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Approximation of extensional fuzzy relations over residuated lattices

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Research report No. 136

2008

Submitted/to appear:

Fuzzy Sets and Systems

Supported by:

MSM6198898701 and 1M0572 of the MŠMT ČR.

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Abstract

We investigate a relationship between extensionality of fuzzy relations and their Lipschitz continuity on generalized metric spaces. The duality of these notions is shown, and moreover, two particular applications of the extensionality property in the field of approximation are given.

1 Introduction

When exploring formal calculi of fuzzy logics, one can find that extensionality is a crucial property for various different problems such as the consistency of special theories [16] or the conservativeness of extended theories, adding new connectives or functional symbols to the language [11, 17], graded properties of fuzzy relations [1] etc. There is a lot of notions, where the extensionality property has its indispensable place, let us mention at least fuzzy functions (an extensional fuzzy relation fulfilling "functionality" property) studied in [15, 8] or [11], which are heavily used in fuzzy control systems.

In this paper, we will study the relationship between extensionality and continuity from the functional point of view. The result of [24] showing the duality between $*$-equivalences and generalized pseudometrics will be used to prove the interrelation between extensionality and 1-Lipschitz continuity on generalized pseudometric spaces in the sense of [23].

We will also provide two applications of the extensionality property in the field of approximation. The first studies Perfilieva's F-transforms [22] over residuated lattices from the point of view of normal forms introduced in [21]. It will be shown that F-transforms (precisely, inverse F-transforms) are the best approximations from the specified set of solutions. Moreover, the approximation theorem will be improved. In the second application of extensionality, or more precisely, extensional fuzzy relations, we use two different formalizations of fuzzy "IF–THEN" rules to approximate a fuzzy function. Both formalizations can be found in [11, 19]; in [19] they are called normal forms, though different from those in [21]. The notion of fuzzy function will be in accordance with [6] and the approximation properties shown later will turn out to be the same as those given in [11] (or analogous to those of [20]). With a single exception, all approximating fuzzy relations are extensional, independently of the choice of a function (fuzzy relation) to be approximated. Therefore, we call them universally extensional. The exceptional case is not universally extensional, but is (in a similar sense) a universal fuzzy function.

2 Basic notions

In the following, we will work with residuated lattices used as the basic structure for fuzzy logics.

Definition 1 *A residuated lattice on L is an algebra*

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

with four binary operations and two constants such that

- $\mathcal{L} = \langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice with the largest element* $\mathbf{1}$ *and the least element* $\mathbf{0}$ *w.r.t. the lattice ordering* $\leq$*,*
- $\mathcal{L} = \langle L, *, \mathbf{1} \rangle$ *is a commutative semigroup with the unit element* $\mathbf{1}$*, i.e.* $*$ *is commutative, associative, and* $\mathbf{1} * x = x$ *for all* $x \in L$*,*
- $*$ *and* $\rightarrow$ *form an adjoint pair, i.e.*

$$z \leq (x \rightarrow y) \quad \textit{iff} \quad x * z \leq y \quad \textit{for all } x, y, z \in L. \tag{2}$$

In the sequel, let us assume $\mathcal{L}$ be a residuated lattice of the form (1). We will call the operation $*$ *product*. Moreover, we define the biresidual operation (bi-residuum)

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

Additionally, assume φ to be an order-reversing bijection from L into itself, and we introduce φ-associated *addition* as follows

$$x \oplus_\varphi y = \varphi^{-1}(\varphi(x) * \varphi(y)). \tag{3}$$

If φ is a strong negation (i.e. $\varphi(\varphi(x)) = x$, $\forall x \in L$) then $\oplus_\varphi$ is dual to $*$ and $\langle *, \oplus_\varphi, \varphi \rangle$ forms a so-called DeMorgan triple.

Convention 2 *In the following, we denote by $\mathcal{L}^\varphi$ the structure $\mathcal{L}$ of the form (1) endowed by an order-reversing bijection φ.*

Throughout the whole text, we will deal with fuzzy relations whose membership functions take values from the support of $\mathcal{L}$ and we denote this fact by $\underset{\sim}{\subset}$. Let $M_1, \ldots, M_n$ be some nonempty sets of objects, $\mathcal{R}_I = \{R_i \underset{\sim}{\subset} M_i^2\}_{i \in I}$ be a system of binary fuzzy relations ordered by $I = \{1, \ldots, n\}$, $f \underset{\sim}{\subset} M_1 \times \ldots \times M_n$ be an n-ary fuzzy relation.

Convention 3 *For the sake of brevity, we will denote $M_1 \times \cdots \times M_n$ by $M^{(n)}$. Moreover, we will write $R(\bar{x}, \bar{y}) = R_1(x_1, y_1) * \cdots * R_n(x_n, y_n)$.*

Definition 4 *We say that f is extensional w.r.t. $\mathcal{R}_I$ if $R(\bar{x}, \bar{y}) * f(\bar{x}) \leq f(\bar{y})$, for each $\bar{x}, \bar{y} \in M^{(n)}$.*

Let us recall that a $*$-equivalence $E \underset{\sim}{\subset} M^2$ is reflexive, symmetric and

$$(\forall x, y, z \in M)(E(x, y) * E(y, z) \leq E(x, z)) \quad - \textit{transitive}$$

binary fuzzy relation on some non-empty set of objects M. Observe that $\leftrightarrow$ is $*$-equivalence on L. In the following we will omit specification $*$ whenever it will be clear from the context.

3 Extensionality versus Lipschitz continuity

It has been shown [24] that $*$-equivalences and pseudo-metrics are dual concepts assuming $*$ to be a t-norm with a continuous additive generator. This construction has been used in [14] (later in [21]) for study of the relationship between extensionality and continuity of fuzzy relations. It turned out that, in this case, the extensionality of some fuzzy relation w.r.t. a $*$-equivalence leads to Lipschitz continuity of its membership function on the induced pseudo-metric space and vice-versa.

In the following, we are interested in relationships between extensionality w.r.t. an $*$-equivalence and Lipschitz continuity on a generalized metric space. To show a duality of these two notions, we will use results of [24] extended to the general setting. There, the scale for measuring the distance as well as domain and range of $*$ being originally $[0, 1]$ will be replaced by L (the support of $\mathcal{L}$).

3.1 Generalized pseudo-metric spaces.

Let us start with the basic notions, which we will use in the sequel. The following definition of generalized metric space has been introduced in [23] in order to deal with different concepts of distance.

Definition 5 *Let $M \neq \emptyset$ be a set of objects, $\mathcal{N} = (N, \circ, \leq)$ an ordered semigroup with the identity element e, and $d : M^2 \to N$ be a mapping.*

$\mathcal{M} = (M, d)$ is said to be a generalized $\mathcal{N}$-pseudometric space and d an $\mathcal{N}$-pseudometric on M if for all $x, y, z \in M$ the following holds true:

$$(a)\ \ d(x, x) = e, \quad (b)\ \ d(x, y) = d(y, x), \quad (c)\ \ d(x, y) \leq d(y, z) \circ d(x, z).$$

Notice that the original name for the above defined $\mathcal{N}$-pseudometric was the generalized metric and our $\mathcal{N}$-pseudometric space was called the generalized metric space. We argue that given notions are more relevant since d does not separate points. Moreover, $\mathcal{N}$-notation relates directly to the chosen structure.

Example 6 *The following are examples of ordered semigroups:*

1. $\mathcal{S}_1 = ([0,1], \oplus, \leq)$, *where* $\oplus$ *is a t-conorm,* $e = 0$, *and* $\leq$ *is the standard ordering.*

2. $\mathcal{S}_2 = ([0,1], T, \geq)$, *where* T *is a t-norm,* $\geq$ *is the reverse of the standard ordering and* $e = 1$.

3. $\mathcal{S}_3 = (\mathbb{R}_+, +, \leq)$. *Hence, any standard metric space is the generalized metric space.*

4. $\mathcal{S}_4 = (L, *, \geq)$, *where* $\mathcal{L}$ *is a residuated lattice on* L, $\geq$ *is a reversed ordering to the lattice ordering* $\leq$, *and* $e = \mathbf{1}$.

5. $\mathcal{S}_5 = (L, \oplus_\varphi, \leq)$, *where* L *is the support of a residuated lattice* $\mathcal{L}$, $\leq$ *is the lattice ordering, and* $e = \mathbf{0}$.

