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"Advanced Methods in Soft Computing"
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A New Approach to a Fuzzy Rule Base Interpolation

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It is very well known that a fuzzy rule base is a characterization of a partially given mapping between fuzzy universes. If fuzzy sets in the fuzzy rule base are defined on the set of reals and fulfil certain conditions then we can say that they determine a partial fuzzy function. For practical applications, it is desirable to interpolate that function in order to compute its values at points (fuzzy or crisp) other than fuzzy sets in antecedents of the fuzzy rule base. Moreover, interpolation requires that in the case of coincidence between input fuzzy sets and fuzzy sets in the antecedents, a computation method should produce fuzzy sets that coincide with corresponding ones in consequences. Perhaps, the most known computation method is the Compositional Rule of Inference (CRI) [3, 4] where a composition is based on a combination of either sup and $*$ or inf and $\rightarrow$ operations. CRI is a realization of the generalized Modus Ponens proposed by Lotfi Zadeh in his early papers [3, 4]. However, CRI has the following flaws:

- it does not interpolate in general.
- it does interpolate, but it may happen that CRI-outputs at other points than those ones in antecedents are trivial fuzzy sets (either zero or equal to the whole universe).

In the paper, we propose a general framework of the interpolation problem stemming from the classical approach which explains a source of those flaws and shows how they can be overcome. We restrict ourselves to those solutions of the interpolation problem which can be expressed by fuzzy relations [1, 2]. We show that in that particular case, the interpolation problem is equivalent to the problem of solvability of a system of fuzzy relation equations. We introduce a notion of a functional fuzzy relation and show, how it can be constructed. We also propose a solution of the interpolation problem in the case of a sparse rule base and at a point which is disjoint with any of fuzzy sets in antecedents of the fuzzy rule base.

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On approximative systems and related structures

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Introduction In [5] we introduced the concept of an L-approximative space generalizing two concepts which are in the focus of interest of many researchers - namely the notion of an L-topological spaces in the sense of C.L. Chang - J.A. Goguen [2], [3] and the notion of a rough set introduced by Z. Pawlak [4]. The principal aim of this talk is to proceed with the study of L-approximative spaces. However now we give preference to a more general concept of an approximative system having the notion of an L-approximative space as a special case.

Definitions Let $L = (L, \wedge, \vee, \leq, *)$ be a *cl*-monoid, that is L is a complete distributive lattice with lower and upper bounds $0, 1 \in L$ endowed with a binary operation $*$ which is distributive over arbitrary joins, see [1]. An *upper approximation operator* on L is a monotone mapping $u : L \to L$ s.t. $u(0) = 0$; $a \leq u(a)$; $u(a \vee b) = u(a) \vee u(b)$; $u(u(a)) = a$ $\forall a, b \in L$. A *lower approximation operator* on L is a monotone mapping $l : L \to L$ s.t. $l(1) = 1$; $a \geq l(a)$; $l(a \wedge b) = l(a) \wedge l(b)$; $l(l(a)) = a$ $\forall a, b \in L$. A triple (L, u, l), where $u : L \to L$ and $l : L \to L$ are upper and lower approximative operators on L resp., is called *an approximative system*. The approximative system (L, u, l) is called *self-dual* if $u(1 \mapsto a) = 1 \mapsto l(a)$ $\forall a \in L$, where $\mapsto$ is the residuation on L induced by $*$. Note that if L is a Girard monoid, then an approximative system (L, u, l) is self-dual iff $l(a) = 1 \mapsto u(1 \mapsto a) \forall a \in L$ iff $u(a) = 1 \mapsto l(1 \mapsto a) \forall a \in L$.

In case when X is a set, $\mathcal{L} = L^X$ and $(\mathcal{L}, u, l)$ is an approximative system, the quadruple (X, L, u, l) is called an *L-approximative space*.

Examples

1. Given a *cl*-monoid L let $u(a) = l(a) = a$ $\forall a \in L$. Then (L, u, l) is an approximative system called *discrete*.
2. Given a *cl*-monoid L let $u(a) = 1$ iff $a \neq 0$ and $u(0) = 0$; $l(a) = 0$ iff $a \neq 1$ and $l(1) = 1$.

Then (L, u, l) is an approximative system called *indiscrete*.
3. Let L be a Heyting algebra, let (X, L, τ_1, τ_2) be an L-bitopological space and let approximation operators $u : L^X \to L^X$, $l : L^X \to L^X$ be defined by $u(A) = cl_1(A)$ and $l(A) = int_2(A) \ \ \forall A \in L^X$. Then (X, L, u, l) is an L-approximative space In paricular, in case $\tau_1 = \tau_2$, that is if (X, L, τ) is an L-topological space [3], the approximative system (L^X, u, l) is self-dual.
4. Let $\rho \subseteq X \times X$ be a reflexive relation on a set X and let $R(x) = \{x' \mid x\rho x'\}$ be the right ρ-class of $x \in X$. Given $A \in 2^X$ let $l(A) = \{x \mid R(x) \subseteq A\}$, and $u(A) = \{x \mid R(x) \cap A \neq \emptyset\}$. Then $u : 2^X \to 2^X$ and $l : 2^X \to 2^X$ are, respectively, upper and lower approximative operators on 2^X, and the approximative system $(2^X, u, l)$ is self dual. Such operators and corresponding 2-approximative spaces $(X, 2, u, l)$ in case when ρ is an equivalence relation were introduced by Z.Pawlak [4] under the name "a rough set".
5. Let L be a cl-monoid, let $\rho : X \times X \to L$ be a reflexive L-relation on a set X and let for every fixed $x \in X$ $\mathcal{R}(x) : X \to X$ be defined by $\mathcal{R}(x)(x') = \rho(x, x')$ $\forall x' \in X$. Further, given $A \in L^X$ let $l(A)(x) = \inf_{x' \in X}(\mathcal{R}(x)(x') \mapsto A(x'))$ and $u(A)(x) = \sup_{x' \in X}(\mathcal{R}(x)(x') \wedge A(x'))$ Then (L^X, l, u) is an approximative system and (X, L, l, u) is an L-approximative space.
Category of approximative systems *The category* **AS** *of approximative systems* has approximative systems (L, u, l) as objects and the morphisms $f : (L_1, u_1, l_1) \to (L_2, u_2, l_2)$ are defined as follows. Let $f : L_2 \to L_1$ be a mapping. Then $f : (L_1, u_1, l_1) \to (L_2, u_2, l_2)$ is a morphism in **AS** if $u_1(f(b)) \leq f(u_2(b))$ and $f(l_2(b)) \leq l_1(f(b))$.
Some properties of the category **AS** will be discussed. Also the connections of **AS** and some related categories, (in particular, the category L-**AS** of L-approximative spaces, the category **R-SET** of rough spaces, and the category L-**R-SET** of L-rough sets) will be considred. In this connection lattice properties of families of approximative systems are studied.

