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# ON LINGUISTIC APPROXIMATION IN THE FRAME OF FUZZY LOGIC DEDUCTION

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## Abstract

This paper presents a new linguistic approximation algorithm and its implementation in the frame of fuzzy logic deduction. The algorithm presented is designed for fuzzy logic deduction mechanism implemented in LFLC (Linguistic Fuzzy Logic Controller).

## keywords:

linguistic approximation, inference mechanism, expert systems

## 1 Introduction

This article deals with the problem of linguistic approximation. By definition, linguistic approximation is a procedure which assigns linguistic expression to a given fuzzy set.

The usual method of obtaining results in fuzzy systems is called defuzzification. It means that as soon as we perform one step of inference mechanism, we defuzzify the obtained fuzzy set and receive one crisp number. However, it is not much in the spirit of fuzzy logic. It is the only possibility in fuzzy control, because we need control action, e.g. electric current or gas flow, as a crisp number in order to perform the next step of the control process.

In other kinds of fuzzy systems, e.g. fuzzy expert systems, decision support systems, etc., we need something different. It is desirable to obtain a linguistic expression better than the crisp number. The former have similar form as the linguistic expressions which enter the inference mechanism. We must use the linguistic approximation for this purpose.

There are several approaches to this problem, e.g. in<sup>4),13)</sup>. In<sup>7)</sup>, a good overview of the approaches presented so far is given. In our approach, we will suppose that fuzzy sets which enter the linguistic approximation algorithm are outputs of a fuzzy logic deduction. If it is the case, then we are allowed to construct the linguistic expression which corresponds to these fuzzy sets by using atomic terms and linguistic modifiers used in the fuzzy logic deduction.

This paper is organized as follows: In Section 2 we present basic theory of fuzzy logic deduction, Section 3 contains description of the linguistic approximation procedure, Section 4 presents two examples, and Section 5 contains some conclusions.



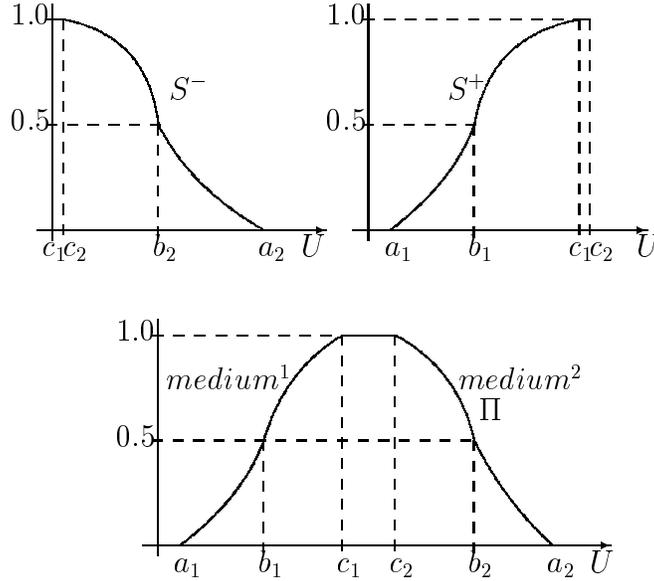


Figure 1: Membership functions of  $S^-$ ,  $S^+$  and  $\Pi$  fuzzy sets. They usually correspond to the meaning of the atomic terms *small*, *big* and *medium*, respectively.

The meanings of the linguistic expressions (1) can be modeled using fuzzy sets with membership functions taken from some class of functions  $\mathcal{Z}(U)$ ,<sup>12)</sup>

$$\mathcal{Z}(U) = \{Z, S \mid Z, S : U \rightarrow [0, 1]\},$$

where members of the class of non-decreasing functions  $Z$  model the meanings of linguistic expressions with atomic terms *small* and  $medium^1$ , and members of the class of non-increasing functions  $S$  model the meanings of linguistic expressions with atomic terms  $medium^2$  and *big*.

