University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

Semicopula-based EQ-algebras

Moataz El-Zekey, Vilém Novák, Radko Mesiar

Research report No. 129

2008

Submitted/to appear:

Fuzzy Sets and Systems

Supported by:

Project MSM 6198898701 of the MMT ČR, SAIA

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-59-7091401 fax: +420-59-6120478
e-mail: vilem.novak@osu.cz

Semicopula-based EQ-algebras

Moataz El-Zekey[*], Vilém Novák[†] and Radko Mesiar[‡]

August 24, 2008

Abstract

Recently, a special algebra called EQ-algebra has been introduced by Vilem Novák in [31], which aims at becoming the algebra of truth values for fuzzy type theory. It has three binary operations — meet, multiplication and fuzzy equality — and a top element. The multiplication, in EQ-algebra, is assumed to be both commutative and associative. In this paper, we generalize the concept of EQ-algebra by excluding both the commutativity and the associativity of the multiplication showing that nothing is lost. We call such type of algebra a *semicopula-based EQ-algebra*. We show that all proved properties of EQ-algebras remain valid and applicable in semicopula-based EQ-algebras and vice versa. Besides these main results, a lot of new and important properties concerning (semicopula-based) EQ-algebras and their special kinds are proved.

Keywords*: EQ-algebra, residuated lattice, fuzzy equality, fuzzy logic, fuzzy type theory, higher-order fuzzy logic, semi-copula*

1 Introduction

Multiple-valued semantics underlying fuzzy logic involve various truth functions of connectives, generalizing the classical Boolean truth functions on

[*]Department of Mathematics and Descriptive Geometry, SvF, Slovak University of Technology, SK-813 68 Bratislava, Slovakia, E-mail Address: m_s_elzekey@hotmail.com, on leave from Benha High Institute of Technology, Benha University, Egypt.

[†]Institute for Research and Applications of Fuzzy Modeling (**IRAFM**), University of Ostrava, 30, dubna 22, 701 03 Ostrava, Czech Republic, E-mail Address: vilem.novak@osu.cz.

[‡]Department of Mathematics and Descriptive Geometry, SvF, Slovak University of Technology, SK-813 68 Bratislava, Slovakia, and **IRAFM**, University of Ostrava, 701 03 Ostrava, Czech Republic, E-mail Address: mesiar@math.sk.

$\{0, 1\}$, i.e. for the values 0, 1 the truth functions must behave classically (see [11] and [16]-[18]). The most widely used classes of truth functions are triangular norms (t-norms) which generalize the notions of conjunction "AND" of classical logic. T-norms are monotone, commutative, associative binary operations on the interval $[0, 1]$ and have neutral element 1 (see [23]).

However, in many practical applications of fuzzy logic it seems clear that one needs more flexibility in the choice of the conjunction: in particular, the associativity and the commutativity of a conjunction may be removed. For example, non-commutativity is important for modeling of common sense reasoning – e.g. "joyful and clever boy" has not the same meaning as "clever and joyful boy" since the stress in natural language is, in general, put to the first component. Also, as underlined in [14], in some cases one works with conjunctions just as binary connectives without any need to extend them to more arguments; as happens e.g. in the inference pattern called generalized modus ponens, then there is no need to assume associativity (see, also, [24]).

Lately, fuzzy logics with non-commutative conjunction have been quite studied extensively in both algebraic and logical form, see e.g. [7], [19], [12] and [20]. In the same spirit, recent investigations have stressed the importance of giving a mathematical foundation also to fuzzy logics with a non-associative conjunction [1], [21], [25]-[29] and [37].

Recently, a special algebra called EQ-algebra has been introduced by Vilem Novák in [31], which aims at becoming the algebra of truth values for fuzzy type theory (FTT) – a higher-order fuzzy logic (see, [30]) which continues the development of classical type theory (cf. [2]). In [35], the study of EQ-algebras have been further deepened. Moreover, the axioms originally introduced in [31] and [32] have been slightly modified. Note that the first version of FTT based on a particular involutive EQ-algebra was proposed in [33] as a generalization of the IMTL-FTT introduced in [30].

From the point of view of logic, the main difference between residuated lattices and EQ-algebras lays in the way how implication operation is obtained. While in residuated lattice, it is obtained from (strong) conjunction, in EQ-algebra it is obtained from equivalence. Though properties of both kinds of algebras are similar, they differ in several essential points.

EQ-algebra has three binary operations — meet $\wedge$, multiplication $\otimes$ and fuzzy equality $\sim$ — and a top element. The multiplication $\otimes$ is assumed to be both commutative and associative. An immediate observation on EQ-algebra is that all its essential properties (see Section 3 in [35]) were proved without any reference to the commutativity and the associativity of the multiplication $\otimes$. In fact, all proved properties of EQ-algebras hold also for the non-associative ones. The readers will notice that transitivity of $\sim$ is the only property where the commutativity of $\otimes$ is used (see [35] Theorem 1 (b)).

In this paper, we generalize the concept of EQ-algebra by excluding both the commutativity and the associativity of the multiplication $\otimes$ showing that nothing is lost. Hence, $\otimes$ can be a quite free operation close to aggregation and EQ-algebras could be even more general. Actually, we need $\otimes$ to be a semi-copula [3], [8], [9], that is, $\otimes$ is isotone w.r.t. $\leq$ with neutral element 1 (i.e., $a \otimes 1 = 1 \otimes a = a$ for all $a \in E$). The concept of semi-copula was already known, in a different context, as conjunctor (a monotone extension of the Boolean conjunction with neutral element 1) or t-seminorm [36]. Notice also that semi-copulas are binary aggregation operators with neutral element 1 (see, e.g. [6]).

We call such type of algebra a *semicopula-based EQ-algebra*, i.e. EQ-algebra in which the multiplication need not be commutative nor associative. Hence, all properties of semicopula-based EQ-algebras are also properties for EQ-algebras. Conversely, we show that all proved properties of EQ-algebras remain valid and applicable in semicopula-based EQ-algebras. Besides these main results, a lot of new and important properties concerning (semicopula-based) EQ-algebras as well as their special kinds are proved.

This paper is organized as follows. Section 2 introduces definitions and fundamental properties of semicopula-based EQ-algebras. In Section 3, several examples of semicopula-based EQ-algebras are presented, including generic ones based on residuated lattices, as well as specific discrete examples. Section 4 studies properties of some special classes of semicopula-based EQ-algebras. Section 5 is dedicated to the study of filters of semicopula-based EQ-algebras. The results are summarized in Section 6.

2 Definition and basic properties

Definition 2.1 *A* semicopula-based *EQ-algebra is an algebra*

$$\mathcal{L} = \langle E, \wedge, \otimes, \sim, 1\rangle$$

of type $(2, 2, 2, 0)$ *where for all* $a, b, c, d \in E$*:*

(E1) $\langle E, \wedge, 1\rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element 1). We put $a \leq b$ iff $a \wedge b = a$, as usual.

(E2) $\otimes$ is a semi-copula on E, i.e. $\otimes$ is isotone w.r.t. $\leq$ and $a \otimes 1 = 1 \otimes a = a$ for all $a \in E$.

(E3) $a \sim a = 1$ (reflexivity axiom)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq (c \sim (d \wedge b))$ (substitution axiom)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence axiom)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity axiom)

(E7) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$ (monotonicity axiom)

(E8) $a \otimes b \leq a \sim b$ (boundedness axiom)

The operation "$\wedge$" is called meet (infimum), "$\otimes$" is called multiplication and "$\sim$" is a fuzzy equality[1].

Note that Definition 2.1 differs from the original definition of EQ-algebras (see [35], Definition 1) in that the multiplication $\otimes$ needs not be commutative nor associative. Hence, EQ-algebras are special cases of semicopula-based EQ-algebras, in which the multiplication $\otimes$ is both commutative and associative.

Clearly, $\leq$ is the classical partial order. We will also put, for $a, b \in E$

$$\tilde{a} = a \sim 1$$

and

$$a \to b = (a \wedge b) \sim a. \tag{1}$$

The derived operation (1) will be called *implication*. Hence, we may rewrite (E6), (E7) into

$$a \to (b \wedge c) \leq a \to b \tag{2}$$
$$a \to b \leq (a \wedge c) \to b \tag{3}$$

respectively.

If $\mathcal{L}$ contains also the bottom element 0, then we may define the unary operation $\neg$ on $\mathcal{L}$ by

$$\neg a = a \sim 0, \qquad a \in E. \tag{4}$$

and call $\neg a$ a *negation* of $a \in E$.

The substitution axiom (E4) can be seen also as a special form of the extensionality (see, e.g. [18]). Note also that axioms (E6) and (E7), in fact, express isotonicity of $\to$ w.r.t. the second variable and antitonicity of $\to$ w.r.t. the first variable.

Lemma 2.1 *Let $\mathcal{L} = \langle E, \wedge, \otimes, \sim, 1\rangle$ be a semicopula-based EQ-algebra. Then the following properties are provable for all a, b, c in E:*

[1] Unlike [31] and [32], which were the first exposition of the idea, axiom (E8) have been weakened and added axiom (E5).

(i) $a \sim b = b \sim a$ (*symmetry*).

(ii) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$ (*transitivity*).

(iii) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$.

(iv) $(a \sim d) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim c)$.

(v) $(c \sim d) \otimes (a \sim b) \leq (a \sim c) \sim (b \sim d)$.

(vi) $b \otimes a \leq a \sim b$.

