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Abstract

On the base of some categorical properties of the power functor for the category of sets, three types of "powerset-like functors" $\mathrm{F} : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ are defined, namely, pre-powerset, powerset and $\mathcal{C}$-powerset functor. A powerset functor for the category $\mathbf{Set}(\Omega)$ of Ω-sets (X, A) over a complete BL-algebra Ω, where $A : X \longrightarrow L$ is a mapping of X to the support of Ω, is introduced. Further, a $\mathcal{C}$-powerset functor for the category $\Omega{-}\mathbf{FSet}$ of Ω-fuzzy sets (X, δ) over a complete BL-algebra, where $\delta : X \times X \longrightarrow L$ is a special similarity relation, is defined. These functors play an important role in the investigation of categorical properties for more complicated functors, which could represent some generalization of the Zadeh's extension principle.

Key words: Category of fuzzy sets, power functors, BL-algebras

1 Introduction

In [10], Zadeh proposed a principle which gives a method how to extend the "crisp" mapping $f : X \longrightarrow Y$ between sets to a "fuzzy" mapping $\hat{f} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ between sets of all fuzzy sets over X and Y with the membership degrees from the interval $[0, 1]$. An analogical extension principle can be also applied in the cases, where we consider the fuzzy sets (more precisely the Ω-fuzzy sets) which membership degrees are interpreted in a truth value structure Ω as e.g. a complete residuated lattice, BL-algebra, MV-algebra, Heyting algebra, or cil-monoid. It is easy to see that if we consider the Boolean

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algebra $\Omega = (\{0,1\}, \vee, \wedge)$, then the mapping $\hat{f}$ obtained by the Zadeh's extension principle coincides with $\mathrm{P}(f)$, where $\mathrm{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ is the common (covariant) power functor from the category **Set**. Since the morphism $\mathrm{P}(f)$ preserves all unions for any mapping f, i.e., $\mathrm{P}(f)(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathrm{P}(f)(A_i)$, then P is also a functor of **Set** to **CSLat** (the category of complete (join) semi-lattices). In [7] (see also [8,9]), Rodabaugh presents several categorical criteria for the functor $\mathrm{P} : \mathbf{Set} \longrightarrow \mathbf{CSLat}$. Their modified versions are as follows: If $\mathrm{G} : \mathbf{CSLat} \longrightarrow \mathbf{Set}$ is the forgetful functor, then

(1) there exists a natural transformation $\eta : \mathrm{I}_{\mathbf{Set}} \longrightarrow \mathrm{G} \circ \mathrm{P}$ (defined by $\eta_X(x) = \{x\}$ for any set X);
(2) $\eta_X : X \to \mathrm{G} \circ \mathrm{P}(X)$ is a universal arrow for any set X;
(3) for any mapping $f : X \longrightarrow Y$ there exists the unique $g : \mathrm{P}(Y) \longrightarrow \mathrm{P}(X)$ such that $\mathrm{P}(f)(A) \leq B$ if and only if $A \leq g(B)$ for any $A \in \mathrm{P}(X)$ and $B \in \mathrm{P}(Y)$, where $\leq$ is the common ordering of sets.

Unfortunately, the Zadeh's extension principle defined as a functor F of **Set** to **CSLat**, which assigns to each set X the set of all Ω-fuzzy sets $\mathcal{F}(X)$, does not satisfy the proposed criteria in general. The reason is simple. There is no natural transformation $\eta : \mathrm{I}_{\mathbf{Set}} \longrightarrow \mathrm{F} \circ \mathrm{P}$ such that η_X is a universal arrow for any set X, if we consider more general truth value structures [1]. Therefore, Rodabaugh proposed to investigate the Zadeh's extension principle, in essence, as a functor $\mathrm{F} : \mathbf{Set} \longrightarrow \mathbf{CSLat}$ such that, for any mapping f in **Set**, the morphism $\mathrm{F}(f)$ lifts $\mathrm{P}(f)$ uniquely and, moreover, there exists the unique lift h of the mapping g that is the right adjoint to $\mathrm{F}(f)$, i.e.,

$$\mathrm{F}(f)(A) \leq B \text{ if and only if } A \leq h(B)$$

is satisfied for any $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, where $\leq$ is the common ordering of Ω-fuzzy sets. Note that the existence and uniqueness of h is closely associated with the α-cut representation of Ω-fuzzy sets (the decomposition theorems) as Rodabaugh presented in [7–9]. Summarizing the previous investigation of Zadeh's extension principle the study of correctness of Zadeh's principle extension can be, in general, divided into two parts. First, this is a correctness of functors (powerset-like functors), which are similar to the power functor P for **Set**, and then a correctness of functors which are "lifts" of powerset-like functors.

Now an interesting question arises, if the Zadeh's extension principle as a functor can be defined in other categories of fuzzy sets in such a way to satisfy the given or modified criteria of correctness. In [5], Močkoř introduced several covariant functors $\mathrm{F} : \Omega{-}\mathbf{FSet} \longrightarrow \Omega{-}\mathbf{FSet}$, which could represent some generalization of Zadeh's extension principle, in the category $\Omega{-}\mathbf{FSet}$ of Ω-

[1] In general, not all Ω-fuzzy sets over X can be expressed by a join of Ω-fuzzy sets $\eta_X(x)$, $x \in X$.

fuzzy sets (X, δ) over MV-algebras, where $\delta : X \times X \longrightarrow L$ is a special similarity relation (see e.g [1,2,4,6]). Their definitions seem to be very natural, but the investigation of their correctness cannot be done by the proposed approach. For example, if we want to follow the proposed idea, then we have to define the category of complete (join) semi-lattices $\mathbf{CSLat}_{\Omega-\mathbf{FSet}}$ over the category $\Omega-\mathbf{FSet}$, i.e., the category, where the objects are complete (join) semi-lattice objects of $\Omega-\mathbf{FSet}$ and the morphisms are such morphisms of $\Omega-\mathbf{FSet}$ which preserves all joins. Moreover, there are some problems, when we want to verify the correctness (after a slight modification) of powerset-like functors in the first part of our investigation and also the correctness of functors which are lifts of powerset-like functors in the second part.

The aim of this paper is to investigate of correctness of powerset-like functors. We propose three types of functors: *pre-powerset*, *powerset* and $\mathcal{C}$-*powerset* functor, where a pre-powerset functor and $\mathcal{C}$-powerset functor ($\mathcal{C}$ denotes a class of subalgebras in which the original category is representable) could be understood as powerset-like functors in the sense of the criteria (1)-(3). These functors then form a base for introducing more general functors representing the Zadeh's extension principle.

The following section is devoted to BL-algebras, which are generated by just one element. In the third section, representations of the categories $\Omega-\mathbf{FSet}$ and $\mathbf{Set}(\Omega)$ are presented. Complete (join) semi-lattice objects in the categories $\Omega-\mathbf{FSet}$ and $\mathbf{Set}(\Omega)$ are illustrated in the fourth section. Final section is devoted to powerset-like functors. We introduce a $\mathcal{C}$-powerset functor for the category $\Omega-\mathbf{FSet}$ and a powerset functor for the category $\mathbf{Set}(\Omega)$.

2 BL-algebras generated by one element

In this paper, the truth value structure will be a complete BL-algebra, i.e., an algebra $\Omega = \langle L, \wedge, \vee, \otimes, \rightarrow, 0_\Omega, 1_\Omega \rangle$, where $\langle L, \wedge, \vee, 0_\Omega, 1_\Omega \rangle$ is a complete lattice, where 0_Ω and 1_Ω denote the least element and the greatest element, respectively, $\langle L, \otimes, 1_\Omega \rangle$ is a commutative monoid and the following conditions are satisfied for any $\alpha, \beta, \gamma \in L$:

(i) $\alpha \otimes \beta \leq \gamma$ if and only if $\alpha \leq \beta \rightarrow \gamma$,
(ii) $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha) = 1_\Omega$,
(iii) $\alpha \otimes (\alpha \rightarrow \beta) = \alpha \wedge \beta$,

where $\leq$ denotes the corresponding lattice ordering. We say that Ω is a linearly ordered or completely distributive BL-algebra, if $\langle L, \wedge, \vee, 0_\Omega, 1_\Omega \rangle$ is a linearly ordered or completely distributive lattice.

Let $\mathbb{N}$ denote the set of all natural numbers (with 0) and $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, where ∞ denotes the symbol of infinity. Let $\leq$ denote the ordering in $\mathbb{N}^*$ and be the natural extension of the ordering in $\mathbb{N}$ (i.e., $n < \infty$ for any $n \in \mathbb{N}$). If Ω is a complete BL-algebra and $\alpha \in L$, then the power of an element α of L is defined by $\alpha^0 = 1_\Omega$, $\alpha^n = \alpha \otimes \alpha^{n-1}$ for any $n \in \mathbb{N} \setminus \{0\}$ and $\alpha^\infty = \bigwedge_{n\in\mathbb{N}} \alpha^n$.

Lemma 1 *Let Ω be a complete BL-algebra and $\alpha \in L$. Then we have*

(i) $\bigwedge_{i\in I} \alpha^{n_i} = \alpha^{\bigvee_{i\in I} n_i}$,
(ii) $\bigvee_{i\in I} \alpha^{n_i} = \alpha^{\bigwedge_{i\in I} n_i}$,
(iii) $\bigvee_{i\in I} \bigwedge_{j\in J} \alpha^{n_{ij}} = \bigwedge_{\gamma\in J^I} \bigvee_{i\in I} \alpha^{n_{i\gamma(i)}}$,
(iv) $\bigwedge_{i\in I} \bigvee_{j\in J} \alpha^{n_{ij}} = \bigvee_{\gamma\in J^I} \bigwedge_{i\in I} \alpha^{n_{i\gamma(i)}}$,

for any index sets I, J.

PROOF. Here, we will prove (i) and (iii). Let $I = \emptyset$. Then $\bigwedge_{i\in I} \alpha^{n_i} = 1_\Omega$. On the other hand, we have $\bigvee_{i\in I} n_i = 0$ and $\alpha^0 = 1_\Omega$. Hence, the equality in (i) is satisfied. Let $I \neq \emptyset$. Since $\alpha^{n_i} \leq \alpha^{n_j}$, whenever $n_j \leq n_i$, we have $\alpha^{\bigvee_{i\in I} n_i} \leq \alpha^{n_j}$ for any $j \in I$. Hence, we obtain $\alpha^{\bigvee_{i\in I} n_i} \leq \bigwedge_{i\in I} \alpha^{n_i}$. Obviously, we have $\bigvee_{i\in I} n_i = n_0 < \infty$ or $\bigvee_{i\in I} n_i = \infty$. In the first case, there exists $j \in I$ such that $n_j = n_0$ and thus $\alpha^{n_j} = \alpha^{\bigvee_{i\in I} n_i} \geq \bigwedge_{i\in I} \alpha^{n_i}$. In the second case, we have $\bigwedge_{i\in I} \alpha^{n_i} = \bigwedge_{n\in\mathbb{N}} \alpha^n = \alpha^\infty = \alpha^{\bigvee_{i\in I} n_i}$, since for any $n \in \mathbb{N}$ there exists $i \in I$ such that $n < n_i$. Hence, the first statement is proved.

In order to prove the third statement it is sufficient to show the following equality

$$\bigwedge_{i\in I} \bigvee_{j\in J} n_{ij} = \bigvee_{\gamma\in J^I} \bigwedge_{i\in I} n_{i\gamma(i)}. \tag{1}$$

The rest of the proof is a straightforward consequent of (i) and (ii). Let $\gamma \in J^I$ be an arbitrary mapping. Then $\bigvee_{j\in J} n_{ij} \geq n_{i\gamma(i)}$ holds for all $i \in I$ and thus $\bigwedge_{i\in I} \bigvee_{j\in J} n_{ij} \geq \bigwedge_{i\in I} n_{i\gamma(i)}$. With regard to an arbitrary choice of γ we obtain $\bigwedge_{i\in I} \bigvee_{j\in J} n_{ij} \geq \bigvee_{\gamma\in J^I} \bigwedge_{i\in I} n_{i\gamma(i)}$. Conversely. First, let us suppose that $\bigvee_{j\in J} n_{ij} = \infty$ holds for all $i \in I$. We can consider two different cases. In the first one, for each $i \in I$, there exists $j \in J$ such that $n_{ij} = \infty$. Here we can obviously define a mapping $\gamma \in J^I$ with $\gamma(i) = j$ if and only if $n_{i\gamma(i)} = \infty$. Hence, we obtain the equality $\bigwedge_{i\in I} \bigvee_{j\in J} n_{ij} = \bigvee_{\gamma\in J^I} \bigwedge_{i\in I} n_{i\gamma(i)} = \infty$. In the second case, we can assume that there exists $i_0 \in I$, where $\bigvee_{j\in J} n_{i_0 j} = \infty$, but $n_{i_0 j} < \infty$ for all $j \in J$. In this case, the following conditions are satisfied

(a) $\forall \gamma \in J^I : \bigwedge_{i\in I} n_{i\gamma(i)} < \infty$,
(b) $\forall \gamma \in J^I \exists \gamma' \in J^I : \bigwedge_{i\in I} n_{i\gamma(i)} < \bigwedge_{i\in I} n_{i\gamma'(i)}$.

Indeed, $n_{i_0\gamma(i_0)} < \infty$ holds for each $\gamma \in J^I$ and (a) is satisfied. Let us suppose that there exists $\gamma^* \in J^I$ such that $\bigwedge_{i\in I} n_{i\gamma(i)} \le \bigwedge_{i\in I} n_{i\gamma^*(i)}$ holds for all mappings $\gamma \in J^I$. Since $\bigvee_{j\in J} n_{ij} = \infty$ for each $i \in I$, we can put $\gamma'(i) = \gamma^*(i)$, if $\gamma^*(i) = \infty$, and $\gamma'(i) = j$, where $n_{i\gamma^*(i)} < n_{ij}$. Evidently, we obtain the mapping $\gamma' \in J^I$ such that $\bigwedge_{i\in I} n_{i\gamma^*(i)} < \bigwedge_{i\in I} n_{i\gamma'(i)}$ holds. Hence, a contradiction and (b) is proved. According to (b), we have $\bigvee_{\gamma\in J^I} \bigwedge_{i\in I} n_{i,\gamma(i)} = \bigvee_{\gamma\in J^I} n_\gamma = \infty$, where $\bigwedge_{i\in I} n_{i\gamma(i)} = n_\gamma$. Further, we suppose that $\bigvee_{j\in J} n_{ij} < \infty$, for some $i \in I$. Let K be a subset of I such that $\bigvee_{j\in J} n_{kj} < \infty$ holds for any $k \in K$ and $\bigvee_{j\in J} n_{lj} = \infty$ holds for any $l \in I \setminus K$. Let us define $n_{kj_k} = \bigvee_{j\in J} n_{kj}$ for all $k \in K$, $n_{lj_l} \ge \bigwedge_{k\in K} n_{kj_k}$ for all $l \in I \setminus K$ and $\gamma^*(i) = j_i$. Then we obtain

$$\bigwedge_{i\in I}\bigvee_{j\in J} n_{ij} \le \bigwedge_{k\in K} n_{kj_k} \wedge \bigwedge_{l\in I\setminus K} n_{lj_l} \le \bigwedge_{i\in I} n_{i\gamma^*(i)} \le \bigvee_{\gamma\in J^I}\bigwedge_{i\in I} n_{i\gamma(i)}.$$

Hence, the equality (1) is satisfied and the proof is finished. □

Let L_α denote the set of all powers of α, i.e., $L_\alpha = \{\alpha^n \mid n \in \mathbb{N}^*\}$. Further, put

$$n \oplus m = \begin{cases} n+m, & n, m \in \mathbb{N}, \\ \infty, & \text{otherwise}, \end{cases} \qquad n \ominus m = \begin{cases} \max(n-m, 0), & n, m \in \mathbb{N}, \\ 0, & m = \infty, \\ \infty, & \text{otherwise}. \end{cases}$$

for any $n, m \in \mathbb{N}^*$.

Theorem 2 *Let $\Omega = \langle L, \wedge, \vee, \otimes, \rightarrow, 0_\Omega, 1_\Omega\rangle$ be a complete BL-algebra and $\alpha \in L$. Then $\Omega_\alpha = \langle L_\alpha, \wedge_\alpha, \vee_\alpha, \otimes_\alpha, \rightarrow_\alpha, \alpha^\infty, 1_\Omega\rangle$, where*

$$\alpha^m \wedge_\alpha \alpha^n = \alpha^m \wedge \alpha^n \tag{2}$$
$$\alpha^m \vee_\alpha \alpha^n = \alpha^m \vee \alpha^n \tag{3}$$
$$\alpha^m \otimes_\alpha \alpha^n = \alpha^{m\oplus n} \tag{4}$$
$$\alpha^m \rightarrow_\alpha \alpha^n = \alpha^{m\ominus n} \tag{5}$$

are defined for any $\alpha^m, \alpha^n \in L_\alpha$, is a complete and completely distributive linearly ordered subalgebra of Ω.