Let the fuzzy sets  $A_{ij}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, r$  and  $B_j$ ,  $j = 1, \dots, r$  represent the meanings of all the natural language expressions  $\mathcal{A}_{ij}$  and  $\mathcal{B}_j$ , respectively, which occur in the generalized modus ponens scheme above. Then the fuzzy set  $B'$  representing the meaning of the conclusion  $\mathcal{B}'$  can be computed using the formula<sup>3),8)</sup>

$$B'y = \bigvee_{x_1 \in U_1, \dots, x_n \in U_n} ((A'_1 x_1 \wedge \dots \wedge A'_n x_n) \otimes \bigwedge_{j=1}^r ((A_{1j} x_1 \wedge \dots \wedge A_{nj} x_n) \rightarrow B_j y)), \quad (2)$$

where  $U_1, \dots, U_n$  are universes of discourse of the antecedent variables,  $A'_i$  are fuzzy sets which represent the observations,  $\otimes$  and  $\rightarrow$  are Łukasiewicz conjunction and implication, respectively.

### 3 Linguistic approximation algorithm

The linguistic approximation is, in general, a method, which assigns linguistic expression to the given fuzzy set. Formally we can describe this method as follows<sup>9)</sup>:

First, we define a similarity relation between fuzzy sets:

**Definition 1** A fuzzy relation

$$\mathcal{R} \subseteq \mathcal{F}(U) \times \mathcal{F}(U),$$

where  $\mathcal{F}(U) = [0, 1]^U$  is the set of all fuzzy sets on  $U$ , is called *similarity relation*, if for all  $A, B \in \mathcal{F}(U)$  the following conditions hold:

1.  $\mathcal{R}\langle A, A \rangle = 1$ ,
2.  $\mathcal{R}\langle A, B \rangle = \mathcal{R}\langle B, A \rangle$ ,
3.  $\text{Supp}(A) \cap \text{Supp}(B) = 0$  implies  $\mathcal{R}\langle A, B \rangle = 0$ ,

where

$$\text{Supp}(A) = \{x \in U \mid Ax > 0\}$$

is the support of the fuzzy set  $A$ .

**Definition 2** Let  $\langle \mathcal{X}, \mathcal{T}(\mathcal{X}), U, G, \mathcal{M} \rangle$  be a linguistic variable<sup>14</sup>, where  $\mathcal{X}$  is name of the variable,  $\mathcal{T}(\mathcal{X})$  is term set,  $U$  is universe,  $G$  and  $\mathcal{M}$  are syntactic and semantic rules, respectively. Let  $A_0 \subseteq U$  be a fuzzy set and  $\mathcal{R} \subseteq \mathcal{F}(U) \times \mathcal{F}(U)$  be a fuzzy relation defined above. Then the term  $\mathcal{A} \in \mathcal{T}(X)$ , for which is  $\mathcal{R}\langle M(\mathcal{A}), A_0 \rangle$  maximal, is called the *linguistic approximation* of the fuzzy set  $A_0$ .

If there are more terms  $\mathcal{A}_i$  with the same value of  $\mathcal{R}\langle M(\mathcal{A}), A_0 \rangle$ , then we should use different similarity measure (possibly more sensitive in this individual situation). Definition 2 states that we search for the term, which meaning is most similar to approximated fuzzy set. There are several problems encountered:

- we have to search not only among atomic and modified atomic terms, but also among composed terms, e.g. *roughly small* **or** *medium*,
- the fuzzy set which enters linguistic approximation can be subnormal,
- the information included in fuzzy set which enters linguistic approximation can be distorted or noisy.

The algorithm which performs linguistic approximation has to take the above mentioned problems into consideration. Algorithms of the linguistic approximation can be divided into two categories<sup>7</sup>):

1. Algorithm performing the entire check of the term set,
2. Algorithm based on the piecewise decomposition of the fuzzy set.