Below we will show that a set with a $*$-equivalence relation is in fact $\mathcal{S}_4$-pseudometric space (from Example 6). If we wish to keep the original ordering or the whole ordered semigroup for a valuation of a distance, we need to define a new generalized pseudometric on the basis of a $*$-equivalence and consider another ordered semigroup (in this case $\mathcal{S}_5$). Let us formulate this claim precisely:

Theorem 7 *[24] Assume* $\mathcal{L}^\varphi$, $M \neq \emptyset$ *and* $\mathcal{S} = (L, \oplus_\varphi, \leq)$. *If* $R \underset{\sim}{\subseteq} M^2$ *is a* $*$*-equivalence then*

$$d_\varphi(x,y) = \varphi^{-1} R(x,y) \tag{4}$$

is an $\mathcal{S}$*-pseudometric, and vice versa, if* $d : M \times M \to L$ *is an* $\mathcal{S}$*-pseudometric then* $R_\varphi \underset{\sim}{\subseteq} M^2$ *defined as* $R_\varphi(x,y) = \varphi(d(x,y))$ *is a* $*$*-equivalence.*

Corollary 8

- *An arbitrary* $*$*-equivalence* $R \underset{\sim}{\subseteq} M^2$ *creates* $\mathcal{S}_4$*-pseudometric space* $\mathcal{M} = (M, R)$, *where the ordered semigroup* $\mathcal{S}_4$ *is from Example 6.*
- *If* $R \underset{\sim}{\subseteq} M^2$ *is a* $*$*-equivalence then* $\mathcal{M} = (M, d_\varphi)$ *is* $\mathcal{S}_5$*-pseudometric space, where* $\mathcal{S}_5$ *is from Example 6, and vice versa.*

An important position in this work has the following Valverde's result related to the representability of a $*$-equivalence by means of an appropriate biresidual operation and a set of fuzzy sets:

Theorem 9 *[24] Let* $\mathcal{L}$ *be a residuated lattice with the support* L *and* $M \neq \emptyset$. *Then* $R \underset{\sim}{\subseteq} M^2$ *is an* $*$*-equivalence if and only if there exists a family* $\{A_i \underset{\sim}{\subseteq} M\}_{i \in J}$ *such that*

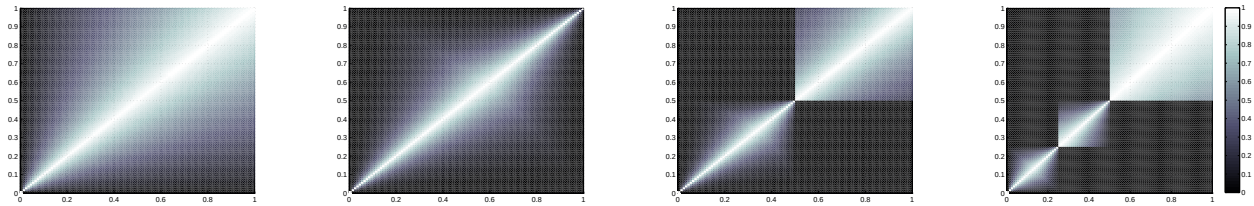
$$R(x,y) = \bigwedge_{i \in J} (A_i(x) \leftrightarrow A_i(y)), \text{ for all } x, y \in M, \tag{5}$$

provided that L *contains all infimas needed in (5).*

Remark 10 *The original Valverde's results formulated in Theorem 7 and 9 were stated for the case of a continuous t-norm* $*$. *However, the proof remains the same in the new setting.*

Example 11 *Let* $\mathcal{L} = \langle [0,1], \cdot, \to_\cdot, 0, 1 \rangle$ *and* $M = [0,1]$. *In this case* $x \leftrightarrow y = \frac{x \wedge y}{x \vee y}$. *Moreover, let us choose the following fuzzy sets (sometimes called score functions):* $A_1(x) = 1 - x^4$, $A_2(x) = x^2$, $A_3(x) = \sin x$, $A_4(x) = 0 \vee (1 - 2 \cdot x)$, $A_5(x) = 0 \vee (1 - 4 \cdot x)$. *Then we may create various* $*$*-equivalences using Theorem 9 due to (5). Some of them are depicted on Figure 1.*

On the other hand, taking arbitrary $*$*-equivalence* $R \underset{\sim}{\subseteq} M^2$, *we can extract a family of fuzzy sets such that (5) holds:* $\{A_c \underset{\sim}{\subseteq} M \mid A_c(x) = R(x,c)\}_{c \in M}$.



(a) $\leftrightarrow$ w.r.t. product t-norm (b) $J = \{1, 2, 3\}$ (c) $J = \{2, 3, 4\}$ (d) $J = \{3, 4, 5\}$

Figure 1: product-equivalences from Example 11 created by $E_J = \bigwedge_{i \in J}(A_i(x) \leftrightarrow A_i(y))$.

3.2 Lipschitz continuity as a dual concept to extensionality.

To avoid any confusion, let us clarify the notion of Lipschitz continuity within the framework of generalized pseudometric spaces.

Definition 12 *Let $\mathcal{N} = (N, \circ, \leq)$ be an ordered semi-group, $k \in N$, $\mathcal{M} = (M, d)$ and $\mathcal{M}' = (M', d')$ be $\mathcal{N}$-pseudometric spaces. We say that $f : M \to M'$ is k-Lipschitz continuous w.r.t. $\mathcal{N}$ if*

$$d'(f(x), f(y)) \leq k \circ d(x, y), \text{ for all } x, y \in M. \tag{6}$$

Below, we are going to show in which sense the Lipschitz continuity and extensionality are dual concepts.

Theorem 13 *Assume $\mathcal{L}^{\varphi}$ and let each $R \in \mathcal{R}_I$ be a $*$-equivalence and $\mathcal{S} = (L, \oplus_{\varphi}, \leq)$.*

- *If $f \underset{\sim}{\subseteq} M^{(n)}$ is extensional w.r.t. $\mathcal{R}_I$ then f is 1-Lipschitz continuous w.r.t. $\mathcal{S}$, i.e.*

$$d(f(\bar{x}), f(\bar{y})) \leq d_n(\bar{x}, \bar{y}), \text{ for all } \bar{x}, \bar{y} \in M^{(n)},$$

where

$$d_n(\bar{x}, \bar{y}) = \varphi^{-1}R_1(x_1, y_1) \oplus_{\varphi} \ldots \oplus_{\varphi} \varphi^{-1}R_n(x_n, y_n) = \bigoplus_{i \in I} \varphi^{-1}R_i(x_i, y_i), \tag{7}$$

$$d(x, y) = \varphi^{-1}(x \leftrightarrow y). \tag{8}$$

- *If $f : M \to M'$ is 1-Lipschitz continuous w.r.t. $\mathcal{S}$ then there exists family of fuzzy sets $\{A_i | \ A_i \underset{\sim}{\subseteq} M'\}_{i \in J}$ such that $\bigwedge_{i \in J} A_i(f(\bar{x}))$ is extensional w.r.t. $R(x, y) = \varphi(d(x, y))$, provided that L contains all infimas that are needed.*

Proof: Observe that $\varphi^{-1}(R_1(x_1, y_1) * \ldots * R_n(x_n, y_n)) = \bigoplus_{i \in I} \varphi^{-1}R_i(x_i, y_i)$ and directly from extensionality of f it follows

$$d(f(\bar{x}), f(\bar{y})) = \varphi^{-1}(f(\bar{x}) \leftrightarrow f(\bar{y})) \leq \varphi^{-1}(R_1(x_1, y_1) * \ldots * R_n(x_n, y_n)) = d_n(\bar{x}, \bar{y}).$$

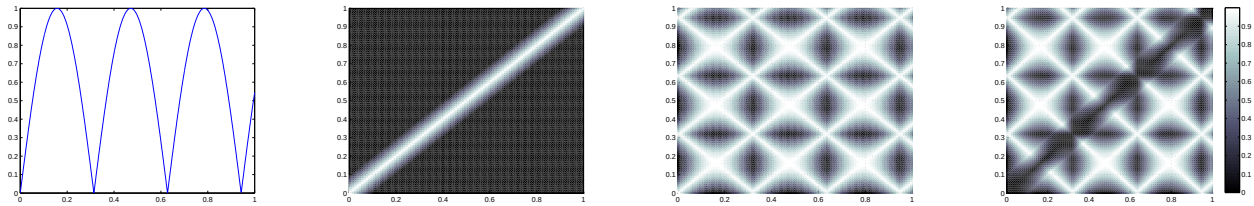
On the other hand, if $f : M \to M'$ is 1-Lipschitz continuous, i.e.

$$d'(f(x), f(y)) \leq d(x, y), \text{ for all } x, y \in M,$$

then by Theorem 7 $\varphi(d(x, y))$ and $\varphi(d'(x, y))$ are $*$-equivalences, moreover, by representation Theorem 9 there exists a family $\{A_i \underset{\sim}{\subseteq} M\}_{i \in J}$ such that $\varphi(d'(x, y)) = \bigwedge_{i \in J}(A_i(x) \leftrightarrow A_i(y))$, for all $x, y \in M$, and hence

$$\varphi(d(x, y)) \leq \varphi(d'(f(x), f(y))) = \bigwedge_{i \in J}(A_i(f(x)) \leftrightarrow A_i(f(y))) \leq \bigwedge_{i \in J} A_j(f(x)) \leftrightarrow \bigwedge_{i \in J} A_j(f(y)).$$

QED



(a) $f(x)$ (b) $l = (x \leftrightarrow y)^{10}$ (c) $r = f(x) \leftrightarrow f(y)$ (d) difference $r - l$

Figure 2: Illustration of Example 14 for $f(x) = |\sin(10x)|$.

