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Abstract

Let X denote a locally compact metric space and $\varphi : X \to X$ be a continuous map. In the 1970s L. Zadeh presented an extension principle, helping us to fuzzify the dynamical system (X, φ), i.e., to obtain a map Φ for the space of fuzzy sets on X. We extend an idea mentioned in [P. Diamond, A. Pokrovskii, *Chaos, entropy and a generalized extension principle*, Fuzzy Sets and Systems **61** (1994)] and we generalize Zadeh's original extension principle.

In this paper we study basic properties, such as the continuity of so-called g-fuzzifications. We also show that, for any g-fuzzification: (i) a uniformly convergent sequence of uniformly convergent maps on X induces a uniformly convegent sequence of continuous maps on the space of fuzzy sets, and (ii) a conjugacy (a semi-conjugacy, resp.) between two discrete dynamical systems can be extended to a conjugacy (a semi-conjugacy, resp.) between fuzzified dynamical systems. Moreover, at the end of this paper we show that there are connections between g-fuzzifications and crisp dynamical systems via set-valued dynamical systems and skew-product (triangular) maps.

Throughout this paper we consider different topological structures in the space of fuzzy sets; namely, the sendograph, endograph and levelwise topologies.

1 Introduction

Let X be a locally compact metric space and $\varphi : X \to X$ be continuous. First,we would like to mention that the terminology and notation we employ is introduced in **Section 2**. Commonly used definition says that the pair (X, φ) forms a *discrete dynamical system*. In the seventies L. Zadeh established a concept often called *Zadeh's extension* that allows us to extend the dynamical system (X, φ) to a (self-) map Φ defined on the space $\mathbb{F}(X)$ of fuzzy sets on X. Then, in 1982 P. E. Kloeden ([14]) defined the sendograph metric on $\mathbb{F}(X)$. He also showed that the map $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ is continuous in the metric topology given by the sendograph metric and proved that there is a closed connection between the *original* dynamical system (X, φ) and the *fuzzified* one $(\mathbb{F}(X), \Phi)$. Unfortunately, since Kloeden's work, the studies on the properties of Zadeh's extension and, consequently, fuzzy dynamical systems induced by Zadeh's extension were reported only occasionally (e.g., [3], [14] etc.).

Thus, the author's original purpose was to study how the different dynamical properties of one of the mentioned dynamical systems are influenced by the dynamical behavior of the second system. Such relations were already studied, but the results were mostly proved under specific assumptions and no connections among the obtained results have been given so far − for instance, some results are obtained for Zadeh's extension of the set $\mathbb{R}^n$ of all n-dimensional real numbers (see [2], [5], [11] or [25]), and other results are proved for the case where X is even compact metric space (see [8] or [20]). Further, not only different spaces but also different metrics have been considered - namely, the endograph (see [8] or [20]), sendograph (see [5] or [14]) and levelwise topologies (see [26] or [27]).

It should also be mentioned that the terminology used in the papers mentioned above is not uniform (see [8] or [20]) as the endograph of a fuzzy set A is denoted $send(A)$ − i.e., Kloeden's original idea (*send* to be an abbreviation of a supported endograph) is denied. Consequently, some statements are considered to be valid even if they are wrong; for instance, [5] contains a mistake when considering the compactness of the space of fuzzy sets on X when this space is equipped with a sendograph topology (($\mathbb{F}(X), d_S)$ is not compact); also, [11] contains a wrong statement on completeness of the space of fuzzy sets equipped with an endograph topology (($\mathbb{F}^1(X), \tau_E$) can be complete), and in [27] they report that ($\mathbb{F}(X), d_\infty$) is separable.

Accordingly, we decided to change our original purpose: instead we will first summarize and complete some basic metric and topological properties of the space of fuzzy sets on X (**Section 3**) when a(n) sendograph, endograph or levelwise topology is considered. In the next part of this paper, each property is studied for any of the mentioned topologies.

Further (**Section 4**), we study the continuity of Zadeh's extension for the various topological spaces. Some particular results concerning this topic are already known (see [2] or [25]), and we generalize these results to the case where X is a locally compact metric space (Theorem 5). However, Zadeh's extension can have an unpleasant property: it can lose information that is carried by the original system (X, φ) (see Example 3). This led us to generalize Zadeh's extension principle in a way that allows us to modify fuzzy sets in each iteration. Probably the first attempt to generalize Zadeh's extension appeared in [8] where either t-norm or t-conorm Γ was used to modify fuzzy sets; also, some basic properties of Γ-fuzzifications were shown. However, the continuity of Γ-fuzzifications were not studied.

In Section 4 we discuss some properties of Γ-fuzzifications in order to show what led us to the notion of a generalized fuzzification (abbr., g-fuzzification). The main part of this paper is devoted to the study of generalized fuzzifications of the original dynamical system (X, φ). In Subsection 4.3 we show that any g-fuzzification is well-defined and we also show that, under some assumptions (Theorems 6 and 7), any g-fuzzification Φ_g is continuous on the space of fuzzy sets, i.e. $(\mathbb{F}(X), \Phi_g)$ defines a discrete fuzzy dynamical system that extends the original (crisp) one.

Moreover, we show that any uniformly convergent sequence of "original" maps on X defines a uniformly convergent sequence of g-fuzzifications (**Section 5**). Then in **Section 6** we prove that fuzzy discrete dynamical systems induced by some g-fuzzification are commuting in the sense of Pederson's definition ([20]) and that such systems have the so-called unions-preserving property.

The next section (**Section 7**) is devoted to g-fuzzifications of two dynamical systems, which are either conjugated or semi-conjugated. The notion of (semi-) conjugacy is very important in topological dynamics. Roughly speaking, conjugacy is the following: if two dynamical systems are conjugated, their dynamical behavior is the same; if they are semi-conjugated, then one of them is a (dynamical) factor of the other system and inherits many dynamical properties. We also show that if the original dynamical systems are conjugated (or semi-conjugated), then their generalized fuzzifications are conjugated (or semi-conjugated) (Theorems 9 and 10).

In the last section, (**Section 8**) some conclusions on our results are drawn. We point out that any generalized fuzzification (i.e., a fuzzy discrete dynamical system) is significantly connected to an appropriate crisp discrete dynamical system in two different ways. One way was already illustrated when some relations between the original system and the fuzzified one were analyzed: any g-fuzzified dynamical system is connected via α-cuts to a set-valued discrete dynamical system induced by the same original system (X, φ). The second way is a new result: we show that under some not very restrictive assumptions any g-fuzzified discrete dynamical system $(\mathbb{F}(X), \Phi_g)$ can be considered as a set-valued crisp discrete dynamical system induced by a certain skew-product (or triangular) map (Proposition 4). Thus, fuzzy discrete dynamical systems induced by generalized fuzzifications are connected to the crisp (set-valued and skew-product) discrete dynamical systems that were studied intensively in the last two decades. At the end of this paper we show that it is possible to consider further generalizations (called skew-product or triangular fuzzifications) only when either the endograph or sendograph topology is considered.

Despite the author's original purpose, we do not study the relations between the original dynamical system and the fuzzified system. This paper mainly serves as a background for further study of such relations. For completeness, it should be emphasized that all of our results are also valid for Zadeh's extension and Γ-fuzzifications since they are (except for some superfluous cases) special cases of the studied g-fuzzifications.

2 Definitions and notation

In this paper, $\mathbb{R}^n$, $\mathbb{N}$ denote the set of n-dimentional real numbers and the set of all integers, respectively. Moreover, I is the closed unit interval $[0, 1] \subseteq \mathbb{R}$ and X is a locally compact metric space. Topological closure of a set $A \subseteq X$ is denoted by $\overline{A}$. Further, by $C(X)$ we denote a class of all continuous maps

$\varphi : X \to X$, and by $C_m(I)$ we consider a class of nondecreasing maps from $C(I)$. Finally, $C'_m(I)$ denotes a family of maps from $C_m(I)$ that are strictly increasing at 0.

Later we will need some notions from topological dynamics. For a given dynamical system (X, φ) and a given point $x \in X$, we denote the n-th *iteration* of the point x inductively by $\varphi^0(x) = x$, $\varphi^{n+1}(x) = \varphi(\varphi^n(x))$ for any $n \in \mathbb{N}$. Then, the sequence $\{\varphi^n(x)\}_{n\in\mathbb{N}}$ of all iterations of x is called a *trajectory* of the point x. Any limit point of the trajectory of the point x is an *ω-limit point* of the point x, and the union $\omega_\varphi(x)$ of all ω-limit points of the point x is an *ω-limit set* of the point x. The iteration of a given set $A \subseteq X$ is defined analogously.

The point $x \in X$ is called *fixed* if $\varphi(x) = x$ or *periodic* if $\varphi^k(x) = x$ for some $k \in \mathbb{N}$. We say that a set $A \subseteq X$ is *φ-invariant* (or *invariant*) if $\varphi(A) \subseteq A$. A dynamical system (X, φ) is *transitive* if, for any two open sets $U, V \subseteq X$, there is $n \in \mathbb{N}$ for which $\varphi^n(U) \cap V \neq \emptyset$.

Given two discrete dynamical systems (X, φ), (Y, ψ) are *topologically conjugated* if there exists a homeomorphism $h : X \to Y$, such that

$$h \circ \varphi = \psi \circ h. \tag{2.1}$$

The systems (X, φ), (Y, ψ) are *semi-conjugated* if there exists a continuous surjection $h : X \to Y$ satisfying the same equality (2.1).

The so-called *skew-product* (or *triangular*) map is a continuous map $F : X \times I \to X \times I$ of the form $(x, y) \mapsto (\varphi(x), g_x(y))$ where $g_x \in C(I)$ for any $x \in I$. Usually, all g_x are *fiber* maps and $\varphi \in C(X)$ is a *base* map. The set $I_x = \{x\} \times I$ is called a *fibre* over the point $x \in X$ and $A_x = A \cap I_x$ is called a *fibre of the set* $A \subseteq X \times I$ over the point $x \in X$. $T(X)$ denotes the space of skew-product maps on $X \times I$ and $T_m(X)$ denotes the space of all skew-product maps whose fiber maps g_x are nondecreasing.

2.1 Metrics spaces of fuzzy sets

Let (X, d) denote a locally compact metric space and let A, B be non-empty closed subsets of X. The *Hausdorff metric* between A and B is usually defined by

$$D_X(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq U_\varepsilon(B) \text{ and } B \subseteq U_\varepsilon(A)\},$$

where

$$U_\varepsilon(A) = \{x \in X \mid D(x, A) < \varepsilon\}, \text{ and } D(x, A) = \inf_{a\in A} d(x, a).$$

By $\mathbb{K}(X)$ we denote the space of all nonempty compact subsets of X equipped by the Hausdorff metric D_X. It is well known ([17]) that $(\mathbb{K}(X), D_X)$ is compact, complete and separable whenever X is compact, complete and separable.