In the first method, we simply compare a given fuzzy set with all the fuzzy sets corresponding to the terms from the term set. Some similarity relation is used for determination of similarity between these fuzzy sets. As a result is taken the term which meaning has the smallest distance from the given one.

The Piecewise Decomposition method in general decomposes the given fuzzy set into several pieces or segments, determines the linguistic term for each of them, and then constructs the result by means of logical connectives<sup>4</sup>).

In our approach, we want the resulting expression to have the form

$$\langle \text{linguistic modifier}_1 \rangle \langle \textit{small} \rangle \textbf{or} \langle \text{linguistic modifier}_2 \rangle \langle \textit{medium} \rangle \textbf{or} \\ \textbf{or} \langle \text{linguistic modifier}_3 \rangle \langle \textit{big} \rangle,$$

where  $\langle \text{linguistic modifier}_i \rangle$ ,  $i = 1, 2, 3$  are linguistic modifiers described in Section 2 including special modifiers *empty* and *ignored*. Modifier *empty* doesn't change the membership function which models the meaning of the atomic term. Modifier *ignored* forces the membership function which model the meaning of modified atomic term to be identically equal to zero. Thus we can obtain linguistic expression with one, two or three linguistic expressions (1) connected by the connective **or**. We divide approximated fuzzy set into several segments and determine the atomic term and the modifier to each of them.

Fuzzy sets which enter linguistic approximation are outputs  $B'$  of fuzzy logic deduction described by formula (2), which is in real situations always performed on discretized universes  $U_1, U_2, \dots, U_n$ . Thus fuzzy set  $B'$  is also discrete and we can in the following consider only fuzzy sets defined on the linearly ordered discrete universe  $U = \{x_1, x_2, \dots, x_m\}$ .

The linguistic approximation algorithm can be described in the following steps:

1. Divide the membership function of the given fuzzy set into segments.
2. Discard "horizontal" segments, which bear no information.
3. Decide, for each segment, whether segment is "peak" or "section".
4. For all segments, if the segment is "peak" then normalize it.
5. Compare segments with meanings of terms from the term set and find term with the highest membership degree of similarity relation.
6. Compose resulting linguistic expression by means of connectives.

**Ad 1:** We divide membership function into segments whose boundaries are defined as follows:

**Definition 3** Let  $A$  be a fuzzy set on universe  $U = \{x_1, x_2, \dots, x_m\}$ . The point  $x_i \in U$  is *boundary* if  $x_i = x_0$  or  $x_i = x_m$  or

$$\text{sgn}(x_{i+1} - x_i) \neq \text{sgn}(x_i - x_{i-1}),$$

where

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}.$$

**Definition 4** Let  $A$  be a fuzzy set from Definition 3 and

$$B = \{b^1, b^2, \dots, b^k\}, \quad k < m$$

be a set of all boundary points. Then sets  $S_i \subseteq U$

$$S_i = \{x_j = b^i, x_{j+1}, \dots, x_{j+l} = b^{i+1}\}, \quad i = 1, 2, \dots, k-1$$

are *segments* of  $A$ .

**Ad 2:** Segment  $S_i$  which satisfies  $Ab^i = Ab^{i+1}$  is called *horizontal*. Horizontal segments bear, in our opinion, no information about resulting linguistic expression, and therefore, they are discarded.

**Ad 3:** Other kinds of segments are: sections and peaks.

**Definition 5** Segment  $S_i$  is a *section* if its first boundary point  $b^i$  satisfies

$$\left| \frac{Ax_{j+1} - 2Ab^i + Ax_{j-1}}{x_{j+1} - x_{j-1}} \right| > \kappa, \quad (3)$$

where  $x_{j-1}$ ,  $x_{j+1}$  are the two closest neighbours of  $b^i$  in  $U$ , or its second boundary point  $b^{i+1}$  satisfies

$$\left| \frac{Ax_{j+1} - 2Ab^{i+1} + Ax_{j-1}}{x_{j+1} - x_{j-1}} \right| > \kappa, \quad (4)$$

where  $x_{j-1}$ ,  $x_{j+1}$  are the two closest neighbours points of  $b^{i+1}$  in  $U$ . If the formulae (3), (4) are not fulfilled then the segment  $S_i$  is a *peak*.