PROOF. One checks easily that α^0 is the greatest element, α^∞ is the least element and the operation $\otimes_\alpha$ is the restriction of $\otimes$ on L_α. Since $(\alpha_1 \otimes \alpha_2) \rightarrow \alpha_3 = \alpha_1 \rightarrow (\alpha_2 \rightarrow \alpha_3)$, then we have $\alpha^m \rightarrow \alpha^n = (\alpha^{m\ominus n} \otimes \alpha^n) \rightarrow \alpha^n = \alpha^{m\ominus n} \rightarrow (\alpha^n \rightarrow \alpha^n) = \alpha^{m\ominus n} = \alpha^m \rightarrow_\alpha \alpha^n$ for any $m, n \in \mathbb{N}^*$. Hence, the operation $\rightarrow_\alpha$ is the restriction of $\rightarrow$ on L_α and thus Ω_α is a BL-subalgebra of Ω. The rest of the proof is a straightforward consequence of Lemma 1 and the fact that $(\mathbb{N}^*, \le)$ is a complete lattice. □

A complete BL-algebra Ω_α defined above is called the *BL-algebra generated by the element* α. It is known that the equality $\beta \otimes \bigwedge_{i\in I} \beta_i = \bigwedge_{i\in I}(\beta \otimes \beta_i)$ for any non-empty I is false in complete BL-algebras in general. Note that this equality is satisfied, for example, in MV-algebras, i.e. BL-algebras, where the law of double negation (i.e., $(\alpha \to 0_\Omega) \to 0_\Omega = \alpha$ for any $\alpha \in L$) is satisfied. BL-algebras generated by one element are further examples of BL-algebras satisfying this equality as the following theorem states.

Theorem 3 *Let Ω be a complete BL-algebra and $\alpha \in L$. Then*

$$\beta \otimes_\alpha \bigwedge_{i\in I} \beta_i = \bigwedge_{i\in I} (\beta \otimes_\alpha \beta_i) \tag{6}$$

holds for any $\beta \in L_\alpha$ and a non-empty index set $\{\beta_i \in L_\alpha \mid i \in I\}$.

PROOF. One checks easily by a direct computation that $n \oplus \bigvee_{i\in I} m_i = \bigvee_{i\in I}(n \oplus m_i)$ holds for any non-empty set I and $n, m_i \in \mathbb{N}^*$, where $i \in I$. Hence, we can write (according to Lemma 1 and Theorem 2)

$$\alpha^n \otimes_\alpha \bigwedge_{i\in I} \alpha^{m_i} = \alpha^n \otimes_\alpha \alpha^{\bigvee_{i\in I} m_i} = \alpha^{n \oplus \bigvee_{i\in I} m_i} =$$
$$\alpha^{\bigvee_{i\in I}(n\oplus m_i)} = \bigwedge_{i\in I} \alpha^{n\oplus m_i} = \bigwedge_{i\in I} (\alpha^n \otimes_\alpha \alpha^{m_i})$$

for any non-empty set I and $n, m_i \in \mathbb{N}^*$, where $i \in I$, and thus (6) is proved. □

3 Categories of fuzzy sets over BL-algebras

This section is devoted to two well-known categories of fuzzy sets, namely, the category $\Omega-$**FSet**, which objects are pairs (X, δ), where $\delta : X \times X \to L$ is a similarity relation, and **Set**(Ω), which objects are pairs (X, A), where $A : X \to L$ is a mapping from X to the support of Ω. We introduce some types of their subobjects and prove that both categories can be represented in the product of their special subcategories in the sense of the following definition.

Definition 4 *Let* **Cat** *be a category with all products and $\mathcal{C} = \{\mathbf{Cat}_i \mid i \in I\}$ be an index family of subcategories of* **Cat**. *We say that* **Cat** *is* representable *in $\mathcal{C}$, if there exists a subcategory* **Rep** *of the product category $\prod_{i\in I} \mathbf{Cat}_i$ which is equivalent to* **Cat**.

It is easy to see that each category **Cat** can be representable in $\mathcal{C} = \{\mathbf{Cat}\}$. This representation of **Cat** is called *trivial*. Otherwise, it is called *nontrivial*.

*3.1 Category $\Omega-$**FSet***

Definition 5 *Let Ω be a complete BL-algebra. An Ω-*valued fuzzy set *(shortly Ω-fuzzy set) is a pair (X, δ), where X is a set and $\delta : X \times X \to \Omega$ is a mapping such that*

$$\begin{aligned} \delta(x, y) &\leq \delta(x, x) \wedge \delta(y, y) && \textit{(strictness)} \\ \delta(x, y) &= \delta(y, x) && \textit{(symmetry)} \\ \delta(x, y) \otimes (\delta(y, y) \to \delta(y, z)) &\leq \delta(x, z) && \textit{(transitivity)} \end{aligned}$$

hold for all $x, y, z \in X$. The mapping δ is called similarity relation.

Definition 6 *The category $\Omega-$**FSet** of Ω-fuzzy sets consists of the following data*

(i) Ω-fuzzy sets as objects,
(ii) structure preserving mappings as morphisms, i.e., $f : (X, \delta) \longrightarrow (Y, \gamma)$ is a morphism, if $f : X \longrightarrow Y$ is a mapping of sets satisfying the following axioms

$$\begin{aligned} \gamma(f(x), f(x)) &\leq \delta(x, x) && \textit{(Strictness)} \\ \delta(x, y) &\leq \gamma(f(x), f(y)) && \textit{(Preservation of Equality)} \end{aligned}$$

for all $x, y \in X$.

If $f : (X, \delta) \longrightarrow (Y, \gamma)$ and $g : (Y, \gamma) \longrightarrow (Z, \rho)$ are two morphisms, then their composition is mapping $g \circ f : X \longrightarrow Z$.

It easy to verify that the composition of morphisms f and g is again a morphism in $\Omega-$**FSet**.

Theorem 7 *The category $\Omega-$**FSet** is complete.*

PROOF. It is obvious that $(L, \wedge)$ is the terminal object To and $(\emptyset, \emptyset)$ is the initial object In. One checks easily that if $\{(X_i, \delta_i) \mid i \in I\}$ is an index set of Ω-fuzzy sets, then

$$\begin{aligned} X &= \{\mathbf{x} \in \prod_{i \in I} X_i \mid (\forall i, j \in I)(\delta_i(x_i, x_i) = \delta_j(x_j, x_j))\}, \\ \delta(\mathbf{x}, \mathbf{y}) &= \bigwedge_{i \in I} \delta_i(x_i, y_i) \end{aligned}$$

and the common set-projections $p_i : \prod_{i \in I} X_i \longrightarrow X_i$ define the product of $\{(X_i, \delta_i) \mid i \in I\}$ in $\Omega-\mathbf{FSet}$. Finally, if

$$(X, \delta) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \gamma),$$

then $B = \{x \in X \mid f(x) = g(x)\}$, $\delta' = \delta|_{B \times B}$ (the restriction of δ on B) and the inclusion mapping $e : B \rightarrowtail X$ define the equalizer of the morphisms f and g in $\Omega-\mathbf{FSet}$. Hence, the category $\Omega-\mathbf{FSet}$ has limits. $\square$

Definition 8 *Let $\Omega = (L, \wedge, \vee, \otimes, \rightarrow, 0_\Omega, 1_\Omega)$ be a complete BL-algebra and $\alpha \in \Omega$. An Ω_α-*valued fuzzy set *(shortly Ω_α-fuzzy set) is an ordered pair (X, δ), where X is a set and $\delta : X \times X \rightarrow L_\alpha$ is a mapping that*

$$\begin{aligned} \delta(x,x) &= \alpha && (\alpha\text{-level property}) \\ \delta(x,y) &\leq \alpha && (\text{strictness}) \\ \delta(x,y) &= \delta(y,x) && (\text{symmetry}) \\ \delta(x,y) \otimes (\alpha \rightarrow \delta(y,z)) &\leq \delta(x,z) && (\alpha\text{-transitivity}) \end{aligned}$$

hold for all $x, y, z \in X$. The mapping δ is called α-similarity relation.

Definition 9 *The category $\Omega_\alpha-\mathbf{FSet}$ consists of the following data*

(i) Ω_α-fuzzy sets as objects,
(ii) $f : (X, \delta) \longrightarrow (Y, \gamma)$ is a morphism, if f is a mapping of X to Y satisfying the preservation of equality.

If $f : (X, \delta) \longrightarrow (Y, \gamma)$ and $g : (Y, \gamma) \longrightarrow (Z, \rho)$ are two morphisms, then their composition is the mapping $g \circ f : X \longrightarrow Z$.

Theorem 10 *Let Ω be a BL-algebra and $\alpha \in L$. Then $\Omega_\alpha-\mathbf{FSet}$ is a complete category that is a full subcategory of $\Omega-\mathbf{FSet}$.*

PROOF. One checks easily that $\mathrm{To} = (\{\alpha\}, \wedge)$ and $\mathrm{In} = (\emptyset, \emptyset)$. The constructions of the products and equalizers are analogous to the constructions presented in the proof of Theorem 7. If $\mathrm{H} : \Omega_\alpha-\mathbf{FSet} \rightarrowtail \Omega-\mathbf{FSet}$ is the inclusion functor and $g : \mathrm{H}(X, \delta) \longrightarrow \mathrm{H}(Y, \gamma)$ is a morphism in $\Omega-\mathbf{FSet}$, then evidently $\mathrm{H}(g) = g$ and thus H is full. Hence, $\Omega_\alpha-\mathbf{FSet}$ is a full subcategory of $\Omega-\mathbf{FSet}$. $\square$

In the following part, we will prove that the category $\Omega-\mathbf{FSet}$ is representable in the family of all subcategories $\Omega_\alpha-\mathbf{FSet}$ in the sense of Definition 4. First, let us define a suitable objects of $\Omega_\alpha-\mathbf{FSet}$ using them the representation will be constructed.

Let X be a set and $R \subseteq X \times X$ be a symmetric relation which satisfies the following additional condition

$$(x, y) \in R \Rightarrow (x, x) \in R \tag{7}$$

for any $x, y \in X$. A relation satisfying the condition (7) is called *conditionally reflexive.* A finite sequence $s = \{s_i\}_{i=1}^n$ such that $s_i = (x_i, x_{i+1}) \in R$ holds for any $i = 1, \ldots, n$, is called a *chain* in R of the *length* n. The length of a chain s is denoted by $|s|$. Let $s = \{s_i\}_{i=1}^n$ be a chain in R. Then s_1 is called the *beginning* and s_n is called the *end* of the chain s. The set of all chains s in R with (x, y) or (y, x) as the beginning of s and (p, q) or (q, p) as the end of s is denoted by $\mathrm{Ch}((x, y), (p, q))$. It is easy to see that for each $(x, y), (p, q) \in R$ we have $s = \{(x, y)\} \in \mathrm{Ch}((x, y), (x, y))$, where $|s| = 1$, and $\mathrm{Ch}((x, y), (p, q)) = \mathrm{Ch}((x, y), (q, p)) = \mathrm{Ch}((y, x), (p, q)) = \mathrm{Ch}((y, x), (q, p))$. Further, there exists a bijective mapping $g : \mathrm{Ch}((x, y), (p, q)) \longrightarrow \mathrm{Ch}((p, q), (x, y))$ such that $|s| = |g(s)|$ for any $s \in \mathrm{Ch}((x, y), (p, q))$. In fact, if $s = \{(x, y), (y, y_1), \ldots, (p, q)\}$ then we put $g(s) = \{(q, p), \ldots, (y_1, y), (y, x)\} \in \mathrm{Ch}((p, q), (x, y))$. Evidently, g is a bijection with the considered condition.

Lemma 11 *Let $(x, y), (p, q), (e, f) \in R$ and $r \in \mathrm{Ch}((x, y), (p, q))$ and $s \in \mathrm{Ch}((p, q), (e, f))$. Then there exists $t \in \mathrm{Ch}((x, y), (e, f))$ such that $|r|+|s| > |t|$.*

PROOF. Let $r \in \mathrm{Ch}((x, y), (p, q))$, $s \in \mathrm{Ch}((p, q), (e, f))$ and $|r| = n$, $|s| = m$. Let us denote $r_{1n} = s_{11} = (p, q)$ and $r_{2n} = s_{21} = (q, p)$ (the possible ends of r and beginnings of s). It is easy to prove that for arbitrary r_{in}, s_{j1}, where $i, j = 1, 2$, there exists $t \in \mathrm{Ch}((x, y), (e, f))$ such that $|r| + |s| > |t|$. $\square$

Lemma 12 *Let $(x, y), (p, q), (e, f) \in R$ and $r \in \mathrm{Ch}((x, y), (p, q))$ and $s \in \mathrm{Ch}((e, f), (p, q))$. Then there exists $t \in \mathrm{Ch}((x, y), (e, f))$ such that $|r|+|s| > |t|$.*

PROOF. Let $s \in \mathrm{Ch}((e, f), (c, d))$. Then there exists $s' \in \mathrm{Ch}((c, d), (e, f))$ such that $|s| = |s'|$. According to lemma 11, there exists $t \in \mathrm{Ch}((a, b), (e, f))$ such that $|r| + |s'| > |t|$ and, hence, $|r| + |s| > |t|$. $\square$

Definition 13 *The category* **Rep** *consists of the following data*

(i) families $\mathbf{R} = \{(R_\alpha, \delta_\alpha) \mid \alpha \in L \ \& \ R_\alpha \subseteq X \times X\}$ as objects, where X is a set, $\delta_\alpha : R_\alpha \times R_\alpha \longrightarrow L_\alpha$ is defined by

$$\forall (x, y), (p, q) \in R_\alpha : \delta_\alpha((x, y), (p, q)) = \bigvee_{s \in \mathrm{Ch}((x,y),(p,q))} \alpha^{|s|}, \tag{8}$$

and, moreover, the following axioms are satisfied for any $\alpha, \beta \in L$ and $x, y, z \in X$

(R1) $R_0 = X \times X$,

(R2) *if* $(x, y) \in R_\alpha$, *then* $(x, x) \in R_\alpha$ *and* $(y, x) \in R_\alpha$ *(i.e.,* R_α *is conditionally reflexive and symmetric relation),*

(R3) *if* $\alpha \leq \beta$, *then* $R_\beta \subseteq R_\alpha$,

(R4) *if*

$$\alpha = \bigvee\{\omega \in L \mid (x, y) \in R_\omega\}, \tag{9}$$

then $(x, y) \in R_\alpha$.

(R5) *if* $(x, y) \in R_\alpha$ *and* $(y, z) \in R_\beta$, *then there exists* $\varepsilon \in L$ *such that* $(x, z) \in R_\varepsilon$ *and*

$$\alpha \otimes ((\bigvee_{\substack{\omega \in L \\ (y,y) \in R_\omega}} \omega) \to \beta) \leq \varepsilon. \tag{10}$$

(ii) a morphism between $\mathbf{R}$ *and* $\mathbf{S}$ *is a family* $\mathbf{f} = \{f_\alpha : R_\alpha \longrightarrow S_\alpha \mid \alpha \in L\}$ *of mappings such that there exists a mapping* $f : X \longrightarrow Y$ *satisfying the following axioms for any* $\alpha \in L$

(R6) $f_\alpha(x, y) = (f(x), f(y))$ *holds for any* $(x, y) \in R_\alpha$,

(R7) $\delta_\alpha((x, y), (p, q)) \leq \gamma_\alpha(f_\alpha(x, y), f_\alpha(p, q))$ *holds for any* $(x, y), (p, q) \in R_\alpha$,

(R8) *the following equality is satisfied*

$$\bigvee\{\alpha \in L \mid (x, x) \in R_\alpha\} = \bigvee\{\beta \in L \mid f_\beta(x, x) \in S_\beta\} \tag{11}$$

for any $x \in X$.

If $\mathbf{f} : \mathbf{R} \longrightarrow \mathbf{S}$ *and* $\mathbf{g} : \mathbf{S} \longrightarrow \mathbf{T}$ *are two morphisms, then their composition is a family* $\mathbf{g} \circ \mathbf{f} = \{g_\alpha \circ f_\alpha : R_\alpha \longrightarrow T_\alpha \mid \alpha \in L\}$ *of mappings.*

One checks easily that $\mathbf{1_R} = \{1_{R_\alpha} \mid \alpha \in L\}$ define the identity morphism in **Rep** and $\mathbf{f} \circ (\mathbf{g} \circ \mathbf{h}) = (\mathbf{f} \circ \mathbf{g}) \circ \mathbf{h}$ is satisfied for any morphisms $\mathbf{h} : \mathbf{R} \longrightarrow \mathbf{S}$, $\mathbf{g} : \mathbf{S} \longrightarrow \mathbf{T}$ and $\mathbf{f} : \mathbf{T} \longrightarrow \mathbf{U}$.

Remark 14 *If* $R_\alpha = \emptyset$, *then evidently* $\delta_\alpha = \emptyset$. *Obviously, the object* $(\emptyset, \emptyset)$ *is the initial object in* $\Omega_\alpha-$**FSet** *(cf. Theorem 7).*

Theorem 15 **Rep** *is a subcategory of the product category* $\prod_{\alpha \in L} \Omega_\alpha-$**FSet**.