Example 14

- *For an illustration, assume $\mathcal{L}$ be the standard Łukasiewicz algebra, $\varphi(x) = 1 - x$, and a fuzzy set $f(x) = |\sin(nx)|$ on $[0,1]$, where $n \in \mathbb{N}$. Using the result of [14], it can be shown that f is extensional w.r.t. $R(x,y) = (x \leftrightarrow y)^n$, see Figure 2 for the case $n = 10$. By Theorem 13, f is Lipschitz continuous w.r.t. $\mathcal{S}' = ([0,1], \oplus_L, \leq)$, where $x \oplus_L y = 1 \wedge (x + y)$.*
- *Going from the other side, $f(x) = \sqrt{x^2+1} - 1$ on $M = [0,1]$ is Lipschitz continuous w.r.t. $\mathcal{S}'' = ([0,1], +, \leq)$, i.e. $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in M$, as well as w.r.t. $\mathcal{S}'$. Hence, using the same φ as above and the second part of Theorem 13, we obtain that f is extensional w.r.t. bi-residuum of Łukasiewicz t-norm. Taking again $\mathcal{L}$ as the standard Łukasiewicz algebra, we can write $x \leftrightarrow y \leq f(x) \leftrightarrow f(y)$.*

4 Extensional fuzzy relations and their approximation

In this section, we will demonstrate the usefulness of the extensionality property on two particular approximation problems. As we have seen, an arbitrary $*$-equivalence can serve as a measure of approximation precision since it is the dual concept to that of pseudo-metric (Theorem 7). In both of the following subsections, we will deal with fuzzy relations in a unified form, aiming at approximation of a specific class of fuzzy relations by means of a $*$-equivalence. Remark that this style of approximation was originally used in [21, 20] and also in [11].

4.1 Approximation by means of F-transforms

In what follows, we will explore F-transforms (more precisely, inverse F-transforms, originally introduced in [22]) based on operations forming a residuated lattice, and show advantages we can gain from the approach focusing on extensionality. It will appear that F-transforms are always extensional w.r.t. fixed $\mathcal{R}$. And this is what we express by the notion of universal extensionality. Moreover, a comparison to the original approach (of Section 5 in [22]) will be given at the end of this subsection.

4.1.1 Formulation of the problem and basic properties

Let $\mathcal{L}$ and $\mathcal{R}_I = \{R_i \subsetneq M_i^2\}_{i \in I}$ be as specified in Section 2, and $\varepsilon \in L$. Moreover, we assume to have only a partial information about $f \subsetneq M^{(n)}$ in the form of a set of samples $\{f(\bar{c}) |\ \bar{c} \in C \subseteq M^{(n)}\}$. The problem (ApproxF) that we are going to solve is formulated as follows:

$$\text{Find } D \subseteq M^{(n)} \text{ and } \tilde{f} \in \mathbb{M}^D_{\vee(\wedge)} \text{ such that } \varepsilon \leq f(\bar{x}) \leftrightarrow \tilde{f}(\bar{x}),\ \forall \bar{x} \in C,$$

where

$$\mathbb{M}^D_\vee = \Big\{ \bigvee_{\bar{d} \in D} (R(\bar{d}, \bar{x}) * g(\bar{d})) \,|\ g \subsetneq M^{(n)} \Big\} \quad \text{and} \quad \mathbb{M}^D_\wedge = \Big\{ \bigwedge_{\bar{d} \in D} (R(\bar{x}, \bar{d}) \to g(\bar{d})) \,|\ g \subsetneq M^{(n)} \Big\}.$$

Due to terminology introduced in [21], we approach this problem from the view point of fuzzy relations of a special structure, the so-called normal forms:

$$\mathrm{DNF}_f^C(\bar{x}) = \bigvee_{\bar{c}\in C} (R(\bar{c},\bar{x}) * f(\bar{c})) \quad \text{and} \tag{9}$$

$$\mathrm{CNF}_f^C(\bar{x}) = \bigwedge_{\bar{c}\in C} (R(\bar{x},\bar{c}) \to f(\bar{c})), \tag{10}$$

are called the disjunctive and conjunctive normal forms for f w.r.t. C, respectively. It is possible to rewrite inverse F-transforms using normal forms and we obtain that $\mathrm{DNF}^D_{\mathrm{CNF}_f^C}$ corresponds to an inverse $F^{\downarrow}$-transform and $\mathrm{CNF}^D_{\mathrm{DNF}_f^C}$ to an inverse $F^{\uparrow}$-transform. Below, we overview known or easily derived results on normal forms.

Lemma 15 *Let $C(\bar{x}) = \bigvee_{\bar{d}\in D} R(\bar{x},\bar{d}) * R(\bar{d},\bar{x})$.*

(a) *The following inequalities are provable for arbitrary $f \subsim M^{(n)}$*

$$\mathrm{DNF}^D_{\mathrm{CNF}_f^C}(\bar{x}) \le f(\bar{x}) \le \mathrm{CNF}^D_{\mathrm{DNF}_f^C}(\bar{x}),\ \forall \bar{x}\in C, \tag{11}$$

$$C(\bar{x}) \le \mathrm{CNF}^D_f(\bar{x}) \to \mathrm{DNF}^D_f(\bar{x}),\ \forall \bar{x}\in M^{(n)}. \tag{12}$$

(b1) *Let $G_i(x,y) = \bigwedge_{\bar{c}\in C}(R_i(c_i,x) \to R_i(c_i,y))$ and $H_i(x,y) = \bigwedge_{\bar{c}\in C}(R_i(y,c_i) \to R_i(x,c_i))$, for $i\in I$. Then,*

$$\begin{matrix}\mathrm{DNF}_f^C\\ \mathrm{CNF}_f^C\end{matrix} \subsim M^{(n)} \text{ is extensional w.r.t. } \begin{matrix}\mathcal{G}_I.\\ \mathcal{H}_I.\end{matrix}$$

(b2) *If each $R\in\mathcal{R}_I$ is transitive then both $\mathrm{DNF}_f^C, \mathrm{CNF}_f^C \subsim M^{(n)}$ are extensional w.r.t. $\mathcal{R}_I$.*

(c) *If $f \subsim M^{(n)}$ is extensional w.r.t. $\mathcal{R}_I$ then*

$$\mathrm{DNF}_f^C(\bar{x}) \le f(\bar{x}) \le \mathrm{CNF}_f^C(\bar{x}), \tag{13}$$

$$C(\bar{x}) \le \mathrm{D(C)NF}_f^D(\bar{x}) \leftrightarrow f(\bar{x}),\ \forall \bar{x}\in M^{(n)}. \tag{14}$$

Proof:

(11) See Theorem 6 and Theorem 8 in [22], where the author used $a*(a\to b)\le b \le a\to(a*b)$. We use the following instance of the above sequence of inequalities

$$R(d,x)*(R(d,x)\to f(x)) \le f(x) \le R(x,d)\to(R(x,d)*f(x)), \forall x\in C.$$

(12)+(b1)+(b2)+(14) See [4], Lemma 6 and Theorem 7.
(13) See [21, 20]. QED

Corollary 16 *Let $\mathcal{G}_I, \mathcal{H}_I$ be as in Lemma 15 (b1).*

(a) $\mathrm{DNF}^D_{\mathrm{CNF}_f^C}$ *is extensional w.r.t. $\mathcal{G}_I$ and* $\mathrm{CNF}^D_{\mathrm{DNF}_f^C}$ *is extensional w.r.t. $\mathcal{H}_I$.*

(b) *Let $C(\bar{x})$ be as above and each $R\in\mathcal{R}_I$ transitive. If $f\subsim M^{(n)}$ is extensional w.r.t. $\mathcal{R}_I$ then we can prove*

$$C(\bar{x}) \le \begin{matrix}\mathrm{DNF}^D_{\mathrm{CNF}_f^C}(\bar{x}) \leftrightarrow f(\bar{x}),\\ \mathrm{CNF}^D_{\mathrm{DNF}_f^C}(\bar{x}) \leftrightarrow f(\bar{x}),\end{matrix} \quad \forall \bar{x}\in C. \tag{15}$$

Proof: (a) Obvious.

(b) Both $\mathrm{CNF}^D_{\mathrm{DNF}^C_f}$ and $\mathrm{DNF}^D_{\mathrm{CNF}^C_f}$ are extensional, thus we can use (14) to show that $C(\bar{x}) \leq \mathrm{D(C)NF}^D_{\mathrm{C(D)NF}^C_f}(\bar{x}) \leftrightarrow \mathrm{C(D)NF}_f(\bar{x})$. And since f is extensional as well, thus we apply (14) second time to prove (15). QED

Based on the known results, we receive the lower estimation of the equivalence (15) under the transitivity requirements. In Subsection 4.1.2, we will try to get rid of the above restriction put on each $R \in \mathcal{R}_I$. We use there some slight generalization of the results from Lemma 15. The main trick stands in taking into account a new collection of binary fuzzy relations $\mathcal{S}_I$ (see (16) below), which is closely related to $\mathcal{R}_I$ and a set of some nodes from $M^{(n)}$. Basic properties of this new collection of relations w.r.t. the normal forms are summarized in the following lemma.