A fuzzy set A on the space X is a function $A : X \to I$. An *α-cut* (or an *α-level set*) $[A]_\alpha$ and a *support* *supp*(A) of a given fuzzy set A is defined usually as -

$$[A]_\alpha = \{x \in X \mid A(x) \geq \alpha\}, \ \alpha \in [0, 1],$$

and

$$supp(A) = \overline{\{x \in X \mid A(x) > 0\}}.$$

Further, we define $\mathbb{F}(X)$ as the system of all upper semi-continuous fuzzy sets $A : X \to I$ having compact supports. Moreover, let

$$\mathbb{F}^\lambda(X) = \{A \in \mathbb{F}(X) \mid A(x) \geq \lambda \text{ for some } x \in X\}$$

and $\mathbb{F}^1(X)$ denotes the system of all *normal* fuzzy sets on X. Finally, we define $\emptyset_X$ as the *empty fuzzy set* ($\emptyset_X(x) = 0$ for each $x \in X$) on the space X and by $\mathbb{F}_0(X)$ the system of all nonempty fuzzy sets.

For any $A \in F(X)$,

$$end(A) = \{(x, a) \in X \times I \mid A(x) \geq a\}$$

the *endograph* and by

$$send(A) = end(A) \cap (supp(A) \times I)$$

denotes *sendograph* (like *supported endograph*) of the fuzzy set A.

Since the fuzzy sets we consider are upper semi-continuous, all sendographs are members of the metric space $(\mathbb{K}(X \times I), D_{X\times I})$ and, consequently, we can define some metrics on $\mathbb{F}_0(X)$ and $\mathbb{F}(X)$ respectively. Namely, the *sendograph* metric

$$d_S(A,B) = D_{X\times I}(send(A), send(B))$$

is defined only for nonempty fuzzy sets $A, B \in \mathbb{F}_0(X)$. The sendograph metric is established only for non-empty fuzzy sets since $send(\emptyset_X) = \emptyset$ and the Hausdorff metric measures the distance between non-empty closed sets. It should be mentioned that probably the most natural extension of the sendograph metric to $\mathbb{F}(X)$ – namely,

$$d_S(A,B) = \max\{\max_{x\in X} A(x), \max_{x\in X} B(x)\}$$

if $A, B \in \mathbb{F}(X)$ and one of them is the empty fuzzy set $\emptyset_X$, does not satisfy the triangle inequality. It is important to realize this if we consider the sendograph metric and the initial space X is compact.

Since $supp(A) \cup supp(B)$ is compact for any $A, B \in \mathbb{F}_0(X)$, the *endograph* metric d_E is defined correctly for any two $A, B \in \mathbb{F}_0(X)$ by

$$d_E(A,B) = D_{X\times I}(end(A), end(B)).$$

Moreover, we use

$$d_E(\emptyset_X, \emptyset_X)$$

and

$$d_E(\emptyset_X, A) = D_{supp(A)\times I}(end(\emptyset_X), end(A)) \text{ for } A \in \mathbb{F}_0(X)$$

in order to obtain a metric on the space $\mathbb{F}(X)$.

Let us define the third *levelwise* metric d_∞ on $\mathbb{F}(X)$ by

$$d_\infty(A,B) = \sup_{\alpha\in(0,1]} D_X([A]_\alpha, [B]_\alpha). \tag{2.2}$$

Similar to the previous case, this metric is correctly defined only for non-empty fuzzy sets $A, B \in \mathbb{F}(X)$ whose maximal values are identical, since the Hausdorff distance D_X is only measured between two non-empty closed subsets of the space X. Thus, we consider the following extension of the Hausdorff metric: D_X:

$$D_X(\emptyset, \emptyset) = 0 \text{ and } D_X(\emptyset, A) = diam\,(X) \text{ for any } A \in \mathbb{K}(X).$$

By using this extension, (2.2) correctly defines the levelwise metric on $\mathbb{F}(X)$. It is obvious that

$$d_\infty(\emptyset_X, \emptyset_X) = 0 \text{ and } d_\infty(\emptyset_X, A) = diam\,(X) \text{ for any } A \in \mathbb{F}_0(X). \tag{2.3}$$

2.2 Different fuzzifications

In fuzzy mathematics a *t-norm* is an associative binary relation $T : I^2 \to I$ satisfying the following three properties: for any $x, y, w, z \in I$,

(i) T is nondecreasing, i.e., $T(x,y) \le T(w,z)$ if $x \le w$ and $y \le z$,

(ii) 1 is the identity of T, i.e., $T(1,x) = T(x,1) = x$,

(iii) T is symmetric, i.e., $T(x,y) = T(y,x)$.

For any t-norm T a *t-conorm* $T^\star : I^2 \to I$ can be defined by

$$T^\star(x,y) = 1 - T(1-x, 1-y) \text{ for } x, y \in I.$$

If Γ is any t-norm or t-conorm, $\Delta_\Gamma(x) = \Gamma(x,x)$ denotes the *diagonal* of Γ.

Let us define some fuzzifications of a given dynamical system (X, φ) formed by a locally compact metric space X and a continuous (self-)map $\varphi : X \to X$. All fuzzifications are defined by formulas sending a given fuzzy set $A \in \mathbb{F}(X)$ to another fuzzy set on X. A *usual fuzzification* (often called *Zadeh's extension*) $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ is defined by

$$(\Phi A)(x) = \sup_{y \in \varphi^{-1}(x)} A(y)$$

for any $A \in \mathbb{F}(X)$ and $x \in X$. Now, if a t-norm or t-conorm Γ is given, we define a Γ-*fuzzification* Φ_Γ ([8]) by

$$(\Phi_\Gamma A)(x) = \sup_{y \in \varphi^{-1}(x)} \Gamma(A(y))$$

for any $A \in \mathbb{F}(X)$ and $x \in X$.

Let us introduce generalized fuzzifications as well. We define $D_m(I)$ as the set of all nondecreasing left-continuous functions $g : I \to I$ for which $g(x) = x$ if $x = 0$ and $x = 1$. We denote $C_m(I)$ as the set of all continuous maps from $D_m(I)$. Let X be a locally compact metric space and $\varphi \in C(X)$. Then, for any $g \in D_m(I)$, we define $\Phi_g : \mathbb{F}(X) \to \mathbb{F}(X)$ by

$$(\Phi_g(A))(x) = \sup_{y \in \varphi^{-1}(x)} \{g(A(y))\} \text{ for any } A \in \mathbb{F}(X),\ x \in X, \tag{2.4}$$

is called a *generalized fuzzification* (also called a g-fuzzification or g-fuzzification if we intend to emphasize the dependence on g) of the crisp dynamical system (X, φ). It is obvious that the usual fuzzification and any reasonable (see Subsection 4.2) Γ-fuzzification are special cases of certain g-fuzzifications (either put $g = id_I$ or $g = g_\Gamma$, resp.).

We also define α-*cut* $[A]^g_\alpha$ of a fuzzy set $A \in \mathbb{F}(X)$ *with respect to* $g \in D_m(I)$ by

$$[A]^g_\alpha = \{x \in supp(A) \mid g(A(x)) \geq \alpha\}.$$

3 Basic topological properties and convergences

In this section we will discuss some basic topological and metric properties on the space of fuzzy sets. Each of the metrics mentioned above (d_S, d_E and d_∞) induces a metric topology (τ_S, τ_E and τ_∞) on $\mathbb{F}(X)$ or $\mathbb{F}_0(X)$, respectively, which is denoted in accordance with the considered metric. The topologies τ_S, τ_E and τ_∞ are called the sendograph, endograph and levelwise topologies. We will discuss (local) compactness, separability and completeness of the space of fuzzy sets X, since the situation has not yet been clarified sufficiently (see Section 1).

3.1 d_S case

This case was first discussed by P. E. Kloeden in [13]. If X is a locally compact metric space, then $(\mathbb{F}_0(X), d_S)$ is also a locally compact metric space. However, P. E. Kloeden showed that the space $(\mathbb{F}_0(X), d_S)$ is not complete – to see this result, take any Cauchy sequence of fuzzy sets "converging" to $\emptyset_X$. According to Lemma 1, there can be some complete subspaces of $(\mathbb{F}_0(X), d_S)$. For completeness, $(\mathbb{F}^\lambda(X), d_S)$ is also locally compact for any $\delta \in (0, 1]$ since it is a closed subset of $(\mathbb{F}_0(X), d_S)$. Finally, it is well known that $(\mathbb{F}_0(X), d_S)$ is separable whenever X is also separable.

If X is a compact metric space, the situation is a little bit different. In terms of the separability, $(\mathbb{F}_0(X), d_S)$ is separable if X is separable ([13]). It is reported in several papers (see, e.g., [7] and [8]) that the compactness of $(\mathbb{F}(X), d_S)$ was declared by the reasoning: "it is straightforward but lengthy to show ...". However, unfortunately, this statement is wrong. In the papers mentioned the authors did not follow the origin of the notation ([13], *send*$(\cdot)$ like *supported endograph*), but used the notation *send*$(\cdot)$ for endographs of fuzzy sets. Thus they considered the endograph topology on $\mathbb{F}(X)$ instead of the sendograph topology and, consequently, the compactness of $(\mathbb{F}(X), d_S)$ has been falsly declared (see, e.g., [5]).

The next simple example shows that even a closed subspace of $(\mathbb{F}_0(X), d_S)$ (namely $(\mathbb{F}^1(X), d_S)$) need not be compact. Before showing this example we recall that a *characteristic* function χ_B of a given set $B \in X$ is defined as usually by

$$\chi_B(x) = 1 \text{ if } x \in B \text{ and } \chi_B(x) = 0 \text{ otherwise.}$$

Example 1 *Let $X = I$ and $A_n(x) = x^n$ for any $n \in \mathbb{N}$.*

Obviously, since $A_n(1) = 1$ for any $n \in \mathbb{N}$ and the sequence $\{A_n\}_{n\in\mathbb{N}}$ decreases otherwise, the only possible limit point in the metric topology τ_S is the characteristic function χ_1 of the point 1. Yet $d_S(A_n, \chi_1) = 1$ for each $n \in \mathbb{N}$ since $supp(A_n) = I$ and $supp(\chi_1) = \{1\}$. Consequently, the sequence $\{A_n\}_{n\in\mathbb{N}}$ has no accumulation point $(\mathbb{F}^1(X), d_S)$. However, this contradicts the fact that any sequence lying in a compact metric space has an accumulation point.