Proof. (i) Using axioms (E3) and (E4), we obtain
$a \sim b = ((a \wedge 1) \sim b) \otimes (a \sim a) \leq b \sim (a \wedge 1) = b \sim a$,
$b \sim a = ((b \wedge 1) \sim a) \otimes (b \sim b) \leq a \sim (b \wedge 1) = a \sim b$.
(ii) By (E4) and item (i) (the symmetry of $\sim$),we have
$(a \sim b) \otimes (b \sim c) = (b \sim a) \otimes (b \sim c) = ((b \wedge 1) \sim a) \otimes (b \sim c) \leq (a \sim (c \wedge 1)) = a \sim c$.
(iii) By (E4), we get
$a \sim d = ((a \wedge b) \sim (a \wedge b)) \otimes (a \sim d) \leq (a \wedge b) \sim (d \wedge b)$.
(iv) By properties (ii), (iii) and the monotonicity of $\otimes$, we get
$(a \sim d) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim (a \wedge b)) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim c)$.
(v) and (vi) Direct from (E5) and (E8), respectively, and the symmetry of $\sim$. ■

Theorem 2.2 *If $\mathcal{L} = \langle E, \wedge, \otimes, \sim, 1 \rangle$ is a semicopula-based EQ-algebra, then $\bar{\mathcal{L}} = \left\langle E, \wedge, \bar{\otimes}, \sim, 1 \right\rangle$ is a semicopula-based EQ-algebra, where $\bar{\otimes}$ is the symmetric of $\otimes$ (defined by $a \bar{\otimes} b = b \otimes a$).*

Proof. Just note that, by Lemma 2.1 (iv)-(vi), $\bar{\mathcal{L}} = \left\langle E, \wedge, \bar{\otimes}, \sim, 1 \right\rangle$ satisfies all the axioms ((E1)-(E8)). ■

Theorem 2.3 *Let $\mathcal{L}$ be semicopula-based EQ-algebra. Then Ψ is a provable formula in $\mathcal{L}$ if and only if $\bar{\Psi}$ is a provable formula in $\mathcal{L}$, where $\bar{\Psi}$ is a formula obtained from Ψ by substituting $\alpha \otimes \beta$ for $\beta \otimes \alpha$ in all of the occurrences of the subformulae $\beta \otimes \alpha$ in Ψ.*

Proof. This follows by complete induction on the length of the proof of Ψ, from Theorem 2.2. ■

Remark 2.1 *It is obvious that all properties of semicopula-based EQ-algebras are also properties for EQ-algebras. Conversely, by Lemma 2.1 and Theorem 2.3, all proved properties of EQ-algebras (in particular, the properties from [35]) hold also for semicopula-based EQ-algebras.*

Definition 2.2 *[35] Let $\mathcal{L}$ be semicopula-based EQ-algebra. We say that it is:*

- semiseparated *if for all $a, b \in E$,*

$$a \sim 1 = 1 \text{ implies } a = 1. \tag{E9}$$

- separated[2] *if for all $a, b \in E$,*

$$a \sim b = 1 \text{ implies } a = b. \tag{E10}$$

- *If $\mathcal{L}$ contain also the bottom element 0, then it is* spanned *if*

$$\tilde{0} = 0 \tag{E11}$$

- good *if for all $a \in E$,*

$$a \sim 1 = a. \tag{E12}$$

- residuated *if for all $a, b, c \in E$,*

$$(a \otimes b) \wedge c = a \otimes b \text{ iff } a \wedge ((b \wedge c) \sim b) = a. \tag{E13}$$

- involutive *(IEQ-algebra) if for all $a \in E$,*

$$\neg\neg a = a. \tag{E14}$$

- lattice semicopula-based EQ-algebra *(a semicopula-based* ℓEQ-algebra*) if it is a lattice and the following substitution axiom holds, for all $a, b, c, d \in E$:*

$$((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c). \tag{E15}$$

A *lattice-ordered* semicopula-based EQ-algebra is an semicopula-based EQ-algebra that is lattice. Note that a semicopula-based EQ-algebra can be lattice-ordered but not semicopula-based ℓEQ-algebra. Obviously, each separated semicopula-based EQ-algebra is semiseparated. Also, if the semicopula-based EQ-algebra is good then it is spanned but note vice-versa. Clearly, (E13) can be written in a classical way as $a \otimes b \leq c$ iff $a \leq b \to c$.

[2]In [30], axiom (E10) of the fuzzy equality $\sim$ is called 1-faithfulness. The term "separated" has been introduced earlier by Höhle [22].

2.1 Properties of EQ-algebras

The following properties have been deduced in EQ-algebras. These remain valid and applicable in semicopula-based EQ-algebras (see [35] and [31] for proofs):

Lemma 2.4 *The following properties hold in all semicopula-based EQ-algebras:*

(a) $a \otimes b \leq a \wedge b \leq a,\ b$ *and* $b \otimes a \leq a \wedge b \leq a,\ b$.

(b) $a = b$ *implies* $a \sim b = 1$.

(c) $(a \to b) \otimes (b \to c) \leq a \to c$. (*transitivity of implication*)

(d) $a \leq \tilde{a} = 1 \to a$ *and* $\tilde{1} = 1$.

(e) $a \to a = 1$. (*i.e.* $\to$ *is reflexive*)

(f) $(a \to b) \otimes (b \to a) \leq a \sim b \leq (a \to b) \wedge (b \to a)$

(g) *If* $a \leq b$ *then* $a \to b = 1$, $a \sim b = b \to a$, $\tilde{a} \leq \tilde{b}$, $c \to a \leq c \to b$ *and* $b \to c \leq a \to c$.

(h) $a \leq b \to c$ *implies* $a \otimes b \leq \tilde{c}$.

(i) $b \leq \tilde{b} \leq a \to b$.

By Lemma 2.4 (c), (e) and (f), the fuzzy relation $\to$ is a fuzzy ordering w.r.t. the fuzzy equality $\sim$ (this notion was studied extensively by Bodenhofer [5]). As mentioned in [35], we can regard an (semicopula-based) EQ-algebra as a set endowed with a classical partial order $\leq$ (and corresponding equality $=$) and a top element 1, and a fuzzy equality $\sim$ together with the a fuzzy ordering $\to$.

The following lemma characterizes a compatibility of $\sim$ with the ordering $\leq$, namely that if "distance" of elements increases (in the sense of $\leq$) then the degree of their equality decreases.

Lemma 2.5 *[35] If* $a \leq b \leq c$ *then* $c \sim a \leq c \sim b$ *as well as* $a \sim c \leq a \sim b$.

Lemma 2.6 *[35] If* $a \to b = 1$ *then* $a \leq b$ *or* $a \sim b = 1$ *or* $a||b$ *(i.e., incomparable).*

According to this simple lemma, it may happen that $a \to b = 1$, $a \sim b < 1$ and $a||b$. Other consequence is that we can have comparable elements a, b such that $a > b$, $a \sim b = 1$ and $a \to b = 1$. Such an ordered couple $\langle a, b\rangle$ will be called *pathological*. An EQ-algebra that does not contain pathological couples is called *regular*.

Let us put

$$a \leftrightarrow b = (a \to b) \wedge (b \to a), \tag{5}$$

$$a \overset{\triangle}{\leftrightarrow} b = (a \to b) \otimes (b \to a). \tag{6}$$

The following lemma characterizes induced fuzzy relations $\leftrightarrow$ and $\overset{\triangle}{\leftrightarrow}$.

Lemma 2.7 *[35] The following holds in every semicopula-based EQ-algebra $\mathcal{L}$:*

(a) $(a \wedge b) \leftrightarrow a = (a \wedge b) \overset{\triangle}{\leftrightarrow} a = a \to b$.

(b) $a \overset{\triangle}{\leftrightarrow} b \leq a \sim b \leq a \leftrightarrow b$.

(c) Both $\leftrightarrow$ as well as $\overset{\triangle}{\leftrightarrow}$ are fuzzy relations fulfilling (E3), (E4) and (E6)–(E8).

(d) If $\mathcal{L}$ is linearly ordered then $a \leftrightarrow b = a \overset{\triangle}{\leftrightarrow} b = a \sim b$.

Lemma 2.8 *[35]Let $\mathcal{L}$ be an semicopula-based EQ-algebra with the bottom element 0. For all $a, b \in E$, it holds that*

(a) $\neg 1 = \tilde{0}$, $\neg 0 = 1$.

(b) $0 \to a = 1$, $\neg a = a \to 0$.

(c) If $a \leq b$ then $\neg b \leq \neg a$.

(d) $\neg \tilde{0} = \neg\neg 1 \leq 1$.

(e) $a \otimes \neg a \leq \tilde{0}$, $\tilde{a} \otimes \tilde{0} \leq \neg a$, $\neg a \otimes \tilde{0} \leq \tilde{a}$ *and* $a \otimes 0 = 0$.

(f) $\neg a \otimes \neg b \leq a \to b$.

(g) $\tilde{0} \leq \neg a$.

Lemma 2.9 *[35]*

(i) A good semicopula-based EQ-algebra is residuated if

$$(a \otimes b) \wedge c = a \otimes b \textit{ implies } a \wedge ((b \wedge c) \sim b) = a$$

for all $a, b, c \in E$.

(ii) A semicopula-based EQ-algebra is good iff

$$a \otimes (a \sim b) \leq b$$

for all $a, b \in E$.