Lemma 17 *Let $C, D, P \subseteq M^{(n)}$ and $\mathcal{R}_I = \{R_i \subsetneq M_i^2\}_{i \in I}$ be a system of binary fuzzy relations ordered by $I = \{1, \ldots, n\}$. Define the new system $\mathcal{S}_I = \{S_i \subsetneq M_i^2\}_{i \in I}$ as follows:*

$$S_i(x, y) = \bigvee_{p \in P} R_i(x, p_i) * R_i(p_i, y). \tag{16}$$

(a) *If each $R \in \mathcal{R}_I$ is transitive then $S_i \leq R_i$, $\forall i \in I$.*

(b) *The following inequality is provable for arbitrary $f \subsetneq M^{(n)}$:*

$$S(\bar{x}, \bar{y}) \leq \mathrm{CNF}^P_f(\bar{x}) \to \mathrm{DNF}^P_f(\bar{y}),\ \forall \bar{x}, \bar{y} \in M^{(n)}. \tag{17}$$

(c) *If $f \subsetneq M^{(n)}$ is extensional w.r.t. $\mathcal{R}_I$ then $f \subsetneq M^{(n)}$ is extensional w.r.t. $\mathcal{S}_I$*

(d) *If $f \subsetneq M^{(n)}$ is extensional w.r.t. $\mathcal{S}_I$ and each $R \in \mathcal{R}_I$ is reflexive then*

$$\mathrm{DNF}^P_f(\bar{x}) \leq f(\bar{x}) \leq \mathrm{CNF}^P_f(\bar{x}),\ \forall \bar{x} \in M^{(n)}, \tag{18}$$

$$S(\bar{x}, \bar{x}) \leq \mathrm{D(C)NF}^P_f(\bar{x}) \leftrightarrow f(\bar{x}),\ \forall \bar{x}, \bar{y} \in M^{(n)}. \tag{19}$$

(e) *If $f \subsetneq M^{(n)}$ is extensional w.r.t. $\mathcal{S}_I$ then $\mathrm{DNF}^D_f(\bar{x}) \leq \mathrm{CNF}^C_f(\bar{y})$, $\forall \bar{x}, \bar{y} \in P$.*

Proof: (a) Obvious.

(b) The proof proceeds in the same way as the one of (12) with the different variables. Since the proof of (12) is not included, we give at least the starting inequality $\underbrace{(R(\bar{p}, \bar{y}) \to R(\bar{p}, \bar{y}))}_{=\mathbf{1}} * (R(\bar{x}, \bar{p}) \to f(\bar{p})) \leq (R(\bar{x}, \bar{p}) * R(\bar{p}, \bar{y})) \to (R(\bar{p}, \bar{y}) * f(\bar{p}))$.

(c) From the extensionality of f w.r.t. $\mathcal{R}_I$ and transitivity of $\to$ we have

$$R(\bar{x}, \bar{p}) * R(\bar{p}, \bar{y}) \leq (f(\bar{x}) \to f(\bar{p})) * (f(\bar{p}) \to f(\bar{y})) \leq f(\bar{x}) \to f(\bar{y}),$$

and hence $\bigvee_{p \in P}(R(\bar{x}, \bar{p}) * R(\bar{p}, \bar{y})) \leq f(\bar{x}) \to f(\bar{y})$. Due to the distributivity of $\vee$ w.r.t. $*$, we finally obtain $S(\bar{x}, \bar{y}) \leq f(\bar{x}) \to f(\bar{y})$.

(d) Both inequalities in (18) follow directly from the extensionality $\begin{matrix} R(\bar{x}, \bar{p}) \\ R(\bar{p}, \bar{x}) \end{matrix} * \underbrace{R(\bar{p}, \bar{p})}_{=\mathbf{1}} \leq \begin{matrix} f(\bar{x}) \to f(\bar{p}) \\ f(\bar{p}) \to f(\bar{x}) \end{matrix}$.

The next inequality (19) is a consequence of (18) and (17).

(e) The inequality follows directly from the extensionality $R(\bar{x}, \bar{p}) * f(\bar{x}) \leq R(\bar{p}, \bar{y}) \to f(\bar{y})$. Since the variables $\bar{x}, \bar{y}$ are separated then we can add the lattice operations, which finishes the proof. QED

4.1.2 Approximation properties of F-transforms

For an arbitrary $f \underset{\sim}{\subseteq} M^{(n)}$, we can certainly show[1]

$$\bigvee_{\bar{d}\in D} (R(\bar{d},\bar{x}) * f(x)) \leq f(x) \leq \bigwedge_{\bar{d}\in D} (R(\bar{x},\bar{d}) \to f(x)),\ \forall x \in C.$$

The task stands in the replacement of $f(x)$ by suitable values so that the approximation is the best possible w.r.t. the available knowledge. In the following proposition[2], we are going to show that the combinations of $\{\mathrm{DNF}_f^D, \mathrm{CNF}_f^C\}$ and $\{\mathrm{CNF}_f^D, \mathrm{DNF}_f^C\}$ have this property.

Proposition 18

- DNF_f^C *is the least element of the following set:*

$$S = \{g \underset{\sim}{\subseteq} M^{(n)} |\ f(\bar{x}) \leq \bigwedge_{d\in D} (R(\bar{x},\bar{d}) \to g(\bar{d})),\ \forall \bar{x} \in C\}. \tag{20}$$

- CNF_f^C *is the greatest element of the following set:*

$$S' = \{g \underset{\sim}{\subseteq} M^{(n)} |\ \bigvee_{d\in D} (R(\bar{d},\bar{x}) * g(\bar{d})) \leq f(\bar{x}),\ \forall \bar{x} \in C\}. \tag{21}$$

Proof: It follows from (11) that DNF_f^C (CNF_f^C) fulfills the inequality inside the specified set S (S'), and hence, $\mathrm{DNF}_f^C \in S$ ($\mathrm{CNF}_f^C \in S'$). Let us show the minimality of the DNF_f^C: from (20), we obtain

$$\begin{aligned}
&f(\bar{x}) \leq \bigwedge_{d\in D} (R(\bar{x},\bar{d}) \to g(\bar{d})) \leq R(\bar{x},\bar{d}) \to g(\bar{d});\ \text{applying adjunction}\\
&R(\bar{x},\bar{d}) * f(\bar{x}) \leq g(\bar{d}),\ \forall \bar{d} \in D, \forall \bar{x} \in C;\ \text{which gives}\\
&\mathrm{DNF}_f^C(\bar{d}) = \bigvee_{\bar{x}\in C} R(\bar{x},\bar{d}) * f(\bar{x}) \leq g(\bar{d}),\ \forall \bar{d} \in D.
\end{aligned}$$

We proceed analogously in the case CNF_f^C: from (21), we have

$$\begin{aligned}
&R(\bar{d},\bar{x}) * g(\bar{d}) \leq \bigvee_{d\in D} (R(\bar{d},\bar{x}) * g(\bar{d})) \leq f(\bar{x}),\\
&g(\bar{d}) \leq \mathrm{CNF}_f^C(\bar{d}) = \bigwedge_{\bar{x}\in C} R(\bar{d},\bar{x}) \to f(\bar{x}) \leq R(\bar{d},\bar{x}) \to f(\bar{x}),\ \forall \bar{d} \in D, \forall \bar{x} \in C,
\end{aligned}$$

which completes the proof of maximality of $\mathrm{CNF}_f^C \in S'$. QED

As it has been shown above, there cannot be better lower approximation of f from the set $\mathbb{M}_\vee^D$ than $\mathrm{DNF}^D_{\mathrm{CNF}_f^C} \in \mathbb{M}_\vee^D$. And analogously, $\mathrm{CNF}^D_{\mathrm{DNF}_f^C} \in \mathbb{M}_\wedge^D$ is the best upper approximation of f. Both results do not consider any boundary for an $\varepsilon-$precision of the approximations. It is dependent on the suitable choice of $\mathcal{R}_I$ and the set of nodes D arising from the following estimation in terms of a $*$-equivalence:

Theorem 19 *Let* $C(\bar{x}) = \bigvee_{\bar{d}\in D} R(\bar{x},\bar{d}) * R(\bar{d},\bar{x})$ *and let* $f \underset{\sim}{\subseteq} M^{(n)}$ *be extensional w.r.t.* $\mathcal{S}_I$ *defined by (16) with* $P = D$*. Then we can prove*

$$C(\bar{x}) \leq \begin{array}{c} \mathrm{DNF}^D_{\mathrm{CNF}_f^C}(\bar{x}) \leftrightarrow f(\bar{x}),\\ \mathrm{CNF}^D_{\mathrm{DNF}_f^C}(\bar{x}) \leftrightarrow f(\bar{x}), \end{array}\ \forall \bar{x} \in C. \tag{22}$$

[1] Considering $\bigvee_{\bar{d}\in D} R(\bar{d},\bar{x}) = \bigvee_{\bar{d}\in D} R(\bar{x},\bar{d}) = \mathbf{1}$ gives the following equality $f(x) = \bigvee_{\bar{d}\in D}(R(\bar{d},\bar{x}) * f(x)) = \bigwedge_{\bar{d}\in D}(R(\bar{x},\bar{d}) \to f(x))$, $\forall x \in C$, because in an arbitrary residuated lattice $\mathbf{1} * a = \mathbf{1} \to a = a$, $\forall a \in L$. Hence, it is reasonable to apply this requirement for the partitioning of the domain of f in the practise.

[2] Compare this result with Lemma 8 and 10 in [22].