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Some aspects of approximation under fuzzy information

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In this paper we generalize the well known notions of "classic" or "crisp" approximation theory (see, e.g. [3]) to the L-fuzzy case. We consider the problem (A, B) of approximation of an operator $B : X \to Y$ defined in a set X and taking values in a normed space Y under information given by an operator $A : X \to L^{\mathbb{R}^n}$, where L is a completely distributive lattice.

By a method for solving the problem (A, B) we mean any operator $\varphi : \mathbb{R}^n \to Y$, which allows us to get an approximation $\varphi(z)$ of the exact value $B(x)$ for each $x \in X$ (see, e.g. [1], [2]) . Our aim is to investigate the error of the method for an element $x \in X_0$. There may exist many different elements $u \in X_0$ coresponding to the same information Ax. Since each of them has the corespondness degree characterized by the value $Ax(z)$, such elements form an L-fuzzy subset of X_0 denoting by U_z:

$$U_z(x) = Ax(z) \quad for\, all \quad x \in X_0; \quad U_z(x) = 0 \quad for\, all \quad x \notin X_0.$$

Thus knowing only z it is impossible to specify which element $x \tilde{\in} U_z$ is being actually approximated and we consider all of them. We investigate the error of approximation by a method φ. This error is given by the L-fuzzy value

$$e(\varphi, z) = \mathcal{S}up_{U_z} \|Bx - \varphi(z)\|_Y,$$

where the supremum $\mathcal{S}up_M$ of a bounded set $M \in L^{\mathbb{R}}$ is defined by the equality

$$\mathcal{S}up_M(v) = \inf_{t<v} \bigl(\sup_{u>t} M(u) \bigr).$$

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Powerset-like Functors in Categories of Fuzzy Sets over BL-algebras

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In [5], Zadeh proposed a principle which proposes a method how to extend the "crisp" mapping $f : X \to Y$ between sets to a "fuzzy" mapping $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ between sets of all fuzzy sets over X and Y with the membership degrees from the interval [0, 1]. The same principle can be also used for fuzzy sets which membership degrees are interpreted in a more general truth values structure Ω as an BL-algebra, MV-algebras, Heyting algebra, or cil-monoid.

If the membership degrees of fuzzy sets are interpreted in the Boolean algebra $\Omega = (\{0,1\}, \wedge, \vee)$, then $\hat{f} = \mathbf{P}(f)$, where $\mathbf{P} : \mathrm{Set} \to \mathrm{Set}$ is the common (covariant) power functor. Since the morphism $\mathbf{P}(f)$ preserves all unions for any mapping f, i.e., $\mathbf{P}(f)(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathbf{P}(f)(A_i)$, the powerset functor $\mathbf{P}$ is also a functor of Set to CSLat (the category of complete (join) semi-lattices). In [3] (see also [4]), Rodabaugh presented several categorical criteria for the functor $\mathbf{P} : \mathrm{Set} \to \mathrm{CSLat}$. Their modified versions are as follows: If $\mathbf{G} : \mathrm{CSLat} \to \mathrm{Set}$ is the forgetful functor, then

(1) there exists a natural transformation $\eta : \mathbf{I}_{\mathrm{Set}} \to \mathbf{G} \circ \mathbf{P}$;

(2) $\eta_X : X \to \mathbf{G} \circ \mathbf{P}(X)$ is a universal arrow for any set X;

(3) for any mapping $f : X \to Y$ there exists the unique $g : \mathbf{P}(Y) \to \mathbf{P}(X)$ such that $\mathbf{P}(f)(A) \leq B$ if and only if $A \leq g(B)$ for any $A \in \mathbf{P}(X)$ and $B \in \mathbf{P}(Y)$, where $\leq$ is the common ordering of sets.

Unfortunately, the Zadeh's extension principle (shortly ZEP) defined as a functor $\mathbf{F}$ of Set to CSLat assigning the set of all fuzzy sets to each set does not satisfy the proposed criteria, in general. The problems arise when a natural transformation $\eta : \mathbf{I}_{\mathrm{Set}} \to \mathbf{F} \circ \mathbf{P}$ is founded for more general truth value structures such that η_X is a universal arrow for any set X. Therefore, Rodabaugh proposed to investigate the ZEP, in essence, as a functor $\mathbf{F}$: Set $\to$ CSLat such that, for any mapping f in Set, the morphism $\mathbf{F}(f)$ lifts $\mathbf{P}(f)$ uniquely and, moreover, there exists the unique lift h of the mapping g that is the right adjoint to $\mathbf{F}(f)$. Note that the existence and uniqueness of h is closely associated with the a-cut representations of fuzzy sets (the decomposition theorems) as Rodabaugh presented in [3].

Summarizing the previous investigation of ZEP the study of its correctness may be divided into two parts. The first part may be devoted to a correctness of functors (powerset-like functors) being similar to the power functor **P** and the second part to a correctness of functors which are "lifts" of powerset-like functors.

Now an interesting question arises, if the ZEP as a functor can be also defined in other categories of fuzzy sets in such a way to satisfy the given criteria of correctness. In [2], Močkoř introduced several covariant functors $\mathbf{F} : \mathrm{FSet} \to \mathrm{FSet}$, which could represent some generalization of ZEP, in the category FSet (see also [1]). Although, the proposed definitions of covariant functors are very natural, it could be shown that they do not satisfy all proposed criteria. This motivates us to search for some modifications of the original approach or even for a new approach to verify whether a functor may be understood as a representation of the ZEP in a given category.