(iii) Each good semicopula-based EQ-algebra is spanned.

(iv) In every good semicopula-based EQ-algebra it holds that

$$a \leq (a \sim b) \sim b.$$

(v) A good semicopula-based EQ-algebra is separated and so, it does not contain pathological couples, i.e. it is regular.

(vi) In every spanned semicopula-based EQ-algebra it holds that $\neg 0 = 1 = \neg\neg 1 = \neg\tilde{0}$.

(vii) Each residuated semicopula-based EQ-algebra is good (and so, separated).

A semicopula-based EQ-algebra $\mathcal{L}$ is *complete* if it is a complete $\wedge$-semilattice. A complete semicopula-based EQ-algebra is a complete lattice-ordered (see, e.g. [4]). Every finite semicopula-based EQ-algebra is lattice-ordered.

Lemma 2.10 *[35]The following holds in every complete EQ-algebra:*

(i) $a \to \wedge_{i\in I} b_i \leq \wedge_{i\in I}(a \to b_i)$.

(ii) $\vee_{i\in I}(a_i \to b) \leq \wedge_{i\in I} a_i \to b$.

Obviously, a complete EQ-algebra is a complete lattice but not necessarily ℓEQ-algebra.

Lemma 2.11 *[35]Let* $\mathcal{L}$ *be an* ℓ*EQ-algebra. Then:*

(i) $a \to b = (a \wedge b) \sim b = (a \vee b) \sim b$.

(ii) $(a \to c) \otimes (b \to c) \leq (a \vee b \to c)$.

(iii) $((b \wedge c) \sim a) \otimes ((b \vee c) \to b) \leq c \sim a$.

(iv) $((a \wedge (b \vee c)) \sim d) \otimes (a \sim b) \leq b \sim d$.

Clearly, a complete residuated ℓEQ-algebra is a complete residuated lattice.

2.2 New properties of semicopula-based EQ-algebras

We have the following new properties of semicopula-based EQ-algebras, and hence they are also new properties of EQ-algebras.

Lemma 2.12 *The following properties are provable in all semicopula-based EQ-algebras:*

(a) $(a \sim b) \otimes (c \sim d) \leq (a \wedge c) \sim (b \wedge d)$.

(b) $a \sim d \leq ((a \wedge b) \sim c) \sim ((d \wedge b) \sim c)$.

(c) $a \sim d \leq (a \sim c) \sim (d \sim c)$.

(d) $a \sim d \leq (b \to a) \sim (b \to d)$.

(e) $a \to d \leq (b \to a) \to (b \to d)$.

(f) $b \to a \leq (a \to d) \to (b \to d)$.

(g) $a \otimes (a \sim b) \leq \tilde{b}$ *and* $(a \sim b) \otimes a \leq \tilde{b}$.

(h) $a \otimes (a \to b) \leq \tilde{b}$ *and* $(a \to b) \otimes a \leq \tilde{b}$ *(weak modus ponens).*

(i) $a \leq b \sim c$ *implies* $a \otimes b \leq \tilde{c}$ *and* $b \otimes a \leq \tilde{c}$.

(j) $a \leq b \to c$ *implies* $a \otimes b \leq \tilde{c}$ *and* $b \otimes a \leq \tilde{c}$.

(k) $(b \to c) \otimes (a \to b) \leq a \to c$.

Proof. (a) By Lemma 2.1 (iii), we have $a \sim b \leq (a \wedge c) \sim (b \wedge c)$ and $c \sim d \leq (b \wedge c) \sim (b \wedge d)$. Hence, by the order properties of $\otimes$ and transitivity of $\sim$ (Lemma 2.1 (ii)), we get

$(a \sim b) \otimes (c \sim d) \leq ((a \wedge c) \sim (b \wedge c)) \otimes ((b \wedge c) \sim (b \wedge d)) \leq (a \wedge c) \sim (b \wedge d)$.

(b) By Lemma 2.1 (iii), we have

$a \sim d \leq (a \wedge b) \sim (d \wedge b) = ((a \wedge b) \sim (d \wedge b)) \otimes (c \sim c) \leq ((a \wedge b) \sim c) \sim ((d \wedge b) \sim c)$ (by E5).

(c) Direct from (b) by putting $b = 1$.

(d) Direct from (b) by putting $c = b$.

(e) By (d), we get

$a \to d = a \sim (a \wedge d) \leq (b \to a) \sim (b \to a \wedge d) \leq (b \to a) \to (b \to a \wedge d) \leq (b \to a) \to (b \to d)$ (by monotonicity of $\to$).

(f) By (c), (E6), (E7) and the hybrid monotonicity properties of $\to$, we get the following chain of inequalities:

$b \to a = (a \wedge b) \sim b \leq ((a \wedge b) \sim ((a \wedge b) \wedge d)) \sim (b \sim ((a \wedge d) \wedge b)) \leq (a \sim (a \wedge d)) \to (b \sim (b \wedge d)) = (a \to d) \to (b \to d)$.

(g) and (h) the first inequality have been proved in [35], the second inequality follows from the first by Theorem 2.3.

(i) and (j) Direct from (g) and (h), respectively, by the monotonicity of $\otimes$.

(k) Direct from Lemma 2.4 (c) by Theorem 2.3. ■

The following Lemma is a direct consequence of the results of Lemma 2.12.

Lemma 2.13 *Let $\mathcal{L}$ be an semicopula-based EQ-algebra with bottom element 0. For all $a, b, c \in E$, it holds that*

(i) $a \to b \leq \neg b \to \neg a$. *Moreover, if it is involutive, then* $a \to b = \neg b \to \neg a$

(ii) $a \sim b \leq \neg a \sim \neg b$. *Moreover, if it is involutive, then* $a \sim b = \neg a \sim \neg b$

(iii) $(a \sim b) \otimes \neg b \leq \neg a$ *and* $\neg b \otimes (a \sim b) \leq \neg a$

(iv) $(a \to b) \otimes \neg b \leq \neg a$ *and* $\neg b \otimes (a \to b) \leq \neg a$

(v) $\neg b \leq (b \to c)$.

(vi) $a \sim d \leq \neg(a \wedge b) \sim \neg(d \wedge b)$.

Lemma 2.14 *Let $\langle b, a \rangle$ be a pathological couple. Then $(b \otimes c) \to (a \otimes c) = 1$ and $(c \otimes b) \to (c \otimes a) = 1$ hold for all $c \in E$.*

Proof. The first identity has been already proved in [35]. The second identity follows from the first one by Theorem 2.3. ■

We say that the *multiplication $\otimes$ is $\to$-isotone* if

$$a \to b = 1 \text{ implies } (a \otimes c) \to (b \otimes c) = 1$$

for all $a, b, c \in E$.

Lemma 2.15 *The following properties hold in semicopula-based EQ-algebras:*

(a) $a \to (b \to a) = 1$.

(b) $(c \to a) \otimes (c \to b) \leq c \to (a \wedge b)$.

(c) *Let* $a \to b = 1$ *and* $c \to d = 1$. *Then* $(a \wedge c) \to (b \wedge d) = 1$.

(d) *If* $a \sim b = 1$ *and* $c \sim d = 1$ *then* $(a \wedge c) \sim (b \wedge d) = 1$.

(e) *If* $a \sim b = 1$ *then* $(a \wedge c) \sim (b \wedge c) = 1$ *as well as* $(a \sim c) \sim (b \sim c) = 1$ *for all* $c \in E$.

(f) *If* $\otimes$ *is* $\rightarrow$*-isotone and* $a \sim b = 1$ *then* $(a \otimes c) \sim (b \otimes c) = 1$ *for all* $c \in E$.

Proof. (a) Since $a \leq b \rightarrow a$, we have $a \wedge (b \rightarrow a) \sim a = a \sim a = 1$.

(b) Since $c \rightarrow a = (c \wedge a) \sim c$, using Axiom (E4) we have

$$((c \wedge a) \sim c) \otimes ((c \wedge b) \sim c) \leq (c \wedge (a \wedge b) \sim c) = c \rightarrow (a \wedge b).$$

(c) From $(a \wedge c) \rightarrow a = 1$ and the assumption we obtain $(a \wedge c) \rightarrow b = 1$ as well as $(a \wedge c) \rightarrow d = 1$. The required inequality then follows from (b).

(d) Direct from Lemma 2.12 (a) and the assumption.

(e) Direct from Lemma 2.1 (iii), Lemma 2.12 (c) and the assumption.

(f) is a consequence of Lemma 2.4 (f) and the assumption. ■

Note that, in general, $\rightarrow$-isotonicity of the multiplication $\otimes$ is not provable.

Lemma 2.16 *Let* $\mathcal{L}$ *be an semicopula-based EQ-algebra.*

(a) *Let* $\mathcal{L}$ *be complete and* $[a, b] \subseteq E$ *be an interval. Then*

$$a \sim b = \bigwedge_{c \in [a,b]} ((a \sim c) \wedge (c \sim b)).$$

(b) *Let* $a \leq b \leq c$. *Then* $a \sim b = 1$ *implies* $a \sim c = b \sim c$ *and* $b \sim c = 1$ *implies* $a \sim c = a \sim b$.

(c) *Let* $a \sim b = 1$. *Then* $\tilde{a} \sim \tilde{b} = 1$. *If* $a \leq b$ *then* $\tilde{a} = \tilde{b}$.

(d) *If* $\tilde{a} = \tilde{b} = 1$ *then* $a \sim b = 1$. *If* $a \sim b = 1$ *and* $\tilde{a} = 1$ *then* $\tilde{b} = 1$.