Concerning the completeness, P. E. Kloeden showed that $(\mathbb{F}_0(X), d_S)$ is not complete. We can show that some subspaces of $(\mathbb{F}_0(X), d_S)$ can be complete.

Lemma 1 *Let X be any locally compact, complete metric space and $\lambda \in (0, 1]$. Then, the metric space $(\mathbb{F}^\lambda(X), d_S)$ is complete.*

Proof. Let us take any Cauchy sequence of fuzzy sets $\{A_n\} \subseteq \mathbb{F}^\lambda(X)$. Since the space $\mathbb{K}(X \times I)$ is complete the Cauchy sequence $\{send(A_n)\}_{n\in\mathbb{N}}$ is convergent and it converges to a non-empty closed subset $\tilde{A}$. Put $A(x) = \max\{y \in I \,|\, (x, y) \in \tilde{A}\}$ for any $x \in supp(\tilde{A})$ and $A(x) = 0$ otherwise.

Clearly, A is a function on X and $\tilde{A} = send(A)$. Moreover, $A \in \mathbb{F}^\lambda(X)$ since $\max_{x\in X} A_n(x) \geq \lambda$ for any $n \in \mathbb{N}$. To finish the proof, it is sufficient to show that the limit point $\tilde{A}$ of $\{A_n\}_{n\in\mathbb{N}}$ represents the graph of an upper semi-continuous function whose support is bounded. Since the sendograph $\tilde{A}$ of the function A is a closed subset of $X \times I$ the function A is upper semi-continuous. To see that $supp(A)$ is bounded, note that the sequence $\{supp(A_n)\}_{n\in\mathbb{N}}$ consists of closed bounded subsets of the space X and hence $\lim supp(A_n) = suppA$ is bounded. □

3.2 d_E case

Let us discuss the situation when X is a locally compact metric space. Then, by using arguments similar to P. E. Kloeden's in [13], we can show that the space $(\mathbb{F}(X), d_E)$ is again separable and locally compact.

Concerning the completeness, we mentioned above (see [10] or [11]) that $(\mathbb{F}^1(\mathbb{R}), d_E)$ is not complete. The next lemma shows that this assertion is not true. The completeness of $(\mathbb{F}(X), d_E)$ depends on the completeness of the space X and the completeness assumption is essential – to make a counterexample, it is sufficient to consider any non-complete space and (as the non-convergent sequence of fuzzy sets) to take the characteristic functions of any nonconvergent Cauchy sequence lying in it.

Lemma 2 *Let X be any locally compact, complete metric space. Then the metric space $(\mathbb{F}(X), d_E)$ is complete. Moreover, $(\mathbb{F}^\lambda(X), d_E)$ is complete for any $\lambda \in (0, 1]$.*

Proof. Let us take any Cauchy sequence of fuzzy sets $\{A_n\}_{n\in\mathbb{N}}$. Then there are two possibilities - either $\{A_n\}_{n\in\mathbb{N}}$ converges to $\emptyset_X$ or $B := \overline{\bigcup supp(A_n)}$ (and, consequently, also $\mathbb{K}(B \times I)$) is compact. Thus the sequence $\{end(A_n)\}_{n\in\mathbb{N}}$ converges in $\mathbb{K}(B \times I)$ to $\tilde{A}$. Put $A(x) = \max\{y \in I \,|\, (x, y) \in \tilde{A}\}$ for any $x \in supp(\tilde{A})$ and $A(x) = 0$ otherwise.

It remains to show that so defined function A is upper semi-continuous and its support $supp(A)$ is bounded. Clearly, since $\tilde{A} = end(A)$ is a closed set, A is upper semi-continuous. To finish the proof we must show that $supp(A)$ is bounded; this is an obvious consequence of the fact that $\{supp(A_n)\}_{n\in\mathbb{N}}$ is a sequence of compact and hence bounded sets and $supp(A) \subseteq \lim supp(A_n)$. We showed that $(\mathbb{F}(X), d_E)$ is complete.

Fix any $\lambda \in (0, 1]$. It is obvious that if $\max A_n \geq \lambda$ for any $n \in \mathbb{N}$, then $\max A \geq \lambda$, i.e., $(\mathbb{F}^\lambda(X), d_E)$ is also complete. □

Now assume that X is a compact metric space. We recall that $(\mathbb{K}(X), D_{X\times I})$ is compact in this case. We have already established above that $(\mathbb{F}(X), d_E)$ is separable and, as mentioned in Section 1, the compactness of $(\mathbb{F}(X), d_E)$ was declared without proof (see, e.g., [6] and [8]). The justification for the compactness is rather easy. Really, since there exists a homeomorphism $(\mathbb{F}(X), d_E) \to (\mathbb{K}(X \times I), D_{X\times I})$, $A \mapsto end(A)$, $\mathbb{F}(X)$ can be considered as a closed subset of $K(X \times I)$. Consequently, the compactness of $(\mathbb{K}(X), D_{X\times I})$ implies the compactness of $(\mathbb{F}(X), d_E)$. Finally, since $(\mathbb{F}(X), d_E)$ is compact, it is also complete.

3.3 d_∞ case

In 1983, Puri and Ralescu ([22]) proved that $(\mathbb{F}^1(\mathbb{R}^n), d_\infty)$ is a complete metric space. Moreover, they also showed that the same space cannot be compact since it is non-separable. The proof showing non-separability and completeness can easily be generalized to $(\mathbb{F}(X), d_\infty)$ if the original space X is complete. The compactness of the general case was studied in [24]. The paper's main result that "$(\mathbb{F}(X), d_\infty)$ is compact if and only if the diameter of the space X is 0" shows that $(\mathbb{F}(X), d_\infty)$ cannot be locally compact.

In accordance with (2.3), it is easy to see that if the space X is complete, then $(\mathbb{F}^\lambda(X), d_\infty)$ is also complete for any $\lambda \in (0, 1]$.

3.4 Summary

Let us summarize the results mentioned above. Below ($\bullet$) denotes the fact that the result is true only if the original space X has the same property.

3.4.1 X is a locally compact metric space

	$(\mathbb{F}_0(X), d_S)$	$(\mathbb{F}(X), d_E)$	$(\mathbb{F}_0(X), d_\infty)$
separability	YES ($\bullet$)	YES ($\bullet$)	NO
local compactness	YES	YES	NO
completeness	NO	YES ($\bullet$)	YES ($\bullet$)

3.4.2 X is a compact metric space

	$(\mathbb{F}_0(X), d_S)$	$(\mathbb{F}(X), d_E)$	$(\mathbb{F}_0(X), d_\infty)$
separability	YES	YES	NO
compactness	NO	YES	NO
completeness	NO	YES ($\bullet$)	YES ($\bullet$)

3.5 Convergences

We define τ_E, τ_S and τ_∞ as the metric topologies given by the endograph, sendograph and level-wise metrics. The relations between different kinds of convergences on the space of fuzzy sets have already been studied by several authors (see also [23], where the convergence in the topology τ_E is called *variational*). We note that there are several results showing that the topologies we study, in addition to some others, coincide on certain subclasses of $\mathbb{F}(X)$, as shown by [24].

Following the results of [23], the τ_E-convergence is implied by both τ_S- and τ_∞-one and the converse implications do not hold. They only assumed the case where $X = \mathbb{R}^n$, but their simple arguments are also valid for any locally compact metric space X.

We also explain relations between τ_∞ and τ_S. The next simple example shows that τ_S-convergence does not imply the τ_∞-one.

Example 2 *Let $X = I$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy sets defined by $A_n(x) = 1 - \frac{x}{n}$ for any $n \in \mathbb{N}$.*

Then obviously $\{A_n\}_{n\in\mathbb{N}}$ converges to χ_I in the sendograph metric. But,

$$D_I\{[\chi_I]_1, [A_n]_1\} = 1$$

for any $n \in \mathbb{N}$ and, consequently, $\{A_n\}_{n\in\mathbb{N}}$ does not converge to χ_I in the level-wise metric.

Finally, it remains to be shown that the τ_∞-convergence implies the τ_S-one. This is implied by the inequality

$$d_S(A,B) \le d_\infty(A,B)$$

which was proved in [5] for arbitrary fuzzy numbers, i.e., $X = \mathbb{R}$ and all α-cuts of fuzzy sets $A, B \in \mathbb{F}(X)$ are convex subsets of $\mathbb{R}$. A simple justification of this inequality, as given in [5], does not use a specific property of fuzzy numbers and, therefore, is valid for any two fuzzy sets $A, B \in \mathbb{F}(X)$ on any locally compact metric space X.

Consequently, we have

$$d_E(A,B) \le d_S(A,B) \le d_\infty(A,B) \tag{3.1}$$

for any $A, B \in \mathbb{F}_0(X)$ since the first inequality in (3.1) is implied directly from the definitions of d_E and d_S.

Later we will need the next lemma, which describes the relation between the sendograph and endograph convergences. This was already studied in 1996 ([23, Proposition 3.9]) for $X = \mathbb{R}^n$. However, as far as we know, the paper which the lemma originally comes from has never been published (G. Greco, M. Moschen, E. Quelho, *On the variational convergence of fuzzy sets*).

Lemma 3 *Let X be any locally compact metric space and $\{A_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{F}_0(X)$. Then, the following two conditions are equivalent:*

(i) *$\{A_n\}_{n\in\mathbb{N}}$ converges to $A \in \mathbb{F}_0(X)$ in τ_S,*

(ii) *$\{A_n\}_{n\in\mathbb{N}}$ converges to $A \in \mathbb{F}_0(X)$ in τ_E and the sequence $\{supp(A_n)\}_{n\in\mathbb{N}}$ converges to $supp(A)$ in the topology given by the Hausdorff metric D_X on $\mathbb{K}(X)$.*

Proof. The simple idea used in Example 1 shows that the convergence of $\{supp(A_n)\}$ to $supp(A)$ in $\mathbb{K}(X)$ is essential for the τ_S-convergence of $\{A_n\}_{n\in\mathbb{N}}$. On the other hand, if $\{supp(A_n)\}_{n\in\mathbb{N}}$ converges to $supp(A)$, then it follows directly from the definitions of d_E and d_S that the τ_E- and τ_S-convergences coincide. □

4 Fuzzifications

4.1 Usual fuzzification

The purpose of this section is to present some basic properties of usual fuzzification. This usual fuzzification was firstly defined by L. Zadeh in 1975. Since that time, only a few papers studying basic properties (like the continuity or the uniform continuity etc.) of the usual fuzzification have been published. We recall some results which we will generalize later.