The aim of my presentation is to investigate the first part of correctness of the ZEP, i.e., the correctness of powerset-like functors. I propose three types of functors: *pre-powerset, powerset and C-powerset functor*, where a pre-powerset functor and C-powerset functor (C denotes a class of subalgebras in which the original category is representable) could be understood as powerset-like functors in the sense of the criteria (1)-(3). These functors then form a base for introducing more general functors representing the ZEP.

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On ordered categories as a framework for fuzzification of algebraic and topological structures[1]

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In the talk we will develop further the topic started in [3], where a fuzzification machinery of algebraic and topological structures was introduced, which amounted to fuzzifying the underlying "set" of a structure in a suitably compatible way, leaving the structure itself crisp. We will show that the algebraic approach generalizes the existing procedures of A. Rosenfeld [2] as well as of internal structures in categories. The topological one, however, falls out of the usual fuzzification machineries mentioned in the literature. The reason is that in contrast to, e.g., [1] we do not fuzzify the concepts themselves but just present a modified version of a given topological structure. On the other hand, our procedure allows the so-called "double fuzzification", i.e., fuzzification of something that is already fuzzified. As an example we provide a fuzzified version of the category of L-topological spaces.

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CONSTRUCTION OF FUZZY LOGIC MODELS IN CATEGORIES OF SETS WITH SIMILARITIES

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We introduce models $\mathcal{E}$ of first order fuzzy logic based on a language J consisting of a set of predicate symbols $P \in \mathcal{P}$, a set of functional symbols $f \in \mathcal{F}$ and a set of classical logical connectives $\{\wedge, \vee, \Rightarrow, \neg, \otimes\}$. These models will be defined in two categories $\mathbf{K}$ of sets with similarity relations (called Ω-sets), i.e. categories with couples (A, δ) as objects, where A is a set and a similarity relations $\delta : A \times A \to \Omega$ is defined with values in a complete MV-algebra Ω and with morphisms defined either as special fuzzy relations between Ω-sets (i.e. $\mathbf{K} = \mathbf{SetR}(\Omega)$) or maps between these sets ($\mathbf{K} = \mathbf{SetF}(\Omega)$).

A model of a language J in a category $\mathbf{K}$ is $\mathcal{E} = ((A, \delta), \{P_{\mathcal{E}} : P \in \mathcal{E}\}, \{f_{\mathcal{E}} : f \in \mathcal{F}\})$, where

(a) (A, δ) is an Ω-set from a category $\mathbf{K}$,

(b) $P_{\mathcal{E}} \subseteq_{\sim\mathbf{K}} (A, \delta) \times \cdots \times (A, \delta)$,

(c) $f_{\mathcal{E}} : (A, \delta) \times \cdots \times (A, \delta) \to (A, \delta)$ is a morphism in a category $\mathbf{K}$,

where a symbol $s \subseteq_{\sim\mathbf{K}} (A, \delta)$ means a *fuzzy set* (in a category $\mathbf{K}$) in (A, δ), i.e. a morphism (in $\mathbf{K}$) $s : (A, \delta) \to (\Omega, \leftrightarrow)$.

In those models $\mathcal{E}$ we investigate constructions of interpretations $\|\psi\|_{\mathcal{E}}$ of formulas ψ in a first order fuzzy logic. We will then prove that interpretations $\|\psi\|_{\mathcal{E}}$ are *fuzzy sets* in some objects of $\mathbf{K}$. We define homomorphisms between models in the category $\mathbf{K}$ and we prove that if $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is a (special) homomorphism of models in a category $\mathbf{K}$ then there is also a relationship between interpretations $\|\psi\|_{\mathcal{E}_1}$ and $\|\psi\|_{\mathcal{E}_2}$ of a formula ψ in models $\mathcal{E}_i$.

Finally, for any model $\mathcal{E}$ based on an Ω-set (A, δ) from the category $\mathbf{K}$ we construct another model $F(\mathcal{E})$ based on a Ω-set $(F(A, \delta), \sigma)$ of all fuzzy sets in (A, δ) with some similarity relation σ and such that a singleton map $\{-\}$ is a strong model homomorphism $\mathcal{E} \to F(\mathcal{E})$.

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On some fuzzy categories of many-valued topological spaces.

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Introduction The concept of a fuzzy category was introduced by A.Šostak in [3] and later was studied in a series of papers see, e.g. [4]. These papers contain many examples of fuzzy categories which appear by *"fuzzifying"* classic categories. On the other hand in [5] we studied some categories of many-valued sets and many-valued topological spaces, in particular, categories **SET**(L), **TOP**(L) and **FTOP**(L). The aim of this talk is to introduce new fuzzy categories $\mathcal{F}$-**SET**(L), $\mathcal{F}$-**TOP**(L) and $\mathcal{F}$-**FTOP**(L) by applying the method of fuzzification [4] to the categories **SET**(L), **TOP**(L) and **FTOP**(L), to discuss some properies of these fuzzy categories and relations between them.

Fuzzy categories First we recall the concept of an (L-)fuzzy category in the form appropriate for our merits. Let $L = (L, \leq, \wedge, \vee, *)$ be a GL-monoid [1], in particular a complete Heyting algebra (when $\wedge = *$) and $\mapsto$ the residuation. *An (L-)fuzzy category* is a pair $(\mathcal{C}, \mu)$ where $\mathcal{C}$ is an ordinary category with the class of objects $\mathcal{O}(\mathcal{C})$, the class of morphisms $\mathcal{M}(\mathcal{C})$, and $\mu : \mathcal{M}(\mathcal{C}) \to L$ an L-subclass of the class of morphism such that:

(i) $\mu(g \circ f) \geq \mu(g) * \mu(f)$ whenever composition $g \circ f$ is defined in the category $\mathcal{C}$;

(ii) for each $X \in \mathcal{O}(\mathcal{C})$ $\mu(e_X) = 1$ where e_X is the identity morphism.