Proof: From Lemma 17(e) and (17), we obtain

$$S(\bar{x},\bar{y}) \leq \mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}) \to \mathrm{DNF}^D_{\mathrm{DNF}^C_f}(\bar{y}) \leq \mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}) \to \mathrm{DNF}^D_{\mathrm{CNF}^C_f}(\bar{y}),\ \forall \bar{x},\bar{y} \in M^{(n)},$$

and together with (11), we receive (22). QED

Hence, to obtain the desired ε-approximation[3], it only remains to choose D w.r.t. $\mathcal{R}_I$ so that $(\forall \bar{x} \in C)(\varepsilon \leq C(\bar{x}))$, which completes the task.

Corollary 20 *Let $\varepsilon \in L, D \subseteq M^{(n)}$ and $f \underset{\sim}{\subseteq} M^{(n)}$ be extensional w.r.t. $\mathcal{S}_I$ defined by (16). If for each $x \in C \exists d \in D:\ \varepsilon \leq R(x,d) * R(d,x)$ then*

$$\varepsilon \leq \begin{array}{l} \mathrm{DNF}^D_{\mathrm{CNF}^C_f}(\bar{x}) \leftrightarrow f(\bar{x}), \\ \mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}) \leftrightarrow f(\bar{x}), \end{array} \quad \forall \bar{x} \in C. \tag{23}$$

Corollary 21 *Let $L = [0,1]$, $*$ be a continuous t-norm, $\varepsilon \in L$, M be a pre-compact (totally bounded) subset over a standard metric space $\mathcal{N} = (N,d)$, $R \underset{\sim}{\subseteq} M$ be reflexive and uniformly continuous. If $f \underset{\sim}{\subseteq} M$ is extensional w.r.t. R then $\forall \varepsilon > 0\ \exists$ finite $D \subseteq M$ such that (23) is valid.*

Proof: Let us use the following abbreviation: $B(c,\alpha) = \{x \in M |\ d(x,c) \leq \alpha\}$. Moreover, let $\omega = 1 - \varepsilon$.

Since $*$ is continuous on [0,1] (which means uniformly continuous) and R is uniformly continuous then $R(x,y) * R(y,x)$ is uniformly continuous as well. Therefore, we may fix $\omega > 0$ and find δ so that $d(x,p) \leq \delta$ implies $|R(x,p) * R(p,x) - \underbrace{R(p,p) * R(p,p)}_{=1}| \leq \omega$, which gives $1 - \omega = \varepsilon \leq R(x,p) * R(p,x)$.

And from the pre-compactness of M, it follows that there exists δ-net, i.e. finite set D of ball centers such that $\bigcup_{p \in D} B(p,\delta) \supseteq M$. Hence, for an arbitrary $x \in C$ there exists $p \in D : c \in B(p,\delta)$ meaning that $d(x,p) \leq \delta$, which implies $\varepsilon \leq R(x,p) * R(p,x)$ and $\varepsilon \leq \bigvee_{p \in D} R(x,p) * R(p,x)$. Together with (22) we receive (23). QED

Whenever we assume each $R \in \mathcal{R}$ to be a $*$-equivalence, we obtain an estimation by means of pseudo-metrics introduced in Theorem 13.

Corollary 22 *Assume $\mathcal{L}^\varphi$, d be given by (8) and d_n by (7). Moreover, let each $R \in \mathcal{R}_I$ be a $*$-equivalence and $\mathcal{S} = (L, \oplus_\varphi, \leq)$. If $f \underset{\sim}{\subseteq} M^{(n)}$ is extensional w.r.t. $\mathcal{S}_I$ defined by (16) then f is 1-Lipschitz continuous w.r.t. $\mathcal{S}$ and*

$$\begin{array}{l} d(\mathrm{DNF}^D_{\mathrm{CNF}^C_f}(\bar{x}), f(\bar{x})) \\ d(\mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}), f(\bar{x})) \end{array} \leq \bigwedge_{\bar{d} \in D} 2d_n(\bar{x},\bar{d}) = \bigwedge_{\bar{d} \in D} d_n(\bar{x},\bar{d}) \oplus d_n(\bar{d},\bar{x}),\ \forall \bar{x} \in C. \tag{24}$$

Proof: Since each $R \in \mathcal{R}_I$ is a $*$-equivalence then also each $S \in \mathcal{S}_I$ is the $*$-equivalence and we can apply Theorem 13. The rest is obvious. QED

Let us consider a simple problem to illustrate the way of approximation by F-transforms.

Example 23 *Let us take the following function*

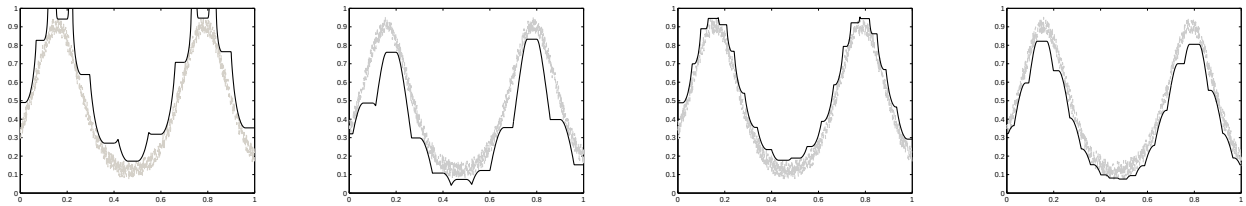
$$f(x) = \frac{1}{3} \cdot e^{\sin(10x)} + Rand(x),\ \forall x \in M = [0,1],$$

where $Rand(x)$ is some additional random noise. Moreover, assume $*$ to be the t-norm generated by $g(x) = (1-x)^2$ and $\mathcal{L}$ be the associated residuated lattice. Take

$$R(x,y) = 1 \wedge (0 \vee (1.2 - 10 \cdot |x-y|)),$$

and create the approximation using 11 and 21 nodes and equidistant discretization $C = \{k \cdot 0.001 |\ k = 1,\ldots,999\}$. The resulting approximations are depicted on Figure 3.

[3]Notice that the choice of $\mathcal{R}_I$ is the limitation for ε as well as for D, e.g. D might become very huge if ε is close to $\mathbf{1}$.



(a) $\mathrm{CNF}^{D}_{\mathrm{DNF}^{C}_{f}}$ (b) $\mathrm{DNF}^{D}_{\mathrm{CNF}^{C}_{f}}$ (c) $\mathrm{CNF}^{D'}_{\mathrm{DNF}^{C}_{f}}$ (d) $\mathrm{DNF}^{D'}_{\mathrm{CNF}^{C}_{f}}$

Figure 3: Approximation of f (grey line) from Example 23 using F-transforms (black line) with 11 (set D) and 21 (set D') nodes.

4.1.3 Comparison to the original approach and an improvement of the approximation theorem

Now, let us compare our approach to the original one of [22]. To do this, let us recall inverse F-transforms from [22] refined for the n-dimensional case:

$$\begin{aligned}
&\text{Assume } f \underset{\sim}{\subseteq} M^{(n)}, C \subseteq M^{(n)}, A_{j_i} \underset{\sim}{\subseteq} M_i, \forall j \in J = \{1, \ldots, k\}, \forall i \in I = \{1, \ldots, n\},\\
&\qquad \text{and } \mathbf{i} \text{ be abbreviation for } i_1, \ldots, i_n,\\
&T_1(\bar{x}) = \bigvee_{\mathbf{i} \in J} A_{\mathbf{i}}(\bar{x}) * [\bigwedge_{\bar{c} \in C} A_{\mathbf{i}}(\bar{c}) \to f(\bar{c})],\\
&T_2(\bar{x}) = \bigwedge_{\mathbf{i} \in J} A_{\mathbf{i}}(\bar{x}) \to [\bigvee_{\bar{c} \in C} A_{\mathbf{i}}(\bar{c}) * f(\bar{c})], \text{, where}\\
&A_{\mathbf{i}}(\bar{x}) = A_{i_1}(x_1) * \ldots * A_{i_n}(x_n), \forall i \in J.
\end{aligned}$$

In [22], T_1 is called inverse $\mathrm{F}^{\downarrow}$-transform and T_2 inverse $\mathrm{F}^{\uparrow}$-transform. If each A_{j_i} is normal and each $\{A_{1_i}, \ldots, A_{k_i}\}$ establishes a semi-partition, i.e.

$$\bigvee_{x \in M_i} A_{j_i}(x) * A_{k_i}(x) \leq \bigwedge_{x \in M_i} A_{j_i}(x) \leftrightarrow A_{k_i}(x), \forall j, k \in J \text{ and } \forall i \in I,$$

then there exists set of nodes $E = \{e_{i_j} \mid i \in J, j \in I\}$ such that the associated system of binary fuzzy relations $\overline{\mathcal{R}}_I = \{\bar{R}_i\}_{i \in I}$ is defined as

$$\bar{R}_i(x, y) = \bigwedge_{j \in J} A_{j_i}(x) \leftrightarrow A_{j_i}(y), \text{ for all } i \in I, \text{ and} \quad R(e_{i_j}, x) = A_{i_j}(x), \text{ for all } i \in J, j \in I.$$