Lemma 4 ([1] or [18]) *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then the usual fuzzification $\Phi : \mathbb{F}(\mathbb{R}^n) \to \mathbb{F}(\mathbb{R}^n)$ is well defined and we have*

$$[\Phi(A)]_\alpha = \varphi([A]_\alpha) \tag{4.1}$$

for any $A \in \mathbb{F}(\mathbb{R}^n)$ and $\alpha \in (0,1]$.

It is obvious that the equality (4.1) holds for any usual fuzzification defined on a given metric space X. Moreover, by the continuity of the original map $\varphi \in C(X)$, $\varphi([A]_\alpha)$ is non-empty, closed and $\varphi(supp(A))$ is compact if $A \in \mathbb{F}_0(X)$. Thus, (4.1) implies that the upper semi-continuity and the compactness of the support of a given fuzzy set is preserved by the usual fuzzification, i.e., Φ defines the map $\mathbb{F}(X) \to \mathbb{F}(X)$.

Papers dealing with the continuity of the usual fuzzification have recently appeared. For instance, in [1] some variants of continuity of the usual fuzzification were studied for different metric topologies on $\mathbb{F}(\mathbb{R}^n)$.

Theorem 1 ([1]) *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be uniformly continuous. Then $\Phi : (\mathbb{F}(\mathbb{R}^n), d_\infty) \to (\mathbb{F}(\mathbb{R}^n), d_\infty)$ is also uniformly continuous.*

It should be mentioned that the converse is also true if the continuity of the usual fuzzification and original map is considered (see [1] or Theorem 6). Theorem 1 was changed and generalized by Román-Flores and his colleagues in recent work (Theorems 3 and 4).

Theorem 2 ([1]) *A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if and only if $\Phi : (\mathbb{F}_0(\mathbb{R}^n), d_S) \to (\mathbb{F}_0(\mathbb{R}^n), d_S)$ is continuous.*

Unfortunately, the proof of Theorem 2 is based on the result ([1], Proposition 4) which is not generally accessible. We note that the proof of the generalization of Theorem 2 (Theorem 7) is not based on technical Proposition 4 from [1].

Theorem 3 [25] *A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if and only if $\Phi : (\mathbb{F}(\mathbb{R}^n), d_\infty) \to (\mathbb{F}(\mathbb{R}^n), d_\infty)$ is continuous.*

Theorem 4 [26] *Let X be any compact metric space. Then a map $\varphi : X \to X$ is continuous if and only if $\Phi : (\mathbb{F}(X), d_\infty) \to (\mathbb{F}(X), d_\infty)$ is continuous.*

The purpose of this section is to generalize the results produced above and to show that the same assertions can be stated regardless of which topology τ_S, τ_E or τ_∞ is considered.

Theorem 5 *Let X be any locally compact metric space and let τ denote one of the topologies τ_∞, τ_S and τ_E. Then $\varphi : X \to X$ is continuous if and only if $\Phi : (\mathbb{F}_0(X), \tau) \to (\mathbb{F}(X)_0, \tau)$ is continuous.*

The proof of this theorem is omitted since the usual fuzzification is a special case of g-fuzzifications which are studied in Subsection 4.3. Theorem 5 is an easy conseqence of Theorems 7 and 6 ($g : I \to I$ is the identity map). We will finish this section by giving an example of the usual fuzzification of certain crisp dynamical systems that can lose information carried by the initial state of a given dynamical system.

Example 3 *Let $X = I$ and $\varphi : I \to I$ be a piece-wise linear map given by $g(0) = 0$, $g(\frac{1}{2}) = 1$ and $g(1) = 0$. This map has the following property: ($\star$) [4, Proposition 45] - for any nondegenerate interval $J \subseteq I$ there is $m \in \mathbb{N}$ such that $\varphi^m(J) = I$.*

Let $((\mathbb{F}(X), d_E), \Phi)$ be a fuzzy dynamical system given by the usual fuzzification of (X, φ). Then, in the crisp system, the trajectories of given points admit different behaviors – from the simplest ones (when $x \in I$ is fixed, periodic, etc.) to the most complex ones (when $\{y \in I \,|\, \varphi^n(x) = y \text{ for some } n \in \mathbb{N}\}$ is dense in I). On the other hand, in the fuzzy system, it follows from ($\star$) and (4.1) that for any nontrivial (i.e., continuous) normal fuzzy set $A \in \mathbb{F}^1(X)$, $\omega_\Phi(A)$ consists only of the fixed point χ_I.

4.2 Γ-fuzzifications

Let us study some basic properties of Γ-fuzzifications. For any $A \in \mathbb{F}(X)$ and $\alpha \in (0, 1]$, an *α-level set* $[A]^\Gamma_\alpha$ with respect to a t-norm or t-conorm Γ is defined by $[A]^\Gamma_\alpha = \{x \in supp(A) \,|\, \Delta_\Gamma(A(x)) \geq \alpha\}$. Γ-fuzzifications were first studied in [8], where several basic properties were Shown; the authors proved the following analogy of (4.1).

Lemma 5 *Let $\varphi \in C(X)$, $A \in \mathbb{F}_0(X)$ and let Γ be any t-norm or t-conorm. Then*

$$\varphi([A]^\Gamma_\alpha) = [\Phi_\Gamma(A)]_\alpha, \ \alpha \in (0, 1], \tag{4.2}$$

if and only if

$$\sup_{x \in \varphi^{-1}(y)} \{\Delta_\Gamma(A(x))\} \tag{4.3}$$

is attained for any $y \in \varphi(supp(A))$.

The condition (4.3) depends only on the diagonal Δ_Γ. Using the assumptions of Lemma 5, the set $B := \varphi^{-1}(y)$ is closed for any $y \in \varphi(supp(A))$ since φ is continuous. Then the point $x_0 = \max_{x \in B}\{A(x)\}$ exists because A is upper semi-continuous. Thus (4.3) depends on $\Delta_\Gamma(x_0)$.

So, since we want to obtain a *reasonable* Γ-fuzzification, i.e., satisfying (4.2), we require $\Delta_\Gamma(x)$ to attain its supremum for any $x \in I$. In this case, the diagonal Δ_Γ induces a map $g_\Gamma : I \to I$, $g_\Gamma(x) = \sup\{\Delta_\Gamma(x)\}$, such that $g_\Gamma(0) = 0$ and $g_\Gamma(1) = 1$. Moreover, since Δ_Γ in nondecreasing for any t-norm or t-conorm Γ, the map g_Γ is nondecreasing too. In fact we have shown that any reasonable Γ-fuzzification induces a map $g : I \to I$ satisfying the conditions mentioned above.

Thus, we can generalize the definition of Γ-fuzzification by using a non-decreasing map $g : I \to I$ for which $g(0) = 0$ and $g(1) = 1$ and define a fuzzification depending on g instead of Δ_Γ. These generalized fuzzifications are introduced in Subsection 4.3. Finally, since Γ-fuzzications are special cases of fuzzifications that are presented in the next subsection, we obtain the following results as consequences of Theorems 6 and 7, Proposition 1 and Example 4. It should be also mentioned that the continuity of Γ-fuzzifications has never been studied before.

Corollary 1 *Let X be any locally compact metric space and let Γ be a t-norm or a t-conorm defining a reasonable Γ-fuzzification. Then if Δ_Γ is not a continuous function, $\Phi_\Gamma : (\mathbb{F}(X), \tau) \to (\mathbb{F}(X), \tau)$ is not continuous whenever $\tau = \tau_E$ or τ_S.*

Proof. See Example 4. $\square$

Corollary 2 *Let X be any locally compact metric space and let Γ be a t-norm or a t-conorm defining a reasonable Γ-fuzzification. Then $\varphi : X \to X$ is continuous if and only if $\Phi_\Gamma : (\mathbb{F}(X), \tau_\infty) \to (\mathbb{F}(X), \tau_\infty)$ is also continuous.*

Proof. See Theorem 6. $\square$

Corollary 3 *Let X be any locally compact metric space and let Γ be a t-norm or a t-conorm for which Δ_Γ is a continuous function. Then $\varphi : X \to X$ is continuous if and only if $\Phi_\Gamma : (\mathbb{F}(X), \tau_E) \to (\mathbb{F}(X), \tau_E)$ is also continuous.*

Moreover, when $\tau = \tau_S$, let Δ_Γ be strictly increasing at 0. Then $\varphi : X \to X$ is continuous if and only if $\Phi_\Gamma : (\mathbb{F}_0(X), \tau_S) \to (\mathbb{F}_0(X), \tau_S)$ is also continuous.

Proof. See Theorem 7. $\square$

It should be also mentioned that Γ-fuzzifications preserve the uniform convergence of uniformly continuous maps – for details, see Propositions 1 and 2.

4.3 Generalized fuzzifications (g-fuzzifications)

First of all, we would like to make two remarks concerning the empty fuzzy set on X.

Remark 1 *Let $g \in C_m(I)$ be such that $g([0, a)) = 0$ for some $a \in (0, 1)$. Then, for any $A \in \mathbb{F}(X)$ with $\max A \le a$, the definition of g-fuzzification implies that $\Phi_g(A) = \emptyset_X$. Thus, $\Phi_g : (\mathbb{F}_0(X), d_S) \to (\mathbb{F}_0(X), d_S)$ is correctly defined only when $g \in C'_m(I)$, i.e., $g \in C_m(I)$ is strictly increasing at 0.*

Remark 2 *If $\tau = \tau_\infty$ or $\tau = \tau_E$ then Φ_g is defined also at $\emptyset_X$ and $\Phi_g(\emptyset_X) = \emptyset_X$. Moreover, the continuity of $\Phi_g : (\mathbb{F}(X), d_\infty) \to (\mathbb{F}(X), d_\infty)$ (resp., $\Phi_g : (\mathbb{F}(X), d_E) \to (\mathbb{F}(X), d_E)$) at the point $\emptyset_X$ follows directly from (2.3), which is due to the fact that g is left-continuous at 0.*

Thus, to study the continuity of Φ_g, it is sufficient to study the continuity of fuzzifications on the space of non-empty fuzzy sets.

Lemma 6 *Let X be a locally compact metric space and $\varphi \in C(X)$. Then*

$$\varphi([A]^g_\alpha) = [\Phi_g(A)]_\alpha \tag{4.4}$$

holds for any $A \in \mathbb{F}_0(X)$, $g \in D_m(I)$ and $\alpha \in (0, 1]$.

Proof. Let the assumptions be fulfilled. We first prove "$\leq$". Take any $x \in \varphi([A]^g_\alpha)$. Then there exists $y_0 \in supp(A)$ for which $g(A(y_0)) \geq \alpha$. Thus, since also $\varphi(y_0) = x$,

$$(\Phi_g(A))(x) = \sup_{y \in \varphi^{-1}(x)} \{g(A(y))\} \geq g(A(y_0)) \geq \alpha,$$

i.e., $x \in [\Phi_g(A)]_\alpha$.