Category SET(L) and fuzzy category $\mathcal{F}$-SET(L) Recall [1] that the objects of the category SET(L) are L-valued sets, that is pairs (X, E) where X is a set and $E : X \times X \to L$ is a mapping s.t. $E(x,x) = 1; E(x,y) = E(y,x); E(x,y)*E(y,z) \leq E(x,z)\ \forall x,y,z \in X$; the morphisms $f : (X, E_X) \to (Y, E_Y)$ are extensional mappings, that is $E_X(x,x') \leq E_Y(f(x), f(x')\ \forall x, x' \in X$. Consider the category $\mathcal{C}(L) = \mathcal{O}b(SET(L)), \mathcal{M}or(SET)$ (that is $\mathcal{C}(L)$ has objects from $SET(L)$ and morphisms from SET. We define the measure of extensionality for a mapping $f : X \to Y$: $\mu(f) = \bigwedge_{x,x' \in X} (E_X(x,x') \mapsto E_Y(f(x), f(x'))$.

The triple $(\mathcal{O}(SET(L)), \mathcal{M}(SET), \mu) =: \mathcal{F} - SET(L)$ is an L-fuzzy category called the L-fuzzification of the category $SET(L)$ of L-valued sets.

Category TOP(L) and fuzzy category $\mathcal{F}$-TOP(L) The method proposed in the previous paragraph can be used to fuzzify the category TOP(L) of L-valued topological spaces and to construct a fuzzy category $\mathcal{F}$-TOP(L) [5].

Category FTOP(L) and fuzzy category $\mathcal{F}$-FTOP(L) The concept of a many valued L-fuzzy topological space as a generalization of the concept of an L-fuzzy topological space [2] and the corresponding category FTOP(L) were introduced in [5]. The objects of this category are triples $(X, E, \mathcal{T})$ where (X, E) is an L-valued set and $\mathcal{T} : L^X \to L$ is an L-fuzzy topology, that is an extensional mapping s.t. $\mathcal{T}(1) = \mathcal{T}(0) = 1; \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V); \mathcal{T}(\vee_i(U_i) \geq \wedge_i \mathcal{T}(U_i)$, and morphisms are extensional mappings $f(X, E_X) \to (Y, E_Y)$ such that $\mathcal{T}_(f^{-1}(V) \geq \mathcal{T}(V)$. As different from the categories considered above, in case of the category FTOP(L) one can regard two properties of morphisms for which the measures are naturally defined - the degree of extensionality and the degree of continuity.

Let $(X, E_X, \mathcal{T}_X)$, $(Y, E_Y, \mathcal{T}_Y)$ be L-valued L-fuzzy topological spaces and $f : X \to Y$ be a mapping of the corresponding underlying sets. The measure the extensionality μ_1 of the mapping $f : (X, E_X) \to (Y, E_Y)$ is defined as above. The measure of continuity μ_2 of the mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is defined by $\mu_2(f) = \bigwedge_{V \in L_E^Y} (\mathcal{T}_Y(V) \mapsto \mathcal{T}_X(f^{-1}(V))$. Further, for a mapping $f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$ we set $\mu(f) = \mu_1(f) \wedge \mu_2(f)$. The resulting structure $\mathcal{F} - \mathbf{FTOP}(L) = (\mathcal{O}(\mathbf{FTOP}(L), \mathcal{M}or(\mathbf{SET}), \mu)$ is an L-fuzzy category called the fuzzification of the category **FTOP**(L).

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A reverse style of logic-based fuzzy topology

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The emergence of first-order [7] and higher-order [8, 1] formal fuzzy logic has enabled to develop *logic-based fuzzy mathematics,* i.e., formal mathematical theories axiomatized in first- or higher-order fuzzy logic. Initial steps in *logic-based fuzzy topology* have been done in [5, 4, 6].

Important features of logic-based fuzzy mathematics are the general gradedness of its theorems and defined notions, and its closeness (in some respects) to classical mathematics, which can give us methodological guidelines for fuzzification of classical notions [2]. However, due to the non-idempotence of conjunction in most fuzzy logics, axiomatic theories over fuzzy logic have several peculiar features that are not met in classical mathematics [3]. These features impose to use a different style of definitions and theorems than is usual in classical mathematics. In particular, it is advantageous to formulate theorems in the form of provable implications ($\vdash \varphi_1 \,\&\, \ldots \,\&\, \varphi_n \to \psi$), rather than a conclusion derived from assumed premises ($\varphi_1, \ldots, \varphi_k \vdash \psi$, or equivalently $\vdash \Delta\varphi_1 \,\&\, \ldots \,\&\, \Delta\varphi_n \to \psi$), as the former are stronger than the latter, although usually not much more difficult to prove. Since multiple occurrences of the same graded premise in a theorem cannot be contracted, the theorems have in fact the form $\vdash \varphi_1^{k_1} \,\&\, \ldots \,\&\, \varphi_n^{k_n} \to \psi$, where the multiplicity k_i indicates how much the conclusion is sensitive to the truth degree of φ_i. Since different theorems need different multiplicities of a premise, the multiplicities should not be fixed in definitions: defined notions rather need to be parameterized by the multiplicities of each conjunct in the defining formula. These parameters then take different values in different theorems; a whole family of parameterized notions is thus defined instead of a single notion.

In logic-based fuzzy topology, these features are in [5, 4] reflected in defining the graded notions of fuzzy topology (based respectively on fuzzily open fuzzy sets, fuzzy systems of fuzzy neighborhoods, and a fuzzy interior operator) as parameterized families of fuzzy predicates $\mathsf{OTop}^{e,v,i,u}$, $\mathsf{NTop}^{k_1,\ldots,k_5}$, and $\mathsf{ITop}^{k_1,\ldots,k_5}$ (as they are conjunctions of 4 resp. 5 independent graded defining conditions; two additional parameters in $\mathsf{OTop}^{e,v,i,u}_{m,n}$ introduced in [4] parameterize further conjuncts inside the condition of union-closedness). Several results on these notions, with varying values of the indices, were given in [5, 4, 6].