In the table below, we summarize the requirements of both approaches that lead to the estimation of the equivalence of T_1, T_2 and $\mathrm{DNF}^{D}_{\mathrm{CNF}^{C}_{f}}, \mathrm{CNF}^{D}_{\mathrm{DNF}^{C}_{f}}$, respectively, to the original function f:

Requirements in [22] (Theorems 10 and 11)	New requirements
$\mathcal{L}$ be a BL-algebra each $A \in \{A_{j_i}\}_{j \in J, i \in I}$ be normal $\{A_{1_i}, \ldots, A_{k_i}\}$ forms a semi-partition $\forall i \in I$ $E \subseteq C$, see p. 1013 in [22] f be extensional w.r.t. $\{\bar{R}_i\}_{i \in I}$	$\mathcal{L}$ be a residuated lattice f be extensional w.r.t. $\{\bar{R}_i\}_{i \in I}$

Hence, to prove (22), we need only one requirement put on f, while in [22], a rather complicated and restrictive approach has been chosen. It means that to prove

$$\bigvee_{\mathbf{i} \in J} A_{\mathbf{i}}(\bar{x}) * A_{\mathbf{i}}(\bar{x}) \leq T_{1(2)}(\bar{x}) \leftrightarrow f(\bar{x}), \ \forall \bar{x} \in C,$$

we have been limited to the stronger algebraic structure, a special type of partition, $E \subseteq C$, additionally requiring normality, and extensionality of f. Notice that most of the requirements follow from the procedure of replacement of fuzzy sets A_{i_j} by $\bar{R}_i$ based on the special set E. Since we do not require anything from R_i, $i \in I$, the whole approach in [22] based on fuzzy sets becomes a special case of our approach and the above inequality follows directly from (22) and the extensionality of f.

Proposition 24 *Let $\overline{\mathcal{R}}_I = \{\bar{R}_i\}_{i\in I}$ is defined as $\bar{R}_i(x,y) = \bigvee_{j\in J} A_{j_i}(x) * A_{j_i}(y)$, for all $i \in I$. If $f \underset{\sim}{\subseteq} M^{(n)}$ is extensional w.r.t. $\overline{\mathcal{R}}_I$ then*

$$\bigvee_{\mathbf{i}\in J} A_{\mathbf{i}}(\bar{x}) * A_{\mathbf{i}}(\bar{x}) \leq \begin{array}{l} T_1(\bar{x}) \leftrightarrow f(\bar{x}), \\ T_2(\bar{x}) \leftrightarrow f(\bar{x}), \end{array} \quad \forall \bar{x} \in C. \tag{25}$$

Proof: (25) is a corollary of Theorem 19, where we specify $\{R_i\}_I$ and D:

- Assume $M^{(n)}$ contains sufficiently many elements to define set D.
- Take $D = \{\bar{d}_i \in M^{(n)} |\ d_{i_j} \neq d_{i'_j}, \forall i, i' \in J, \forall j \in I\}$.
- Define $R(\bar{x}, \bar{d}_i) = R(\bar{d}_i, \bar{x}) = A_{\mathbf{i}}(\bar{x})$ and $R(\bar{x}, \bar{y}) = \mathbf{0}$, $\forall \bar{y} \in M^{(n)} \setminus D$.

In the case of lacking elements to construct set D, which can hypothetically arise because there is no restriction to the number of sets A_{i_j}, we can artificially extend $M^{(n)}$ to reach the required cardinality. QED

On the contrary, any fuzzy relation $R \underset{\sim}{\subseteq} M^2$ can be viewed as a collection of fuzzy sets $\{R(c,x)\}_{c\in C}$, where $C \subseteq M$. Hence, both approaches are equivalent. But the investigation of this problem shows that focusing directly on extensionality and searching for universally extensional fuzzy relations yields a significant improvement of the approximation theorem and brings also some new results.

4.2 Approximation of extensional fuzzy function

Another nice approach to an approximation of a given function is based on a relational representation of that function. We will work with two particular fuzzy relations that again have a unified form inspired by formalization of a collection of "IF–THEN" rules—in our case, a vague description of a functional dependency. One of these relations appears to be a universally extensional fuzzy relation and the other is a universal fuzzy function. A connecting element is a fuzzy relation over the considered function which is a fuzzy function as well as an extensional fuzzy relation (under some restriction): we can call it an extensional fuzzy function.

4.2.1 Formulation of the problem

Let $M^{(n+1)} = M^{(n)} \times M$, $f\colon M^{(n)} \to M$, $I' = \{1, 2, \ldots, n+1\}$, each $R \in \mathcal{R}_{I'}$ be $*$-equivalence, and $\varepsilon \in L$. The problem (ApproxR) that we are going to solve can be formulated as follows:

Find $D \subseteq M^{(n)}$ and $\tilde{F} \underset{\sim}{\subseteq} M^{(n+1)}$ such that $\varepsilon \leq \tilde{F}(\bar{x}, y) \leftrightarrow R_{n+1}(f(\bar{x}), y)$, $\forall \bar{x} \in C$, $\forall y \in M$,

either in the form

$$\mathrm{DNF}_f^D(\bar{x}, y) = \bigvee_{\bar{d}\in D} R(\bar{x}, \bar{d}) * R_{n+1}(y, f(\bar{d})), \text{ or} \tag{26}$$

$$\overline{\mathrm{CNF}}_f^D(\bar{x}, y) = \bigwedge_{\bar{d}\in D} R(\bar{x}, \bar{d}) \to R_{n+1}(y, f(\bar{d})). \tag{27}$$

Note that DNF_f^D of the form (26) is a special case of $\mathrm{DNF}_{\tilde{f}}^D$ in (9), where $\tilde{f} \equiv \mathbf{1}$, while $\overline{\mathrm{CNF}}_f^D$ of (27) differs from $\mathrm{CNF}_{\tilde{f}}^D$ in (10) because there is no possibility of choosing $\tilde{f}$ so that $\overline{\mathrm{CNF}}_f^D$ would become its special case. Both relations are used to formalize fuzzy rules expressing functional dependency. The second formalization relates closely to the functionality property [1] (or see (F.2) in [6]).

Definition 25 *Let $\mathcal{R} = \{R_1 \underset{\sim}{\subseteq} M_1, \ldots, R_{n+1} \underset{\sim}{\subseteq} M_{n+1}\}$ and $F \underset{\sim}{\subseteq} M^{(n+1)}$. A fuzzy relation F is said to be a fuzzy function w.r.t. $\mathcal{R}_{I'}$ if*

$$R(\bar{x}, \bar{x}') * F(\bar{x}, y) * F(\bar{x}', y') \leq R_{n+1}(y, y'),\ \forall \bar{x}, \bar{x}' \in M^{(n)},\ \forall y, y' \in M_{n+1}. \tag{28}$$

By the functionality property we mean the above inequality.

This form of the functionality property appears also in [2, 1] and it can be interpreted as follows: any two images of the indistinguishable inputs $\bar{x}$ and $\bar{x}'$ under F are indistinguishable.

Remark 26 *It is possible to introduce a new normal form for $f \underset{\sim}{\subseteq} M^{(n+1)}$ based on the functionality property*

$$\overline{\mathrm{CNF}}_f^C(x, y) = \bigwedge_{\bar{c} \in C} (f(\bar{c}) \to (R_1(c_1, x) \to R_2(c_2, y))), \tag{29}$$

where, for the sake of brevity, we have assumed $n = 1$. The generalization to $n > 1$ is obvious.

Now, it is clear that (27) is a special case of (29) because $\mathbf{1} \to a = a$, $\forall a \in L$.

It is also possible to introduce extensionality for crisp functions. In accordance with [2], we call it compatibility.

Definition 27 *Let $\mathcal{R} = \{R_1 \underset{\sim}{\subseteq} M_1, \ldots, R_{n+1} \underset{\sim}{\subseteq} M_{n+1}\}$ and $f \colon M_1 \times \ldots \times M_n \to M_{n+1}$. A function f is said to be compatible with $\mathcal{R}_{I'}$ if*

$$R(\bar{x}, \bar{y}) \leq R_{n+1}(f(\bar{x}), f(\bar{y})),\ \forall \bar{x}, \bar{y} \in M^{(n)}. \tag{30}$$

4.2.2 Approximation of fuzzy relation over compatible function

Below, we summarize the properties of (26), (27)[4] and $R_{n+1}(f(\bar{x}), y)$.