Proof. (a) is a direct consequence of Lemma 2.5 and the properties of infimum.

(b) $a \sim c \leq b \sim c$ follows from Lemma 2.5a). Furthermore, by transitivity of $\sim$ (Lemma 2.1 (ii)) and the assumption we have $(a \sim b) \otimes (b \sim c) = (b \sim c) \leq a \sim c$. The proof of the second part is the same.

(c) The first statement follows from $(a \sim b) \otimes (1 \sim 1) = 1 \leq (a \sim 1) \sim (b \sim 1)$.The second statement follows from Lemma 2.5.

(d) follows from the previous statements. ■

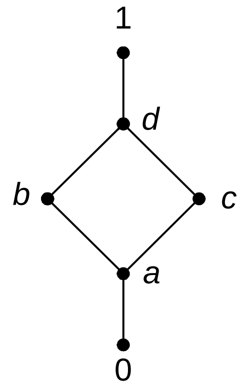

Figure 1: Six elements EQ-algebra

3 Examples of semicopula-based EQ-algebras

In [35], a specific discrete example of an (commutative and associative) EQ-algebra has been given. By slight modification of that example, we can get the following non-commutative and non-associative version of EQ-algebra:

Example 3.1 *Figure 1, a six-element semicopula-based EQ-algebra is depicted. The multiplication and the fuzzy equality are defined as follows:*

$\otimes$	**0**	**a**	**b**	**c**	**d**	**1**
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	b	b
c	0	0	a	0	a	c
d	0	0	a	a	b	d
1	0	a	b	c	d	1

$\sim$	**0**	**a**	**b**	**c**	**d**	**1**
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Note that $b \otimes d = b$ but $d \otimes b = a$, also, $c \otimes (b \otimes d) = a$ but $(c \otimes b) \otimes d = 0$. Hence, $\otimes$ is non-commutative and non-associative conjunction. Also, this algebra contains a pathological couple, namely $\langle 1, d \rangle$, and therefore not regular, and hence also not separated.

Example 3.2 *Consider $E = \{0, a, b, c, 1\}$ to be a five elements chain. The following multiplication and the fuzzy equality define a linearly ordered semicopula-based EQ-algebra.*

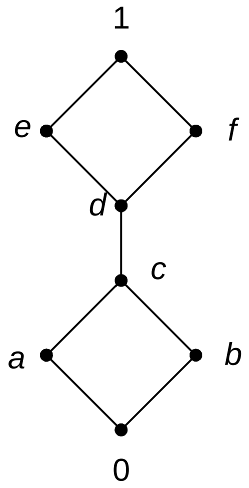

Figure 2: Eight elements IEQ-algebra

⊗	**0**	**a**	**b**	**c**	**1**
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	b	b
c	0	0	0	c	c
1	0	a	b	c	1

∼	**0**	**a**	**b**	**c**	**1**
0	1	a	0	0	0
a	a	1	a	a	a
b	0	a	1	b	b
c	0	a	b	1	c
1	0	a	b	c	1

Note that ⊗ is non-commutative but associative conjunction. This algebra is good.

Example 3.3 *[33] Example of a finite non-trivial non-residuated IEQ-algebra is the following: its (semi)lattice structure is in Figure 2. Multiplication and fuzzy equality are defined as follows:*

⊗	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

∼	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	1	e	f	d	c	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	e	d	1	f
1	0	a	b	c	d	e	f	1

Example 3.4 *The following multiplication defines a semicopula-based IEQ-algebra (its (semi)lattice structure, see Figure 2, and its fuzzy equality is the same as in Example 3.3):*

$\otimes$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	a	a	a	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

Example 3.5 *Example of a linearly ordered 6-element EQ-algebra which is not residuated is the following:*

$\otimes$	**0**	**a**	**b**	**c**	**d**	**1**
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	**0**	**a**	**b**	**c**	**d**	**1**
0	1	c	b	a	0	0
a	c	1	b	a	a	a
b	b	b	1	b	b	b
c	a	a	b	1	c	c
d	0	a	b	c	1	d
1	0	a	b	c	d	1

Example 3.6 *The following fuzzy equality defines a linearly ordered IEQ-algebra (the multiplication is the same as above):*

$\sim$	**0**	**a**	**b**	**c**	**d**	**1**
0	1	c	b	a	0	0
a	d	1	b	a	a	a
b	c	d	1	b	b	b
c	b	c	d	1	c	c
d	a	b	c	d	1	d
1	0	a	b	c	d	1

$\to$	**0**	**a**	**b**	**c**	**d**	**1**
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	d	1	1	1	1
c	b	c	d	1	1	1
d	a	b	c	d	1	1
1	0	a	b	c	d	1

There are many other examples of non-trivial finite (semicopula-based) EQ-algebras including linearly ordered ones. Therefore, in general neither non-linear nor linear (semicopula-based) EQ-algebras coincide with residuated lattices.

Let $\Lambda = (E, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ be a residuated lattice (the definition and a lot of properties of residuated lattices can be found in [34]). We may introduce two kinds of biresiduation operation:

$$a \Leftrightarrow b = (a \Rightarrow b) \wedge (a \Rightarrow b) \tag{7}$$

$$a \overset{\triangle}{\Leftrightarrow} b = (a \Rightarrow b) \otimes (a \Rightarrow b) \tag{8}$$

Both operations are natural interpretations of equivalence since they are reflexive, symmetric, and transitive in the following sense:

$$(a \boxdot b) \otimes (b \boxdot c) \leq (a \boxdot c)$$

for all $a, b, c \in E$ where $\boxdot \in \{\Leftrightarrow, \overset{\triangle}{\Leftrightarrow}\}$. The biresiduation operation (7) has been used in the development of FTT (see [30]).

Example 3.7 *Let $\Lambda = \langle E, \wedge, \vee, \otimes, \Rightarrow, 0, 1\rangle$ be a residuated lattice.*

(i) [35] The algebra $\mathcal{L}_\Lambda = \langle E, \wedge, \otimes, \Leftrightarrow, 1\rangle$ is a separated and residuated EQ-algebra. If $\mathcal{L}$ is linearly ordered then also $\overset{\wedge}{\mathcal{L}}_\Lambda = \left\langle E, \wedge, \otimes, \overset{\triangle}{\Leftrightarrow}, 1\right\rangle$ is a separated and residuated EQ-algebra (since both $\Leftrightarrow$ and $\overset{\triangle}{\Leftrightarrow}$ coincide).

(ii) Let $* \leq \otimes$ be an isotone monoidal operation on E. Then $\mathcal{L}'' = \langle E, \wedge, *, \Leftrightarrow, 1\rangle$ is a separated semicopula-based EQ-algebra in which $*$ is associative but not necessary commutative. If $* < \otimes$ then $\mathcal{L}''$ is not residuated because we can have $a * b \leq c < a \otimes b$.

(iii) Let $\odot \leq \otimes$ be a semi-copula on E. Then both $\mathcal{L}^{\#} = \langle E, \wedge, \odot, \Leftrightarrow, 1\rangle$ and $\mathcal{L}^{\#\#} = \left\langle E, \wedge, \bar{\odot}, \Leftrightarrow, 1\right\rangle$ are (not necessary commutative nor associative) semicopula-based EQ-algebras, where $\bar{\odot}$ is the symmetric of $\odot$ (defined by $a \mathbin{\bar{\odot}} b = b \odot a$).

(iv) Let (m, n) be a pair of weak negations on E (i.e. they are order reversing and satisfy, for all a, $a \leq m(n(a))$, $a \leq n(m(a))$ and $n(1) = m(1) = 0$ [15]). Define $\odot$ on E by

$$a \odot b = \begin{cases} 0, & a \leq m(b) \\ a \otimes b, & \text{otherwise} \end{cases} = \begin{cases} 0, & b \leq n(a) \\ a \otimes b, & \text{otherwise} \end{cases}$$

It is easy to see that $\odot \leq \otimes$. Also, $\odot$ need not to be commutative nor associative, hence $\mathcal{L}' = \langle E, \wedge, \odot, \Leftrightarrow, 1\rangle$ is (not necessarily commutative nor associative) semicopula-based EQ-algebra. Moreover, if the pair of weak negations (m, n) is compatible with $\otimes$ (see [15]) then $\odot$ is associative and hence $\mathcal{L}'$ is associative (but not commutative) semicopula-based EQ-algebra. Furthermore, if $n = m$ then $\odot$ will be commutative.

Note that the semicopula-based EQ-algebras in Example 3.7 (ii)-(iv) are non-residuated (even if the multiplication are both commutative and associative). These examples give us a justification why in EQ-algebras adjunction property which strictly couples $\rightarrow$ and $\otimes$ is relaxed. The following example is an instance of Example 3.7:

Example 3.8 *Let $E = [0,1]$ and define a product $\otimes$ and a residuum $\Rightarrow$ on E as follows:*

$$a \otimes b = \begin{cases} 0, & a+b \leq \frac{1}{2} \\ \min\{a,b\}, & a+b > \frac{1}{2} \end{cases}$$

$$a \Rightarrow c = \begin{cases} 1, & a \leq c \\ \max\{\frac{1}{2}-a, c\}, & a > c \end{cases}$$

Then $\Lambda = \langle E, \wedge, \vee, \otimes, \Rightarrow, 0, 1 \rangle$ is a residuated lattice.