The value of the constant  $\kappa$  is important for the proper distinguishing between sections and peaks. If  $\kappa$  is too big, then the segment of type peak can be recognized as section and vice versa. We obtained the best results with  $\kappa = 0.5$ . Informally, the peak is the segment which corresponds to “upper” part of membership function, i.e. the part in which the global maximum of membership function is included, and the section is the segment which corresponds to that part of the membership function which does not include global maximum.

**Ad 4:** If the segment is of the type “peak” and is subnormal, i.e. the maximal membership degree is smaller than 1, then we have to normalize it. The reason is that meanings of terms from our term set are all normal fuzzy sets. The simplest way to normalize the “peak” is to find out the maximal membership degree  $M = \bigvee_{x \in S_i} Ax$  in the segment  $S_i$  and to modify all membership degrees using the formula

$$A_{mod}x = \frac{Ax}{M}, \quad \forall x \in S_i$$

**Ad 5:** The term most appropriate to the given segment can be obtained as follows: we generate fuzzy sets which correspond to all the possible linguistic terms with the same atomic term. As a atomic terms we consider *small*, *big*, *medium*<sup>1</sup> and *medium*<sup>2</sup>, as described in Section 2. The most appropriate term we determine as term which has membership function most similar to given segment.

The following formula can be used for computation of the similarity relation<sup>9)</sup>:

$$\mathcal{R}\langle A, B \rangle = 1 - \frac{\frac{1}{\#(\text{Supp}A \cup \text{Supp}B)} \sum_{x \in \text{Supp}A \cup \text{Supp}B} f(Ax - Bx)}{\frac{1}{\#(\text{Supp}A)} \sum_{x \in \text{Supp}A} f(Ax) + \frac{1}{\#(\text{Supp}B)} \sum_{x \in \text{Supp}B} f(Bx)}, \quad (5)$$

where  $\#(C)$  denotes the number of elements of the set  $C$ , and

$$f : [-1, 1] \rightarrow [0, 1]$$

is an even continuous measurable function increasing on  $[0, 1]$ .

As the most appropriate term we choose that with maximal value of  $\mathcal{R}$ . Functions  $f(x)$  are chosen from the family of functions  $\mathcal{F}_e = \{|x|, x^2, x^4, \dots\}$ . We start with  $f(x) = |x|$  and if there are more than one term with the same maximal value of  $\mathcal{R}$ , we use the subsequent  $f \in \mathcal{F}_e$ .

**Ad 6:** In the last step of algorithm we compose segments with assigned atomic terms *medium*<sup>1</sup> and *medium*<sup>2</sup> to one *medium* term, and then compose all partial terms by means of connective **or**.

## 4 Example

In this section, we present our linguistic approximation algorithm working on two fuzzy sets, depicted in Figures 2 and 3.

The shape of the membership function of fuzzy set in Figure 2 suggests us that the corresponding linguistic expression should be composed of two partial expressions with atomic terms *medium* and *big* connected by **or**. Linguistic approximation algorithm indeed gives us the expected result.

At first, we perform Step 1 of our algorithm, division to segments. Result is shown in Figure 2 and labeled as S1, S2, S3, S4. Then, horizontal segment S2 is discarded (Step 2). All remaining segments S1, S3 and S4 are of type “section” (Step 3), i.e. they don’t include the global maximum of the membership function. Because the normalization (Step 4) is performed only for segments of type “peak”, it is not used here.