To prove "$\geq$," fix any $x \in [\Phi_g(A)]_\alpha$. Then $\varphi^{-1}(x)$ is a non-empty closed subset of X. Moreover, since A is upper semi-continuous and g is nondecreasing, there exists $a := \max\{g(A(\varphi^{-1}(x)))\}$. Thus there is $y_0 \in supp(A)$ such that $g(A(y_0)) = a$. By the choice of x, $g(A(y_0)) \geq \alpha$, i.e., $y_0 \in [A]^g_\alpha$. Finally, since $y_0 \in \varphi^{-1}(x)$, $x \in \varphi([A]^g_\alpha)$. □

Lemma 7 *Let X be any locally compact metric space and $\varphi \in C(X)$. Then, for any $g \in D_m(I)$, any g-fuzzification Φ_g is correctly defined.*

Proof. Let $A \in \mathbb{F}_0(X)$. Since the case $\Phi_g(A) = \emptyset_X$ is not interesting (see Remarks 1 and 2), we may assume that $\Phi_g(A) \neq \emptyset_X$. We must show that $\Phi_g(A)$ is an upper semi-continuous function having compact support. First, we show that $\Phi_g(A)$ is upper semi-continuous. It is sufficient to show that $[\Phi_g(A)]_\alpha$ is closed for any $\alpha \in (0, 1]$. So, fix $\alpha \in (0, 1]$ and $A \in \mathbb{F}(X)$.

We may assume that $[\Phi_g(A)]_\alpha \neq \emptyset$. By (4.4), it suffices to show that $[A]^g_\alpha$ is closed since φ is continuous. So, let $\{x_n\} \subseteq [A]^g_\alpha$ be a sequence converging to x_0. Since A is upper semi-continuous, $\liminf A(x_n) \leq A(x_0)$. Then, by the choice of $\{x_n\}_{n\in\mathbb{N}}$, and since g is nondecreasing and left-continuous,

$$\alpha \leq \liminf g(A(x_n)) = g(\liminf A(x_n)) \leq g(A(x_0)),$$

i.e., $x_0 \in [A]^g_\alpha$.

Thus we have shown that $[A]^g_\alpha$ (and hence $\varphi([A]^g_\alpha)$) is closed. Moreover, since $[A]^g_\alpha \subseteq supp(A)$ and $supp(A)$ is compact, $[A]^g_\alpha$ is compact. Therefore, the continuity of φ and (4.4) imply that $[\Phi_g(A)]_\alpha$ is compact too. Finally, since any compact subset of X is also bounded, we have shown that any α-cut of $\Phi_g(A)$ is bounded.

It remains to be shown that $supp(\Phi_g(A))$ is bounded. If $supp(\Phi_g(A)) \neq \emptyset$, then we may assume that, for any decreasing sequence of $\{\alpha_n\} \subseteq (0, 1]$ converging to 0, there is an increasing sequence $\{[\Phi_g(A)]_{\alpha_n}\}_{n\in\mathbb{N}}$ of non-empty closed subsets of X. Since $\lim[\Phi_g(A)]_{\alpha_n} = supp(\Phi_g(A))$ and each $[\Phi_g(A)]_{\alpha_n}$ is bounded, $supp(\Phi_g(A))$ is also bounded. □

The next example shows that the continuity of $g \in D_m(I)$ is necessary for the continuity of Φ_g if we consider either the sendograph or endograph topologies.

Example 4 *Consider any locally compact metric space X and a map $\varphi = id_X$. For instance, we may assume that g is not continuous at x_0 from the left, i.e., there are $a, b \in I$ for which $g(x_0) = b$, $\lim_{x \to x_0^-} g(x) = a$ and $a < b$. Let τ denotes either τ_E or τ_S.*

Now consider a strictly increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq I$ converging to x_0. Take any compact set $B \subseteq X$ and put $A_n = x_n\chi_B$ for any $n \geq 0$. Then $\{A_n\}_{n\in\mathbb{N}}$ converges to A_0 in τ. But

$$\max_{x\in X} \Phi_g(A_n)(x) \leq a < b = \max_{x\in X} \Phi_g(A_0)(x).$$

Thus Φ_g is not continuous if the topology τ is considered. If g is not continuous at x_0 from the right the construction is similar.

We have shown that the discontinuity of g implies the discontinuity of the induced g-fuzzifications. On the other hand, if g is continuous, then an analogy of Theorem 5 can be proposed. Moreover, for the case of levelwise topology, the continuity of g on I does not need to be assumed.

Lemma 8 *Let $g \in D_m(I)$. Then for any $\alpha \in (0, 1]$ there is $c \in [0, 1]$ such that $[A]^g_\alpha = [A]_c$.*

Proof. Obviously, there are three possibilibies:

(i) $g^{-1}(\alpha)$ is a singleton. In this case, $c = g^{-1}(\alpha)$.

(ii) $g^{-1}(\alpha)$ is a nondegenerate closed interval $[c,d]$ and c is so given.

(iii) $g^{-1}(\alpha) = \emptyset$. This means that there is a point $x_0 \in I$ at which g is not continuous, i.e., $\lim_{x\to^- x_0} g(x) = a < b = g(x_0)$. In this case, $[A]^g_\alpha = [A]^g_b$ and this case can be reduced either to (i) or to (ii). □

Theorem 6 *Let X be any locally compact metric space and let $g \in D_m(I)$. Then $\varphi : X \to X$ is continuous if and only if $\Phi_g : (\mathbb{F}(X), \tau_\infty) \to (\mathbb{F}(X), \tau_\infty)$ is also continuous.*

Proof. With respect to Remark 2, let $\{A_n\}_{n\in\mathbb{N}}$ be a convergent sequence of non-empty fuzzy sets and let A_0 be the limit point of this sequence. Since $A_0 \neq \emptyset_X$, $supp(A_i)$ is compact for any $i \in \mathbb{N}$, $K := \bigcup_{i\in\mathbb{N}} supp(A_i)$ and also $\varphi(K)$ are compact. Again, by the continuity of φ, the map $\tilde{\varphi} : \mathbb{K}(K) \to \mathbb{K}(\varphi(K))$ defined for any $B \in \mathbb{K}(K)$ by

$$\tilde{\varphi}(B) = \{y \in \varphi(K) \,|\, \varphi(x) = y \text{ for some } x \in B\}$$

is uniformly continuous ([17]). This means that for any $\varepsilon > 0$ there is $\delta > 0$ satisfying, for any $\tilde{A}, \tilde{B} \in \mathbb{K}(K)$,

$$D_X(\tilde{A}, \tilde{B}) < \delta \quad \Rightarrow \quad D_X(\tilde{\varphi}(\tilde{A}, \tilde{\varphi}(\tilde{B}))) < \varepsilon. \tag{4.5}$$

To show the continuity of Φ_g we need to show the uniform continuity in α-cuts. Let $\varepsilon > 0$ be fixed and let $\delta > 0$ be given by (4.5). Since $\{A_n\}_{n\in\mathbb{N}}$ converges to A_0 in τ_∞, there is $n_0 \in \mathbb{N}$ such that, $d_\infty(A_n, A_0) \geq \delta$ for any $n \geq n_0$. So, for any $\alpha \in (0,1]$ and $n \geq n_0$,

$$D_X([A_n]_\alpha, [A_0]_0) < \delta.$$

Again by (4.5),

$$D_X(\tilde{\varphi}([A_n]_\alpha), \tilde{\varphi}([A_0]_\alpha)) = D_X(\varphi([A_n]_\alpha), \varphi([A_0]_\alpha)) < \varepsilon$$

holds for any $\alpha > 0$ and $n \geq n_0$. Thus, see (4.4) and Lemma 8, we have finished.

The converse implication is obviously true, since the original dynamical system (X, φ) can be isometrically embedded to $(\mathbb{F}(X), d_\infty)$ by $i : X \to \mathbb{F}(X)$, $x \to \chi_x$. In this case obviously, $\Phi_g(\chi_x) = \chi_{\varphi(x)}$ and $d_\infty(\chi_x, \chi_y) = d_X(x, y)$ for any $x, y \in X$. Thus, for any $\{x_n\} \subseteq X$ converging to $x_0 \in X$ there exists a sequence $\{\chi_{x_n}\} \subseteq \mathbb{F}(X)$ converging to χ_{x_0}. Now the continuity of Φ_g implies that $\{\chi_{\varphi(x_n)}\} \in \mathbb{F}(X)$ converges to $\chi_{\varphi(x_0)}$, i.e., $\{\varphi(x_n)\} \in X$ converges to $\varphi(x_0)$. □

Theorem 7 *Let X be any locally compact metric space and let $g \in C_m(I)$. Then $\varphi : X \to X$ is continuous if and only if $\Phi_g : (\mathbb{F}(X), \tau_E) \to (\mathbb{F}(X), \tau_E)$ is continuous.*

Moreover, if $g \in C'_m(I)$, then $\varphi : X \to X$ is continuous if and only if $\Phi_g : (\mathbb{F}_0(X), \tau_S) \to (\mathbb{F}_0(X), \tau_S)$ is continuous.

Proof. Let τ denotes one of τ_S, τ_E. With respect to Remark 2, let $\{A_n\} \subseteq \mathbb{F}_0(X)$ be a sequence of non-empty fuzzy sets convergent in the topology τ and let $A_0 \neq \emptyset_X$ be its limit point. Similarly to the proof of Theorem 6, we obtain a compact set $K \subseteq X$ containing $\bigcup supp(A_n)$. Consider the product map $\kappa := (\varphi, g) : K \times I \to \varphi(K) \times I$. Since κ is continuous and $K \times I$ is non-empty and compact, the set-valued map $\tilde{\kappa} : \mathbb{K}(K \times I) \to \mathbb{K}(\varphi(K) \times I)$ induced from the map κ is uniformly continuous.

For the rest of the proof we will identify fuzzy sets by their sendographs or endographs. To finish the proof it is sufficient to show that

$$\tilde{\kappa}(send(A)) = send(\Phi_g(A)) \text{ for any } A \in \mathbb{F}(K), \tag{4.6}$$

or

$$\tilde{\kappa}(end(A)) = end(\Phi_g(A)) \text{ for any } A \in \mathbb{F}(K). \tag{4.7}$$

Really, if (4.6) or (4.7) is true the g-fuzzification Φ_g can be considered as the restriction of $\tilde{\kappa}$ to $(\mathbb{F}_0(K), \tau)$. Finally, since $(\mathbb{F}_0(K), \tau)$ can be considered as a subspace of $(\mathbb{K}(K \times I), D_{K\times I})$, the uniform continuity of $\tilde{\kappa}$ implies the continuity of Φ_g.