Define $\odot$ and $\bar{\odot}$ on $[0,1]$ by [27]

$$a \odot b = \begin{cases} 0, & 2a+b \leq 1 \\ \min\{a,b\}, & 2a+b > 1 \end{cases}, \qquad a \bar{\odot} b = \begin{cases} 0, & a+2b \leq 1 \\ \min\{a,b\}, & a+2b > 1 \end{cases}$$

Both $\odot$ and $\bar{\odot}$ are isotone monoidal operations on $[0,1]$, but they are not commutative. It is direct to see that $\odot$, $\bar{\odot} \leq \otimes$. Also, it is direct to see that

$$a \Leftrightarrow c = a \overset{\triangle}{\Leftrightarrow} c = \begin{cases} 1, & a = c \\ \max(\frac{1}{2}-a, c), & a > c \\ \max(\frac{1}{2}-c, a), & a < c \end{cases}$$

Hence, $\mathcal{L} = \langle E, \wedge, \otimes, \Leftrightarrow, 1 \rangle$ is an EQ-algebra while both $\mathcal{L}' = \langle E, \wedge, \odot, \sim, 1 \rangle$ and $\mathcal{L}'' = \left\langle E, \wedge, \bar{\odot}, \sim, 1 \right\rangle$ are associative semicopula-based EQ-algebras.

Moreover, Let $* : [0,1]^2 \rightarrow [0,1]$ be the nilpotent minimum of Fodor [13], given by

$$a * b = \begin{cases} 0, & a+b \leq 1 \\ \min\{a,b\}, & a+b > 1 \end{cases}$$

It is direct to see that $* \leq \otimes$, hence $\mathcal{L}^{\#} = \langle E, \wedge, *, \Leftrightarrow, 1 \rangle$ is a separated EQ-algebra.

4 Properties of special semicopula-based EQ-algebras

Lemma 4.1 *The following properties are provable in all good semicopula-based EQ-algebras:*

(a) $a \leq (a \to b) \to b$.

(b) $a \leq b \to c$ *iff* $b \leq a \to c$

(c) $a \to (b \to c) \leq (a \otimes b) \to c$ *and* $a \to (b \to c) \leq (b \otimes a) \to c$.

(d) $a \to (b \to c) = b \to (a \to c)$ *(Exchange principle).*

(e) *For all indexed families* $\{a_i\}$ *in* E*, provided that* $\{a_i\}$ *has supremum in* E*, we have*

$$\bigvee_i a_i \to c = \bigwedge_i (a_i \to c).$$

(f) $(a \sim b) \otimes a \leq a \wedge b$ *and* $a \otimes (a \sim b) \leq a \wedge b$.

(g) $(a \to b) \otimes a \leq a \wedge b$ *and* $a \otimes (a \to b) \leq a \wedge b$.

(h) $a \leq b \to c$ *implies* $a \otimes b \leq c$ *and* $b \otimes a \leq c$.

Proof. (a) In good EQ-algebra, $a \leq (a \sim b) \sim b$. Hence,
$a \leq (a \sim (a \wedge b)) \sim (a \wedge b) \leq (a \to b) \to (a \wedge b) \leq (a \to b) \to b$.

(b) $a \leq b \to c$ implies $a \to c \geq (b \to c) \to c \geq b$ (by (a)). Similarly, $b \leq a \to c$ implies $a \leq b \to c$.

(c) By (a), we have
$b \leq (b \to c) \to c$ and $a \leq (a \to (b \to c)) \to (b \to c)$. Hence, by monotonicity of $\otimes$ and the transitivity of $\to$, we get
$a \otimes b \leq ((a \to (b \to c)) \to (b \to c)) \otimes ((b \to c) \to c) \leq (a \to (b \to c)) \to c$. Hence, by (b) we get the first inequality. The second inequality follows from the first one by Theorem 2.3.

(d) By (a) and the order properties of $\to$, we get
$a \to (b \to c) \geq ((a \to c) \to c) \to (b \to c) \geq b \to (a \to c)$ (By Lemma 2.12 (e)).

(e) Suppose $\bigvee_i a_i$ exist in E. By order properties of $\to$, we find that $\bigvee_i a_i \to c$ is a lower bound in E for the indexed family $\{a_i \to c\}_i$. Let δ be another lower bound for this family. Then $\delta \leq a_i \to c$ for all i, equivalently, using item (b), $\forall i$, $a_i \leq \delta \to c$, that is $\bigvee_i a_i \leq \delta \to c$. A second use of (b) yields $\delta \leq \bigvee_i a_i \to c$. This completes the proof that $\bigvee_i a_i \to c$ is the infimum in E for the indexed family $\{a_i \to c\}_i$.

(f), (g) and (h) are direct from Lemma 2.12 (g), (h) and (j), respectively.
■

The following Lemma is a direct consequence of the results of Lemma 4.1.

Lemma 4.2 *Let $\mathcal{L}$ be a good semicopula-based EQ-algebra with the bottom element* 0*. For all $a, b, c \in E$, it holds that*

(a) $a \leq \neg b$ *iff* $b \leq \neg a$ *implies* $a \otimes b = 0$ *and* $b \otimes a = 0$.

(b) $a \leq \neg\neg a$.

(c) $\neg\neg\neg a = \neg a$

(d) $a \to \neg b = b \to \neg a$.

(e) $a \to \neg b \leq \neg(a \otimes b)$ *and* $a \otimes b \leq \neg(a \to \neg b)$.

(f) $a \otimes \neg a = 0$ *and* $\neg a \otimes a = 0$.

(g) For all indexed families $\{a_i\}$ in E, provided that $\{a_i\}$ has supremum in E, we have

$$\neg(\bigvee_i a_i) = \bigwedge_i (\neg a_i).$$

Lemma 4.3 *The following identities hold in all good semicopula-based EQ-algebras: For all $b \in E$,*

(a) $b = \bigwedge_{c\in E}((b \to c) \to c) = \bigwedge_{c\in E}((b \sim c) \sim c)$.

(b) $b = \bigvee_{a\in E}((a \to b) \otimes a) = \bigvee_{a\in E}(a \otimes (a \to b))$.

(c) $b = \bigvee_{a\in E}((a \sim b) \otimes a) = \bigvee_{a\in E}(a \otimes (a \sim b))$.

Proof. (a) $b \leq \bigwedge_{c\in E}((b \to c) \to c)$ (by Lemma 4.1 (a)) $\leq (b \to b) \to b = 1 \to b = b$. Hence, equality holds. The proof of the second identity is similar using Lemma 2.9 (iv).

(b) $b = (b \to b) \otimes b \leq \bigvee_{a\in E}((a \to b) \otimes a) \leq b$ (by Lemma 4.1 (g)). The proof of the second identity follows from the first by Theorem 2.3.

(c) The proof is similar to (b) using Lemma 4.1 (f). ■

Lemma 4.4 *If $\mathcal{L} = \langle E, \wedge, \otimes, \sim, 1\rangle$ is a residuated semicopula-based EQ-algebra, then, for all $a, b, c \in E$*

(a) $a \leq b \to c$ *iff* $b \leq a \to c$ *iff* $a \otimes b \leq c$.

(b) $a \leq b \to (a \otimes b)$ *and* $a \leq b \to (b \otimes a)$.

(c) $a \to (b \to c) = (a \otimes b) \to c$.

(d) $b \to c \leq (a \otimes b) \to (a \otimes c)$.

(e) $a \otimes b = b \otimes a$ *(i.e.* $\otimes$ *is commutative).*

(f) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ *(i.e.* $\otimes$ *is associative).*

(g) For all indexed families $\{c_i\}$ *in* E*, provided that* $\{c_i\}$ *has infimum in* E*, we have*
$$a \to \bigwedge_i c_i = \bigwedge_i (a \to c_i).$$

(h) $\mathcal{L}$ *is a residuated EQ-algebra.*

Proof. Firstly, a residuated EQ-algebra is good (see Lemma 2.9 (vii)).
(a) From the assumption and Lemma 4.1 (b).
(b) By applying (a) on the trivial identity $a \otimes b = a \otimes b$.
(c) By Lemma 4.1 (c), we need only to prove that
$$(a \otimes b) \to c \leq a \to (b \to c). \tag{*}$$
By (b), Lemma 4.1 (d), Lemma 4.1 (a) and the order properties of $\to$, we have
$$a \leq b \to (a \otimes b) \leq b \to (((a \otimes b) \to c) \to c) = ((a \otimes b) \to c) \to (b \to c).$$
This renders (*) via residuation.
(d) By (b), the order properties of $\to$, (c) and Lemma 4.1 (d), we get
$b \to c \leq b \to (a \to (a \otimes c)) = (a \otimes b) \to (a \otimes c)$.
(e) By (a), Commutativity is direct by the following equivalences:
$a \otimes b \leq c$ iff $a \leq b \to c$ iff $b \leq a \to c$ iff $b \otimes a \leq c$.
(f) By (b) and (d), we get
$c \leq b \to (b \otimes c) \leq (a \otimes b) \to (a \otimes (b \otimes c))$.
Hence, by (a), we get $(a \otimes b) \otimes c \leq a \otimes (b \otimes c)$.
Thus, by Theorem 2.3, we have $(a \bar{\otimes} b) \bar{\otimes} c \leq a \bar{\otimes} (b \bar{\otimes} c)$; that is $a \otimes (b \otimes c) \leq (a \otimes b) \otimes c$. This completes the proof.
(g) By (a), the proof follows the same steps as in the case of Lemma 4.1 (e).
(h) Direct from (e) and (f). ■

Notice that, by Lemma 4.4 (h), a residuated semicopula-based EQ-algebra is a residuated EQ-algebra, i.e. a commutative and associative one. Hence, we have the following corollary from [35].