Here I want to argue that the large number of indices involved in the definitions of fuzzy topological notions suggests that an even more radical approach should be taken in developing logic-based fuzzy topology (and possibly all logic-based fuzzy mathematics)—namely, that we should turn from the direct to a reverse style of constructing the theory. Classical mathematical theories derive their theorems from a fixed set of axioms; i.e., they proceed in the *direct* way from the axioms to their consequences. *Reverse mathematics* (see [9, §I.9]) strives to find the weakest natural conditions ensuring a desired conclusion (i.e., it proceeds in the *reverse* direction, from the desired conclusions to their sufficient or necessary preconditions). In the reverse style of logic-based fuzzy topology, thus, rather than trying to define a fixed notion of fuzzy topology (with a large number of parameters) and studying its properties that follow from its defining conditions, we would try to find relationships between various 'topologically flavored' properties of fuzzy structures. That is, we would not fix a set of axioms in advance, but search for conditions ensuring each particular desired property, case by case.

The conviction that this style of logic-based fuzzy topology is better than the direct one stems from the observation that one can hardly delimit a set of conditions that would capture a good notion of (logic-based) topological space. The non-equivalent ways of defining fuzzy topology by OTop, NTop, and ITop (and the unmanageable number of possible variants and parameters of the definitions if better-behaved topological spaces are considered, e.g., those satisfying various compactness or separation properties) suggest that there is no privileged set of axioms that should be adopted for fuzzy topologies. Rather, there are various independent graded properties (of fuzzy families of fuzzy sets) that are 'topologically flavored'—for example, union-closedness, the finite subcovering property, various separation properties, properties related to continuity, etc. The subject of logic-based fuzzy topology should be the study of mutual relationships between such properties, and searching for (in the optimal case, minimal) multisets of preconditions that ensure such properties. The reverse-style of logic-based fuzzy topology can thus be briefly characterized as the study of topological properties rather than topological spaces (for an illustration, cf. the theorems of [5], some of which depend on only one or even none of the defining conditions of OTop).

A few examples of theorems in reverse-style logic-based fuzzy topology will be given. The issue of fuzzy topologies with a fuzzy set of points, whose introduction is rather problematic under the logic-based approach, will also be discussed.

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Fuzzy Class Theory: A State of the Art*

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It is indisputable that mathematical structures containing vague concepts have a broad range of applications; therefore they have been intensively investigated during the last four decades. The 'subject' studying these structures is, maybe unfortunately, called *Fuzzy mathematics*.

There is an ongoing project of the Prague research group in fuzzy logic, directed towards developing the *logic-based* fuzzy mathematics, i.e., an 'alternative' mathematics build formally analogously as the classical mathematics but not in the classical logic but in a suitable formal fuzzy logic. First steps in the development thereof were enabled by recent results in mathematical fuzzy logic, especially by the emergence of higher-order fuzzy logics, proposed by Libor Běhounek and the present author, see [2]. Our approach leads not only to an axiomatization, but also to a systematic study utilizing proof-theoretic and model-theoretic methods. Moreover, the unified formalism makes an interconnection of particular disciplines of fuzzy mathematics possible and provides the formal foundations of (part of) fuzzy mathematics, see [1] for more details about relation of traditional and logic-based fuzzy mathematics.

The core of the project is a formulation of certain formalistic methodology (see [3]), proposing the foundational theory, and studying the particular disciplines of fuzzy mathematics within this theory using our methodology. The proposed foundational theory is called Fuzzy Class Theory (FCT) and it is a first-order theory over multi-sorted predicate fuzzy logic, with a very natural axiomatic system which approximates nicely Zadeh's original notion of fuzzy set [6]. An important feature of the theory is the gradedness of all defined concepts, which makes it more genuinely fuzzy than traditional approaches.

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In this talk I describe the current state of the project and illustrate its features on the example of formal fuzzy topology developed by Běhounek and Kroupa in [5, 4]. For more details about the project see its webpage `www.cs.cas.cz/hp`.

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Geometrical methods in the investigation of residuated structures

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Abstract

First we shall give an overview on the main geometrical aspects of the study of residuated lattices. Next, we present the structural description of both e-involutive uninorms on $[0,1]$ and e-involutive finite involutive uninorm chains. This description involves a striking new construction, called skew-symmetrization, in which one has to leave the accustomed residuated setting, and has to enter a co-residuated setting too. So, for the description of residuated structures one needs as well co-residuation; it is a surprising observation in the theory of residuated lattices; a theory which goes back to 70 years.

1 Introduction

Residuated lattices and substructural logics are subjects of intense investigation. Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, fuzzy logics, linear logic and their non-commutative versions. Equivalent algebraic counterparts of substructural logics are classes of residuated lattices. The main topic of this paper is to give a structural description for e-involutive uninorms on the real unit interval $[0,1]$ and finite e-involutive uninorm chains, which are particular residuated lattices.

2 Preliminaries

For any binary operation $\circledast$ (on a poset) which is commutative and non-decreasing one can define its *residuum* $\to_\circledast$ by $x \circledast y \le z \iff x \to_\circledast z \ge y$. The displayed equivalence is often referred to as *adjointness conditions*. If $\to_\circledast$ exists, it has an equivalent description, namely, $\to_\circledast$ is the unique binary operation on the poset such that we have $x \to_\circledast y = \max\{z \mid x \circledast z \le y\}$.

Let $\mathcal{C} = \langle X, \le, \perp, \top\rangle$ be a bounded poset. A *involution* over $\mathcal{C}$ is an order reversing bijection on X such that its composition by itself is the identity map of X. Involutions are continuous in the order topology of $\mathcal{C}$. *T-conorms* (resp. *t-norms*) over $\mathcal{C}$ are commutative monoids on X with unit element $\perp$ (resp. $\top$). T-conorms and t-norms are *duals* of one another. That is, for any involution $'$ and t-conorm $\oplus$ over $\mathcal{C}$, the operation $\odot$ on X defined by $x \odot y = (x' \oplus y')'$ is a t-norm over $\mathcal{C}$. Vice versa, for any involution $'$ and t-norm $\odot$ over $\mathcal{C}$, the operation $\oplus$ on X defined by $x \oplus y = (x' \odot y')'$ is a t-conorm over $\mathcal{C}$. *Uninorms* over $\mathcal{C}$ [10, 2] are commutative monoids on X with unit element e (which may be different from $\perp$ and $\top$). Every uninorm over $\mathcal{C}$ has an *underlying t-norm* $\odot$ and *t-conorm* $\oplus$ acting on the subdomains $[\perp, e]$ and $[e, \top]$, respectively. That is, for any uninorm $\circledast$ over $\mathcal{C}$, its restriction to $[\perp, e]$ is a t-norm over $[\perp, e]$, and its restriction to $[e, \top]$ is a t-conorm over $[e, \top]$.