PROOF. First, let us prove that $(R_\alpha, \delta_\alpha)$ is an object of $\Omega_\alpha-$**FSet**. Let $(x, y), (p, q), (e, f) \in R_\alpha$. Since $(x, y) \in \mathrm{Ch}((x, y), (x, y))$, then we have

$$\delta_\alpha((x, y), (x, y)) = \bigvee_{s \in \mathrm{Ch}((x,y),(x,y))} \alpha^{|s|} = \alpha^1 = \alpha.$$

Hence, δ_α satisfies α-level property and is strict. If $\mathrm{Ch}((x, y), (p, q)) = \emptyset$, then obviously $\mathrm{Ch}((p, q), (x, y)) = \emptyset$. Hence, $\delta_\alpha((x, y), (p, q)) = \delta_\alpha((p, q), (x, y)) =$

α^∞. If $\mathrm{Ch}((x,y),(p,q)) \neq \emptyset$, then there exists a bijective mapping

$$g : \mathrm{Ch}((x,y),(p,q)) \longrightarrow \mathrm{Ch}((x,y),(p,q))$$

with $|s| = |g(s)|$ and thus δ_α is symmetric. Finally, if $\mathrm{Ch}((x,y),(p,q)) = \emptyset$ or $\mathrm{Ch}((p,q),(e,f)) = \emptyset$, then one checks that

$$\delta_\alpha((x,y),(p,q)) \otimes (\alpha \to \delta_\alpha((p,q),(e,f))) = \alpha^\infty \leq \delta_\alpha((x,y),(e,f))$$

and δ_α is α-transitive. If $\mathrm{Ch}((x,y),(p,q)) \neq \emptyset \neq \mathrm{Ch}((p,q),(e,f))$, then we have

$$\begin{gathered}\delta_\alpha((x,y),(p,q)) \otimes (\alpha \to \delta_\alpha((p,q),(e,f))) = \\ \bigvee_{r\in \mathrm{Ch}((x,y),(p,q))} \alpha^{|r|} \otimes (\alpha \to \bigvee_{s\in \mathrm{Ch}((p,q),(e,f))} \alpha^{|s|}) = \\ \bigvee_{r\in \mathrm{Ch}((x,y),(p,q))} \bigvee_{s\in \mathrm{Ch}((p,q),(e,f))} \alpha^{|r|} \otimes (\alpha \to \alpha^{|s|}) = \\ \bigvee_{r\in \mathrm{Ch}((x,y),(p,q))} \bigvee_{s\in \mathrm{Ch}((p,q),(e,f))} \alpha^{|r|+|s|-1} \leq \bigvee_{t\in \mathrm{Ch}((x,y),(e,f))} \alpha^{|t|} = \delta_\alpha((x,y),(e,f))\end{gathered}$$

where the inequality follows from Lemmas 11 and 12. Hence, δ_α is α-transitive and $(R_\alpha, \delta_\alpha)$ is an object of $\Omega_\alpha-$**FSet**. Further, let us prove that f_α is a morphism of $\Omega_\alpha-$**FSet**. This is, however, a straightforward consequence of (R7) and the fact that $\delta_\alpha((x,y),(x,y)) = \alpha = \gamma_\alpha((p,q),(p,q))$ for any $(x,y) \in R_\alpha$ and $(p,q) \in S_\alpha$. Moreover, it is easy to see that 1_{R_α} defines the identity morphism $1_{(R_\alpha,\delta_\alpha)}$ and if $f_\alpha : (R_\alpha,\delta_\alpha) \longrightarrow (S_\alpha,\gamma_\alpha)$ and $g_\alpha : (S_\alpha,\gamma_\alpha) \longrightarrow (T_\alpha,\vartheta_\alpha)$ are two morphisms, then the composition $g_\alpha \circ f_\alpha : (R_\alpha,\delta_\alpha) \longrightarrow (T_\alpha,\vartheta_\alpha)$ is again a morphism in $\Omega_\alpha-$**FSet**. Hence, **Rep** is a subcategory of the product category $\prod_{\alpha\in L} \Omega_\alpha-$**FSet**. □

Lemma 16 *Let (X,δ) be any object of $\Omega-$**FSet**. Then $\mathbf{R} = \{(R_\alpha,\delta_\alpha) \mid \alpha \in \Omega\}$, where $R_\alpha = \{(x,y) \in X\times X \mid \delta(x,y) \geq \alpha\}$ and $\delta_\alpha : R_\alpha \times R_\alpha \longrightarrow L_\alpha$ is defined by (8), is an object of the category* **Rep**.

PROOF. Obviously, $R_0 = X \times X$, $R_\alpha \subseteq X \times X$ and if $(x,y) \in R_\alpha$, then $\delta(x,x) \geq \delta(y,x) = \delta(x,y) \geq \alpha$. Hence, **R** is well defined and (R1) and (R2) are satisfied. If $\alpha \geq \beta$ and $(x,y) \in R_\alpha$, then $\delta(x,y) \geq \alpha \geq \beta$ and thus $(x,y) \in R_\beta$. Hence, (R3) is satisfied. Let $x,y \in X$ be arbitrary elements. If $(x,y) \in R_\omega$, then $\delta(x,y) = \alpha \geq \omega$. Hence, we obtain $\delta(x,y) \geq \bigvee\{\omega \in L \mid (x,y) \in R_\omega\}$. Since $(x,y) \in R_\alpha$, then $\delta(x,y) \leq \bigvee\{\omega \in L \mid (x,y) \in R_\omega\}$ and thus $\delta(x,y) = \bigvee\{\omega \in L \mid (x,y) \in R_\omega\}$. A straightforward consequence of this equality is the validity of (R4). Let $\alpha,\beta \in L$, $(x,y) \in R_\alpha$ and $(y,z) \in R_\beta$. Put $\varepsilon = \alpha \otimes (\delta(y,y) \to \beta)$. It is easy to see that

$$\delta(x,z) \geq \delta(x,y) \otimes (\delta(y,y) \to \delta(y,z)) \geq \alpha \otimes ((\bigvee_{\substack{\omega\in L\\(y,y)\in R_\omega}} \omega) \to \beta) = \varepsilon,$$

where $\delta(y,y) = \bigvee\{\omega \in L \mid (y,y) \in R_\omega\}$ was proved above. Then $(x,z) \in R_\varepsilon$ and ε fulfills (10). Hence, (R5) is satisfied and **R** is an object of **Rep**. □

Lemma 17 *Let* $\mathbf{f} : \mathbf{R} \longrightarrow \mathbf{S}$ *be a morphism in* **Rep**. *Then there exists the unique mapping* $f : X \longrightarrow Y$*, where* X *and* Y *are the corresponding sets to* **R** *and* **S**, *respectively.*

PROOF. Let $\mathbf{f}$ be determined by two mappings $f, g : X \longrightarrow Y$. Then $(f(x), f(x)) = f_0(x,x) = g_0(x,x) = (g(x), g(x))$ holds for any $x \in X$ and thus $f = g$. Hence, there exists the unique mapping f determining $\mathbf{f}$. □

Theorem 18 *The categories* $\Omega-$**FSet** *and* **Rep** *are equivalent.*

PROOF. First, let us define a covariant functor $\mathrm{F} : \Omega-\mathbf{FSet} \longrightarrow \mathbf{Rep}$ as follows

(i) $\mathrm{F}(X,\delta) = \{(R_\alpha, \delta_\alpha) \mid \alpha \in L\}$, where $R_\alpha = \{(x,y) \in X \times X) \mid \delta(x,y) \geq \alpha\}$ and δ_α is defined by (8),

(ii) $\mathrm{F}(f) = \{f_\alpha \mid \alpha \in L\}$, where $f_\alpha : R_\alpha \longrightarrow S_\alpha$ defined by $f_\alpha(x,y) = (f(x), f(y))$.

According to Lemma 16, the family $\mathrm{F}(X,\delta)$ is an object of **Rep**. Let $f : (X,\delta) \longrightarrow (Y,\gamma)$ be a morphism in $\Omega-$**FSet**. If $R_\alpha = \emptyset$, then $f_\alpha = \emptyset$. If $(x,y) \in R_\alpha$, then $\gamma(f(x), f(y)) \geq \delta(x,y) \geq \alpha$ which implies $f_\alpha(x,y) \in S_\alpha$. Hence, f_α is correctly defined. Moreover, (R6) is trivially satisfied. If $s = \{(x_1,x_2), \ldots, (x_{n-1}, x_n)\}$ is a chain in R_α, then

$$f_\alpha(s) = \{f_\alpha(x_1,x_2), \ldots, f_\alpha(x_{n-1}, x_n)\}$$

is a chain in S_α. Hence, we obtain

$$\begin{aligned}\delta_\alpha((x,y),(p,q)) &= \bigvee_{s \in \mathrm{Ch}((x,y),(p,q))} \alpha^{|s|} \\ &\leq \bigvee_{f(s) \in \mathrm{Ch}(f_\alpha(x,y)), f_\alpha(p,q))} \alpha^{|f(s)|} \leq \gamma_\alpha(f_\alpha(x,y), f_\alpha(p,q))\end{aligned}$$

and (R7) is satisfied. Since $\delta(x,x) = \gamma(f(x), f(x))$, then

$$\bigvee\{\alpha \in L \mid (x,x) \in R_\alpha\} = \bigvee\{\beta \in L \mid f_\beta(x,x) \in S_\beta\}.$$

Hence, (R8) is satisfied and $\mathrm{F}(f)$ is a morphism in **Rep**. Let us consider two morphisms $f : (X,\delta) \longrightarrow (Y,\gamma)$ and $g : (Y,\gamma) \longrightarrow (Z,\vartheta)$ of $\Omega-$**FSet**. Obviously, we have $(g \circ f)_\alpha(x,y) = ((g \circ f)(x), (g \circ f)(y)) = g_\alpha(f(x), f(y)) = (g_\alpha \circ f_\alpha)(x,y)$ for any $(x,y) \in R_\alpha$ and, hence, $\mathrm{F}(g \circ f) = \mathrm{F}(g) \circ \mathrm{F}(f)$. Moreover, if $1_{(X,\delta)}$ is the identity morphism, then we can write $(1_{(X,\delta)})_\alpha(x,y) =$

$(1_{(X,\delta)}(x), 1_{(X,\delta)}(y)) = (x, y)$ for any $\alpha \in L$ and $(x, y) \in R_\alpha$. Hence, $\mathrm{F}(1_{(X,\delta)}) = \mathbf{1}_{\mathrm{F}(X,\delta)}$ and F is a covariant functor.

Further, let us define a covariant functor $\mathrm{G} : \mathbf{Rep} \longrightarrow \Omega-\mathbf{FSet}$ as follows

(i) $\mathrm{G}(\mathbf{R}) = (X, \delta)$, where $R_0 = X \times X$ and $\delta : X \times X \to L$ is defined by $\delta(x, y) = \bigvee\{\alpha \in L \mid (x, y) \in R_\alpha\}$,

(ii) $\mathrm{G}(\mathbf{f}) = f$, where $f(x) = f_{0_\Omega}(x, x)$.

Since $(x, y) \in R_\alpha$ implies $(x, x) \in R_\alpha$ and $(y, x) \in R_\alpha$, then we have

$$\delta(x, y) = \bigvee\{\alpha \in L \mid (x, y) \in R_\alpha\} \leq \bigvee\{\alpha \in L \mid (x, x) \in R_\alpha\} = \delta(x, x)$$

and similarly

$$\delta(x, y) = \bigvee\{\alpha \in L \mid (x, y) \in R_\alpha\} = \bigvee\{\alpha \in L \mid (y, x) \in R_\alpha\} = \delta(y, x).$$

Hence, the relation δ is strict and symmetric. Let $x, y, z \in X$, then

$$\delta(x, y) \otimes (\delta(y, y) \to \delta(y, z)) = \Big(\bigvee_{\substack{\alpha \in L \\ (x,y) \in R_\alpha}} \alpha\Big) \otimes \Big(\bigvee_{\substack{\omega \in L \\ (y,y) \in R_\omega}} \omega \to \bigvee_{\substack{\beta \in L \\ (y,z) \in R_\beta}} \beta\Big) =$$
$$\bigvee_{\substack{\alpha \in L \\ (x,y) \in R_\alpha}} \bigvee_{\substack{\beta \in L \\ (y,z) \in R_\beta}} (\alpha \otimes ((\bigvee_{\substack{\omega \in L \\ (y,y) \in R_\omega}} \omega) \to \beta)) \leq \bigvee_{\substack{\varepsilon \in L \\ (x,z) \in R_\varepsilon}} \varepsilon = \delta(x, z),$$

where the inequality follows from (R5). Hence, (X, δ) is an object of $\Omega-\mathbf{FSet}$. According to Lemma 17, $\mathrm{F}(\mathbf{f})$ is the unique mapping which corresponds to $\mathbf{f}$. Let $x, y \in X$, then we have

$$\delta(x, x) = \bigvee_{\substack{\alpha \in L \\ (x,x) \in R_\alpha}} \alpha = \bigvee_{\substack{\beta \in L \\ f_\beta(x,x) \in S_\beta}} \beta = \gamma(f(x), f(x)) = \gamma(\mathrm{G}(\mathbf{f})(x), \mathrm{G}(\mathbf{f})(x))$$

according to (R8) and

$$\delta(x, y) = \bigvee_{\substack{\alpha \in L \\ (x,y) \in R_\alpha}} \delta_\alpha((x, y), (x, y)) \leq \bigvee_{\substack{\alpha \in \Omega \\ (x,y) \in R_\alpha}} \gamma_\alpha(f_\alpha(x, y), f_\alpha(x, y)) \leq$$
$$\bigvee_{\substack{\beta \in L \\ f_\beta(x,y) \in S_\alpha}} \gamma_\beta(f_\beta(x, y), f_\beta(x, y)) = \gamma(f(x), f(y)) = \gamma(\mathrm{G}(\mathbf{f})(x), \mathrm{G}(\mathbf{f})(y)).$$

Hence, $\mathrm{G}(\mathbf{f})$ is a morphism in $\Omega-\mathbf{FSet}$. Moreover, obviously $\mathrm{G}(\mathbf{g} \circ \mathbf{f}) = g \circ f = \mathrm{G}(\mathbf{g}) \circ \mathrm{G}(\mathbf{f})$ and $\mathrm{G}(\mathbf{1_R}) = 1_X = 1_{\mathrm{G}(\mathbf{R})}$. Thus G is a covariant functor.

Finally, we will prove that $\mathrm{G} \circ \mathrm{F} = 1_{\Omega-\mathbf{FSet}}$ and $\mathrm{F} \circ \mathrm{G} = 1_{\mathbf{Rep}}$. Let (X, δ) be an arbitrary object of $\Omega-\mathbf{FSet}$. Obviously, $\mathrm{G} \circ \mathrm{F}(X, \delta) = (X, \delta')$. Let $x, y \in X$. We have shown that $\delta(x, y) = \bigvee\{\omega \in L \mid (x, y) \in R_\omega\}$, where $R_\omega = \{(x, y) \mid \delta(x, y) \geq \alpha\}$ (see the proof of Theorem 16). If $f : (X, \delta) \to (Y, \gamma)$ is a morphism in $\Omega-\mathbf{FSet}$, then trivially $\mathrm{G} \circ \mathrm{F}(f) = f$ and thus $\mathrm{G} \circ \mathrm{F} = 1_{\Omega-\mathbf{FSet}}$.

Let $\mathbf{R} = \{(R_\alpha, \delta_\alpha) \mid \alpha \in L\}$ be an object in **Rep** and let $\mathrm{F}\circ\mathrm{G}(\mathbf{R}) = \{(R'_\alpha, \delta'_\alpha) \mid \alpha \in L\}$. If $(x,y) \in R_\alpha$, then $\delta(x,y) = \bigvee\{\omega \in L \mid (x,y) \in R_\omega\} \geq \alpha$ and thus $(x,y) \in R'_\alpha$, i.e. $R_\alpha \subseteq R'_\alpha$. If $(x,y) \in R'_\alpha$ and $\delta(x,y) = \beta$, then $(x,y) \in R_\beta$, according to (R4), and thus $(x,y) \in R_\alpha$, according to (R3), i.e. $R'_\alpha \subseteq R_\alpha$. Hence, we obtain $R_\alpha = R'_\alpha$ and also $\delta_\alpha = \delta'_\alpha$, according to the definition. If $\mathbf{f} : \mathbf{R} \longrightarrow \mathbf{S}$ is a morphism in **Rep**, then trivially $\mathrm{F} \circ \mathrm{G}(\mathbf{f}) = \mathbf{f}$ and thus $\mathrm{F} \circ \mathrm{G} = 1_{\mathbf{Rep}}$. Thus the categories $\Omega-$**FSet** and **Rep** are equivalent and the proof is finished. □

Theorem 19 *$\Omega-$**FSet** is representable in $\mathcal{C} = \{\Omega_\alpha-$**FSet** $\mid \alpha \in L\}$.*

PROOF. This is a straightforward consequence of Theorem 18. □

3.2 Category $\mathbf{Set}(\Omega)$

Definition 20 *Let $\Omega = (L, \wedge, \vee, \otimes, \rightarrow, 0_\Omega, 1_\Omega)$ be a complete residuated lattice. An Ω*-valued set *(shortly Ω-set) is an ordered pair (X, A), where X is a set and $A : X \to L$ is a mapping.*

Definition 21 *The category $\mathbf{Set}(\Omega)$ consists of the following data*

(i) Ω-sets as objects,
(ii) $f : (X, A) \to (Y, B)$ is a morphism, if $f : X \to Y$ is a mapping such that $A(x) \leq B(f(x))$ holds for any $x \in X$.

If $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ are two morphisms, then their composition is usual composition of mappings $g \circ f : X \longrightarrow Z$.