Corollary 4.5 *Let* $\mathcal{L} = \langle E, \wedge, \vee, \otimes, \sim, 1, 0 \rangle$ *be a (complete) residuated* ℓ*EQ-algebra with the bottom element. Then* $\mathcal{L}' = \langle E, \wedge, \vee, \otimes, \to, 1, 0 \rangle$ *is a (complete) residuated lattice where* $\to$ *is defined by* $a \to b = (a \wedge b) \sim a$.

Note that, by Lemma 4.4 and Corollary 4.5, it turns out that EQ-algebra (commutative or not) gives rise to commutative residuated lattices. This is logically justified since the fuzzy equality $\sim$ remains symmetric and so, there is only one implication derived from it - unlike non-commutative residuated lattices where there are always two of them and consequently, formal fuzzy logic based on such algebras has also two implications. This gives us an indication for a possible further generalization of the concept of EQ-algebra via dropping the symmetry of the fuzzy equality $\sim$. As we shall show in a future article, if non-symmetric fuzzy equality $\sim$ is allowed, its implication $\rightarrow$ would have split into a right implication $\rightarrow_R$ and a left implication $\rightarrow_L$. Accordingly, in some of its occurrences, $\rightarrow$ would have to be replaced by $\rightarrow_R$, and in other ones by $\rightarrow_L$. A similar splitting would take place in the fuzzy equality $\sim$, causing the whole algebra to be even more complicated. We prefer to postpone this added complexity, until an actual need for it materializes.

Lemma 4.6 *The following properties are provable in all lattice-ordered and good semicopula-based EQ-algebras:*

(a) $a \vee b \leq ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$.

(b) $a \vee b \leq ((a \sim b) \rightarrow b) \wedge ((a \sim b) \rightarrow a)$.

(c) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$.

(d) $(a \rightarrow c) \otimes (b \rightarrow c) \leq (a \vee b) \rightarrow c$.

Proof. (a) By Lemma 4.1 (a) and the properties of $\rightarrow$ we have $a \leq (a \rightarrow b) \rightarrow b$ as well as $b \leq (a \rightarrow b) \rightarrow b$ which gives $a \vee b \leq (a \rightarrow b) \rightarrow b$. Similarly, we prove that $a \vee b \leq (b \rightarrow a) \rightarrow a$ which gives the required inequality.

(b) Direct from (a) by the fact that $a \sim b \leq a \rightarrow b$ and the order properties of $\rightarrow$.

(c) Direct from Lemma 4.1 (e).

(d) By (c), $(a \rightarrow c) \otimes (b \rightarrow c) \leq (a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$. ■

Lemma 4.7 *Let $\mathcal{L}$ be a lattice-ordered and good semicopula-based EQ-algebra with bottom element* 0. *For all $a, b, c \in E$, it holds that*

(a) $\neg(a \vee b) = \neg a \wedge \neg b$.

(b) $a \vee b \leq \neg(\neg a \wedge \neg b)$.

(c) $\neg a \vee \neg b \leq \neg(a \wedge b)$.

(d) $a \sim b \leq (\neg a \wedge \neg c) \sim (\neg b \wedge \neg c) = \neg(a \vee c) \sim \neg(b \vee c)$.

(e) $((a \vee b) \sim c) \otimes (d \sim a) \leq \neg(d \vee b) \sim \neg c$.

(f) $(d \sim a) \otimes ((a \vee b) \sim c) \leq \neg(d \vee b) \sim \neg c$.

Proof. (a) Direct from Lemma 4.6 (c).
(b) Direct from (a) and Lemma 4.2 (b).
(c) Direct from Lemma 2.10 (ii).
(d) By Lemma 2.13 (ii), Lemma 2.1 (iii) and item (a), we get the following
$a \sim b \leq \neg a \sim \neg b \leq (\neg a \wedge \neg c) \sim (\neg b \wedge \neg c) \leq \neg(a \vee c) \sim \neg(b \vee c)$.
(e) By Lemma 2.13 (ii), item (a) and Axiom (E4), we get the following
$((a \vee b) \sim c) \otimes (d \sim a) \leq (\neg(a \vee b) \sim \neg c) \otimes (\neg d \sim \neg a) = ((\neg a \wedge \neg b) \sim \neg c) \otimes (\neg d \sim \neg a) \leq (\neg d \wedge \neg b) \sim \neg c = \neg(d \vee b) \sim \neg c$.
(f) Direct from (e) and Theorem 2.3. ■

Lemma 4.8 *The following properties are provable in all semicopula-based ℓEQ-algebras:*

(a) $a \sim d \leq (a \vee b) \sim (d \vee b)$.

(b) $(a \sim d) \otimes ((a \vee b) \sim c) \leq ((d \vee b) \sim c)$.

(c) $(a \sim b) \otimes (c \sim d) \leq (a \vee c) \sim (b \vee d)$.

Proof. (a) By (E15), we get
$a \sim d = ((a \vee b) \sim (a \vee b)) \otimes (a \sim d) \leq (a \vee b) \sim (d \vee b)$.
(b) By (a), the transitivity of $\sim$ and the monotonicity of $\otimes$, we get
$(a \sim d) \otimes ((a \vee b) \sim c) \leq ((d \vee b) \sim (a \vee b)) \otimes ((a \vee b) \sim c) \leq ((d \vee b) \sim c)$.
(c) By (a), we have $a \sim b \leq (a \vee c) \sim (b \vee c)$ and $c \sim d \leq (b \vee c) \sim (b \vee d)$. Hence, by the monotonicity of $\otimes$ and transitivity of $\sim$ (Lemma 2.1 (ii)), we get
$(a \sim b) \otimes (c \sim d) \leq ((a \vee c) \sim (b \vee c)) \otimes ((b \vee c) \sim (b \vee d)) \leq (a \vee c) \sim (b \vee d)$. ■

Proposition 4.9 *The following is equivalent.*

(i) A semicopula-based EQ-algebra $\mathcal{L}$ is separated.

(ii) $a \leq b$ *iff* $a \to b = 1$ *for all* $a, b \in E$.

Proof. (i) implies (ii): is proved in [35]. Conversely, let (ii) be hold and assume that $a \sim b = 1$. Hence, by the properties of $\rightarrow$, $a \sim b \leq a \rightarrow b$ and $a \sim b \leq b \rightarrow a$. Thus, $a \rightarrow b = 1$ and $b \rightarrow a = 1$, that is, $a \leq b$ and $b \leq a$ which gives $a = b$. ■

This means that the implication operation $\rightarrow$ in a separated EQ-algebra precisely reflects the ordering $\leq$ and so, the multiplication $\otimes$ is $\rightarrow$-isotone in it.

Proposition 4.10 *The following is equivalent.*

(i) A semicopula-based EQ-algebra $\mathcal{L}$ is good.

(ii) $a \otimes (a \sim b) \leq b$ for all $a, b \in E$.

(iii) $a \otimes (a \rightarrow b) \leq b$ for all $a, b \in E$.

(iv) $1 \rightarrow b = b$ for all $b \in E$.

Proof. (i) $\iff$ (ii): is proved in [35] (see Lemma 2.9 (ii)). Also, by Lemma 4.1 (g), (i) implies (iiii).

(iii) implies (ii) and (iv): Assume (iii) holds, hence, $a \otimes (a \sim b) \leq a \otimes (a \rightarrow b) \leq b$. Also, $b \leq 1 \rightarrow b = 1 \otimes (1 \rightarrow b) \leq b$, i.e. $1 \rightarrow b = b$.

(iv) implies (i): $b = 1 \rightarrow b = (1 \wedge b) \sim 1 = b \sim 1$. ■

Proposition 4.11 *The following is equivalent.*

(i) A semicopula-based EQ-algebra $\mathcal{L}$ is residuated.

(ii) A semicopula-based EQ-algebra $\mathcal{L}$ is good and

$$(a \otimes b) \rightarrow c \leq a \rightarrow (b \rightarrow c)$$

holds for all $a, b, c \in E$.

(iii) A semicopula-based EQ-algebra $\mathcal{L}$ is separated and

$$(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c)$$

holds for all $a, b, c \in E$.

(iv) A semicopula-based EQ-algebra $\mathcal{L}$ is good and

$$a \rightarrow b \leq (a \otimes c) \rightarrow (b \otimes c)$$

holds for all $a, b, c \in E$.

(v) A semicopula-based EQ-algebra $\mathcal{L}$ is good and

$$a \leq b \to (a \otimes b)$$

holds for all $a, b \in E$.

Proof. (i) implies (ii), (iii), (iv) and (v): By Lemma 4.4, Lemma 2.9 (vii) and Lemma 4.1 (c).

(ii) implies (iii): Direct by Lemma 4.1 (c) and the fact that a good EQ-algebra is separated.

(iii) implies (i) and, therefore, also implies (iv) and (v).

(iv) implies (v): Let (iv) hold. Then, $a = 1 \to a \leq (1 \otimes b) \to (a \otimes b) = b \to (a \otimes b)$.