In the Step 5, we compare segments S1 and S3, whose membership function is increasing with fuzzy sets which model the meaning of linguistic expressions (1) with atomic terms *small* and *medium*<sup>1</sup>, and segment S4 whose membership function is decreasing with fuzzy sets which model the meaning of linguistic expressions (1) with atomic term *medium*<sup>2</sup> and *big*. The comparison is performed by means of formula (5). We obtained the following results: S1 corresponds to *more or less medium*<sup>1</sup>, S3 to *roughly big* and S4 to *more or less medium*<sup>2</sup>. As the last step, we compose the final linguistic expression: *more or less medium or roughly big*.

Linguistic expression describing fuzzy set depicted in Figure 3 should be also composed of expressions with atomic terms *medium* and *big*, but segments S2, S4 and S6 are of type “peak”. Because all these segments are normal, no normalization is necessary. Segments S1, S3, S5 and S7 are horizontal. Steps 5 and 6 of linguistic approximation algorithm gives us the final result *quite roughly medium or big*.

## 5 Conclusion

Applications of linguistic approximation can be found whenever we need linguistic description of results given by fuzzy systems. The easy interpretability of fuzzy systems is

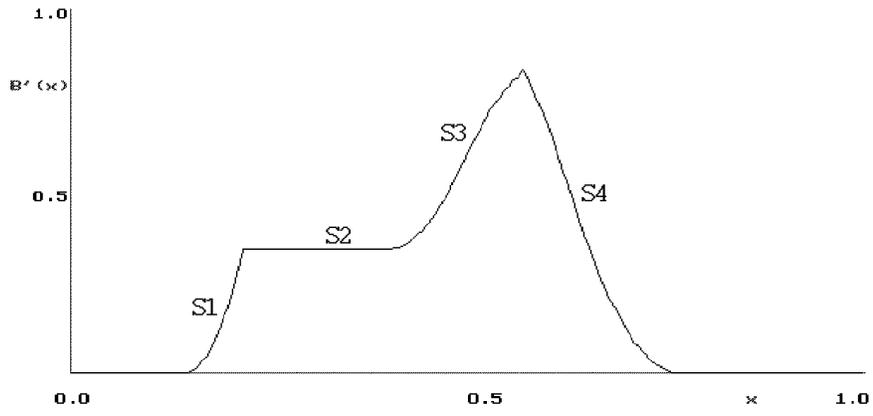


Figure 2: Example 1 fuzzy set

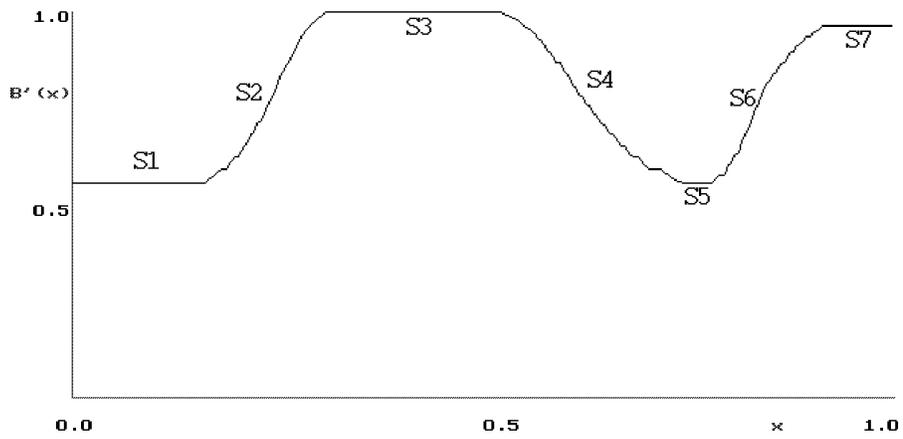


Figure 3: Example 2 fuzzy set

one of their crucial features, and linguistic approximation allows to interpret results given by them.

The most promising is the field of expert systems and decision support systems, where we don't need crisp (defuzzified) value, but rather linguistic expression which can be easily understood by humans. Other fields of application can be found in fuzzy data analysis and fuzzy modeling<sup>13)</sup>, various kinds of medical systems, and even in some types of fuzzy control<sup>2),1)</sup>.

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