Theorem 22 *The category $\mathbf{Set}(\Omega)$ is complete.*

PROOF. It is obvious that a pair $(\{0\}, \chi_{\{0\}})$ is the terminal object and the pair $(\emptyset, \emptyset)$ is the initial object of $\mathbf{Set}(\Omega)$. One checks easily that is $\{(X_i, A_i) \mid i \in I\}$ is a family of Ω-sets, then $(\prod_{i\in I} X_i, A)$, $A(\mathbf{x}) = \bigwedge_{i\in I} A_i(x_i)$ and the common set-projections define the product of $\{(X_i, A_i) \mid i \in I\}$. Finally, if

$$(X, A) \underset{g}{\overset{f}{\rightrightarrows}} (Y, B),$$

then $Z = \{x \in X \mid f(x) = g(x)\}$, $C = A|_Z$ (the restriction of A on Z) and the inclusion mapping $e : Z \rightarrowtail X$ define the equalizer of the morphisms f and g in $\mathbf{Set}(\Omega)$. Hence, the category $\mathbf{Set}(\Omega)$ has limits. □

Definition 23 *Let* $\Omega = (L, \wedge, \vee, \otimes, \rightarrow, 0_\Omega, 1_\Omega)$ *be a complete residuated lattice and* $\alpha \in L$*. An* Ω_α-set *is a ordered pair* (X, A)*, where* X *is a set and* $A : X \longrightarrow L_\alpha$*.*

Definition 24 *The category* $\mathbf{Set}(\Omega_\alpha)$ *consists of the following data*

(i) Ω_α*-sets as objects,*
(ii) $f : (X, A) \longrightarrow (Y, B)$ *is a morphism, if* $f : X \longrightarrow Y$ *is a mapping with* $A(x) \leq B(f(x))$*.*

If $f : (X, A) \longrightarrow (Y, B)$ *and* $g : (Y, B) \longrightarrow (Z, C)$ *are two morphisms, then their composition is usual composition of mappings* $g \circ f : X \longrightarrow Z$*.*

Theorem 25 *Let* Ω *be a BL-algebra and* $\alpha \in L$*. Then* $\mathbf{Set}(\Omega_\alpha)$ *is a complete category that is a full subcategory of the category* $\mathbf{Set}(\Omega)$*.*

PROOF. It is obvious. □

Let us introduce a special subcategory **Rep** of the category $\prod_{\alpha \in L} \mathbf{Set}(\Omega_\alpha)$ that will represent the category $\mathbf{Set}(\Omega)$.

Definition 26 *The category* **Rep** *consists of the following data*

(i) *families* $\mathbf{X} = \{(X_\alpha, A_\alpha) \mid \alpha \in L\}$ *as objects, where* X_α *is a set,* $A_\alpha : X_\alpha \longrightarrow \{\alpha\}$ *and the following axioms are satisfied for any* $\alpha, \beta \in L$ *and* $x \in X$
  (S1) $X_\alpha \subseteq X_\beta$*, whenever* $\alpha \geq \beta$*,*
  (S2) if

$$\alpha = \bigvee \{\omega \in L \mid x \in X_\omega\}, \tag{12}$$

  then $x \in X_\alpha$*,*

(ii) *a family* $\mathbf{f} = \{f_\alpha : X_\alpha \longrightarrow Y_\alpha \mid \alpha \in L\}$ *of mappings is a morphism, if there exists a mapping* $f : X_{0_\Omega} \longrightarrow Y_{0_\Omega}$ *such that* $f_\alpha = f|_{X_\alpha}$ *for any* $\alpha \in L$*.*

The composition of morphisms is defined as the composition of the corresponding mappings.

Theorem 27 **Rep** *is a subcategory of the product category* $\prod_{\alpha \in L} \mathbf{Set}(\Omega_\alpha)$*.*

PROOF. It is obvious. □

Theorem 28 *The categories* $\mathbf{Set}(\Omega)$ *and* **Rep** *are equivalent.*

PROOF. First, let us introduce a covariant functor $\mathrm{F} : \mathbf{Set}(\Omega) \longrightarrow \mathbf{Rep}$ as follows

(i) $F(X, A) = \{(X, A_\alpha) \mid \alpha \in L\}$, where $X_\alpha = \{x \in X \mid A(x) \geq \alpha\}$ and $S_\alpha(x) = \alpha$ for any $x \in X_\alpha$,

(ii) $F(f) = \{f_\alpha \mid \alpha \in L\}$, where $f_\alpha = f|_{X_\alpha}$.

Obviously, $F(X, A)$ is an object of $\mathbf{Set}(\Omega)$. Let $f : (X, A) \longrightarrow (Y, B)$ be a morphism in $\mathbf{Set}(\Omega)$ and $\alpha \in L$. Since $f_\alpha(x) = f(x)$ for any $x \in X_\alpha$ and $A(x) \leq B(f(x))$, then obviously $A_\alpha(x) \leq B_\alpha(f(x))$. One checks that F is a covariant functor.

Further, let us define a covariant functor $\mathrm{G} : \mathbf{Rep} \longrightarrow \mathbf{Set}(\Omega)$ as follows

(i) $\mathrm{G}(\mathbf{X}) = (X, A)$, where $X = X_0$ and $A : X \longrightarrow L$ is defined by

$$A(x) = \bigvee_{\substack{\alpha \in L \\ x \in X_\alpha}} A_\alpha(x), \tag{13}$$

(ii) $\mathrm{G}(\mathbf{f}) = f_0$.

Obviously, $\mathrm{G}(\mathbf{X})$ is a object of $\mathbf{Set}(\Omega)$. Let $\mathbf{f} : \mathbf{X} \longrightarrow \mathbb{Y}$ be an arbitrary morphism and $x \in X$. Then we have

$$A(x) = \bigvee_{\substack{\alpha \in L \\ x \in X_\alpha}} A_\alpha(x) \leq \bigvee_{\substack{\alpha \in L \\ f(x) \in Y_\alpha}} B_\alpha(f_\alpha(x)) = B(\mathrm{G}(\mathbf{f})(x)),$$

since $A_\alpha(x) \leq B_\alpha(f_\alpha(x))$ holds for any $x \in X_\alpha$. Hence, $\mathrm{G}(\mathbf{f})$ is a morphism in $\mathbf{Set}(\Omega)$. One checks easily that G is a covariant functor.

Finally, we will prove that $\mathrm{G} \circ \mathrm{F} = 1_{\mathbf{Set}(\Omega)}$ and $\mathrm{F} \circ \mathrm{G} = 1_{\mathbf{Rep}}$. Let (X, A) be an arbitrary object of $\mathbf{Set}(\Omega)$ and put $\mathrm{G} \circ \mathrm{F}(X, A) = (X', A')$. Obviously, $X' = X_0 = X$. Since $A(x) \geq A_\alpha(x)$ for any $\alpha \in L$ such that $x \in X_\alpha$, then

$$A(x) \geq \bigvee_{\substack{\alpha \in L \\ x \in X_\alpha}} A_\alpha(x)$$

holds for any $x \in X$. The inverse inequality is a direct consequence of $x \in X_{A(x)}$. Hence, we obtain

$$A(x) = \bigvee_{\substack{\alpha \in L \\ x \in X_\alpha}} A_\alpha(x) = A'(x)$$

for any $x \in X$ and thus $\mathrm{G} \circ \mathrm{F}(X, A) = (X, A)$. According to the definitions of the functors F and G, we have $\mathrm{G} \circ \mathrm{F}(f) = f$ and thus $\mathrm{G} \circ \mathrm{F} = 1_{\mathbf{Set}(\Omega)}$. Let $\mathbf{X} = \{(X_\alpha, A_\alpha) \mid \alpha \in L\}$ be an object of $\mathbf{Rep}$ and put $\mathrm{F} \circ \mathrm{G}(\mathbf{X}) = \{(X'_\alpha, A'_\alpha) \mid \alpha \in L\}$. It is easy to see that $X_\alpha \subseteq X'_\alpha$ for any $\alpha \in L$. Let $x \in X'_\alpha$. Then

$A(x) \geq \alpha$ implies $x \in X_\alpha$, where $(S2)$ is used in the case that $A(x) = \alpha$. Hence, $X_\alpha = X'_\alpha$. Since $A_\alpha(x) = A'_\alpha(x) = \alpha$ for any X_α, we have $\mathrm{F} \circ \mathrm{G}(\mathbf{X}) = \mathbf{X}$. Obviously, we have $\mathrm{F} \circ \mathrm{G}(\mathbf{f}) = F(f_0) = \mathbf{f}$ and thus $\mathrm{F} \circ \mathrm{G} = 1_{\mathbf{Rep}}$. Hence, the categories $\mathbf{Set}(\Omega)$ and $\mathbf{Rep}$ are equivalent and the proof is finished. □

Theorem 29 $\mathbf{Set}(\Omega)$ *is representable in* $\mathcal{C} = \{\mathbf{Set}(\Omega_\alpha) \mid \alpha \in L\}$.

PROOF. This is a straightforward consequence of Theorem 28. □

4 Complete semi-lattice objects in categories of fuzzy sets

4.1 Basic definitions

In the following part, we will present an internal definition of complete semi-lattice objects in a category $\mathbf{Cat}$. Let us suppose that a complete category $\mathbf{Cat}$ is given.

Definition 30 *Let* X *be an object of* $\mathbf{Cat}$*. A pair* $(X, \vee)$*, where* $\vee : X \times X \longrightarrow X$ *is a morphism of* $\mathbf{Cat}$*, is a* semi-lattice object *of the category* $\mathbf{Cat}$*, if the following diagrams (axioms) are satisfied*

$$\begin{array}{ccccc} (X \times X) \times X & \xrightarrow{\vee \times 1_X} & X \times X & \xrightarrow{\vee} & X \\ \cong \downarrow & & & & \downarrow 1_X \\ X \times (X \times X) & \xrightarrow{1_X \times \vee} & X \times X & \xrightarrow{\vee} & X \end{array} \qquad \textit{(associativity)}$$

$$\begin{array}{ccc} X \times X & \xrightarrow{\vee} & X \\ r \downarrow & & \downarrow 1_X \\ X \times X & \xrightarrow{\vee} & X \end{array} \qquad \textit{(commutativity)}$$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ 1_X \downarrow & & \downarrow \vee \\ X & \xrightarrow{1_X} & X \end{array} \qquad \textit{(idempotence)}$$

where r *is the twist morphism, i.e.,* $\pi_1 \circ r = \pi_2$ *and* $\pi_2 \circ r = \pi_1$ *hold, and* Δ *is the diagonal morphism, i.e.,* $\pi_1 \circ \Delta = \pi_2 \circ \Delta = 1_X$ *holds.*

Definition 31 *Let* $(X, \vee)$ *be a semi-lattice object in* $\mathbf{Cat}$ *and* To *be the terminal object. We say that* $(X, \vee)$ *has the* least element *or* greatest element*, if* *there exist the unique morphisms* $\bot : \mathrm{To} \longrightarrow X$ *or* $\top : \mathrm{To} \longrightarrow X$ *such that, for any morphism* $x : \mathrm{To} \longrightarrow X$*, the diagrams*

$$\begin{array}{ccc} \mathrm{To} \times \mathrm{To} & \xrightarrow{\bot \times x} & X \times X \\ {\scriptstyle \bot \times x}\downarrow & & \downarrow{\scriptstyle \vee} \\ X \times X & \xrightarrow{p_2} & X \end{array} \quad or \quad \begin{array}{ccc} \mathrm{To} \times \mathrm{To} & \xrightarrow{\top \times x} & X \times X \\ {\scriptstyle \top \times x}\downarrow & & \downarrow{\scriptstyle \vee} \\ X \times X & \xrightarrow{p_1} & X, \end{array} \tag{14}$$

where p_2 *or* p_1 *are the corresponding projections, commute, respectively. A semi-lattice object with the least or greatest element is denoted by* $(X, \vee, \bot)$ *or* $(X, \vee, \top)$*, respectively.*

Note that if we consider a semi-lattice object $(X, \vee)$ in the category of sets such that there exists $\bot : \{x\} \longrightarrow X$ holding $\bot(x) \vee y = y$ for any $y \in X$, then $\bot(x)$ is the least element in X. Moreover, the morphism $\vee$ represents a join operation. Hence, a semi-lattice object $(X, \vee, \bot)$ could be also interpreted as a *join semi-lattice object with the least element*. Let us define complete (join) semi-lattice objects in **Cat**. Let I be a non-empty set and X^I denote a product of I copies of X, i.e., $X^I = \prod_I X$. Note that X^I is usually called a *power* of X (see [3]). For any $i \in I$, we define a morphism (an *i-diagonal morphism*) $\xi_i : X^I \to X \times X^I$ such that the following diagrams commute:

$$\begin{array}{ccccc} X \times X^I & \xleftarrow{\xi_i} & X^I & \xrightarrow{\xi_i} & X \times X^I \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle 1_{X^I}}\downarrow & & \downarrow{\scriptstyle p_2} \\ X & \xleftarrow{q_i} & X^I & \xrightarrow{1_{X^I}} & X^I \end{array} \tag{15}$$

where p_1, p_2 and q_i are the projections of the corresponding products. If X is a set and $\{a_j\}_{j \in I} \in X^I$, then obviously $\xi_i((a_j)_{j\in I}) = (a_i, (a_j)_{j\in I})$.

Definition 32 *A semi-lattice object* $(X, \vee, \bot)$ *in* **Cat** *is* complete*, if, for the empty set* I*,* $\bigvee_I = \bot$ *and, for any non-empty set* I*, there exists a morphism* $\bigvee_I : X^I \longrightarrow X$ *such that*

(i) the diagram

$$\begin{array}{ccccc} X^I & \xrightarrow{\xi_i} & X \times X^I & \xrightarrow{1_X \times \bigvee_I} & X \times X \\ {\scriptstyle \bigvee_I}\downarrow & & & & \downarrow{\scriptstyle \vee} \\ X & & \xrightarrow{\quad 1_X \quad} & & X, \end{array} \tag{16}$$

commutes for any $i \in I$*,*

(ii) if $f : X^I \longrightarrow X$ is a morphism satisfying (i), then the diagram

$$\begin{array}{ccccc} X^I & \xrightarrow{\Delta} & X^I \times X^I & \xrightarrow{\bigvee_I \times f} & X \times X \\ {\scriptstyle \bigvee_I}\downarrow & & & & \downarrow{\scriptstyle \vee} \\ X & & \xrightarrow{\quad 1_X \quad} & & X. \end{array} \tag{17}$$

commutes.

Remark 33 *If $(X, \vee, \perp)$ is a (join) semi-lattice object in the category of sets, then (i) states that for any non-empty set I there exists an upper bound $\bigvee_{i\in I} x_i$ (i.e., $x_i \leq_X \bigvee_{i\in I} x_i$ for any $i \in I$) and (ii) states that this upper bound is the least upper bound (i.e., if there is $x \in X$ such that $a_i \leq_X x$ for any $i \in I$, then $\bigvee_{i\in I} x_i \leq_X x$).*

Definition 34 *A category $\mathbf{CSLat_{Cat}}$ (*complete semi-lattice category over **Cat***) consists of the following data*

*(i) complete (join) semi-lattice objects $(X, \vee, \perp)$ of the category **Cat** as objects,*

*(ii) $f : (X, \vee_X, \perp_X) \longrightarrow (Y, \vee_Y, \perp_Y)$ is a morphism, if $f : X \longrightarrow Y$ is a morphism of **Cat** such that the diagrams (the first one for any non-empty set I)*

$$\begin{array}{ccc} X^I & \xrightarrow{\bigvee_{X,I}} & X \\ {\scriptstyle \prod_I f}\downarrow & & \downarrow{\scriptstyle f} \\ Y^I & \xrightarrow{\bigvee_{Y,I}} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{To} & \xrightarrow{\perp_X} & X \\ & {\scriptstyle \perp_Y}\searrow & \downarrow{\scriptstyle f} \\ & & Y. \end{array} \tag{18}$$

commute.

*4.2 Complete join semi-lattice objects in $\Omega-$**FSet***

Let (X, δ) be an object of $\Omega-\mathbf{FSet}$. An Ω-subset of (X, δ) is an Ω-fuzzy set $\mathbb{A} = (A, \delta_A)$, where $A \subseteq X$ and $\delta_A = \delta|_A$ is the restriction of δ on A. The set of all Ω-subsets of (X, δ) is denoted by $\mathrm{Sub}(X, \delta)$. In the following text, for simplicity, we omit the index A in δ_A and write only δ. Let $\widehat{\delta} : \mathrm{Sub}(X, \delta)^2 \longrightarrow L$ be a relation defined by

$$\begin{aligned} \hat{\delta}(\mathbb{A}, \mathbb{B}) = \bigvee_{a\in A} \delta(a, a) \otimes \bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a, x) \to \bigvee_{b\in B} \delta(b, x)) \wedge \\ \bigvee_{b\in B} \delta(b, b) \otimes \bigwedge_{y\in X} (\bigvee_{b\in B} \delta(b, y) \to \bigvee_{a\in A} \delta(a, y)). \end{aligned} \tag{19}$$

for any $\mathbb{A}, \mathbb{B} \in \mathrm{Sub}(X, \delta)$.