(v) implies (i): Let (v) hold and assume that $a \otimes b \leq c$. Hence, by the assumption and the order properties of $\to$, we get $b \to c \geq b \to a \otimes b \geq a$. On the other hand, since $\mathcal{L}$ is good, $a \leq b \to c$ implies $a \otimes b \leq c$ (by Lemma 4.1 (h)). ■

Lemma 4.12 *Let $\mathcal{L}$ be a good semicopula-based EQ-algebra. Then the following holds for all $a, b \in E$:*

(a) Axiom (E8) is provable from the other axioms.

(b) If $0 \in E$ and $a \sim b = \neg a \sim \neg b$ then $a = \neg\neg a$.

(c) $a \otimes (b \to c) \leq b \to (a \to c)$.

(d) The multiplication $\otimes$ is $\to$-isotone.

Proof. (a) follows immediately from axiom (E5) by putting $b = d = 1$ and using the fact that $\mathcal{L}$ is good.

(b) Since $\mathcal{L}$ is good, $a \leq \neg\neg a$ by Lemma 4.2 (b). Vice-versa, using axiom (E8), the fact that $\mathcal{L}$ is good and the assumption we obtain

$\neg\neg a = ((a \sim 0) \sim 0) \otimes (1 \sim 1) \leq ((a \sim 0) \sim 1) \sim (0 \sim 1) = (\neg a \sim 1) \sim \neg 1 = \neg a \sim \neg 1 = a \sim 1 = a$.

(c) Direct from Axiom (E8) and Lemma 4.1 (d).

(d) follows immediately from Lemma 2.9 (v). ■

Note that the proof of (b) requires $\mathcal{L}$ to be good. Therefore, the properties of contraposition and double negation are not, in general, equivalent in EQ-algebras.

Proposition 4.13 *Let $\mathcal{L}$ be a good and complete semicopula-based EQ-algebra. Put*

$$a \odot b = \bigwedge\{c | a \leq b \to c\}. \tag{9}$$

Then $\odot$ is a commutative semi-copula and $\otimes \leq \odot$. Moreover, if $\to$ satisfies the following identity:

$$a \to \bigwedge_{j \in J} b_j = \bigwedge_{j \in J} (a \to b_j) \tag{10}$$

for all subfamilies $\{b_j\}_{j \in J}$ of E. Then $\odot$ is associative and, hence, $\mathcal{L}' = \langle E, \wedge, \vee, \odot, \to, 0, 1\rangle$ is a complete residuated lattice and $\mathcal{L}'' = \langle E, \wedge, \vee, \odot, \sim, 0, 1\rangle$ is a complete residuated EQ-algebra.

Proof. Monotonicity of $\odot$ follows immediately from the order properties of $\to$. Commutativity follows immediately from Lemma 4.1 (b). Also, $a \odot 1 = \bigwedge\{c | a \leq c\} = a$. Now, let us denote $C = \{c | a \leq b \to c\}$. By Lemma 4.1 (h), $a \otimes b \leq c$ holds for every $c \in C$, i.e, $a \otimes b \leq \bigwedge C = a \odot b$.

Now, if $\to$ satisfies (10) then, by Lemma 4.1 (e), $\to$ satisfies following identity:

$$\bigvee_{i \in I} a_i \to \bigwedge_{j \in J} b_j = \bigwedge_{i,j} (a \to b_j) \tag{11}$$

for all subfamilies $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ of E. The rest follows from (11), the order properties of $\to$ (Lemma 2.4(g)), Proposition 4.9 (ii), Proposition 4.10 (iv) and the Exchange principle (Lemma 4.1(d)), it can be derived from the known results in literature (see, e.g. [29] and [10])). ■

Of course, if $\mathcal{L}$ in Proposition 4.13 is residuated then $\otimes = \odot$.

Lemma 4.14 *[33] Let $\mathcal{L}$ be a semicopula-based IEQ-algebra.*

(a) Put $a \vee b = \neg(\neg a \wedge \neg b)$. Then $\vee$ is a supremum of a, b and so, $\mathcal{L}$ is a lattice.

(b) $a \sim b = \neg a \sim \neg b$.

(c) $\mathcal{L}$ is good, spanned and separated semicopula-based ℓEQ-algebra.

(d) $\neg b \to \neg a = a \to b$.

Proof. (a) Since $\neg a \wedge \neg b \leq \neg a$, $\neg b$, we have $a, b \leq a \vee b$. Let $a, b \leq c \leq a \vee b$. Then $\neg a \wedge \neg b \leq \neg c \leq \neg a$, $\neg b$ and we conclude that $c = a \vee b$.

(b) Direct from Lemma 2.13 (ii).

(c) We will show that $\mathcal{L}$ is good and so, it is also spanned and separated. For any $a \in E$ it holds that

$a \leq \tilde{a} = (a \sim 1) \otimes (0 \sim 0) \leq (a \sim 0) \sim (0 \sim 1) = \neg a \sim 0 = \neg\neg a = a$.

Furthermore, using Lemma 4.7 (e) and item (b), property (E15) is fulfilled.

(d) By Lemma 4.2 (d), we get

$\neg b \to \neg a = a \to \neg\neg b = a \to b$. ■

Clearly, every semicopula-based IEQ-algebra is also regular.

Lemma 4.15 *Let $\mathcal{L}$ be a semiseparated semicopula-based EQ-algebra.*

(a) If $a = 1$ and $a \to b = 1$ then $b = 1$.

(b) $\mathcal{L}$ contains no pathological couple $\langle 1, a\rangle$. On the other hand, each regular semicopula-based EQ-algebra is semiseparated.

Proof. (a) follows from Lemma 2.12 (h) and semiseparateness.

(b) is obvious. ■

The property (a) is an algebraic counterpart of classical modus ponens.

5 Filters in semicopula-based EQ-algebras

Since EQ-algebras are not necessary commutative, the definition of a filter in [35] is slightly modified as follows:

Definition 5.1 *Let $\mathcal{L} = \langle E, \wedge, \otimes, \sim, 1\rangle$ be a semicopula-based EQ-algebra. A subset $F \subset E$ is called a* filter *of $\mathcal{L}$ if for all $a, b, c \in E$ it holds that:*

(i) $1 \in F$,

(ii) if $a, b \in F$ then $a \otimes b \in F$,

(iii) if $a, a \to b \in F$, then $b \in F$,

(iv) if $a \to b \in F$ then $(a \otimes c) \to (b \otimes c) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$

Note that the above definition differs from the original one (see [35]), in that it requires $\otimes$ to satisfy the two conditions $(a \otimes c) \to (b \otimes c) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$ rather than one condition as in [35].

Lemma 5.1 *The singleton $\{1\}$ is a filter in any good semicopula-based EQ-algebra, and is contained in any other filter.*

Proof. Let $F = \{1\}$. It is obvious that properties (i) and (ii) in the filter definition are fulfilled. If both a and $a \to b$ are in F, then $a = 1$ and $a \to b = 1$. Thus, by Lemma 4.15 (b), $b = 1 \in F$, i.e. property (iii) is fulfilled. Also, If $a \to b \in F$ then $a \to b = 1$. Consequently, $a \leq b$ (by the fact that a good EQ-algebra is separated) and, hence, $(a \otimes c) \leq (b \otimes c)$ and $(c \otimes a) \leq (c \otimes b)$ (by monotonicity of $\otimes$). Thus, $(a \otimes c) \to (b \otimes c) = 1 \in F$ and $(c \otimes a) \to (c \otimes b) = 1 \in F$. ■

Lemma 5.2 *[35] Let F be a filter of a semicopula-based EQ-algebra $\mathcal{L}$. For all $a, b \in E$ it holds that*

(a) If $a \in F$ and $a \leq b$ then $b \in F$.

(b) If $a, a \sim b \in F$, then $b \in F$.

(c) If $0 \in E$ then a filter $F \subset E$ is proper iff $0 \notin F$.

Lemma 5.3 *Let F be a filter of a semicopula-based EQ-algebra $\mathcal{L} = \langle E, \wedge, \otimes, \sim, 1 \rangle$, then the following four conditions are equivalent:*

(i) $a \sim b \in F$.

(ii) $a \overset{\triangle}{\leftrightarrow} b \in F$.

(iii) $a \leftrightarrow b \in F$.

(iv) $a \to b \in F$ and $b \to a \in F$.

Proof. (i) $\Rightarrow$ (ii) and (iii): are proved in [35].
(ii) $\Rightarrow$ (i) and (iii): by the fact that $a \overset{\triangle}{\leftrightarrow} b \leq a \sim b \leq a \leftrightarrow b$.
(iii) $\Rightarrow$ (iv): by the fact that $a \leftrightarrow b \leq a \to b$ and $a \leftrightarrow b \leq b \to a$.
(iv) $\Rightarrow$ (ii): direct by property (ii) of a filter. ■

Lemma 5.4 *Let F be a filter of a semicopula-based EQ-algebra $\mathcal{L}$, $a \sim b \in F$ and $a' \sim b' \in F$. Then the following holds true:*

(a) $a \overset{\triangle}{\leftrightarrow} b \in F$, $a \leftrightarrow b \in F$ and

(b) $(a \wedge a') \sim (b \wedge b') \in F$.

(c) $(a \sim a') \sim (b \sim b') \in F$.

(d) $(a \otimes c) \sim (b \otimes c) \in F$ for all $c \in E$.

(e) $(c \otimes a) \sim (c \otimes b) \in F$ for all $c \in E$.

(f) If $\mathcal{L}$ is a semicopula-based ℓEQ-algebra then $(a \vee a') \sim (b \vee b') \in F$.