Let us prove (4.6). By the definition, for any $x \in supp(\Phi_g(A))$,

$$(\Phi_g(A))(x) = \sup_{y\in\varphi^{-1}(x)} \{g(A(y))\}.$$

Since $\varphi^{-1}(x)$ is non-empty and closed there is a point $y_0 \in supp(A)$ for which $\max A|_{\varphi^{-1}(x)} = A(y_0)$. Thus $(\Phi_g(A))(x) = g(A(y_0))$. Finally it is obvious $\kappa(y_0, A(y_0)) = (x, g(A(y_0)))$ and, by the choice of y_0,

$$\tilde{\kappa}(send(A)) \cap I_x = \{(x, z) \in \varphi(K) \times I \,|\, z \leq (\Phi_g(A))(x)\}.$$

The proof of (4.7) similar.

The proof of the converse is easy. As in the previous proof, there is an isometrical embedding $i : X \to \mathbb{F}_0(X)$, $i(x) = \chi_x$. Obviously, $\Phi_g(x) = \chi_{\varphi(x)}$ and $d_X(x, y) = d_E(\chi_x, \chi_y) = d_S(\chi_x, \chi_y)$ for any $x, y \in X$. □

5 Uniform convergence

It was shown in [19] that the usual fuzzification preserves the uniform continuity if fuzzy-number valued functions are considered. The main purpose of this section is to generalize this result to g-fuzzifications and to show that functions other than fuzzy-number valued functions might be considered. For completeness, it was shown in [19] that the uniform continuity cannot be replaced by the continuity. In the proof of Proposition 1 we will need the next lemma, which is a simple generalization of the result proved by Dubois and Prade ([9]).

Lemma 9 *Let X be any locally compact metric space and $\{\varphi_n\} \subseteq C(X)$ be a sequence of uniformly continuous maps that uniformly converges to $\varphi_0 \in C(X)$. Then the sequence $\{\tilde{\varphi}_n\} \in C(\mathbb{K}(X))$ consists of uniformly continuous maps and converges to $\tilde{\varphi}_0 \in C(\mathbb{K}(X))$.*

Proof. Let the assumptions be fulfilled. The uniform continuity of any $\tilde{\varphi}_n$, $n \geq 0$, is an easy consequence of the uniform continuity of the original map φ_n and the definition of the Hausdorff metric on $\mathbb{K}(X)$.

We show that the sequence $\{\tilde{\varphi}_n\}_{n\in\mathbb{N}}$ uniformly converges to $\tilde{\varphi}_0$. Let $\varepsilon > 0$ be fixed. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ uniformly converges to φ_0 there is $m \in \mathbb{N}$ such that, for any $n > m$,

$$D_X(\varphi_n(x), \varphi_0(x)) < \varepsilon \text{ for any } x \in X.$$

Thus, for any $B \in \mathbb{K}(X)$, the last sentence imply that $\varphi_0(B) \subseteq U_\varepsilon(\varphi_n(B))$ and also $\varphi_n(B) \subseteq U_\varepsilon(\varphi_0(B))$ for any $n > m$. By the definitions of the Hausdorff metric on X and the induced map on $\mathbb{K}(X)$,

$$D_X(\tilde{\varphi_n}(B), \tilde{\varphi}_0(B)) < \varepsilon$$

for any $B \in \mathbb{K}(X)$ and $n > m$. □

Proposition 1 *Let X be a locally compact metric space, $g \in D_m(I)$ and let the metric topology τ_∞ be considered on $\mathbb{F}(X)$. Denote by $\Phi_{g,n}$ the g-fuzzification induced by a given map φ_n for $n \geq 0$.*

If $\{\varphi_n\} \subseteq C(X)$ is a sequence of uniformly continuous maps that uniformly converges to φ_0, then a sequence of g-fuzzifications $\{\Phi_{g,n}\}_{n\in\mathbb{N}}$ uniformly converges to $\Phi_{g,0}$.

Proof. We have shown in Theorem 6 that $\Phi_{g,n} : \mathbb{F}(X) \to \mathbb{F}(X)$ is continuous for any $n > 0$. Since $\Phi_{g,n}(\emptyset_X) = \emptyset_X$ for any $n \in \mathbb{N}$ it remains to show that $\{\Phi_{g,n}\}_{n\in\mathbb{N}}$ converges uniformly to $\Phi_{g,0}$ on $\mathbb{F}_0(X)$, i.e., for any $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that for any $n > m$

$$d_\infty(\Phi_{g,n}(A), \Phi_{g,0}(A)) < \varepsilon \text{ for any A} \in \mathbb{F}_0\text{(X)}. \tag{5.1}$$

Therefore, we fix $\varepsilon > 0$. Employing the assumptions, each φ_n is unifomly continuous. By Lemma 9 the sequence $\{\tilde{\varphi}_n\} \subseteq C(X)$ is a uniformly convergent sequence of uniformly continuous maps. From this result it follows that there is $m \in \mathbb{N}$ such that, for any $n > m$,

$$D_X(\tilde{\varphi}_n(B), \tilde{\varphi}_0(B)) \text{ for any } B \in \mathbb{K}(X). \tag{5.2}$$

Since α-cuts of any fuzzy set $A \in \mathbb{F}(X)$ are connected with set-valued maps, see 4.2 and Lemma 8, (5.2) implies (5.1) directly from the definition of the metric d_∞. □

Proposition 2 *Let X be a locally compact metric space and let $g \in C_m(I)$ if $\tau = \tau_E$ or $g \in C'_m(I)$ if $\tau = \tau_S$. Denote by $\Phi_{g,n}$ the g-fuzzification induced by a given map φ_n for $n \geq 0$.*

If $\{\varphi_n\} \subseteq C(X)$ is a sequence of uniformly continuous maps that uniformly converges to φ_0, then a sequence of g-fuzzifications $\{\Phi_{g,n}\}_{n\in\mathbb{N}}$ uniformly converges to $\Phi_{g,0}$ on $\mathbb{F}(X)$ if $\tau = \tau_E$ (resp., on $\mathbb{F}_0(X)$ if $\tau = \tau_S$).

Proof. First we consider the topology d_S on $\mathbb{F}_0(X)$. The crux of this proof is similar to that mentioned in Theorem 7. A map $g : I \to I$ is uniformly continuous – just as any continuous map defined on a compact metric space. Since a product map consisting of two uniformly continuous maps is uniformly continuous, any map $\kappa_n = (\varphi_n, g) : X \times I \to X \times I$ is uniformly continuous for any $n \geq 0$. Moreover, it is easy to see that also $\{\kappa_n\}_{n\in\mathbb{N}}$ uniformly converges to κ_0. Consequently, by Lemma 9, the sequence of set-valued maps $\{\tilde{\kappa}_n\} \subseteq \mathbb{K}(X \times I)$ uniformly converges to $\tilde{\kappa}_0 \in \mathbb{K}(X \times I)$. In fact, this completes the proof since the uniform convergence is preserved on any subspace of $\mathbb{K}(X \times I)$; since (4.6) was shown, $(\mathbb{F}(X), d_S)$ can be considered as a subspace of $\mathbb{K}(X \times I)$.

If the topology τ_E on $\mathbb{F}(X)$ is taken, the proof is analogous since $(\mathbb{F}_0(X), d_E)$ Can also be considered as subset of $\mathbb{K}(X \times I)$ and $\Phi_{g,n}(\emptyset_X) = \emptyset_X$ for any $n \in \mathbb{N}$. □

6 Further properties

Fuzzified dynamical systems have a property that is not typical of crisp systems. For example, fuzzy sets represent imprecise values that are used for some application. For instance, fuzzy set $A =$"approximately 5" can be considered as the union or addition of two fuzzy sets $B =$"approximately but not bigger than 5" and $C =$"approximately but not smaller than 5". Then, regardless of how the input is represented, we want to obtain the same result, i.e. the output of the system can be obtained either as an output given the fuzzy set A, or as the union of outputs depending on the fuzzy sets B and C. More formally, if $\varphi : \mathbb{F}(X) \to \mathbb{F}(Y)$ is the considered system and $A = B \cup C$ then

$$\varphi(A) = \varphi(B) \cup \varphi(C).$$

Consequently, many fuzzy systems also represent their own set-valued system. This property can be generalized as follows: given two spaces X and Y, we say that a system $\varphi : \mathbb{F}_0(X) \to \mathbb{F}_0(Y)$ has the *unions-preserving property* (shortly *UP-property*) if

$$end(\varphi(A)) = \bigcup_i end(\varphi(A_i)) \tag{6.1}$$

where $\{A_i\}$ is any (even uncountable) system of fuzzy sets such that

$$end(A) = \bigcup_i end(A_i). \tag{6.2}$$

It is clear that the sendographs of fuzzy sets could be used in this definition instead of the endographs. The next lemma shows that any discrete fuzzy dynamical system that is a generalized fuzzification of any crisp dynamical system has the UP-property. On the other side, it is very easy to construct an example (see Example 5) of a non-fuzzified system that satisfies the UP-property.

Lemma 10 *Let $(\mathbb{F}_0(X), \Phi_g)$ be any g-fuzzification of a system (X, φ) and let $g \in D_m(I)$. Then $(\mathbb{F}_0(X), \Phi_g)$ has the UP-property.*

Proof. Let the assumptions be fulfilled and take any $A \in \mathbb{F}(X)$. Let $\{A_i\} \subseteq \mathbb{F}(X)$ be any system of fuzzy sets satisfying (6.2). The case $\Phi_g(A) \neq \emptyset_X$ is trivial. So. let $x \in supp(\Phi_g(A))$. Then

$$\Phi_g(A(x)) = \sup_{y \in \varphi^{-1}(x)} \{g(A(y))\}.$$

Consider now $y_0 \in supp(A)$ attaining $\sup_{y \in \varphi^{-1}(x)}\{g(A(y))$. By (6.2) we have $A_i(y_0) \leq A(y_0)$. Moreover, there exists a non-empty index set J such that $A_j(y_0) = A(y_0)$ for each $j \in J$. Thus, for any $j \in J$,

$$\Phi_g(A)(x) = \sup_{y \in \varphi^{-1}(x)} \{g(A(y))\} = g(A(y_0)) \leq \sup_{y \in \varphi^{-1}(x)} \{g(A_j(y))\} = \Phi_g A_j(x),$$

Finally, since $A(y) \geq A_i(y)$ for any $y \in X$ and any i,

$$\Phi_g(A)(x) = \sup_{y \in \varphi^{-1}(x)} \{g(A(y))\} \geq \sup_{y \in \varphi^{-1}(x)} \{g(A_i(y))\} = \Phi_g(A_i)(x).$$

The proof is finished. □

Example 5 *Let $X = \overline{\{\frac{1}{n} \,|\, n \in \mathbb{N}\}}$. A map $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ is defined for $A \in \mathbb{F}(X)$ as follows - $\Phi(A) = A$ if $\min\{supp(A)\} = 0$. Otherwise, for any $A \in \mathbb{F}(X)$ put $b := \max_{x \in X}\{A(x)\}$ and $a := \frac{1}{n+1}$ if $\frac{1}{n} = \min\{supp(A)\}$. Then let*

$$\Phi(A)(x) = \begin{cases} A(x) & \text{if } x \neq a, \\ A(x) = b & \text{if } x = a. \end{cases}$$

It is obvious that this map is continuous regardless of which topology is used (τ_S, τ_E or τ_∞). Moreover, since any fuzzy set of the form $c\chi_{\{x\}}$, $x \neq 0$, $c \in (0,1]$, is mapped to a fuzzy set whose support consists of exactly two points of the set X, the map Φ cannot be a fuzzification of a crisp dynamical system.