Definition 1 $\langle X, \le, \perp, \top, e, f, \circledast\rangle$ is called an *involutive uninorm algebra* if $\mathcal{C} = \langle X, \le, \perp, \top\rangle$ is a bounded poset, $\circledast$ is a uninorm over $\mathcal{C}$ with unit element e, for every $x \in X$, $x \to_\circledast f = \max\{z \in X \mid x \circledast z \le f\}$ exists, and for every $x \in X$, we have $(x \to_\circledast f) \to_\circledast f = x$. If $\mathcal{C}$ is a chain, we call $\langle X, \le, \perp, \top, e, f, \circledast\rangle$ an involutive uninorm chain. In an involutive uninorm algebra one can define an order-reversing involution by $x' = x \to_\circledast f$.

Definition 2 $\langle X, \le, \perp, \top, e, \circledast\rangle$ is called an *e-involutive uninorm algebra* if $\langle X, \le, \perp, \top, e, e, \circledast\rangle$ is an involutive uninorm algebra.

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3 Results

Definition 3 For any binary operation $\bullet$ (on a poset) which is commutative and non-decreasing one can define its *co-residuum* $\leftarrow_\bullet$ by $x \bullet y \geq z \iff x \leftarrow_\bullet z \leq y$. If $\leftarrow_\bullet$ exists it has an equivalent description: $\leftarrow_\bullet$ is the unique binary operation on the poset such that $x \leftarrow_\bullet y = \min\{z \mid x \circledast z \leq y\}$. In case $\bullet$ is associative, we have $x \leftarrow_\bullet (y \leftarrow_\bullet z) = (x \bullet y) \leftarrow_\bullet z$ and $x \leftarrow_\bullet (y \leftarrow_\bullet z) = y \leftarrow_\bullet (x \leftarrow_\bullet z)$. We shall call the displayed equivalence and equalities co-adjointness condition, co-exportation law, and co-exchange principle, respectively.

Definition 4 For any commutative residuated chain $\langle X, \leq, \oplus, \to_\oplus, 1\rangle$, define $\boldsymbol{\oplus} : X \times X \to X$ by $x \boldsymbol{\oplus} y = \inf\{u \oplus v \mid u \geq x, v \geq y, (u,v) \neq (x,y)\}$, and call it the *skewed pair* of $\oplus$. For any commutative co-residuated chain $\langle X, \leq, \boldsymbol{\oplus}, \to_{\boldsymbol{\oplus}}, 1\rangle$, define $\oplus : X \times X \to X$ by $x \oplus y = \sup\{u \boldsymbol{\oplus} v \mid u \leq x, v \leq y, (u,v) \neq (x,y)\}$, and call it the *skewed pair* of $\oplus$. Call $(\oplus, \boldsymbol{\oplus})$ a *skew pair.*

Definition 5 Let $(L_2, \leq)$ be a chain and $L_1 \subseteq L_2$. Let $(L_1, \oplus, \to_\oplus, \leq, \top)$ be a commutative residuated chain and $'$ be an order reversing involution on L_2. The operation $\odot$ is said to be *dual* to $\oplus$ with respect to $'$ if $\odot$ is a binary operation on $(L_1)' = \{x' \mid x \in L_1\}$ given by $x \odot y = (x' \oplus y')'$. We say that the operation $\boldsymbol{\odot}$ is *skew dual* to $\oplus$ with respect to $'$ if $\boldsymbol{\odot}$ is the skewed pair of $\odot$.

Definition 6 Let $\mathcal{C} = \langle X, \leq, \perp, \top\rangle$ be a bounded chain, and $'$ be an involution on X with fixed point $e \in X$. For any left-continuous t-conorm $\oplus$ on $[e, \top]$, define its skew symmetrization $\oplus_{\underline{\mathbf{s}}} : X \to X$ as follows.

$$x \oplus_{\underline{\mathbf{s}}} y = \begin{cases} x \oplus y & \text{if } x, y \in [e, \top] \\ (x \to_\oplus y')' & \text{if } x \in [e,\top] \text{ and } y \in [\perp, e] \text{ and } x \leq y' \\ (y \to_\oplus x')' & \text{if } x \in [\perp, e] \text{ and } y \in [e,\top] \text{ and } x \leq y' \\ (x' \boldsymbol{\oplus} y')' & \text{if } x, y \in [\perp, e] \\ x' \leftarrow_{\boldsymbol{\oplus}} y & \text{if } x \in [\perp, e] \text{ and } y \in [e,\top] \text{ and } x \geq y' \\ y' \leftarrow_{\boldsymbol{\oplus}} x & \text{if } x \in [e,\top] \text{ and } y \in [\perp, e] \text{ and } x \geq y' \end{cases}, \tag{1}$$

where $\boldsymbol{\oplus}$ denotes the skewed pair of $\oplus$.
For any left-continuous t-norm $\odot$ on $[\perp, e]$, define its skew symmetrization $\odot_{\overline{\mathbf{s}}} : X \to X$ as follows.

$$x \odot_{\overline{\mathbf{s}}} y = \begin{cases} (x' \boldsymbol{\odot} y')' & \text{if } x, y \in [e, \top] \\ x' \leftarrow_{\boldsymbol{\odot}} y & \text{if } x \in [e,\top] \text{ and } y \in [\perp, e] \text{ and } x \leq y' \\ y' \leftarrow_{\boldsymbol{\odot}} x & \text{if } x \in [\perp, e] \text{ and } y \in [e,\top] \text{ and } x \leq y' \\ (y \to_\odot x')' & \text{if } x \in [e,\top] \text{ and } y \in [\perp, e] \text{ and } x \geq y' \\ (x \to_\odot y')' & \text{if } x \in [\perp, e] \text{ and } y \in [e,\top] \text{ and } x \geq y' \\ x \odot y & \text{if } x, y \in [\perp, e] \end{cases}, \tag{2}$$

where $\boldsymbol{\odot}$ denotes the skewed pair of $\odot$.