Lemma 35 *Let (X, δ) be an object of $\Omega-$**FSet**. Then the pair $(\mathrm{Sub}(X, \delta), \hat{\delta})$, where $\hat{\delta}$ is defined by (19), is an object of the category $\Omega-$**FSet**.*

PROOF. One check easily that $\hat{\delta}(\mathbb{A}, \mathbb{A}) = \bigvee_{a\in A} \delta(a, a)$. Hence, we have

$$\hat{\delta}(\mathbb{A},\mathbb{A}) \wedge \hat{\delta}(\mathbb{B},\mathbb{B}) = \bigvee_{a\in A} \delta(a,a) \wedge \bigvee_{b\in A} \delta(b,b) \geq \hat{\delta}(\mathbb{A},\mathbb{B})$$

for any $\mathbb{A}, \mathbb{B} \in \mathrm{Sub}(X, \delta)$ and thus $\hat{\delta}$ is strict. Since the definition of $\hat{\delta}$ is symmetric, $\hat{\delta}$ is symmetric. Since $a \otimes (a \to b) = a \wedge b$ and $(a \to b) \otimes (b \to c) \leq a \to c$ hold for any $a, b, c \in L$, then we have

$$\begin{gathered}
\hat{\delta}(\mathbb{A},\mathbb{B}) \otimes (\hat{\delta}(\mathbb{B},\mathbb{B}) \to \hat{\delta}(\mathbb{B},\mathbb{C})) \leq \\
\Big(\big(\bigvee_{a\in A} \delta(a,a) \otimes \bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a,x) \to \bigvee_{b\in B} \delta(b,x))\big) \wedge \bigvee_{b\in B} \delta(b,b)\Big) \otimes \\
\Big(\bigvee_{b\in B} \delta(b,b) \to (\bigvee_{b\in B} \delta(b,b) \otimes \bigwedge_{y\in X} (\bigvee_{b\in B} \delta(b,y) \to \bigvee_{c\in C} \delta(c,y)))\Big) \leq \\
\bigvee_{b\in B} \delta(b,b) \otimes \Big(\bigvee_{b\in B} \delta(b,b) \to (\bigvee_{a\in A} \delta(a,a) \otimes \bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a,x) \to \bigvee_{b\in B} \delta(b,x))\Big) \otimes \\
\Big(\bigvee_{b\in B} \delta(b,b) \to (\bigvee_{b\in B} \delta(b,b) \otimes \bigwedge_{y\in X} (\bigvee_{b\in B} \delta(b,y) \to \bigvee_{c\in C} \delta(c,y)))\Big) = \\
\Big(\bigvee_{b\in B} \delta(b,b) \to (\bigvee_{a\in A} \delta(a,a) \otimes \bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a,x) \to \bigvee_{b\in B} \delta(b,x)))\Big) \otimes \\
\bigvee_{b\in B} \delta(b,b) \otimes \bigwedge_{y\in X} (\bigvee_{b\in B} \delta(b,y) \to \bigvee_{c\in C} \delta(c,y)) \leq \\
\bigvee_{a\in A} \delta(a,a) \otimes \Big(\bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a,x) \to \bigvee_{b\in B} \delta(b,x)) \otimes \bigwedge_{y\in Y} (\bigvee_{b\in B} \delta(b,y) \to \bigvee_{c\in C} \delta(c,y))\Big) \leq \\
\bigvee_{a\in A} \delta(a,a) \otimes \bigwedge_{x\in X} \Big((\bigvee_{a\in A} \delta(a,x) \to \bigvee_{b\in B} \delta(b,x)) \otimes (\bigvee_{b\in B} \delta(b,y) \to \bigvee_{c\in C} \delta(c,y))\Big) \leq \\
\bigvee_{a\in A} \delta(a,a) \otimes \bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a,x) \to \bigvee_{c\in C} \delta(c,x)).
\end{gathered}$$

Analogously, we obtain

$$\hat{\delta}(\mathbb{A},\mathbb{B}) \otimes (\hat{\delta}(\mathbb{B},\mathbb{B}) \to \hat{\delta}(\mathbb{B},\mathbb{C})) \leq \bigvee_{c\in C} \delta(c,c) \otimes \bigwedge_{x\in X} (\bigvee_{c\in C} \delta(c,x) \to \bigvee_{a\in A} \delta(a,x)).$$

Hence, $\hat{\delta}$ is transitive and $(\mathrm{Sub}(X, \delta), \hat{\delta})$ is an object of $\Omega-$**FSet**. $\square$

Let $(Y, \gamma) = (\mathrm{Sub}(X, \delta), \hat{\delta}) \times (\mathrm{Sub}(X, \delta), \hat{\delta})$ denote the product. Let us define a mapping $\cup : Y \longrightarrow \mathrm{Sub}(X, \delta)$ for any $(\mathbb{A}, \mathbb{B}) \in Y$ (recall that $\hat{\delta}(\mathbb{A}, \mathbb{A}) = \hat{\delta}(\mathbb{B}, \mathbb{B})$ holds in this case) as follows

$$\mathbb{A} \cup \mathbb{B} = (A, \delta) \cup (B, \delta) = (A \cup B, \delta). \tag{20}$$

Lemma 36 *The mapping* $\cup$ *defined by (20) is a morphism in* $\Omega-\mathbf{FSet}$.

PROOF. Let us put $(Y,\gamma) = (\mathrm{Sub}(X,\delta),\hat{\delta}) \times (\mathrm{Sub}(X,\delta),\hat{\delta})$. It is easy to verify that $\cup : Y \longrightarrow \mathrm{Sub}(X,\delta)$ is a morphism in $\Omega-\mathbf{FSet}$ if and only if the following conditions

$$\hat{\delta}(\mathbb{A}_1,\mathbb{B}_1)\wedge\hat{\delta}(\mathbb{A}_2,\mathbb{B}_2) = \gamma((\mathbb{A}_1,\mathbb{A}_2),(\mathbb{B}_1,\mathbb{B}_2)) \leq \hat{\delta}(\mathbb{A}_1\cup\mathbb{A}_2,\mathbb{B}_1\cup\mathbb{B}_2), \tag{21}$$

$$\hat{\delta}(\mathbb{A}_1,\mathbb{A}_1)\wedge\hat{\delta}(\mathbb{A}_2,\mathbb{A}_2) = \gamma((\mathbb{A}_1,\mathbb{A}_2),(\mathbb{A}_1,\mathbb{A}_2)) \geq \hat{\delta}(\mathbb{A}_1\cup\mathbb{A}_2,\mathbb{A}_1\cup\mathbb{A}_2) \tag{22}$$

hold for any $(\mathbb{A}_1,\mathbb{A}_2),(\mathbb{B}_1,\mathbb{B}_2) \in Y$. Let $(\mathbb{A}_1,\mathbb{A}_2),(\mathbb{B}_1,\mathbb{B}_2) \in Y$ be arbitrary elements. According to the definition of $\hat{\delta}$, we have

$$\bigvee_{a\in A_1}\delta(a,a) = \bigvee_{b\in A_2}\delta(b,b) \quad \text{and} \quad \bigvee_{a\in B_1}\delta(a,a) = \bigvee_{b\in B_2}\delta(b,b). \tag{23}$$

Hence, we can write

$$\begin{aligned}
\hat{\delta}(\mathbb{A}_1,\mathbb{B}_1)\wedge\hat{\delta}(\mathbb{A}_2,\mathbb{B}_2) &= \bigwedge_{i=1}^{2}\Big(\bigvee_{a\in A_i}\delta(a,a)\otimes\bigwedge_{x\in X}(\bigvee_{a\in A_i}\delta(a,x)\to\bigvee_{b\in B_i}\delta(b,x))\wedge \\
&\bigvee_{b\in B_i}\delta(b,b)\otimes\bigwedge_{y\in X}(\bigvee_{b\in B_i}\delta(b,y)\to\bigvee_{a\in A_i}\delta(a,y))\Big) \leq \\
&\Big(\bigvee_{i=1}^{2}\bigvee_{a\in A_i}\delta(a,a)\otimes\bigwedge_{x\in X}\bigwedge_{i=1}^{2}(\bigvee_{a\in A_i}\delta(a,x)\to\bigvee_{i=1}^{2}\bigvee_{b\in B_i}\delta(b,x))\Big)\wedge \\
&\Big(\bigvee_{i=1}^{2}\bigvee_{b\in B_i}\delta(b,b)\otimes\bigwedge_{y\in X}\bigwedge_{i=1}^{2}(\bigvee_{b\in B_i}\delta(b,y)\to\bigvee_{i=1}^{2}\bigvee_{a\in A_i}\delta(a,y))\Big) = \\
&\Big(\bigvee_{i=1}^{2}\bigvee_{a\in A_i}\delta(a,a)\otimes\bigwedge_{x\in X}(\bigvee_{i=1}^{2}\bigvee_{a\in A_i}\delta(a,x)\to\bigvee_{i=1}^{2}\bigvee_{b\in B_i}\delta(b,x))\Big)\wedge \\
&\Big(\bigvee_{i=1}^{2}\bigvee_{b\in B_i}\delta(b,b)\otimes\bigwedge_{y\in X}(\bigvee_{i=1}^{2}\bigvee_{b\in B_i}\delta(b,y)\to\bigvee_{i=1}^{2}\bigvee_{a\in A_i}\delta(a,y))\Big) = \\
&\bigvee_{a\in A_1\cup A_2}\delta(a,a)\otimes\bigwedge_{x\in X}(\bigvee_{a\in A_1\cup A_2}\delta(a,x)\to\bigvee_{b\in B_1\cup B_2}\delta(b,x))\wedge \\
&\bigvee_{b\in B_1\cup B_2}\delta(b,b)\otimes\bigwedge_{y\in X}(\bigvee_{b\in B_1\cup B_2}\delta(b,y)\to\bigvee_{a\in A_1\cup A_2}\delta(a,y)) = \hat{\delta}(\mathbb{A}_1\cup\mathbb{A}_2,\mathbb{B}_1\cup\mathbb{B}_2),
\end{aligned}$$

where the equalities $\alpha\otimes(\alpha_1\wedge\alpha_2) = (\alpha\otimes\alpha_1)\wedge(\alpha\otimes\alpha_2)$ and $(\alpha_1\to\alpha)\wedge(\alpha_2\to\alpha) = (\alpha_1\vee\alpha_2)\to\alpha$ holding in each BL-algebra are applied. Hence, (21) is satisfied. According to (23), we have

$$\begin{aligned}
\hat{\delta}(\mathbb{A}_1,\mathbb{A}_1)\wedge\hat{\delta}(\mathbb{A}_2,\mathbb{A}_2) &= \bigvee_{a\in A_1}\delta(a,a)\wedge\bigvee_{b\in A_2}\delta(b,b) = \\
\bigvee_{a\in A_1}\delta(a,a)\vee\bigvee_{b\in A_2}\delta(b,b) &= \bigvee_{c\in A_1\cup A_2}\delta(c,c) = \hat{\delta}(\mathbb{A}_1\cup\mathbb{A}_2,\mathbb{A}_1\cup\mathbb{A}_2).
\end{aligned}$$

Hence, (22) is satisfied and $\cup$ is a morphism in $\Omega-\mathbf{FSet}$. $\square$

Theorem 37 *A pair* $((\mathrm{Sub}(X,\delta),\hat{\delta}),\cup)$ *is a semi-lattice object in* $\Omega-\mathbf{FSet}$.

PROOF. This is a straightforward consequence of Lemmas 35 and 36 and the fact that $\cup$ is the associative, commutative and idempotent operation. □

Unfortunately, we cannot prove that the pair $((\mathrm{Sub}(X,\delta),\hat{\delta}),\cup)$ is, in general, a complete (join) semi-lattice object of $\Omega-\mathbf{FSet}$. One obstacle is that the equality $\alpha\otimes\bigwedge_{i\in I}\alpha_i=\bigwedge_{i\in I}(\alpha\otimes\alpha_i)$ is not true in all BL-algebras. Hence, we cannot prove that, for each non-empty set I, there exists a morphism $\bigcup_I:(\mathrm{Sub}(X,\delta),\hat{\delta})^I\longrightarrow(\mathrm{Sub}(X,\delta),\hat{\delta})$ in $\Omega-\mathbf{FSet}$ satisfying the considered conditions of Definition 32. A further obstacle is with the construction of the morphism $\bot:\mathrm{To}\longrightarrow(\mathrm{Sub}(X,\delta),\hat{\delta})$ (recall that $\mathrm{To}=(L,\wedge)$). Nevertheless, if we restrict ourselves on the categories $\Omega_\alpha-\mathbf{FSet}$, then the previous construction with a slight modification leads to complete semi-lattice objects. Recall that $\mathrm{To}=(\{\alpha\},\wedge)$ is the terminal object in $\Omega_\alpha-\mathbf{FSet}$ (see Theorem 10).

Theorem 38 *Let* (X,δ) *be an object of* $\Omega_\alpha-\mathbf{FSet}$. *Then the triplet*

$$((\mathrm{Sub}(X,\delta),\hat{\delta}_\alpha),\cup,\bot),$$

where, for any $\mathbb{A},\mathbb{B}\in\mathrm{Sub}(X,\delta)$, *we have*

$$\begin{aligned}\hat{\delta}_\alpha(\mathbb{A},\mathbb{B})=\alpha\otimes\Big(&\bigwedge_{x\in X}(\bigvee_{a\in A}\delta(a,x)\rightarrow\bigvee_{b\in B}\delta(b,x))\wedge\\&\bigwedge_{y\in X}(\bigvee_{b\in B}\delta(b,y)\rightarrow\bigvee_{a\in A}\delta(a,y))\Big)\end{aligned}\tag{24}$$

$\cup$ *is defined by (20) and* $\bot:(\{\alpha\},\wedge)\longrightarrow(\mathrm{Sub}(X,\delta),\hat{\delta}_\alpha)$ *is defined by* $\bot(\alpha)=(\emptyset,\emptyset)$, *is a complete (join) semi-lattice object of the category* $\Omega_\alpha-\mathbf{FSet}$.

PROOF. Since $\delta(a,x),\delta(b,x)\in L_\alpha$ for any $x\in X$, then $\hat{\delta}_\alpha(\mathbb{A},\mathbb{B})\in L_\alpha$ and thus

$$\hat{\delta}_\alpha:\mathrm{Sub}(X,\delta)\times\mathrm{Sub}(X,\delta)\longrightarrow L_\alpha.$$

One checks easily that $\hat{\delta}_\alpha(\mathbb{A},\mathbb{B})\leq\hat{\delta}_\alpha(\mathbb{A},\mathbb{A})=\alpha$ for any $\mathbb{A},\mathbb{B}\in\mathrm{Sub}(X,\delta)$. Hence, $\hat{\delta}_\alpha$ satisfies the α-level property and strictness. Since the definition of $\hat{\delta}_\alpha$ is symmetric, then $\hat{\delta}_\alpha$ is symmetric. Moreover, if we replace $\bigvee_{a\in A}\delta(a,a)$ and $\bigvee_{b\in B}\delta(b,b)$ by α for any $A,B\subseteq X$ (also for $A=\emptyset$ and $B=\emptyset$) in the proof of Lemmas 35 and 36, then we obtain the proof of the transitivity of $\hat{\delta}_\alpha$ and the fact that $\cup$ is a morphism in $\Omega_\alpha-\mathbf{FSet}$. Hence, the pair $((\mathrm{Sub}(X,\delta),\hat{\delta}_\alpha),\cup)$ is a semi-lattice object of $\Omega_\alpha-\mathbf{FSet}$.

Let us prove that the triplet $((\mathrm{Sub}(X,\delta),\hat{\delta}_\alpha),\cup,\bot)$ is a complete (join) semi-lattice object of $\Omega_\alpha-\mathbf{FSet}$. Let us put $(Y,\gamma)=\mathrm{Sub}(X,\delta)^I$ for a non-empty set I and define a mapping $\bigcup_I : Y \longrightarrow \mathrm{Sub}(X,\delta)$ for each $(\mathbb{A}_i)_{i\in I} \in Y$ as follows

$$\bigcup_I(\mathbb{A}_i)_{i\in I} = (\bigcup_{i\in I} A_i, \delta). \tag{25}$$

In the following text, we will write $\bigcup_{i\in I} \mathbb{A}_i$ instead of $\bigcup_I(\mathbb{A}_i)_{i\in I}$. It easy to verify that $\bigcup_I : Y \longrightarrow \mathrm{Sub}(X,\delta)$ is a morphism in $\Omega_\alpha-\mathbf{FSet}$ for any non-empty I if and only if

$$\bigwedge_{i\in I} \hat{\delta}_\alpha(\mathbb{A}_i, \mathbb{B}_i) \le \hat{\delta}_\alpha(\bigcup_{i\in I} \mathbb{A}_i, \bigcup_{i\in I} \mathbb{B}_i) \tag{26}$$

holds for any non-empty index sets $(\mathbb{A}_i)_{i\in I}, (\mathbb{B}_i)_{i\in I} \in Y$. According to Theorem 3, we can write

$$\begin{aligned}\bigwedge_{i\in I} \hat{\delta}_\alpha(\mathbb{A}_i, \mathbb{B}_i) = \bigwedge_{i\in I} \Big((\alpha \otimes \bigwedge_{x\in X} (\bigvee_{a\in A_i} \delta(a,x) \to \bigvee_{b\in B_i} \delta(b,x)))\wedge \\ (\alpha \otimes \bigwedge_{x\in X} (\bigvee_{b\in B_i} \delta(b,x) \to \bigvee_{a\in A_i} \delta(a,x)))\Big) \le \\ \Big(\alpha \otimes \bigwedge_{x\in X} (\bigvee_{i\in I} \bigvee_{a\in A_i} \delta(a,x) \to \bigvee_{i\in I} \bigvee_{b\in B_i} \delta(b,x))\Big)\wedge \\ \Big(\alpha \otimes \bigwedge_{x\in X} (\bigvee_{i\in I} \bigvee_{b\in B_i} \delta(b,x) \to \bigvee_{i\in I} \bigvee_{a\in A_i} \delta(a,x))\Big) = \\ \alpha \otimes \bigwedge_{x\in X} (\bigvee_{a\in\bigcup_{i\in I} A_i} \delta(a,x) \to \bigvee_{b\in\bigcup_{i\in I} B_i} \delta(b,x))\wedge \\ \alpha \otimes \bigwedge_{x\in X} (\bigvee_{b\in\bigcup_{i\in I}} \delta(b,x) \to \bigvee_{a\in\bigcup_{i\in I}} \delta(a,x)) = \hat{\delta}_\alpha(\bigcup_{i\in I} \mathbb{A}_i, \bigcup_{i\in I} \mathbb{B}_i)\end{aligned}$$

for any $(\mathbb{A}_i)_{i\in I}, (\mathbb{B}_i)_{i\in I} \in Y$. Hence, $\bigcup_I : Y \longrightarrow \mathrm{Sub}(X,\delta)$ is a morphism in $\Omega_\alpha-\mathbf{FSet}$ for any non-empty set I. Moreover, for any $(\mathbb{A}_i)_{i\in I} \in Y$ and $j \in I$, we have

$$\mathbb{A}_j \cup \bigcup_{i\in I} \mathbb{A}_i = (A_j \cup \bigcup_{i\in I} A_i, \delta) = (\bigcup_{i\in I} A_i, \delta) = \bigcup_{i\in I} \mathbb{A}_i,$$

which makes the diagram (16) commutative. If $f : Y \longrightarrow \mathrm{Sub}(X,\delta)$ be a morphism making the diagram (16) commutative, then $A_i \subseteq f((A_i)_{i\in I})$ for any $i \in I$ and thus $\bigcup_{i\in I} A_i \subseteq f((A_i)_{i\in I})$, i.e., $\bigcup_{i\in I} \mathbb{A}_i \cup f((\mathbb{A}_i)_{i\in I})) = \bigcup_{i\in I} \mathbb{A}_i$. Hence, the diagram (17) commutes. Since $\wedge(\alpha,\alpha) = \alpha = \delta_\alpha((\emptyset,\emptyset),(\emptyset,\emptyset)) = \delta_\alpha(\bot(\alpha),\bot(\alpha))$, then $\bot$ is a morphism in $\Omega_\alpha-\mathbf{FSet}$. If $x(\alpha) = (A,\delta)$ for a morphism $x : \mathrm{To} \longrightarrow (\mathrm{Sub}(X,\delta), \hat{\delta}_\alpha)$, then $\bot(\alpha) \cup x(\alpha) = (\emptyset,\emptyset) \cup (A,\delta) = (\emptyset \cup A, \delta) = (A,\delta) = x(\alpha)$. Hence, the left diagram in (14) commutes and thus $((\mathrm{Sub}(X,\delta), \hat{\delta}_\alpha), \cup, \bot)$ is a complete object of $\Omega_\alpha-\mathbf{FSet}$. $\square$