Proof. (a)-(d) are proved in [35].

(e) By assumption and Lemma 5.3 (iv), it hold that $a \to b \in F$ and $b \to a \in F$, and hence, by property (iv) of a filter, also $(c \otimes a) \to (c \otimes b) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$. Thus, again by Lemma 5.3, this is equivalent to $(c \otimes a) \sim (c \otimes b) \in F$.

(f) By Lemma 4.8 (c), we have

$(a \sim b) \otimes (a' \sim b') \leq (a \vee a') \sim (b \vee b')$.

Since the left-hand side belongs to F, due to property (ii) of a filter. Hence, also the right-hand side $(a \vee a') \sim (b \vee b') \in F$. ■

Given a filter $F \subset E$, as usual, the following relation on E is an equivalence relation:

$$a \approx_F b \text{ iff } a \sim b \in F \tag{12}$$

As an immediate consequence of Lemma 5.4, we have the following Theorem:

Theorem 5.5 *[35] Let F be a filter of a semicopula-based (ℓ)EQ-algebra $\mathcal{L}$. The relation $\approx_F$ is a congruence relation on $\mathcal{L}$.*

We define a factor-algebra

$$\mathcal{L}/F = \langle E/F, \wedge_F, \otimes_F, \sim_F, 1\rangle \tag{13}$$

in the standard way as follows. The support is $E/F = \{[a] | a \in E\}$. The operations are defined by $[a] \wedge_F [b] = [a \wedge b]$ and similarly for the other operations. The top element is [1].

The ordering in $\mathcal{L}/F$ will be defined using the derived meet operation as follows:

$[a] \leq [b]$ iff $[a] \wedge_F [b] = [a]$ iff $a \wedge b \approx_F a$ iff $a \wedge b \sim a = a \to b \in F$.

Theorem 5.6 *[35] Let F be a filter of a semicopula-based (ℓ)EQ-algebra $\mathcal{L}$. The factor algebra $\mathcal{L}/F$ is a separated semicopula-based (ℓ)EQ-algebra and $f : a \mapsto [a]$ is a homomorphism of $\mathcal{L}$ onto $\mathcal{L}/F$.*

6 Conclusion

In this paper, we generalized the concept of EQ-algebra by excluding both the commutativity and the associativity of the multiplication $\otimes$ showing that nothing is lost. Hence, $\otimes$ can be a quite free operation close to aggregation and EQ-algebras could be even more general. Actually, we need $\otimes$ to be a semi-copula. We called such type of algebra a *semicopula-based EQ-algebra*, i.e. EQ-algebra in which the multiplication need not be neither commutative nor associative. Therefore, all properties of semicopula-based EQ-algebras are also properties of EQ-algebras. Conversely, we have shown that all proved properties of EQ-algebras remain valid and applicable in semicopula-based EQ-algebras. Besides these main results, a lot of new and important properties concerning (semicopula-based) EQ-algebras and their special kinds are proved. By these new derived properties of (semicopula-based) EQ-algebras, we can say that these algebras are good candidate as algebras of truth values specifically for FTT. Of course, much algebraic study must still be done to be able to develop FTT based on (semicopula-based) EQ-algebras.

Acknowledgement 1 *This work was done when the first author was a visiting researcher at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava. He gratefully acknowledges the grant obtained from SAIA under the National Scholarship Program (NSP) for the mobility of Teachers and Researchers that made this visit possible. The research was also supported by the project MSM 6198898701 of the MŠMT ČR.*

References

[1] A.A. Abdel-Hamid, N.N. Morsi, Associatively tied implications, Fuzzy Sets and Systems 136 (2003) 291-311.

[2] Andrews, P. (2002). An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof. Dordrecht: Kluwer.

[3] B. Bassan and F. Spizzichino: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J. Multivariate Anal. 93 (2005), 313–339.

[4] Blyth, T. (2005), Lattices and Ordered Algebraic Structures. London: Springer.

[5] U. Bodenhofer, A similarity-based generalization of fuzzy orderings preserving the classical axioms, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems 8 (2000), 593–610.

[6] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods. In: Aggregation Operators. New Trends and Applications (T. Calvo, R. Mesiar, and G. Mayor, eds.), Physica–Verlag, Heidelberg 2002, pp. 3–106.

[7] A. Di Nola, G. Georgescu, A. Iorgulescu, Pseudo-BL algebras I, II, Multiple-Valued Logic 8 (no. 5-6) (673–714) (2002) 717–750.

[8] F. Durante, C. Sempi, Semicopulæ, Kybernetika (Prague) 41 (no. 3) (2005), 315-328.

[9] F. Durante, J.J. Quesada-Molina and C. Sempi, Semicopulas: characterizations and applicability, Kybernetika (Prague) 42 (no. 3) (2006), 287-302.

[10] F. Durante, E.P. Klement, R. Mesiar, C. Sempi, Conjunctor and their residual implicators: Characterizations and construction methods, Mediterr. j. math. 4 (2007), 343-356.

[11] Esteva, F. and L. Godo: 2001, 'Monoidal t-norm based logic: towards a logic for left-continuous t-norms'. Fuzzy Sets and Systems 124, 271–288.

[12] P. Flondor, G. Georgescu, A. Iorgulescu, Pseudo-t-norms and pseudo-BL algebras, Soft Comput. 5 (2001) 355–371.

[13] J.C. Fodor, Contrapositive symmetry of fuzzy implications, Fuzzy Sets and Systems 69 (1995) 141-156.

[14] J. Fodor and T. Keresztfalvi, Nonstandard conjunctions and implications in fuzzy logic. Int. Journal Approx,. Reasoning 12 (1995) 69-84

[15] G. Georgescu, A. Popescu, Non-commutative fuzzy structures and pairs of weak negations, Fuzzy Sets and Systems 143 (2004) 129–155.

[16] S. Gottwald, A Treatise on Many-Valued Logic, Research Studies Press, Baldock, 2001.

[17] S. Gottwald, P. Hájek, Triangular norm-based mathematical fuzzy logics, in: E.P. Klement, R. Mesiar, Eds., Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms, Elsevier, Amsterdam, (2005), 257-299.

[18] Hájek, P., Metamathematics of Fuzzy Logic. Dordrecht: Kluwer,1998.

[19] P. Hájek, Observations on non-commutative fuzzy logic, Soft Comput. 8 (2003) 38–43.

[20] P. Hájek, Fuzzy logics with non-commutative conjunction, J. Logic Comput. 13 (2003) 469–479.

[21] P. Hájek, R. Mesiar, On copulas, quasicopulas and fuzzy logic, Soft Computing 12 (2008)1239-1243.

[22] Höhle, U., 'Quotients with respect to similarity relations'. Fuzzy Sets and Systems 27 (1988), 31–34.

[23] E. P. Klement, R. Mesiar, and E. Pap, Triangular norms, Kluwer Academic Publishers, Dordrecht, 2000.

[24] Mayuka F. Kawaguchi, Masaaki Miyakoshi, Composite fuzzy relational equations with non-commutative conjunctions, Inform. Sci. 110 (no. 1-2) (1998), 113-125.

[25] N.N. Morsi, Propositional calculus under adjointness, Fuzzy Sets and Systems 132 (2002) 91-106.

[26] N.N. Morsi, E.A. Aziz Mohammed, M.S. El-Zekey, Propositional calculus for adjointness lattices, Mathware Soft Comput. 9 (2002) 5–23.

[27] N.N. Morsi, W. Lotfallah, M. El-Zekey, The logic of tied implications, Part 1: Properties, applications and representation, Fuzzy Sets and Systems 157 (5) (2006) 647-669.

[28] N.N. Morsi, W. Lotfallah, M. El-Zekey, The logic of tied implications, Part 2: Syntax, Fuzzy Sets and Systems 157 (15) (2006) 2030-2057. Corrigendum in: Fuzzy Sets and Systems 157 (17) (2006) 2416-2417.

[29] N.N. Morsi, E.M. Roshdy, Issues on adjointness in multiple-valued logics, Information Sciences 176 (19) (2006) 2886-2909.

[30] Novák, V., 'On Fuzzy Type Theory', Fuzzy Sets and Systems 149 (2005), 235–273.

[31] V. Novák, EQ-algebras: primary concepts and properties, Proc. Ninth Czech-Japan Seminar, Kitakyushu & Nagasaki, Graduate School of Information, Waseda University, 2006, pp. 219–223.

[32] V. Novák, EQ-algebras in progress, in: Theoretical Advances and Applications of Fuzzy Logic and Soft Computing (O. Castillo, et al., eds.), Advances in Soft Computing 42, Springer, 2007, pp. 876–884.

[33] V. Novák, Principal fuzzy type theories for fuzzy logic in broader sense, in: Proc. Twelfth Internat. Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems (Málaga, Spain), 2008.

[34] V. Novák, I. Perfilieva and J. Mockor, Mathematical Principles of Fuzzy Logic, Kluwer, Boston, 1999.

[35] V. Novák, B. De Baets, EQ-algebras, Fuzzy Sets and Systems (2008), to appear.

[36] F. Suárez García and P. Gil Álvarez, Two families of fuzzy integrals, Fuzzy Sets and Systems 18 (1986), 67-81.

[37] Ensuk Yang: Weakly associative monoidal (uninorm) logic: toward a fuzzy-relevance logic and a non-associative fuzzy logic. Submitted.