodor, R. R. Yager, A. Rybalov, Structure of uninorms, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (1997) 411-427.
- [7] R. A. Gordon, The Integrals of Lebesgue, Denjoy, Perron and Henstock, American Mathematical Society, Providence, 1994.
- [8] M. Grabisch, H. T. Nguyen, E. A., Walker, Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.
- [9] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
- [10] E. P. Klement, R. Mesiar, E. Pap, Integration with respect to decomposable measures based on a conditionally distributive semiring on the unit interval, Internationasl Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 8/6 (2000) 701-717.
- [11] V. N. Kolokoltsov, V. P. Maslov, Idempotent Analysis and Its Applications, Kluwer Academic Publishers, Dordrecht, 1997.
- [12] P. E. Kopp, Martingales and Stochastic Integrals, Cambridge Univ. Press, Cambridge, 1984.
- [13] K. Lendelova On the pseudo-Lebesgue-Stieltjes integral, Novi Sad J. Math. 36 (2006) 123-134.
- [14] A. Marková-Stupňanová, A note on  $g$ -derivative and  $g$ -integral, Tatra Mt. Math. Publ. 8 (1996) 71-76.
- [15] V. P. Maslov, S. N. Samborskij, Idempotent Analysis, in: Advances in Soviet Mathematics (V.P. Maslov and S.N. Samborskij, eds.), Vol 13, American Mathematical Society, Providence, 1992.
- [16] R. Mesiar, Pseudo-linear integrals and derivatives based on a generator  $g$ , Tatra Mt. Math. Publ. 8 (1996) 67-70.
- [17] E. Pap,  $g$ -calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23 (1993) 145-156.
- [18] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.
- [19] E. Pap, Solving nonlinear equations by non-additive measures, Nonlinear Analysis 30 (1997) 31-40.
- [20] E. Pap, Pseudo-additive measures and their applications, in: Handbook of Measure Theory (E. Pap, ed.), Volume II, pp. 1403-1465, Elsevier, North-Holland 2002.
- [21] E. Pap, D. Vivona, Non-commutative and non-associative pseudo-analysis and its applications on nonlinear partial differential equations, J. Math. Anal. Appl. 246/2 (2000) 390-408.
- [22] I. Štajner-Papuga, T. Grbić, M. Daňková, Riemann-Stieltjes type integral based on generated pseudo-operations, Novi Sad J. Math. 36 (2006) 111-124.
- [23] Z. Wang, G.J. Klir, Fuzzy Measure Theory, Plenum Press, New York, 1992.