Let (X, A) be an object of the category $\mathbf{Set}(\Omega)$. An Ω-*subset* of (X, A) is an Ω-set $\mathbb{Y} = (Y, A_Y)$, where $A_Y = A|_Y$ is the restriction of A on Y. Obviously, if $Y = \emptyset$, then $A_Y = \emptyset$ is the mapping. The set of all Ω-subsets of (X, A) is denoted by $\mathrm{Sub}(X, A)$. In the following text, for simplicity, we will omit the index Y in A_Y and write only A. Let us define a mapping $\hat{A} : \mathrm{Sub}(X, A) \to L$ as follows

$$\hat{A}(Y, A) = \bigwedge_{x \in Y} A(x). \tag{27}$$

Lemma 39 *Let* (X, A) *be an object of* $\mathbf{Set}(\Omega)$*. Then the pair* $(\mathrm{Sub}(X, A), \hat{A})$ *is an object of the category* $\mathbf{Set}(\Omega)$*.*

PROOF. It is obvious. □

Let us put $(Z, B) = (\mathrm{Sub}(X, A), \hat{A})^I$ for a non-empty set I and define a mapping $\bigcup_I : Z \longrightarrow \mathrm{Sub}(X, A)$ for any $(\mathbb{Y}_i)_{i \in I} \in Z$ as follows

$$\textstyle\bigcup_I (\mathbb{Y}_i)_{i \in I} = (\displaystyle\bigcup_{i \in I} Y_i, A). \tag{28}$$

In the following text, we will write $\bigcup_{i \in I} \mathbb{Y}_i$ instead of $\bigcup_I (\mathbb{Y}_i)_{i \in I}$.

Lemma 40 *The mapping* $\bigcup_I$ *defined by (28) is a morphism in* $\mathbf{Set}(\Omega)$ *for any non-empty set* I*.*

PROOF. Let I be a non-empty set. One checks easily that $\bigcup_I$ is a morphism in $\mathbf{Set}(\Omega)$ for any non-empty set I if and only if

$$\bigwedge_{i \in I} \hat{A}(\mathbb{Y}_i) \le \hat{A}(\bigcup_{i \in I} \mathbb{Y}_i) \tag{29}$$

holds for any $(\mathbb{Y}_i)_{i \in I} \in Z$. Let $(\mathbb{Y}_i)_{i \in I} \in Z$ be a non-empty index set. Then

$$\bigwedge_{i \in I} \hat{A}(\mathbb{Y}_i) = \bigwedge_{i \in I} \bigwedge_{y \in Y_i} A(y) = \bigwedge_{y \in \bigcup_{i \in I} Y_i} A(y) = \hat{A}(\bigcup_{i \in I} \mathbb{Y}_i).$$

Hence, $\bigcup_I$ is a morphism in $\mathbf{Set}(\Omega)$ for any non-empty set I. □

Let us put $(Z, B) = (\mathrm{Sub}(X, A), \hat{A}) \times (\mathrm{Sub}(X, A), \hat{A})$ and define a mapping $\cup : Z \longrightarrow \mathrm{Sub}(X, A)$ by $\cup = \bigcup_I$, where $I = \{1, 2\}$.

Theorem 41 *Let (X, A) be an object of* $\mathbf{Set}(\Omega)$*. Then the triplet*

$$((\mathrm{Sub}(X, A), \hat{A}), \cup, \perp),$$

where $\hat{A}$ and $\cup$ are defined above and $\perp : (\{0\}, \chi_{\{0\}}) \longrightarrow (\mathrm{Sub}(X, A), \hat{A})$ is defined by $\perp(0) = (\emptyset, \emptyset)$, is an complete (join) semi-lattice object of $\mathbf{Set}(\Omega)$*.*

PROOF. According to Lemmas 39 and 40, $(\mathrm{Sub}(X, A), \hat{A})$ is an object and $\cup$ is a morphism in $\mathbf{Set}(\Omega)$, respectively. Moreover, $\cup$ is obviously associative, commutative and idempotent and thus $((\mathrm{Sub}(X, A), \hat{A}), \cup)$ is a semi-lattice object in $\mathbf{Set}(\Omega)$.

Further, let us show that $((\mathrm{Sub}(X, A), \hat{A}), \cup, \perp)$ is a complete (join) semi-lattice object in $\mathbf{Set}(\Omega)$. According to Lemma 40, there exists a morphism $\bigcup_I$ in $\mathbf{Set}(\Omega)$ for any non-empty set I. Moreover, this morphism makes evidently the diagrams (16) and (17) commutative. Further, we have $\chi_{\{0\}}(0) = 1_\Omega = \bigwedge_{x \in \emptyset} A(x) = \hat{A}(\emptyset, \emptyset) = \hat{A}(\perp(0))$ and thus $\perp : (\{0\}, \chi_{\{0\}}) \longrightarrow (\mathrm{Sub}(X, A), \hat{A})$ is a morphism in $\mathbf{Set}(\Omega)$. Obviously, if $x(0) = (Y, A)$ defines a mapping $x : (\{0\}, \chi_{\{0\}}) \longrightarrow (\mathrm{Sub}(X, A), \hat{A})$, then $\perp(0) \cup x(0) = (\emptyset, \emptyset) \cup (Y, A) = (\emptyset \cup Y, A) = (Y, A) = x(0)$. Hence, the left diagram in (14) commutes and the triplet $((\mathrm{Sub}(X, A), \hat{A}), \cup, \perp)$ is a complete (join) semi-lattice object in $\mathbf{Set}(\Omega)$. □

5 Powerset-like functors

5.1 Basic definitions

In the following part, we will consider a discrete category $\mathbf{Cat}$, i.e., the category with forgetful functor $\mathrm{Z} : \mathbf{Cat} \to \mathbf{Set}$. Recall that a set P equipped with a reflexive, antisymmetric and transitive binary relation $\leq$ is called *partially order set*. The category of all partially order sets $(P, \leq)$ as objects and order preserving mappings as morphisms will be denoted by **Poset**. It is well known that a partially order set $(P, \leq)$ can be understood as a category, in which the elements of P are objects and there is just one morphism between p and p', if $p \leq p'$. If

$$(P, \leq_P) \underset{g}{\overset{f}{\rightleftarrows}} (Q, \leq_Q) \tag{30}$$

are two order preserving mappings, then they are also covariant functors. It is easy to show that f *(regarded as a functor) is a left adjoint to* g (or equivalently

g *is a right adjoint to* f) if and only if

$$f(p) \leq q \quad \text{if and only if} \quad p \leq g(p). \tag{31}$$

holds for any $p \in P$ and $q \in Q$. If f is a left adjoint to g, then we write $f \vdash g$. For more information about adjoints of functors, we refer to [3].

Definition 42 *A covariant functor* $\mathrm{F} : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *is called a* pre-powerset functor*, if there exist a covariant functor* $\mathrm{G} : \mathbf{Cat} \longrightarrow \mathbf{Poset}$ *such that the following diagram is commutative*

$$\begin{CD} \mathbf{Cat} @>{\mathrm{F}}>> \mathbf{Cat} \\ @V{\mathrm{G}}VV @VV{\mathrm{Z}}V \\ \mathbf{Poset} @>{\mathrm{Z}'}>> \mathbf{Set}, \end{CD} \tag{32}$$

where Z *and* Z' *are the forgetful functors, and a contravariant functor* $\mathrm{G}^{\mathrm{op}} : \mathbf{Cat} \longrightarrow \mathbf{Poset}$ *satisfying the following conditions*

(i) $\mathrm{G}(a) = \mathrm{G}^{\mathrm{op}}(a)$ *holds for any object* a *of* $\mathbf{Cat}$*,*
(ii) $\mathrm{G}^{\mathrm{op}}(f)$ *(regarded as a functor) is the unique right adjoint to* $\mathrm{G}(f)$*, i.e.,* $\mathrm{G}(f) \vdash \mathrm{G}^{\mathrm{op}}(f)$*, for any morphism* f *of* $\mathbf{Cat}$ *.*

Definition 43 *A covariant functor* $\mathrm{F} : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *is called a* powerset functor*, if* F *is a pre-powerset functor and there exists a functor* P *such that the following diagram is commutative*

$$\begin{CD} \mathbf{Cat} @>{\mathrm{F}}>> \mathbf{Cat} \\ @V{\mathrm{P}}VV @VV{\mathrm{I}_{\mathbf{Cat}}}V \\ \mathbf{CSLat}_{\mathbf{Cat}} @>{\mathrm{Z}}>> \mathbf{Cat}, \end{CD} \tag{33}$$

where Z *is the forgetful functor,* $\mathrm{I}_{\mathbf{Cat}}$ *is the identity functor, and* P *is a left adjoint to* Z*.*

Definition 44 *Let* $\mathbf{Cat}$ *be representable in* $\mathcal{C}$*. Then a covariant functor* $\mathrm{F} : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *is called a* $\mathcal{C}$-powerset functor*, if* F *is a pre-powerset functor and* $\mathrm{F}_i : \mathbf{Cat}_i \longrightarrow \mathbf{Cat}_i$ *is a powerset functor for any* $\mathbf{Cat}_i \in \mathcal{C}$*, where* F_i *is determined from the following commutative diagram*

$$\begin{CD} \mathbf{Cat}_i @>{\mathrm{F}_i}>> \mathbf{Cat}_i \\ @VVV @VVV \\ \mathbf{Cat} @>{\mathrm{F}}>> \mathbf{Cat}. \end{CD} \tag{34}$$

Remark 45 *It is easy to see that if* F *is a* $\mathcal{C}$*-powerset functor and* **Cat** *is trivially representable in* $\mathcal{C}$ *then* F *is a powerset functor.*

5.2 *$\mathcal{C}$-powerset functor for* $\Omega-$**FSet**

The aim of this paragraph is to show that $\mathrm{F}: \Omega-\mathbf{FSet} \longrightarrow \Omega-\mathbf{FSet}$ defined by

(i) $\mathrm{F}(X,\delta) = (\mathrm{Sub}(X,\delta),\hat{\delta})$,
(ii) if $f:(X,\delta) \longrightarrow (Y,\gamma)$ is a morphism, then

$$\mathrm{F}(f):(\mathrm{Sub}(X,\delta),\hat{\delta}) \longrightarrow (\mathrm{Sub}(Y,\gamma),\hat{\gamma})$$

is given by $\mathrm{F}(f)(A,\delta) = (f(A),\gamma)$,

is a $\mathcal{C}$-powerset functor. Since the proof of this statement is too long, we divide it to several lemmas and propositions.

Lemma 46 *Let* $f:(X,\delta) \to (Y,\gamma)$ *be a morphism in* $\Omega-$**FSet** *and* $(A,\delta) \in \mathrm{Sub}(X,\delta)$*. Then*

$$\bigvee_{a'\in f(A)} \gamma(a',y) = \bigvee_{x\in X} \Big((\bigvee_{a\in A} \delta(a,x)) \otimes (\gamma(f(x),f(x)) \to \gamma(f(x),y)) \Big) \qquad (35)$$

holds for any $y \in Y$*.*

PROOF. Since $\gamma(f(x),f(y)) \geq \delta(x,y)$ and $\gamma(f(x),f(x)) = \delta(x,x)$ for any $x,y \in X$, then we have

$$\bigvee_{a'\in f(A)} \gamma(a',y) = \bigvee_{a\in A} \gamma(f(a),y)$$
$$\geq \bigvee_{x\in X}\bigvee_{a\in A} \Big(\gamma(f(a),f(x)) \otimes (\gamma(f(x),f(x)) \to \gamma(f(x),y))\Big) \geq$$
$$\bigvee_{x\in X} \Big((\bigvee_{a\in A} \delta(a,x)) \otimes (\gamma(f(x),f(x)) \to \gamma(f(x),y)) \Big) \geq$$
$$\bigvee_{x\in A} \Big((\bigvee_{a\in A} \delta(a,x)) \otimes (\gamma(f(x),f(x)) \to \gamma(f(x),y)) \Big) \geq$$
$$\bigvee_{x\in A} (\delta(x,x) \otimes (\delta(x,x) \to \gamma(f(x),y))) =$$
$$\bigvee_{x\in A} (\delta(x,x) \wedge \gamma(f(x),y)) = \bigvee_{a'\in f(A)} \gamma(a',y),$$

where the inequality $\gamma(x,y) \geq \gamma(x,z) \otimes (\gamma(z,z) \to \gamma(z,y))$ and $\bigvee_{i\in I}(\alpha \otimes a_i) = \alpha \otimes (\bigvee_{i\in I} a_i)$ are used. Hence, the equality (35) is proved. □

Proposition 47 F *is a covariant functor.*

PROOF. Let (X, δ) be an object of $\Omega-$**FSet**. According to Lemma 35, $\mathrm{F}(X, \delta)$ is an object of $\Omega-$**FSet** . Let $f : (X, \delta) \longrightarrow (Y, \gamma)$ be a morphism and (A, δ), $(B, \delta) \in \mathrm{Sub}(X, \delta)$. According to Lemma 46, we can write

$$\bigvee_{a' \in f(A)} \gamma(a', a') \otimes \bigwedge_{y \in Y} \Big(\bigvee_{a' \in f(A)} \gamma(a', y) \to \bigvee_{b' \in f(B)} \gamma(b', y) \Big) =$$

$$\bigvee_{a \in A} \delta(a, a) \otimes \bigwedge_{y \in Y} \Big(\big(\bigvee_{x \in X} (\bigvee_{a \in A} \delta(a, x)) \otimes (\gamma(f(x), f(x)) \to \gamma(f(x), y)) \big) \to$$

$$\big(\bigvee_{z \in X} (\bigvee_{b \in B} \delta(b, z)) \otimes (\gamma(f(z), f(z)) \to \gamma(f(z), y)) \big) \Big) \geq$$

$$\bigvee_{a \in A} \delta(a, a) \otimes \bigwedge_{y \in Y} \bigwedge_{x \in X} \bigvee_{z \in X} \Big(\big((\bigvee_{a \in A} \delta(a, x)) \otimes (\gamma(f(x), f(x)) \to \gamma(f(x), y)) \big) \to$$

$$\big((\bigvee_{b \in B} \delta(b, z)) \otimes (\gamma(f(z), f(z)) \to \gamma(f(z), y)) \big) \Big) \geq$$

$$\bigvee_{a \in A} \delta(a, a) \otimes \bigwedge_{y \in Y} \bigwedge_{x \in X} \Big(\big((\bigvee_{a \in A} \delta(a, x)) \otimes (\gamma(f(x), f(x)) \to \gamma(f(x), y)) \big) \to$$

$$\big((\bigvee_{b \in B} \delta(b, x)) \otimes (\gamma(f(x), f(x)) \to \gamma(f(x), y)) \big) \Big) \geq$$

$$\bigvee_{a \in A} \delta(a, a) \otimes \bigwedge_{y \in Y} \bigwedge_{x \in X} \big(\bigvee_{a \in A} \delta(a, x) \to \bigvee_{b \in B} \delta(b, x) \big) =$$

$$\bigvee_{a \in A} \delta(a, a) \otimes \bigwedge_{x \in X} \big(\bigvee_{a \in A} \delta(a, x) \to \bigvee_{b \in B} \delta(b, x) \big),$$

where $\bigwedge_{i \in I} (\alpha_i \to \alpha) = (\bigvee_{i \in I} \alpha_i) \to \alpha$, $\bigvee_{i \in I} (\alpha \to \alpha_i) \leq \alpha \to (\bigvee_{i \in I} \alpha_i)$ and $(\alpha \to \beta) \leq (\alpha \otimes \gamma) \to (\beta \otimes \gamma)$ are applied. Analogously, we have

$$\bigvee_{b' \in f(B)} \gamma(b', b') \otimes \bigwedge_{y \in Y} \Big(\bigvee_{b' \in f(B)} \gamma(b', y) \to \bigvee_{a' \in f(A)} \gamma(a', y) \Big)$$
$$\geq \bigvee_{b \in B} \delta(b, b) \otimes \bigwedge_{x \in X} \big(\bigvee_{b \in B} \delta(b, x) \to \bigvee_{a \in A} \delta(a, x) \big).$$

Hence, we obtain $\hat{\gamma}(\mathrm{F}(f)(A, \delta), \mathrm{F}(f)(B, \delta)) \geq \hat{\delta}((A, \delta), (B, \delta))$. Moreover, we have

$$\hat{\gamma}(\mathrm{F}(f)(A, \delta), \mathrm{F}(f)(A, \delta)) = \bigvee_{a' \in f(A)} \gamma(a', a') =$$
$$\bigvee_{a \in A} \gamma(f(a), f(a)) = \bigvee_{a \in A} \delta(a, a) = \hat{\delta}((A, \delta), (A, \delta))$$

and thus $\mathrm{F}(f)$ is a morphism in $\Omega-$**FSet**. One checks easily that $\mathrm{F}(f \circ g) = \mathrm{F}(f) \circ \mathrm{F}(g)$ and $\mathrm{F}(1_{(X,\delta)}) = 1_{\mathrm{F}(X,\delta)}$. Hence, F is a covariant functor. □

Proposition 48 F *is a pre-powerset functor.*

PROOF. Let us define $\mathrm{G}, \mathrm{G}^{\mathrm{op}} : \Omega-\mathbf{FSet} \longrightarrow \mathbf{Poset}$ as follows

(i) $\mathrm{G}(X,\delta) = \mathrm{G}^{\mathrm{op}}(X,\delta) = (\mathrm{Sub}(X,\delta), \subseteq)$, where $(A,\delta) \subseteq (B,\delta)$, if $A \subseteq B$,
(ii) if $f : (X,\delta) \longrightarrow (Y,\gamma)$ is a morphism, then

$$\mathrm{G}(f) : (\mathrm{Sub}(X,\delta), \subseteq) \longrightarrow (\mathrm{Sub}(Y,\gamma), \subseteq)$$

is given by $\mathrm{G}(f)(A,\delta) = (f(A),\gamma)$ for any $(A,\delta) \in \mathrm{Sub}(X,\delta)$,
(iii) if $f : (X,\delta) \longrightarrow (Y,\gamma)$ is a morphism, then

$$\mathrm{G}^{\mathrm{op}}(f) : (\mathrm{Sub}(Y,\gamma), \subseteq) \longrightarrow (\mathrm{Sub}(X,\delta), \subseteq)$$

is given by $\mathrm{G}^{\mathrm{op}}(f)(B,\gamma) = (f^{-1}(B),\delta)$ for any $(B,\gamma) \in \mathrm{Sub}(Y,\gamma)$.