Theorem 1 *Any e-involutive uninorm chain $\langle L, \leq, \perp, \top, e, \circledast\rangle$ which has a dense set of continuity points can be described as the skew symmetrization of its underlying t-conorm or t-norm. That is, $\circledast = \oplus_{\underline{\mathbf{s}}} = \odot_{\overline{\mathbf{s}}}$, where $\oplus$, $\odot$, $\boldsymbol{\oplus}$ and $\boldsymbol{\odot}$ denotes the underlying t-conorm and t-norm of $\circledast$, and their co-residuated pairs, respectively.*

Corollary 1 *For any e-involutive uninorm chain $\circledast$ which has a dense set of continuity points, its underlying t-norm and t-conorm form a skew dual pair with respect to $'$. Further, $\circledast$ is self skew dual with respect to $'$.*

Corollary 2 *Any e-involutive uninorm (on $[0,1]$) can be described as the skew symmetrization of its underlying t-conorm or t-norm.*

Corollary 3 *Any finite e-involutive uninorm chain can be described as the skew symmetrization of its underlying t-conorm or t-norm.*

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Representable (Non-commutative) EQ(R)-algebras

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Recently [3], a special algebra called *EQ-algebra* has been introduced which aims at becoming the algebra of truth values for fuzzy type theory (FTT). It has three basic operations – *meet* $\wedge$, *multiplication* $\otimes$ and *fuzzy equality* $\sim$ – and a *top element* 1, while the *implication* $\rightarrow$ is derived from fuzzy equality $\sim$. In [4], the study of EQ-algebras have been further deepened. Moreover, the axioms originally introduced in [3] have been slightly modified.

It is interesting that the implication and multiplication, in EQ-algebras, are no more closely tied by the adjuction and so, this algebra generalizes residuated lattice and hence the multiplication has weaker properties. Consequently, as shown by El-Zekey et al. in [2], we can refrain from commutativity and associativity of the multiplication generalizing the concept of EQ-algebra. The resulting algebra has been called a *semicopula-based EQ-algebra*, i.e. EQ-algebra in which the multiplication need not be commutative nor associative.

We continue in this paper the study of (semicopula-based) EQ-algebras, begun in [3], [4] and [2]. In the first, we review the basic definitions and properties of (semicopula-based) EQ-algebras and their special kinds with more details and more examples. We introduce prelinear (semicopula-based) EQ-algebras, in analogy with the prelinear structures of Esteva, Godo, Hájek and Höhle. We show that every prelinear and good (semicopula-based) EQ-algebra is a good (semicopula-based) ℓEQ-algebra; i.e. lattice EQ-algebra. Moreover, we obtain a lot of results which are analogous to the ones obtained for BL and MTL. However, we show that not all the properties of linearly ordered (good) EQ-algebras can be obtained for prelinear (good) EQ-algebras. For example, a multiplication $\otimes$ in a prelinear good (semicopula-based) EQ-algebra may or may not preserve finite meets and/or finite joins in each argument.

Consequently, we define and begin the investigation of a very important special class of semicopula-based EQ-algebras, namely (non-commutative) EQ-algebras with *(R) condition* (i.e. with *residuation* condition) or a *(non-commutative) EQ(R)-algebra* for short; we show that the reduct $\langle E, \otimes, 1\rangle$ of a (non-commutative) EQ(R)-algebra is also a reduct of a (non-commutative) residuated lattice. One of the main results of this paper is to characterize the class of all (non-commutative) EQ(R)-algebras that may be represented as subalgebras of products of linearly ordered (non-commutative) EQ(R)-algebras. Such algebras are called *representable*. We show that prelinearity alone does not characterize

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the representable class of all good (non-commutative) EQ(R)-algebras.

We introduce and study in depth the prefilters, filters and the congruences of semicopula-based EQ-algebras and show that many properties of lattices of (pre)filters of non-commutative residuated lattices can be obtained for lattices of (pre)filters of good semi-copula based EQ-algebras, with the main result that the lattice of filters (which form a complete sublattice of the lattice of all prefilter) are in bijective correspondence with the lattice of congruences. We show also that, in the case of good (non-commutative) EQ(R)-algebras, these lattices are distributive. This allows us to characterize representable good (non-commutative) EQ(R)-algebras, on lines of C. J. van Alten's characterization of representable integral (non-commutative) residuated lattices [5].

We prove that representable good non-commutative EQ(R)-algebras can be characterized by the identity

$$(a \to b) \vee (d \to (d \otimes (c \to ((b \to a) \otimes c)))) = 1 \tag{1}$$

or, equivalently, by

$$((a \to b) \to u) \leq [(d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u] \to u. \tag{2}$$

Inequality (2) can be chosen to characterize representable good non-commutative EQ(R)-algebras, apparently because it is free from lattice operations.

Consequently, We prove also that representable good EQ(R)-algebras can be characterized by the identity

$$((a \to b) \to u) \leq [(c \to ((b \to a) \otimes c)) \to u] \to u \tag{3}$$

The proof consists in showing that if a good (non-commutative) EQ(R)-algebra satisfies (3) (satisfies (2), respectively), then every minimal prime prefilter is a filter.

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Examples of finite EQ-algebras

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EQ-algebra is a special algebra which has three binary operations - meet, multiplication, fuzzy equality - and a top element. This new algebra has been developed particularly for fuzzy type theory (FTT), i.e. higher-order fuzzy logic. Other algebraic structures, such as IMTL-, BL-, MV-algebra (special kinds of residuated lattices) which were used as structures of truth values for FTT till now, are not so natural for it as EQ-algebra. Main reason is that the fundamental connective in EQ-algebra as well as in FTT is fuzzy equality while the fundamental connective in the above mentioned algebras is implication and fuzzy equality (equivalence) is only a derived operation.