Another notion concerning fuzzy discrete dynamical systems was recently established ([20]). S. M. Pederson defined so-called *commuting* fuzzifications by a nonconstant continuous map between a given crisp and fuzzy discrete dynamical system. It is necessary to emphasize that even if only the sendograph topology on $\mathbb{F}(X)$ is considered in [20], the following notion can also be stated for the endograph and levelwise topologies.

The original definition was established for compact metric spaces, but we assume that X is a locally compact metric space. For given discrete dynamical systems ((X, φ) and $(\mathbb{F}_0(X), \Phi)$) we say that $\Phi(\varphi) : \mathbb{F}_0(X) \to \mathbb{F}_0(X)$ is a *commuting fuzzification* of φ if there is a nonconstant continuous map $\Phi : X \to \mathbb{F}(X)$ for which

$$\Phi(\varphi(x)) = (\Phi(\varphi))(\Phi(x)) \text{ for any } x \in X. \tag{6.3}$$

The last equality says that the diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\to} & X \\ \Phi \downarrow & & \downarrow \Phi \\ \mathbb{F}_0(X) & \overset{\Phi(\varphi)}{\to} & \mathbb{F}_0(X) \end{array} \tag{6.4}$$

commutes.

We have several remarks concerning this definition. In our opinion, the assumption that Φ is nonconstant is not necessary and the justification of this assumption mentioned in [20] is not valid[1]. In any case, it is obvious that any g-fuzzification is commuting if $g \in D_m(I)$ and for any of the topologies considered in this article. Really, let (X, φ) be a discrete dynamical system and τ denotes any of τ_S, τ_E and τ_∞. Following the notation of the Pederson's definition the map $\Phi_g : X \to \mathbb{F}_0(X)$ defined by $x \mapsto \chi_{\{x\}}$ is obviously continuous in τ - in fact, Φ_g is an isometrical embedding of X to $\mathbb{F}_0(X)$. Moreover, since $\Phi_g(\chi_x) = \chi_{\varphi(x)}$ for any $x \in X$ and $g \in D_m(I)$, the diagram (6.4) commutes. Thus we have the following result.

[1]Without repeating Pederson's argument (page 364 of [20]) the conclusion that "every continuous fuzzification of φ is commuting" is not right. Instead, it should be written that "every continuous fuzzification of φ having v as a fixed point is commuting".

Theorem 8 *Let X be any locally compact metric space and $\varphi \in C(X)$.*

(i) If $g \in D_m(I)$ then g-fuzzification $((\mathbb{F}(X), \tau_\infty), \Phi_g)$ of (X, φ) is commuting.

(ii) If $g \in C_m(I)$ then g-fuzzification $((\mathbb{F}(X), \tau_E), \Phi_g)$ of (X, φ) is commuting.

(iii) If $g \in C'_m(I)$ then g-fuzzification $((\mathbb{F}_0(X), \tau_S), \Phi_g)$ of (X, φ) is commuting.

Seemingly, well established notions of conjugacy or semiconjugacy are more convenient for describing relations between a given crisp dynamical system and its g-fuzzification. It was already shown in [14] that a dynamical system $(\mathbb{F}_0(X), \Phi)$ is an extension of the original system (X, φ) if we consider the sendograph topology. However, the dynamical systems we consider (i.e., fuzzy dynamical systems given by a g-fuzzification) are not conjugated because they have different cardinalities.

Moreover, the next example shows that there is no semi-conjugacy between the fuzzified and original systems.

Example 6 *Let X be any compact metric space without the fixed point property, and let $\varphi \in C(X)$ be a map that does not have a fixed point on X (for instance, the unit circle $\mathbb{S}^1$ and an irrational rotation $\mathbb{S}^1 \to \mathbb{S}^1$). Let $g \in C'_m(I)$ and $(\mathbb{F}_0(X), \Phi_g)$ be a g-fuzzification of (X, φ).*

Since X is a compact space, there exists a minimal subset U of X, i.e., U is closed, non-empty, $\varphi(U) = U$ and no proper subset of U has the same three properties. Then, obviously by the UP-property, χ_U is a fixed point of Φ_g. Moreover, it is well-known that any semi-conjugacy $h : \mathbb{F}_0(X) \to X$ maps fixed points of $(\mathbb{F}_0, \Phi_g)$ to fixed points of the factor (X, φ). Accordingly, for the choice of the space X, there is no semi-conjugacy $h : \mathbb{F}_0(X) \to X$.

7 Topological (semi-)conjugacy

The main aim of this section is to show that a given (semi-)conjugacy between two dynamical systems can be naturally extended to discrete dynamical systems that are fuzzified in the same way (by the same fuzzification). Similarly, as in the previous section, we obtain different results for the different topologies.

Theorem 9 *Let X, Y be locally compact metric spaces. Let (X, φ), (Y, ψ) be discrete dynamical systems that are conjugated by $h : X \to Y$ and let $g \in D_m(I)$. Then, a map $H : (\mathbb{F}(X), \tau_\infty) \to (\mathbb{F}(Y), \tau_\infty)$, is defined by*

$$H(\emptyset_X) = \emptyset_Y \text{ and } [(H(A))]_\alpha = h([A]_\alpha))$$

for any $A \in \mathbb{F}_0(X)$ and $\alpha \in (0, 1]$, conjugates $(\mathbb{F}(X), \Phi_g)$ to $(\mathbb{F}(Y), \Psi_g)$.

If h is only a semi-conjugacy, then H is a semi-conjugacy too.

Proof. Let the assumptions be fulfilled. It is obvious that the map H is correctly defined, since any α-cut of any upper semi-continuous fuzzy set is closed, any upper semicontinuous fuzzy set $A \in \mathbb{F}_0(X)$ is mapped to an upper semicontinuous fuzzy set, and the uniform continuity of the set-valued map $\tilde{h}([A]_\alpha) = h([A]_\alpha)$ on *supp*(A) assures that *supp*$(H(A))$ is also compact.

First we prove that $H : (\mathbb{F}(X), \tau_\infty) \to (\mathbb{F}(Y), \tau_\infty)$ is continuous. Since the continuity of H at $\emptyset_X$ is obvious, we follow the proof of Theorem 6. For any sequence $\{A_n\} \subseteq \mathbb{F}_0(X)$ converging to $A \in \mathbb{F}_0(X)$ there is a compact set $K \subseteq X$ containing support of all fuzzy sets A_n. Then, for any $\alpha \in (0, 1]$, the uniform continuity of the set-valued map $\tilde{h} : K \to h(K)$ assures that the sequence $\{A_n\}_{n\in\mathbb{N}}$ converges to A uniformly in each α-cut and, consequently, $\{H(A_n)\}_{n\in\mathbb{N}}$ converges to $H(A)$ in the topology τ_∞. Moreover, the surjectivity of h (and of the set-valued map $\tilde{h}$, respectively) easily implies that H is surjective too.

To finish the proof it is necessary to show that for any $A \in \mathbb{F}_0(X)$,

$$H(\Phi_g(A)) = \Psi_g(H(A)). \tag{7.1}$$

We show that (7.1) coincide in each α-cut where $\alpha \in (0, 1]$. Thus let $\alpha \in (0, 1]$ and $A \in \mathbb{F}_0(X)$ be fixed. By the definition of g-fuzzification (see also Lemma 8),

$$[\Phi_g(A)]_{g(\alpha)} = \varphi([A]_\alpha)$$

and by the definition of the map H,

$$[H(B)]_{g(\alpha)} = h([B]_{g(\alpha)}).$$

Consequently,

$$[H(\Phi_g(A))]_{g(\alpha)} = h([\Phi_g(A)]_{g(\alpha)}) = h(\varphi([A]_\alpha)). \tag{7.2}$$

Analogously we get

$$[\Psi_g(H(A))]_{g(\alpha)} = \psi([H(A)]_\alpha = \psi(h([A]_\alpha)). \tag{7.3}$$

Finally since h is a conjugacy ($\psi \circ h = h \circ \varphi$)

$$h(\varphi([A]_\alpha)) = \psi(h([A]_\alpha)) \text{ for any } \alpha \in (0,1].$$

Consequently, (7.2) and (7.3) imply (7.1).

In fact we have shown that $H : (\mathbb{F}(X), \tau_\infty) \to (\mathbb{F}(Y), \tau_\infty)$ is a semi-conjugacy whenever $h : X \to Y$ is a semi-conjugacy. To prove that $H : (\mathbb{F}(X), \tau_\infty) \to (\mathbb{F}(Y), \tau_\infty)$ is a conjugacy, we can analogously use the continuity and the surjectivity of $h^{-1} : Y \to X$ and we get that the map $H^{-1} : (\mathbb{F}(Y), \tau_\infty) \to (\mathbb{F}(X), \tau_\infty)$ is also continuous and surjective. The validity of (7.1) is preserved. □

Theorem 10 *Let (X, φ), (Y, ψ) be discrete dynamical systems (semi-)conjugated by $h : X \to Y$. Let a map $H : \mathbb{F}(X) \to \mathbb{F}(Y)$ be defined by*

$$H(\emptyset_X) = \emptyset_Y, \text{ and } [(H(A))]_\alpha = h([A]_\alpha))$$

for any $A \in \mathbb{F}_0(X)$ and $\alpha \in (0,1]$.

(i) If $g \in C_m(I)$ then H (semi-)conjugates $((\mathbb{F}(X), d_E), \Phi_g)$ to $((\mathbb{F}(Y), d_E), \Psi_g)$.