Obviously, both mappings $\mathrm{G}(f)$ and $\mathrm{G}^{\mathrm{op}}(f)$ preserve the partial order $\subseteq$ and thus they are morphisms in **Poset**. Moreover, the diagram (32) commutes. Hence, it is easy to see that G is a covariant functor and G^{op} is a contravariant functor of $\Omega-$**FSet** to **Poset**. As we have mentioned $\mathrm{G}(f)$ and $\mathrm{G}^{\mathrm{op}}(f)$ can be understood as functors

$$(\mathrm{Sub}(X,\delta), \subseteq) \underset{\mathrm{G}^{\mathrm{op}}(f)}{\overset{\mathrm{G}(f)}{\rightleftarrows}} (\mathrm{Sub}(Y,\gamma), \subseteq).$$

for any morphism $f : (X,\delta) \longrightarrow (Y,\gamma)$. Let us prove that $\mathrm{G}^{\mathrm{op}}(f)$ is the unique adjoint to $\mathrm{G}(f)$, i.e., for any $(A,\delta) \in \mathrm{Sub}(X,\delta)$ and $(B,\gamma) \in \mathrm{Sub}(Y,\gamma)$, there is

$$\mathrm{G}(f)(A,\delta) \subseteq (B,\gamma) \quad \text{if and only if} \quad (A,\delta) \subseteq \mathrm{G}^{\mathrm{op}}(f)(B,\gamma)$$

and $\mathrm{G}^{\mathrm{op}}(f)$ is the unique functor with such property. Obviously, if $\mathrm{G}(f)(A,\delta) \subseteq (B,\gamma)$, then $f(A) \subseteq B$. Hence, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$ and thus $(A,\delta) \subseteq \mathrm{G}^{\mathrm{op}}(f)(B,\gamma)$. Analogously, we can prove the opposite implication and thus $\mathrm{G}(f) \vdash \mathrm{G}^{\mathrm{op}}(f)$ holds for any morphism f in $\Omega-$**FSet**.

Let $g : (\mathrm{Sub}(Y,\gamma), \subseteq) \longrightarrow (\mathrm{Sub}(X,\delta), \subseteq)$ be a functor such that $\mathrm{G}(f)(A,\delta) \subseteq (B,\gamma)$ if and only if $(A,\delta) \subseteq g(B,\gamma)$. If we put $(A,\delta) = g(B,\gamma)$, then we obtain $\mathrm{G}(f)(g(B,\gamma)) \subseteq (B,\gamma)$. Hence, we obtain $g(B,\gamma) \subseteq \mathrm{G}(f)(B,\gamma)$. Analogously, we can prove that $\mathrm{G}(f)(B,\gamma) \subseteq g(B,\gamma)$ and thus $\mathrm{G}^{\mathrm{op}}(f)$ is the unique right adjoint to $\mathrm{G}(f)$. Hence, F is a pre-powerset functor. □

In the subsection 4.2, we showed that complete (join) semi-lattices objects can be constructed in the subcategories $\Omega_\alpha-$**FSet** (for $\alpha \in L$) of the category $\Omega-$**FSet**. Hence, the functor F cannot be a powerset functor. Nevertheless, we proved that the category $\Omega-\mathbf{FSet}$ is representable in $\mathcal{C} = \{\Omega_\alpha-\mathbf{FSet} \mid \alpha \in L\}$. This enables us to investigate further properties of the functor F over the subcategories $\Omega_\alpha-\mathbf{FSet}$ in the sense of Definition 44. Note that the construction of complete (join) semi-lattice objects in $\Omega-\mathbf{FSet}$ is not the main obstacle in the proof of that F is not the powerset functor. Further problems are, for example, that the structure Ω is not completely distributive or $\delta(x,x) \neq \delta(y,y)$ holds for some $x, y \in X$ in general.

Lemma 49 *Let (X,δ) be an object of $\Omega_\alpha-\mathbf{FSet}$. Then*

$$\delta(a,b) = \alpha \otimes (\bigwedge_{x\in X}(\delta(a,x) \to \delta(b,x)) \wedge \bigwedge_{y\in X}(\delta(b,y) \to \delta(a,y))) \tag{36}$$

holds for any $a, b \in X$.

PROOF. It easy to see that

$$\begin{aligned}\alpha \otimes (\bigwedge_{x\in X}(\delta(a,x) \to \delta(b,x)) \wedge \bigwedge_{x\in X}(\delta(b,x) \to \delta(a,x))) \leq \\ \alpha \otimes \Big((\delta(a,a) \to \delta(b,a)) \wedge (\delta(b,b) \to \delta(a,b))\Big) = \\ \alpha \wedge \delta(a,b) \wedge \delta(b,a) = \delta(a,b).\end{aligned}$$

Conversely. Since $\delta(x,a) \otimes (\alpha \to \delta(a,b)) \leq \delta(b,x)$ is satisfied for any $x \in X$, then we have

$$\alpha \to \delta(a,b) \leq \bigwedge_{x\in X}(\delta(x,a) \to \delta(b,x)) = \bigwedge_{x\in X}(\delta(a,x) \to \delta(b,x))$$

and thus

$$\delta(a,b) = \alpha \otimes (\alpha \to \delta(a,b)) \leq \alpha \otimes \bigwedge_{x\in X}(\delta(a,x) \to \delta(b,x)).$$

Analogously, we have $\delta(a,b) \leq \alpha \otimes \bigwedge_{x\in X}(\delta(b,x) \to \delta(a,x)$. The inequality follows from $\alpha \otimes (\alpha_1 \wedge \alpha_2) = (\alpha \otimes \alpha_1) \wedge (\alpha \otimes \alpha_2)$ holding in each BL-algebra and thus (36) is proved. □

Proposition 50 *Let $\alpha \in L$ and $\mathrm{P} : \Omega_\alpha-\mathbf{FSet} \longrightarrow \mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$ be defined by*

(i) $\mathrm{P}(X,\delta) = ((\mathrm{Sub}(X,\delta),\hat{\delta}),\cup,\perp)$, *where* $((\mathrm{Sub}(X,\delta),\hat{\delta}),\cup,\perp)$ *is defined in Theorem 38,*

(ii) if $f : (X,\delta) \longrightarrow (Y,\gamma)$ is a morphism, then

$$\mathrm{P}(f) : ((\mathrm{Sub}(X,\delta),\hat{\delta}),\cup,\perp_{\mathrm{Sub}(X,\delta)}) \longrightarrow ((\mathrm{Sub}(Y,\gamma),\hat{\gamma}),\cup,\perp_{\mathrm{Sub}(Y,\gamma)})$$

is given by $\mathrm{P}(f)(A,\delta) = (f(A),\gamma)$ for any $(A,\delta) \in \mathrm{Sub}(X,\delta)$.

Then P *is a covariant functor.*

PROOF. According to Theorem 38, $\mathrm{P}(X,\delta)$ is an object of $\mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$. Analogously to the proof of Proposition 47, we can show that $\mathrm{P}(f)$ is a morphism in $\Omega_\alpha-\mathbf{FSet}$ for any morphism f in $\Omega_\alpha-\mathbf{FSet}$. Let $\{(A_i,\delta) \mid i \in I\}$ be a non-empty index set. Then we have

$$\mathrm{P}(f)(\bigcup_{i\in I}(A_i,\delta)) = (f(\bigcup_{i\in I} A_i),\gamma) = (\bigcup_{i\in I} f(A_i),\gamma) = \bigcup_{i\in I} \mathrm{P}(f)(A_i,\delta).$$

Hence, the left diagram in (18) commutes for any non-empty set I. Moreover, we have $f \circ \perp_{\mathrm{Sub}(X,\delta)}(\alpha) = f(\emptyset,\emptyset) = (f(\emptyset),\emptyset) = (\emptyset,\emptyset) = \perp_{\mathrm{Sub}(Y,\gamma)}(\alpha)$ and thus the right diagram in (18) also commutes. Hence, $\mathrm{P}(f)$ is a morphism in $\mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$. One checks easily that $\mathrm{P}(f\circ g) = \mathrm{P}(f)\circ \mathrm{P}(g)$ and $\mathrm{P}(1_{(X,\delta)}) = 1_{\mathrm{P}(X,\delta)}$ and thus P is a covariant functor. □

Proposition 51 *Let* $\mathrm{Z} : \mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}} \longrightarrow \Omega_\alpha-\mathbf{FSet}$ *be the forgetful functor. Then* $\eta : \mathrm{I}_{\Omega_\alpha-\mathbf{FSet}} \longrightarrow \mathrm{Z}\circ\mathrm{P}$ *defined by* $\eta_{(X,\delta)}(x) = (\{x\},\delta)$ *is a natural transformation, i.e., the following diagram*

$$\begin{array}{ccc} (X,\delta) & \xrightarrow{\quad f \quad} & (Y,\gamma) \\ {\scriptstyle \eta_{(X,\delta)}}\downarrow & & \downarrow{\scriptstyle \eta_{(Y,\delta)}} \\ \mathrm{Z}\circ\mathrm{P}(X,\delta) & \xrightarrow{\mathrm{Z}\circ\mathrm{P}(f)} & \mathrm{Z}\circ\mathrm{P}(Y,\gamma) \end{array} \tag{37}$$

is commutative for any morphism f *of the category* $\Omega_\alpha-\mathbf{FSet}$.

PROOF. Let (X,δ) be an object of $\Omega_\alpha-\mathbf{FSet}$ and $a,b\in X$. Then, according to the definition of $\hat{\delta}_\alpha$ in (24) and the equality (36), we have

$$\hat{\delta}_\alpha(\eta_{(X,\delta)}(a),\eta_{(X,\delta)}(b)) = \hat{\delta}_\alpha((\{a\},\delta),(\{b\},\delta)) = \\ \alpha\otimes\Big(\bigwedge_{x\in X}(\delta(a,x)\rightarrow\delta(b,x))\wedge\bigwedge_{y\in X}(\delta(b,y)\rightarrow\delta(a,y))\Big) = \delta(a,b).$$

Moreover, we have

$$\hat{\delta}(\eta_{(X,\delta)}(a),\eta_{(X,\delta)}(a)) = \alpha\otimes\bigwedge_{x\in X}(\delta(a,x)\rightarrow\delta(a,x)) = \alpha\otimes 1_\Omega = \alpha = \delta(a,a)$$

for any $a\in X$. Hence, the mapping $\eta_{(X,\delta)}$ is a morphism of the category $\Omega_\alpha-\mathbf{FSet}$. Further, if $x\in X$, then $\mathrm{Z}\circ\mathrm{P}(f)\circ\eta_{(X,\delta)}(x) = (\{f(x)\},\gamma) = \eta_{(Y,\gamma)}\circ f(x)$. Hence, the diagram (37) commutes and η is a natural transformation. □

Proposition 52 *Let $\alpha \in L$ and (X, δ) be an object of $\Omega_\alpha-$***FSet***. Then the morphism $\eta_{(X,\delta)} : (X, \delta) \longrightarrow \mathrm{Z} \circ \mathrm{P}(X, \delta)$ defined above is a universal arrow, i.e.,*

$$\begin{array}{ccccc} (X,\delta) & \xrightarrow{\eta_{(X,\delta)}} & \mathrm{Z}\circ\mathrm{P}(X,\delta) & \qquad & \mathrm{P}(X,\delta) \\ & \searrow^{\forall f\ (2)} \quad \circ\ (4) & \Big\downarrow \mathrm{Z}(\hat{f}) & & \Big\downarrow \exists!\hat{f}\ (3) \\ & & (Y,\gamma) & & \forall((Y,\gamma),\vee,\perp_Y)\ (1). \end{array} \tag{38}$$

PROOF. Let (X, δ) be an object of $\Omega_\alpha-$**FSet** and $((Y, \gamma), \vee, \perp)$ be an object of $\mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$. If $f : (X, \delta) \longrightarrow (Y, \gamma)$ is a morphism of $\Omega_\alpha-$**FSet**, then let us define

$$\hat{f} : ((\mathrm{Sub}(X, \delta), \hat{\delta}), \cup, \perp_{\mathrm{Sub}(X,\delta)}) \longrightarrow ((Y, \gamma), \vee, \perp_Y)$$

by

$$\hat{f}(A, \delta) = \begin{cases} \bigvee_{a\in A} f(a), \text{ if } A \neq \emptyset, \\ \perp_Y(\alpha), \qquad \text{otherwise.} \end{cases} \tag{39}$$

First, let us show that $\hat{f}$ is a morphism in $\mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$. Let $(A, \delta), (B, \delta) \in \mathrm{Sub}(X, \delta)$ such that $A = \emptyset$ or $B = \emptyset$. If $A = B = \emptyset$, then $\hat{\delta}_\alpha((A, \delta), (B, \delta)) = \alpha = \gamma(\perp(\alpha), \perp(\alpha)) = \gamma(\hat{f}(A, \delta), \hat{f}(B, \delta))$. If $A \neq \emptyset$, then one checks easily that $\hat{\delta}_\alpha((A, \delta), (B, \delta)) = \alpha \otimes (\alpha \to 0_\Omega) = 0_\Omega$. Hence, we obtain $\hat{\delta}_\alpha((A, \delta), (B, \delta)) \leq \gamma(\hat{f}(A, \delta), \hat{f}(B, \delta))$. The same result could be obtained for $B \neq \emptyset$. Now, let us suppose that $(A, \delta), (B, \delta) \in \mathrm{Sub}(X, \delta)$ such that $A \neq \emptyset$ and $B \neq \emptyset$. Then we can write

$$\hat{\delta}_\alpha((A, \delta), (B, \delta)) \leq \alpha \otimes \bigwedge_{x\in X} (\bigvee_{a\in A} \delta(a, x) \to \bigvee_{b\in B} \delta(b, x)) \leq$$
$$\alpha \otimes (\bigwedge_{x\in A} \bigvee_{a\in A} \delta(a, x) \to \bigwedge_{x\in A} \bigvee_{b\in B} \delta(b, x)) =$$
$$\alpha \otimes (\alpha \to \bigwedge_{x\in A} \bigvee_{b\in B} \delta(b, x)) = \bigwedge_{x\in A} \bigvee_{b\in B} \delta(x, b).$$

Analogously, we can prove that $\hat{\delta}_\alpha((A, \delta), (B, \delta)) \leq \bigwedge_{y\in B} \bigvee_{a\in A} \delta(a, y)$. Hence, we obtain

$$\hat{\delta}_\alpha((A, \delta), (B, \delta)) \leq \bigwedge_{a\in A} \bigvee_{b\in B} \gamma(f(a), f(b)) \wedge \bigwedge_{b\in B} \bigvee_{a\in A} \gamma(f(a), f(b)). \tag{40}$$

A straightforward consequence of the fact that, for any non-empty set I, the mapping $\bigvee_I$ is a morphism in $\Omega_\alpha-$**FSet** is the inequality $\bigwedge_{i\in I} \gamma(a_i, b_i) \leq$

$\gamma(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$ holding for any non-empty index sets $\{a_i \in Y \mid i \in I\}$ and $\{b_i \in Y \mid i \in I\}$. Since Ω_α is a complete and completely distributive BL-algebra (see Theorem 2), then

$$\begin{aligned}
\hat{\delta}((A,\delta),(B,\delta)) \leq \bigwedge_{a\in A}\bigvee_{b\in B}\gamma(f(a),f(b)) \wedge \bigwedge_{b\in B}\bigvee_{a\in A}\gamma(f(a),f(b)) = \\
\bigvee_{\pi\in B^A}\bigwedge_{a\in A}\gamma(f(a),f(\pi(a))) \wedge \bigvee_{\varrho\in A^B}\bigwedge_{b\in B}\gamma(f(\varrho(a)),f(b)) \leq \\
\bigvee_{\pi\in B^A}\gamma(\bigvee_{a\in A} f(a), \bigvee_{b\in\pi(A)} f(b)) \wedge \bigvee_{\varrho\in A^B}\gamma(\bigvee_{a\in\varrho(B)} f(a), \bigvee_{b\in B} f(b)) = \\
\bigvee_{\pi\in B^A}\bigvee_{\varrho\in A^B}\Big(\gamma(\bigvee_{a\in A} f(a), \bigvee_{b\in\pi(A)} f(b)) \wedge \gamma(\bigvee_{a\in\varrho(B)} f(a), \bigvee_{b\in B} f(b))\Big) \leq \\
\bigvee_{\pi\in B^A}\bigvee_{\varrho\in A^B}\gamma(\bigvee_{a\in A} f(a) \vee \bigvee_{a\in\varrho(B)} f(a), \bigvee_{b\in\pi(A)} f(b) \vee \bigvee_{b\in B} f(b)) = \\
\bigvee_{\pi\in B^A}\bigvee_{\varrho\in A^B}\gamma(\bigvee_{a\in A} f(a), \bigvee_{b\in B} f(b)) = \gamma(\bigvee_{a\in A} f(a), \bigvee_{b\in B} f(b)) = \gamma(\hat{f}(A,\delta),\hat{f}(B,\delta)).
\end{aligned}$$

Hence, $\hat{f}$ is a morphism in $\Omega_\alpha-\mathbf{FSet}$.