Proposition 28 *Let $F(\bar{x}, y) = R_{n+1}(f(\bar{x}), y)$, $C(\bar{x}) = \bigvee_{\bar{c} \in D} R(\bar{x}, \bar{c}) * R(\bar{x}, \bar{c}) = \bigvee_{\bar{c} \in D} R(\bar{x}, \bar{c})^2$. It can be proved that*

- DNF_f^D *is extensional w.r.t. $\mathcal{R}_{I'}$,*
- $\overline{\mathrm{CNF}}_f^D$ *is fuzzy function w.r.t. $\mathcal{P}_{I'}$, where $P_{n+1} = R_{n+1}$ and $P_i = \bigvee_{d \in D} R_i(x_i, d_i) * R_i(y_i, d_i), \forall i \neq n+1$.*

If f is compatible with $\mathcal{R}_{I'}$ then

- *F is extensional fuzzy function w.r.t. $\mathcal{R}_{I'}$,*

and moreover, $\forall \bar{x} \in M^{(n)}$ and $\forall y, y' \in M$:

$$\mathrm{DNF}_f^D(\bar{x}, y) \leq F(\bar{x}, y) \leq \overline{\mathrm{CNF}}_f^D(\bar{x}, y). \tag{31}$$

Proof: The extensionality of DNF_f^D follows from:

$$\begin{aligned} R(\bar{x}, \bar{y}) * R_{n+1}(x, y) &\leq \bigwedge_{d \in D} (R(\bar{x}, d) * R_{n+1}(x, f(d))) \to (R(d, \bar{y}) * R_{n+1}(f(d), y)) \leq, \\ &\leq (R(\bar{x}, d) * R_{n+1}(x, f(d))) \to (R(d, \bar{y}) * R_{n+1}(f(d), y)). \end{aligned}$$

The functionality of $\overline{\mathrm{CNF}}_f^D$ follows from:

$$\begin{aligned} &\overline{\mathrm{CNF}}_f^D(\bar{x}, x) * \overline{\mathrm{CNF}}_f^D(\bar{y}, y) \leq (R(\bar{x}, d) \to R_{n+1}(x, f(d))) * (R(d, \bar{y}) \to R_{n+1}(f(d), y)) \leq, \\ &\leq (R(\bar{x}, d) * R(d, \bar{y})) \to (R_{n+1}(x, f(d)) * R_{n+1}(f(d), y)) \leq (R(\bar{x}, d) * R(d, \bar{y})) \to R_{n+1}(x, y). \end{aligned}$$

[4] Additional results can be found in e.g. [2, 6, 7].

The extensionality of F follows from the compatibility of f:

$$R(\bar{x},\bar{y}) * R_{n+1}(x,y) * R_{n+1}(x,f(\bar{x})) \leq R_{n+1}(f(\bar{x}),f(\bar{y})) * R_{n+1}(y,f(\bar{x})) \leq R_{n+1}(y,f(\bar{y})).$$

The functionality of F proves the following sequences of inequalities: $R(\bar{x},\bar{x}')*R_{n+1}(y,f(\bar{x}))*R_{n+1}(y',f(\bar{x}')) \leq R_{n+1}(f(\bar{x}),f(\bar{x}')) * R_{n+1}(y,f(\bar{x})) * R_{n+1}(y',f(\bar{x}')) \leq R_{n+1}(f(\bar{x}'),y) * R_{n+1}(y',f(\bar{x}')) \leq R_{n+1}(y,y')$.

(31) $\mathrm{DNF}_f^D(\bar{x},y) \leq F(\bar{x},y)$ follows form the extensionality:

$$\begin{aligned} R(\bar{x},\bar{x}') * R_{n+1}(y,y') * R_{n+1}(y',f(\bar{x}')) &\leq R(\bar{x},\bar{x}') * R_{n+1}(y,f(\bar{x}')) \\ &\leq \bigvee_{\bar{x}'\in D} R(\bar{x},\bar{x}') * R_{n+1}(y,f(\bar{x}')) \leq R_{n+1}(y,f(\bar{x})) = F(\bar{x},y). \end{aligned}$$

Further, $F(\bar{x},y) \leq \overline{\mathrm{CNF}}_f^D(\bar{x},y)$ follows from the functionality property (28):

$$R(\bar{x},\bar{x}') * R_{n+1}(f(\bar{x}),y) \leq R_{n+1}(y,f(\bar{x}')) \leq R_{n+1}(f(\bar{x}'),y') \to R_{n+1}(y,y'),$$

and hence by adjunction

$$R_{n+1}(f(\bar{x}),y) \leq \bigwedge_{\bar{x}'\in D} R(\bar{x},\bar{x}') \to R_{n+1}(y,f(\bar{x}')) \leq R(\bar{x},\bar{x}') \to R_{n+1}(y,f(\bar{x}')).$$

QED

From the above proposition it follows that F is a fuzzy function in the sense of [11], i.e. F is extensional w.r.t. $\mathcal{R}_{I'}$ and fulfills the property of generalized functionality. In our formalism, we prefer to separate extensionality from functionality and to say that F is an extensional fuzzy function. Additionally, we see that DNF is an universally extensional fuzzy relation and $\overline{\mathrm{CNF}}$ is an universal fuzzy function. Further, we give the estimation of an equivalence between F and DNF ($\overline{\mathrm{CNF}}$).

Theorem 29 *Let $F(\bar{x},y) = R_{n+1}(f(\bar{x}),y)$, $C(\bar{x}) = \bigvee_{\bar{c}\in D} R(\bar{x},\bar{c})^2$, and f be compatible with $\mathcal{R}_{I'}$. Then $\forall \bar{x} \in M^{(n)}$ and $\forall y,y' \in M$:*

$$C(\bar{x}) \leq \mathrm{DNF}_f^D(\bar{x},y) \leftrightarrow F(\bar{x},y), \tag{32}$$

$$C(\bar{x}) \leq \overline{\mathrm{CNF}}_f^D(\bar{x},y) \leftrightarrow F(\bar{x},y). \tag{33}$$

Proof:

(32)–(33)

$$(R(\bar{x},\bar{c}) \to R_{n+1}(y,f(\bar{c}))) * (\underbrace{R(\bar{x},\bar{c}) \to R(\bar{x},\bar{c})}_{\mathbf{1}}) \leq R(\bar{x},\bar{c})^2 \to (R(\bar{x},\bar{c}) \to R_{n+1}(y,f(\bar{c})))$$

$$\bigvee_{\bar{c}\in D} R(\bar{x},\bar{c})^2 \leq \bigwedge_{\bar{c}\in D} (R(\bar{x},\bar{c}) \to R_{n+1}(y,f(\bar{c}))) \to \bigvee_{\bar{c}\in D} (R(\bar{x},\bar{c}) \to R_{n+1}(y,f(\bar{c}))); \text{ i.e.}$$

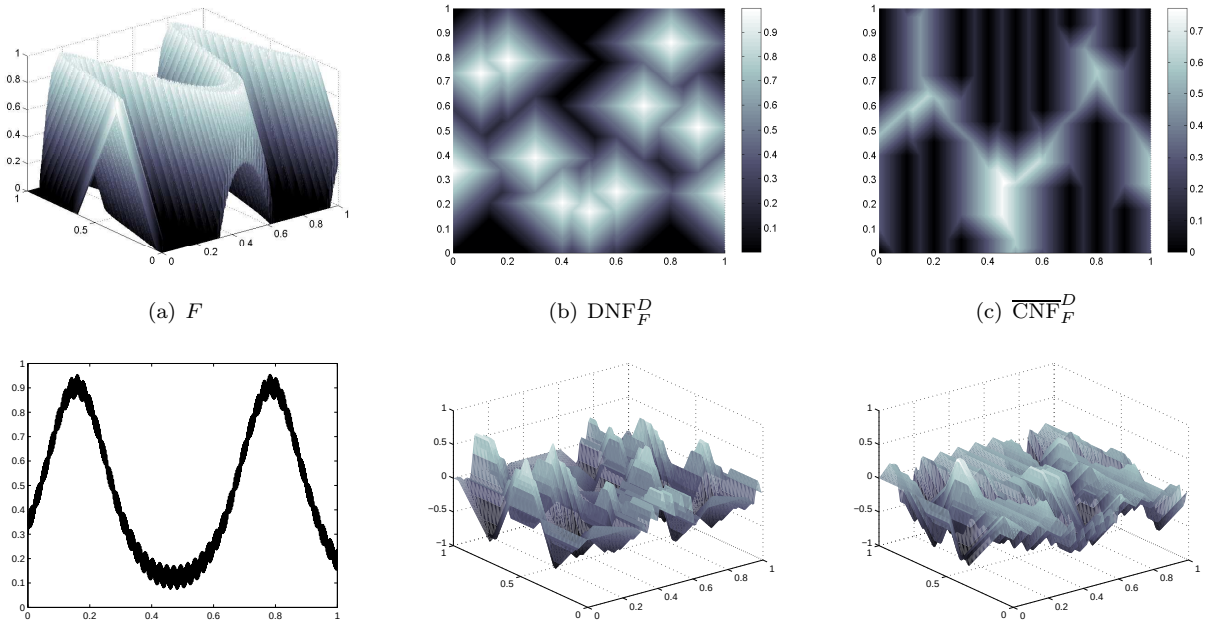
$C(\bar{x}) \leq \overline{\mathrm{CNF}}_f^D(\bar{x},y) \to \mathrm{DNF}_f^D(\bar{x},y)$, and using (31), we come to (32) as well as (33). QED

Hence, by means of DNF and $\overline{\mathrm{CNF}}$ we can "approximate" F with an arbitrary precision within the range restricted only by the choice of $\mathcal{R}_I$ and D. This fact is expressed by inequalities (32) and (33).