EQ-algebra was presented in [1] for the first time. However, the original axioms was slightly modified and the work on this algebra was continued in [3] and also later in [2].

The main aim of this contribution is to present various types of examples of finite EQ-algebras and to show properties which are or are not fulfilled in them. We will take an interest in the following properties of EQ-algebras: spanned, good, involutive, non-residuated, separated, semiseparated and others. We will also show several examples of delta operation and filters of EQ-algebras at the end of my talk.

All examples of EQ-algebras have been found with the help of special software which has been developed in IRAFM of the University of Ostrava. In fact, this software checks axioms of EQ-algebras on finite sets.

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EQ-fuzzy Logics

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Abstract

A new class of fuzzy logics with set of truth values belonging to a class of algebras called EQ-algebras is presented. The latter was introduced for the first time in [9]:

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle.$$

The basic binary operations are meet ($\wedge$), multiplication ($\otimes$), and fuzzy equality ($\sim$). The $\mathbf{1}$ is a top element w.r.t. the classical ordering: $a \leq b$ iff $a \wedge b = a$.

The concept of EQ-algebra is motivated by a fuzzy type theory (FTT) (a higher-order fuzzy logic) since until now, the truth values in FTT were assumed to form either of an IMTL-, BL-, or MV-algebra, all of them being special kinds of residuated lattices in which the basic operations are the monoidal operation (multiplication) and its residuum ($\rightarrow$). Recall that residuum is a natural interpretation of implication in fuzzy logic. The equivalence is then interpreted by a biresiduum $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ which is a derived operation. The basic connective in FTT, however, is a fuzzy equality and, therefore, it is not natural to interpret it by a derived operation. Consequently, residuated lattices seem not to be natural algebras of truth values for FTT. This defect is removed by the metioned class of EQ-algebras. From the algebraic point of view, EQ-algebras generalize, in a certain sense, the residuated lattices and so, they become an interesting class of algebraic structures as such.

A lot of various results obtained recently demonstrate that EQ-algebras have enough properties that make them a proper algebraic structure for the development of formal fuzzy logics which we will call *EQ-fuzzy logics*. The corresponding EQ-algebra must be good, i.e. $a \sim \mathbf{1} = a$ for any $a \in E$. However, the mutliplication $\otimes$ does not need to be commutative, i.e. it can be either a semicopula or a non-commutative associative monoidal operation (cf. [2]).

We will present a basic EQ-fuzzy logic having 12 axioms whose semantics is formed by good semicopula-based EQ-algebras. Then further axioms are introduced which can be added to the basic ones to obtain several more special kinds of EQ-fuzzy logics: falsity $\perp$ together with ex-falso quodlibet axiom (note that the basic EQ-fuzzy logic has no falsity and so, no negation), double negation leading to IEQ-fuzzy logic, prelinearity axiom, commutativity axiom and residuation axiom which, when present,

leads to fuzzy logics equivalent with the core fuzzy logics due to Hájek and Cintula [6]. We will also mention the EQ-fuzzy type theory.

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Applied Commonsense Reasoning in Fuzzy Logic

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In this talk we show how theoretical systems of fuzzy type theory [2], theory of evaluative linguistic expressions [4] and perception-based logical deduction [3] can be applied to solve commonsense problems, namely a detective case of Lt. Columbo. We concentrate mainly on general and mathematical principles of various components mentioned above needed for solving Columbo's case. Moreover, we also discuss the role of nonmonotonical reasoning [1] in this context. Finally we outline some directions of further research.

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On a fuzzy valued measure and integral

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Let L be a completely distributive lattice and L^X be the set of all L-fuzzy subsets of a set X. The operations on L-sets are defined by using the minimum t - norm, the corresponding t - conorm and an involution.

We consider such t-norm based clases of L-sets as semirings and tribes [1], such L-fuzzy valued functions as elementary measures, exterior measures and measures. For our purposes we decided in favour of the L-fuzzy real numbers, as they were first defined by B.Hutton [2].

The aim of the present work is to construct an L-fuzzy valued measure of L-sets [3] and to introduce an L-fuzzy valued integral. We generalize the well known construction of the classic measure theory to the L-fuzzy case.

We construct an L-fuzzy valued measure by extending a given crisp finite measure ν defined on a σ-algebra Φ of crisp subsets of X. In order to do this we describe a semiring and an elementary measure defined on this semiring on the basis of Φ and ν. Then we extend the elementary measure to the exterior measure defined on L^X and obtain the tribe Σ of measurable L-sets. To get the L-fuzzy valued measure we restrict the exterior measure to Σ.

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T-norm based operations with L-fuzzy numbers

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This paper deals with a fuzzy analogue of a real number. In the literature on fuzzy mathematics one can find several different schemes for defining fuzzy numbers. We consider the notion originating to B.Hutton's paper [1] and later developed by several authors (see e.g. [2], [3], [4]).

Let L be a completely distributive lattice with lower and upper bounds $0_L, 1_L \in L$. An L-fuzzy real number is a function $z : \mathbb{R} \to L$ such that

(i) z is non-increasing;

(ii) $\inf_x z(x) = 0_L$, $\sup_x z(x) = 1_L$;

(iii) z is left semi-continuous, i.e. $\inf_{t<x} z(t) = z(x)$.

The set of all fuzzy real numbers is called the fuzzy real line and is denoted by $\mathbb{R}(L)$. The operations of L-fuzzy addition and L-fuzzy multiplication as they are defined in [3] are jointly continuous extensions of addition and multiplication from the real line $\mathbb{R}$ to the L-fuzzy real line $\mathbb{R}(L)$.

The aim of the present paper is to present an alternative definition for arithmetic operations with L-fuzzy numbers. For this aim we use the extension of an aggregation operator by means of a t-norm. Basic algebraic properties of these arithmetic operations are discussed. Examples illustrating the role of a t-norm in the definition of operations are given.

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