(ii) If $g \in C'_m(I)$ then H (semi-)conjugates $((\mathbb{F}_0(X), d_S), \Phi_g)$ to $((\mathbb{F}_0(Y), d_S), \Psi_g)$.

Proof. Let the assumptions be fulfilled. First we show that $H : (\mathbb{F}(X), \tau) \to (\mathbb{F}(Y), \tau)$ is continuous if τ is either τ_S or τ_E. Consider the map $\kappa := (h, id_I) : X \times I \to Y \times I$. Since κ is continuous the set-valued map $\tilde{\kappa} : \mathbb{K}(X \times I) \to \mathbb{K}(Y \times I)$ is also continuous in the metric topology induced by the Hausdorff metric.

Since the metric d_S on $\mathbb{F}_0(X)$ is induced from the Hausdorff metric on $\mathbb{K}(X \times I)$, the set $B := \{send(A) \mid A \in \mathbb{F}(X)\}$ is a subset of $\mathbb{K}(X \times I)$. Clearly, since there is a one-to-one correspondence between fuzzy sets and their sendographs, $H : (\mathbb{F}_0(X), \tau_S) \to (\mathbb{F}_0(Y), \tau_S)$ can be considered as $\tilde{\kappa}|_B$. Finally, the continuity of $\tilde{\kappa}|_B$ implies the continuity of $H : (\mathbb{F}_0(X), \tau_S) \to (\mathbb{F}_0(Y), \tau_S)$.

Now, let $\tau = \tau_E$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence convergent to A_0 in τ. Then, there are two possibilities. If $A_0 = \emptyset_X$, then the continuity of H is easily obtained by the definition of H. If $A_0 \neq \emptyset_X$, then

$$B = \overline{\bigcup_{n\in\mathbb{N}} supp(A_n)}$$

and, consequently, $\mathbb{K}(B \times I)$ are also compact. Moreover, $\tilde{\kappa} : \mathbb{K}(B \times I) \to \mathbb{K}(\varphi(B) \times I)$ is continuous. Analogous to Theorem 7,

$$\tilde{\kappa}(end(A)) = end(H(A)) \text{ for } A \in \mathbb{F}(X).$$

Thus, the continuity of $\tilde{\kappa}$ implies the continuity of H on $\{A \in \mathbb{F}(X) \mid supp(A) \subseteq B\}$. Thus, by the choice of B, $\{H(A_n)\}_{n\in\mathbb{N}}$ converges to $H(A)$.

In the proof of Theorem 9, (7.1) was proved. Since the surjectivity of H is obvious, we have shown that H semiconjugates $(\mathbb{F}(X), \Phi_\Gamma)$ to $(\mathbb{F}(Y), \Psi_\Gamma)$. However, H is a conjugacy because $h : X \to Y$ is a bijection and, consequently, $\tilde{\kappa}$ is a bijection too. □

8 Possible connections to crisp discrete dynamical systems

In this section we mention some approaches to further the study of fuzzified discrete dynamical systems. Even though the notion of usual fuzzification (known also as Zadeh's extension) has been known for more than thirty years, there are only a few papers that study the connections between the original crisp discrete dynamical system and the fuzzified system. For instance, the chaotic behavior of fuzzified systems was studied when the sendograph metric on the space of fuzzy sets was considered [for instance [8]]. Other notions connected to chaotic behavior were also studied in the case of the levelwise metric [26] but, seemingly, there is no connection between these papers.

One possible connection can be elucidated by the fact that the topologies considered in this paper are comparable (see (3.1)). This immediatelly provides some relations among fuzzified systems. For instance, an immediate consequence of (3.1) is the following statement.

Proposition 3 *Let X be any locally compact metric space and $g \in C_m(I)$. If a dynamical system $((\mathbb{F}(X), \tau_\infty), \Phi_g)$ is transitive, then $((\mathbb{F}(X), \tau_E), \Phi_g)$ is also transitive.*

Proof. Following the definition of the transitivity, let U, V be any open subsets of $(\mathbb{F}(X), \tau_E)$. By (3.1), there are open subsets U', V' of $(\mathbb{F}(X), \tau_\infty)$ such that $U' \subseteq U$ and $V' \subseteq V$. Since $((\mathbb{F}(X), \tau_\infty), \Phi_g)$ is transitive, $\Phi_g^n(U') \cap V' \neq \emptyset$ for some $n \in \mathbb{N}$. Consequently, by the choice of U', V', also $\Phi_g^n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. $\square$

Now we are going to discuss two connections between fuzzy discrete dynamical systems and the original system. The first connection is easy to see from Lemma 8 and (6) – any generalized fuzzification is connected to the dynamics of the set-valued system induced by the original map via α-cuts. This approach is already known and was used, for instance, in [26] where fuzzy discrete dynamical systems induced by the usual fuzzification were studied.

Another possibility is to consider generalized fuzzy discrete dynamical systems as set-valued dynamical systems induced by suitable skew-product maps. Skew-product maps form some of the simplest extensions of original systems and have been intensively studied approximately since 1990 (see, e.g., [15] or [16] for further information). In this section, we want to show that some generalized fuzzifications can be expressed by a suitable skew-product map, and this approach allows us to simplify the computation of the fuzzified system.

In the rest of this section we identify fuzzy sets with their endographs. Following this identification, $end(A)(x) := \max\{y \in I \mid (x, y) \in endA\}$.

Proposition 4 *Let X be a locally compact metric space and $\varphi \in C(X)$. Then, for any $g \in C_m(I)$, there is a unique skew-product map $F : X \times I \to X \times I$ such that, $F(end(A))(x) = \Phi_g(A)(x)$ for any $A \in \mathbb{F}(X)$ and $x \in X$.*

Proof. Let $F : X \times I \to X \times I$ be a skew-product map for which φ is the base map and all fiber maps g_x are equal to g.

Take any $A \in \mathbb{F}(X)$ and $y \in X$. It remains to show that

$$F(end(A))(x) = \Phi_g(A)(x). \tag{8.1}$$

By the definition of g-fuzzification,

$$\Phi_g(A)(x) = \sup_{y \in \varphi^{-1}(x)} \{g(A(y))\}.$$

By the continuity of φ, $\varphi^{-1}(x)$ is closed and, consequently, there is $y_0 \in \varphi^{-1}(x)$ for which

$$\Phi_g(A)(x) = \max_{y \in \varphi^{-1}(x)} \{g(A(y))\} = g(A(y_0)). \tag{8.2}$$

On the other hand, by the choice of y_0,

$$F(end(A))(x) = \bigcup_{y \in \varphi^{-1}(x)} F(A_y) = F(A_{y_0}), \tag{8.3}$$

i.e., $F(end(A))(x) = g(A(y_0))$. Thus, (8.2) together with (8.3) imply (8.1). $\square$

However, even if we use generalized fuzzifications of a given crisp system (X, φ), their skew-product representations are very simple – in each case, all fiber maps g_x of the map F are equal to $g \in C_m(I)$. In general, skew product maps can be more complex and this could lead to the idea of a further generalization of g-fuzzifications by using skew-product maps.

We establish the definition of a skew-product fuzzification, but we also show that it makes sense to consider this kind of fuzzification only if the levelwise metric on the space of fuzzy sets is not considered. So, let a locally compact metric space X and $\varphi \in C(X)$ form a discrete dynamical system. Consider also a skew-product map $F \in T_m(X)$ for which $g_x(0) = (0)$ for any $x \in X$. Then a *skew-product* or *triangular* F-fuzzification (shortly F-fuzzification) of the original system (X, φ) is a map $\Phi_F : \mathbb{F}(X) \to \mathbb{F}(X)$ given by

$$\tilde{F}(end(A))(x) = \Phi_F(A)(x). \tag{8.4}$$

Here we study the continuities of the skew-product fuzzification $\Phi_F : (\mathbb{F}(X), \tau) \to (\mathbb{F}(X), \tau)$ if τ is equal to any of τ_S, τ_E or τ_∞. With respect to Remark 1, for the case of the sendograph topology, we consider F-fuzzifications for which all fiber maps belong to $C'_m(I)$ ($F \in T'_m(X)$). The next proposition shows that F-fuzzifications are continuous if either the sendograph or endograph topology on the space of fuzzy sets is taken. However, Example 7 shows that Φ_F need not be continuous in the case of levelwise topology.

Proposition 5 *Let X be a locally compact metric space, $\varphi \in C(X)$ and $F \in T_m(X)$ (resp. $F \in T'_m(X)$). If $\tau = \tau_E$ (resp. $\tau = \tau_S$) then the skew-product F-fuzzification $\Phi_F : \mathbb{F}(X) \to \mathbb{F}(X)$ (resp. $\Phi_F : \mathbb{F}_0(X) \to \mathbb{F}_0(X)$) is continuous if the topology τ is considered.*

Proof. If $\tau = \tau_S$ then the continuity of Φ_F is obvious, since Φ_F is a restriction of a continuous map $\tilde{F}$ to a subspace of $(\mathbb{K}(X \times I), D_{X \times I})$, namely, to the subspace of all sendographs of fuzzy sets.

Consider now the case of the endograph topology. Any sequence $\{A_n\}_{n\in\mathbb{N}}$ of fuzzy sets converges either to $A_0 = \emptyset_X$ (and the continuity of Φ_F is obvious) or $B = \overline{\bigcup_{n\in\mathbb{N}} supp(A_n)}$ is compact. Then, similar to the proof of Theorem 10 the continuity of Φ_F on $\mathbb{F}_0(X)$ is implied by the continuity of $\tilde{F}$ on $\mathbb{K}(B \times I)$. $\square$

Example 7 *Let $X = I$ and let $\varphi = id_I$. Consider any linear function $f : I \to I$ such that $f(0) = a \neq b = f(1)$ for some $a, b \in [0, 1]$. Then, for each $x \in X$, let $g_x : I \to I$ be defined as a linear function such that $g_x(0) = 0$ and $g_x(1) = f(x)$.*

Now we consider any sequence of pairwise disjoint points $\{x_n\} \subseteq X$ converging to $x_0 \in X$. Obviously, the sequence of fuzzy sets $\{\chi_{x_n}\}_{n\in\mathbb{N}}$ converges to χ_{x_0} in the levelwise topology. But, $\max(\Phi_F(\chi_{x_n})) = f(x_n)$ for any $n \in \mathbb{N}$. Thus,

$$\max\{\Phi_F(\chi_{x_n})\} \neq \max\{\Phi_F(\chi_{x_0})\}$$

for any $n \in \mathbb{N}$, and this is equivalent to

$$d_\infty(\Phi_F(\chi_{x_n}), \Phi_F(\chi_{x_0})) = diam X = 1 \text{ for any } n \in \mathbb{N},$$

i.e., Φ_F is not continuous for levelwise topology.

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