Remark 30 *A proof of the fact that F is an extensional fuzzy function can be found in various papers or books (see e.g. [6, 2]). And the inequalities (31)–(33) have been already proved in [11] (see Theorem 7.2.8 and Lemma 7.2.12) for an arbitrary fuzzy function $G \underset{\sim}{\subseteq} M^{(n+1)}$, where the functionality was considered in the following form:*

$$G(\bar{x},y) * G(\bar{x},y') \leq R_{n+1}(y,y').$$

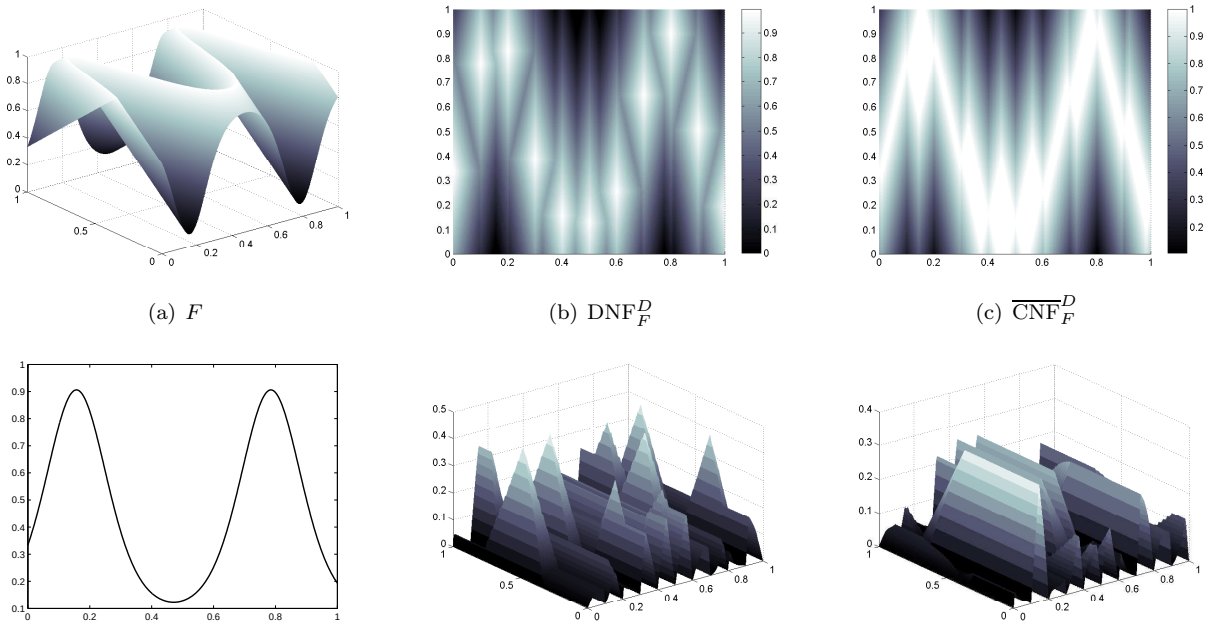
But $F(\bar{x},y) = R_{n+1}(f(\bar{x}),y)$ is the fuzzy function, where the functionality property is generalized. Because all $R \in \mathcal{R}_{I'}$ are reflexive, the generalized functionality implies the functionality in Hájek's sense in this case. However, (31)–(33) are consequences of the results in [11]. Despite of this fact, we have decided to provide the reader with proofs of these claims to keep the compactness of the whole text.



(a) F (b) DNF_F^D (c) $\overline{\mathrm{CNF}}_F^D$

(d) f (e) $F - \mathrm{DNF}_F^D$ (f) $\overline{\mathrm{CNF}}_F^D - F$

Figure 4: Approximation of F (not a fuzzy function) from Example 34(a) and its D(C)NF.



(a) F (b) DNF_F^D (c) $\overline{\mathrm{CNF}}_F^D$

(d) f (e) $F - \mathrm{DNF}_F^D$ (f) $\overline{\mathrm{CNF}}_F^D - F$

Figure 5: Approximation of F from Example 34(b).

We can also give an analogous sequence of corollaries for (32) and (33) as in Subsection 4.1.2.

Corollary 31 *Let $F(\bar{x},y) = R_{n+1}(f(\bar{x}),y)$, $C(\bar{x}) = \bigvee_{\bar{c}\in D} R(\bar{x},\bar{c})^2$, and $f\colon M \to M$ be compatible w.r.t. $\mathcal{R}_{I'}$. Moreover, let $\varepsilon \in L$, $D \subseteq M^{(n)}$. If for each $x \in M^{(n)} \exists d \in D:\ \varepsilon \le R(x,d) * R(d,x)$ then*

$$\varepsilon \le \begin{array}{l} \mathrm{DNF}_f^D(\bar{x},y) \leftrightarrow F(\bar{x},y), \\ \overline{\mathrm{CNF}}_f^D(\bar{x},y) \leftrightarrow F(\bar{x},y), \end{array} \quad \forall \bar{x} \in M^{(n)}, \forall y \in M. \tag{34}$$

Corollary 32 *Let $L = [0,1]$, $*$ be a continuous t-norm, $\varepsilon \in L$, M be a pre-compact (totally bounded) subset over a standard metric space $\mathcal{N} = (N,d)$, $R \underset{\sim}{\subseteq} M$ be reflexive and uniformly continuous. If $f\colon M \to M$ is compatible with $\mathcal{R}_{I'}$ then $\forall \varepsilon > 0\ \exists$ finite $D \subseteq M$ such that (34) is valid.*

Proof: Analogous to Corollary 21. QED

Corollary 33 *Assume $\mathcal{L}^{\varphi}$, d be given by (8) and d_n by (7). Moreover, let $F(\bar{x},y) = R_{n+1}(f(\bar{x}),y)$ and $\mathcal{S} = (L, \oplus_{\varphi}, \le)$. If $f\colon M^{(n)} \to M$ is compatible with $\mathcal{R}_I$ then F is 1-Lipschitz continuous w.r.t. $\mathcal{S}$ and*

$$\begin{array}{l} d(\mathrm{DNF}_f^D(\bar{x},y), F(\bar{x},y)) \\ d(\overline{\mathrm{CNF}}_f^D(\bar{x},y), F(\bar{x},y)) \end{array} \le \bigwedge_{\bar{d}\in D} 2d_n(\bar{x},\bar{d}) = \bigwedge_{\bar{d}\in D} d_n(\bar{x},\bar{d}) \oplus d_n(\bar{d},\bar{x}),\ \forall \bar{x} \in M^{(n)}, \forall y \in M. \tag{35}$$

Proof: From Proposition 28, it follows that if f is compatible then F is extensional and hence, we can use Theorem 13. The rest is obvious. QED

We finish the section by an illustrative example.

Example 34 *Let $\mathcal{L}$ be the standard Łukasiewicz algebra and moreover*

(a) $f(x) = \frac{1}{3} \cdot e^{\sin(10x)} + 0.05 \cdot \sin(2000x)$, $M = [0,1]$, *and take* $R_1(x,y) = R_2(x,y) = (x \leftrightarrow y)^3$.

Observing Figure 4(e) and 4(f), we recognize that inequalities (31) are violated. This is caused by the fact that f is not compatible w.r.t. $\{R_1, R_2\}$. Hence, $F(x,y) = (f(x) \leftrightarrow y)^3$ cannot be approximated by DNF_F^D ($\overline{\mathrm{CNF}}_F^D$) maintaining the lower (upper) approximation property.

If $R_1(x,y) = (x \leftrightarrow y)^{103}$ then f would preserve compatibility w.r.t. $\{R_1, R_2\}$. Such a high power follows from the estimation of the derivative of $0.05 \cdot \sin(2000x)$ (the noisy part of f).

(b) $f(x) = \frac{1}{3} \cdot e^{\sin(10x)}$, $M = [0,1]$, *and take* $R_1(x,y) = (x \leftrightarrow y)^5$, $R_2(x,y) = x \leftrightarrow y$. *Then f is compatible w.r.t. $\{R_1, R_2\}$. It follows from the Lipschitz continuity of f on the standard metric space with the Lipschitz constant $K = 5$.*

Figure 5 demonstrates an approximation of $F(x,y) = f(x) \leftrightarrow y$ by DNF_F^D and $\overline{\mathrm{CNF}}_F^D$, where the set D consists of 11 equidistantly distributed nodes over M.

5 Conclusions

We have shown the duality between extensionality and 1-Lipschitz continuity on a special pseudo-metric space. The whole study is based on the original work of L. Valverde [24], where the link of duality between $*$-equivalences and pseudo-metrics has been established.

Extensionality is widely used in different fields. To show its applicability, we have chosen fuzzy approximation in the sense of I. Perfilieva [21]. The first application shows the advantage of the extensionality-based approach to fuzzy transforms. It significantly reduced the number of requirements needed in order to estimate the precision of approximation by means of an equivalence relation. In the second application, we have focused on creating a fuzzy relation over the function in concern and its approximation with an arbitrary precision. In both cases universally extensional fuzzy sets (relations) were formed and results from [21] were straightforwardly applied to prove the final estimation. By these two applications we wanted to highlight the need for an investigation of universally extensional fuzzy sets (relations) and its interesting outcomes.

Acknowledgement: The author is grateful to Irina Perfilieva, Antonín Dvořák and the anonymous referee for their pertinent comments that brought significant improvement to this article.

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