Further, we have

$$\hat{f}(\bigcup_{i\in I}(A_i,\delta)) = \hat{f}(\bigcup_{i\in I} A_i,\delta) = \bigvee_{a\in\bigcup_{i\in I} A_i} f(a) = \bigvee_{i\in I}\bigvee_{a\in A_i} f(a) = \bigvee_{i\in I}\hat{f}(A_i,\delta)$$

for any non-empty set I. Hence, the left diagram in (18) commutes for any non-empty set I. Moreover, $\hat{f}\circ\perp_{\mathrm{Sub}(X,\delta)}(\alpha) = \hat{f}(\emptyset,\emptyset) = \perp_Y(\alpha)$. Hence, the right diagram in (18) commutes and thus $\hat{f}$ is a morphism in $\mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$.

Further, we have $\mathrm{Z}(\hat{f})\circ\eta_{(X,\delta)}(x) = \hat{f}(\{x\},\delta) = \bigvee_{a\in\{x\}} f(a) = f(x)$ which makes the diagram in (38) commutative. Finally, we have to show that the morphism $\hat{f}$ is the unique extension of f. Let $g : \mathrm{P}(X,\delta) \longrightarrow ((Y,\gamma),\vee,\perp)$ be a morphism such that the diagram in (38) is commutative. Then, for any $(A,\delta)\in\mathrm{Sub}(X,\delta)$ such that $(A,\delta)\neq(\emptyset,\emptyset)$, we can write

$$\begin{aligned}
g(A,\delta) = g(\bigcup_{a\in A}(\{a\},\delta)) = \bigvee_{a\in A} g(\{a\},\delta) = \bigvee_{a\in A}\hat{f}(\{a\},\delta) = \\
\hat{f}(\bigcup_{a\in A}(\{a\},\delta)) = \hat{f}(A,\delta).
\end{aligned}$$

Moreover, we have $g(\emptyset,\emptyset) = g(\perp_{\mathrm{Sub}(X,A)}(\alpha)) = g\circ\perp_{\mathrm{Sub}(X,A)}(\alpha) = \perp_Y(\alpha) = \hat{f}(\emptyset,\emptyset)$, since g is a morphism in $\mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$. Hence, $g=\hat{f}$ and the proof is finished. □

Theorem 53 F *is a $\mathcal{C}$-powerset functor.*

PROOF. Obviously, if (X, δ) is an object of $\Omega_\alpha-\mathbf{FSet}$, then $\mathrm{F}(X, \delta)$ is an object of $\Omega_\alpha-\mathbf{FSet}$. Let us denote F_α the restriction of F on $\Omega_\alpha-\mathbf{FSet}$, i.e., $\mathrm{F}_\alpha : \Omega_\alpha-\mathbf{FSet} \longrightarrow \Omega_\alpha-\mathbf{FSet}$ is defined by $\mathrm{F}_\alpha(X, \delta) = \mathrm{F}(X, \delta)$ for any object (X, δ) and $\mathrm{F}_\alpha(f) = \mathrm{F}(f)$ for any morphism f of the category $\Omega_\alpha-\mathbf{FSet}$.

According to Theorem 19 and Proposition 48, the category $\Omega-\mathbf{FSet}$ is representable in $\mathcal{C}$ and F is a pre-powerset functor, respectively. It is easy to see that, for any $\alpha \in L$, there exists a functor $\mathrm{P} : \Omega_\alpha-\mathbf{FSet} \longrightarrow \mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}}$ (according to Proposition 50) making the following diagram

$$\begin{array}{ccc} \Omega_\alpha-\mathbf{FSet} & \xrightarrow{\mathrm{F}_\alpha} & \Omega_\alpha-\mathbf{FSet} \\ \mathrm{P}\big\downarrow & & \big\downarrow \mathrm{I}_{\Omega_\alpha-\mathbf{FSet}} \\ \mathbf{CSLat}_{\Omega_\alpha-\mathbf{FSet}} & \xrightarrow{\mathrm{Z}} & \Omega_\alpha-\mathbf{FSet}, \end{array}$$

commutative. Since a straightforward consequence of Propositions 51 and 52 is that P is a left adjoint to Z (i.e., $\mathrm{P} \vdash \mathrm{Z}$), then F_α is a powerset functor for any $\alpha \in L$. Hence, F is a $\mathcal{C}$-powerset functor and the proof is finished. $\square$

5.3 Powerset functor for $\mathbf{Set}(\Omega)$

Analogously to the previous subsection, the aim of this part is to show that $F : \mathbf{Set}(\Omega) \longrightarrow \mathbf{Set}(\Omega)$ defined by

(i) $\mathrm{F}(X, A) = (\mathrm{Sub}(X, A), \hat{A})$,
(ii) if $f : (X, A) \longrightarrow (Y, B)$ is a morphism, then

$$\mathrm{F}(f) : (\mathrm{Sub}(X, A), \hat{\delta}) \longrightarrow (\mathrm{Sub}(Y, B), \hat{B})$$

is given by $\mathrm{F}(f)(Z, A) = (f(Z), B)$ for any $(Z, A) \in \mathrm{Sub}(X, A)$.

is a powerset functor. Again, the proof is divided to several propositions.

Proposition 54 F *is a covariant functor.*

PROOF. According to Theorem 41, $\mathrm{F}(X, A)$ is an object of $\mathbf{Set}(\Omega)$. If $f : (X, A) \longrightarrow (Y, B)$ is a morphism in $\mathbf{Set}(\Omega)$, then

$$\hat{A}(Z, A) = \bigwedge_{x \in Z} A(x) \le \bigwedge_{x \in Z} A'(f(x)) = \bigwedge_{y \in f(Z)} B(y) = \hat{B}(f(Z), B) = \hat{B}(\mathrm{F}(f)(Z, B))$$

for any $(Z, A) \in \mathrm{Sub}(X, A)$. Hence, $\mathrm{F}(f)$ is a morphism in $\mathbf{Set}(\Omega)$. One checks easily that $\mathrm{F}(f \circ g) = \mathrm{F}(f) \circ \mathrm{F}(g)$ and $\mathrm{F}(1_{(X,A)}) = 1_{\mathrm{F}(X,A)}$. Hence, F is a covariant functor. □

Proposition 55 F *is a pre-powerset functor.*

PROOF. Let us define $\mathrm{G}, \mathrm{G}^{\mathrm{op}} : \mathbf{Set}(\Omega) \longrightarrow \mathbf{Poset}$ as follows

(i) $\mathrm{G}(X, A) = \mathrm{G}^{\mathrm{op}}(X, A) = (\mathrm{Sub}(X, A), \subseteq)$, where $(Z, A) \subseteq (Z', A)$, if $Z \subseteq Z'$,

(ii) if $f : (X, A) \longrightarrow (Y, B)$ is a morphism in $\mathbf{Set}(\Omega)$, then

$$\mathrm{G}(f) : (\mathrm{Sub}(X, A), \subseteq) \longrightarrow (\mathrm{Sub}(Y, B), \subseteq)$$

is given by $\mathrm{G}(f)(Z, A) = (f(Z), B)$ for any $(Z, A) \in \mathrm{Sub}(X, A)$,

(iii) if $f : (X, A) \longrightarrow (Y, B)$ is a morphism in $\mathbf{Set}(\Omega)$, then

$$\mathrm{G}^{\mathrm{op}}(f) : (\mathrm{Sub}(Y, B), \subseteq) \longrightarrow (\mathrm{Sub}(X, A), \subseteq)$$

is given by $\mathrm{G}^{\mathrm{op}}(f)(Z, B) = (f^{-1}(Z), A)$ for any $(Z, B) \in \mathrm{Sub}(Y, B)$.

Obviously, both mappings $\mathrm{G}(f)$ and $\mathrm{G}^{\mathrm{op}}(f)$ preserve partial order $\subseteq$ and thus they are morphisms in **Poset**. Moreover, the diagram (32) commutes. Analogously to the proof of Proposition 48, one verifies that $\mathrm{G}^{\mathrm{op}}(f)$ is the unique right adjoint to $\mathrm{G}(f)$ for any morphism f in $\mathbf{Set}(\Omega)$ and thus F is a pre-powerset functor. □

Proposition 56 *Let* $\mathrm{P} : \mathbf{Set}(\Omega) \longrightarrow \mathbf{CSLat}_{\mathbf{Set}(\Omega)}$ *be defined by*

(i) $\mathrm{P}(X, A) = (\mathrm{Sub}(X, A), \hat{A}, \cup, \perp)$*, where* $(\mathrm{Sub}(X, A), \hat{A}, \cup, \perp)$ *is defined in Theorem 41,*

(ii) if $f : (X, A) \longrightarrow (Y, B)$ *is a morphism, then*

$$\mathrm{P}(f) : ((\mathrm{Sub}(X, A), \hat{A}), \cup, \perp_{\mathrm{Sub}(X,A)}) \longrightarrow ((\mathrm{Sub}(Y, B), \hat{B}), \cup, \perp_{\mathrm{Sub}(Y,B)})$$

is given by $\mathrm{P}(f)(Z, A) = (f(Z), B)$ *for any* $(Z, A) \in \mathrm{Sub}(X, A)$.

Then P *is a covariant functor.*

PROOF. This is analogous to the proof of Proposition 50. □

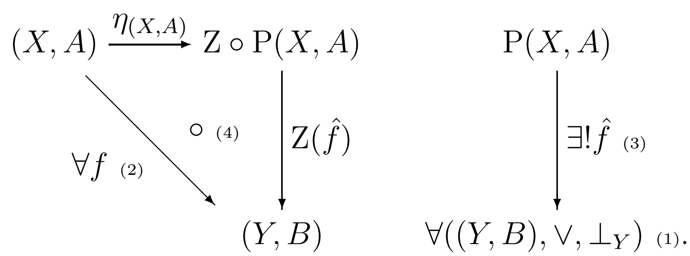
Proposition 57 *Let* $\mathrm{Z} : \mathbf{CSLat}_{\mathbf{Set}(\Omega)} \longrightarrow \mathbf{Set}(\Omega)$ *be the forgetful functor. Then* $\eta : \mathrm{I}_{\mathbf{Set}(\Omega)} \longrightarrow \mathrm{Z} \circ \mathrm{P}$ *defined by* $\eta_{(X,A)}(x) = (\{x\}, A)$ *is a natural transformation.*

PROOF. Let (X, A) be an object of $\mathbf{Set}(\Omega)$ and $x \in X$. Then

$$A(x) = \bigwedge_{y \in \{x\}} A(y) = \hat{A}(\{x\}, A) = \hat{A}(\eta_{(X,A)}(x))$$

and thus $\eta_{(X,A)}$ is a morphism in $\mathbf{Set}(\Omega)$. Moreover, if $f : (X, A) \longrightarrow (Y, B)$ is a morphism in $\mathbf{Set}(\Omega)$ (see the diagram 37 and replace objects of $\Omega_\alpha-\mathbf{FSet}$ by objects of $\mathbf{Set}(\Omega)$), then we have $\eta_{(Y,B)} \circ f(x) = (\{f(x)\}, B) = (f(\{x\}), B) = \mathrm{Z}\circ\mathrm{P}(f)(\{x\}, A) = \mathrm{Z}\circ\mathrm{P}(f)\circ\eta_{(X,A)}(x)$. Hence, η is a natural transformation. $\square$

Proposition 58 *Let* (X, A) *be an object of the category* $\mathbf{Set}(\Omega)$*. Then the morphism* $\eta_{(X,A)} : (X, A) \longrightarrow \mathrm{Z} \circ \mathrm{P}(X, A)$ *is a universal arrow, i.e.,*



(41)

PROOF. Let (X, A) be an object of $\mathbf{Set}(\Omega)$ and $((Y, B), \vee, \perp_Y)$ be an object of $\mathbf{CSLat}_{\mathbf{Set}(\Omega)}$. If $f : (X, A) \longrightarrow (Y, B)$ is a morphism in $\mathbf{Set}(\Omega)$, then we define

$$\hat{f}(Z, A) = \begin{cases} \bigvee_{x \in Z} f(x), & \text{if } Z \neq \emptyset, \\ \perp_Y(0), & \text{otherwise,} \end{cases} \tag{42}$$

for any $(Z, A) \in \mathrm{Sub}(X, A)$. First, we will prove that $\hat{f}$ is a morphism in $\mathbf{CSLat}_{\mathbf{Set}(\Omega)}$. Since f is a morphism in $\mathbf{Set}(\Omega)$ and $\bigwedge_{i \in I} B(y_i) \leq B(\bigvee_{i \in I} y_i)$ holds for any non-empty index set $\{y_i \in Y \mid i \in I\}$, then we can write

$$\hat{A}(Z, A) = \bigwedge_{x \in Z} A(x) \leq \bigwedge_{x \in Z} B(f(x)) \leq B(\bigvee_{x \in Z} f(x)) = B(\hat{f}(Z, A))$$

for any $(Z, A) \in \mathrm{Sub}(X, A) \setminus \{\emptyset, \emptyset\}$. Since $\perp_Y : (\{0\}, \chi_{\{0\}}) \longrightarrow (Y, B)$ is a morphism in $\mathbf{Set}(\Omega)$, then $\chi_{\{0\}}(0) = 1_\Omega \leq B(\perp_Y(0))$. Hence, if $(Z, A) = (\emptyset, \emptyset)$, then $\hat{A}(Z, A) = 1_\Omega = B(\perp_Y(0)) = B(f(Z, A))$ and thus $\hat{f}$ is a morphism in $\mathbf{Set}(\Omega)$.

Further, we have

$$\hat{f}(\bigcup_{i \in I}(X_i, A)) = \hat{f}((\bigcup_{i \in I} X_i, A)) = \bigvee_{x \in \bigcup_{i \in I} X_i} f(x) = \bigvee_{i \in I} \bigvee_{x \in X_i} f(x) = \bigvee_{i \in I} \hat{f}(X_i, A)$$

for any non-empty set I. Hence, the left diagram in (18) commutes for any non-empty set I. Moreover, $\hat{f} \circ \perp_{\mathrm{Sub}(X,A)}(0) = \hat{f}(\emptyset, \emptyset) = \perp_Y(0)$. Hence, the right diagram in (18) commutes and thus $\hat{f}$ is a morphism in $\mathbf{CSLat}_{\mathbf{Set}(\Omega)}$.

Further, we have

$$\mathrm{Z}(\hat{f}) \circ \eta_{(X,A)}(x) = \hat{f}(\{x\}) = \bigvee_{y \in \{x\}} f(y) = f(x)$$

and the diagram in (41) commutes. Finally, we have to prove that $\hat{f}$ is the unique extension of f. Let $g : \mathrm{P}(X, A) \longrightarrow ((Y, B), \vee, \perp_Y)$ be a morphism such that the diagram in (41) is commutative. Then, for any $(Z, A) \in \mathrm{Sub}(X, A)$ such that $(Z, A) = (\emptyset, \emptyset)$, we can write

$$g(Z, A) = g(\bigcup_{x \in Z} (\{x\}, A)) = \bigvee_{x \in Z} g(\{x\}, A) = \bigvee_{x \in Z} f(x) = \hat{f}(Z, A).$$

Moreover, if $(Y, A) = (\emptyset, \emptyset)$, then we have $g(Y, A) = g(\perp_{\mathrm{Sub}(X,A)}(0)) = g \circ \perp_{\mathrm{Sub}(X,A)}(0) = \perp_Y(0) = \hat{f}(Y, A)$, since g is a morphism in $\mathbf{Set}(\Omega)$. Hence, $g = \hat{f}$ and the proof is finished. $\square$

Theorem 59 F *is a powerset functor.*

PROOF. According to Proposition 55, F is a pre-powerset functor. A straightforward consequence of Propositions 57 and 58 is that P is a left adjoint to Z (i.e., $\mathrm{P} \vdash \mathrm{Z}$), then F is a powerset functor for any $\alpha \in L$. Hence, F is a powerset functor and the proof is finished. $\square$

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