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Research report No. 126

2008

Submitted/to appear:

Fuzzy Sets and Systems

Supported by:

Grant 201/04/1033 of the GA ČR and project MSM 6198898701 of the MŠMT ČR

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Abstract

The paper is a contribution to the model theory of fuzzy logic in narrow sense. We focus on the classical omitting types theorem and generalize it for predicate fuzzy logics with traditional syntax. First we prove this theorem for MTL∀-fuzzy predicate logic with countable language. Then we extend it to all core fuzzy logics with the language of arbitrary cardinality.

Key words: Model theory, fuzzy logic with traditional syntax, MTL-algebra,, MTL∀-fuzzy logic, core fuzzy logic.

1 Introduction

Mathematical fuzzy logic is now a mature theory whose fundamental problems seem to be already solved. Its further development can be seen in two directions. The first one is to study particular properties of fuzzy logic and to search specific problems to be solved, especially in connection with the main goal of them — to develop a mathematical theory providing tools for modeling of the vagueness phenomenon. The second direction is to study generalization of important classical results that may put a different light of mathematical logic in general and may also be useful for the above task. Among such results certainly belongs the classical omitting type theorem in model theory. Recall that this theorem, among others, enables to extend the power of classical logic by characterizing properties that are too complicated to be expressed by one formula only, and so, a set of

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[1] The research was partially supported by grant 201/04/1033 of the GA ČR and by the project MSM 6198898701 of the MŠMT ČR.

formulas is needed. Thus, the omitting types theorem may help to extend the power of fuzzy logic still more.

Recall that there are several systems of fuzzy logic in narrow sense. Most of them have traditional syntax and many-valued interpretation. Outstanding role among these systems plays basic (predicate) fuzzy logic (BL∀ from now) introduced by P. Hájek in [3]. This logic has been further generalized to MTL-fuzzy logic by F. Esteva and L. Godo in [2]. MTL-logic is a basis of a wide class of fuzzy logics called *core fuzzy logics* (BL-fuzzy logic is among them). This concept has been introduced by P. Hájek and P. Cintula in [4] where also the model theory for predicate core fuzzy logics has been elaborated.

This paper continues the model theory of the MTL∀ and core predicate fuzzy logics. For both logics, we present generalization of the classical omitting type theorem.

It should be noted that there is also a *fuzzy logic with evaluated syntax* ($\mathrm{Ev_Ł}$ from now) introduced by J. Pavelka in [12] and extended to first order by V. Novák in [8]. It is specific for $\mathrm{Ev_Ł}$ that the set of truth values must be either finite of the standard Łukasiewicz MV-algebra with the support $[0,1]$. $\mathrm{Ev_Ł}$ goes beyond core fuzzy logics by generalizing also the syntax, i.e. each formula is added its *syntactic evaluation*. This means, besides others, that the language contains logical (truth) constants (i.e. names of the truth values) for either all, or at least countably many truth values (cf. [10]). This trick has the following outcomes: first, it enables us to consider axioms that may not be fully convincing and so, we may accept them only in some degree. A good example is axiomatic characterization of the sorites paradox (see [5,11]) where the axiom stating that "if n stones do not form a heap then $n+1$ stones also do not form a heap" cannot be taken as convincing. Furthermore, it enables us to put reasoning about concrete degrees also in the syntax. At the same time, introducing degrees into syntax is a natural generalization of the traditional syntax since the latter can be simply taken as "boolean evaluated syntax". The detailed presentation of $\mathrm{Ev_Ł}$ can be found in [11] where also its model theory has been established. Two special results in model theory of $\mathrm{Ev_Ł}$ were published: [9] concerns the joint consistency theorem and [7] presents the omitting type theorem.

This paper is organized as follows: Section 2 contains a brief overview of MTL∀- and core fuzzy logics. Section 3 introduces generalization of the concepts of the omitting types theory in MTL∀ and contains a detailed proof of the main theorem. Section 4 extends this result to core fuzzy logics.

2 Preliminaries

2.1 MTL∀ predicate fuzzy logics

The development of any logic is essentially influenced by the structure of its truth values. This role in MTL-fuzzy logic is played by MTL-algebras.

Definition 1
An MTL-algebra is a structure $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ *such that for all* $a, b, c \in L$:

(i) $\langle L, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ *is a bounded lattice,*
(ii) $\langle L, \otimes, \mathbf{1}\rangle$ *is a commutative monoid,*
(iii) $a \leq b \to c$ *iff* $a \otimes b \leq c$, *(adjunction),*
(iv) $(a \to b) \vee (b \to a) = \mathbf{1}$, *(prelinearity).*

If the lattice reduct of $\mathcal{L}$ is linearly ordered then we say that $\mathcal{L}$ is an MTL-chain. The *negation* operation is defined by $\neg a = a \to \mathbf{0}$ and the (weak) *biresiduation* by $a \leftrightarrow b = (a \to b) \otimes (b \to a)$.

A *predicate language* J of MTL∀ consists of *object variables* $x, y, \ldots$, a set of *objects constants* $\mathrm{OC}(J) = \{u, v, \ldots\}$, a non-empty set of n-ary *predicates* $\mathrm{Pred}(J) = \{P, Q, \ldots\}$ †), *logical connectives* $\boldsymbol{\wedge}, \boldsymbol{\&}, \boldsymbol{\Rightarrow}$, *logical (truth) constants* $\bot, \top$, and *quantifiers* $\forall, \exists$.

The other connectives are defined as follows:

$$A \boldsymbol{\vee} B \text{ is } ((A \boldsymbol{\Rightarrow} B) \boldsymbol{\Rightarrow} B) \boldsymbol{\wedge} ((B \boldsymbol{\Rightarrow} A) \boldsymbol{\Rightarrow} A),$$
$$\neg A \text{ is } A \boldsymbol{\Rightarrow} \bot,$$
$$A \boldsymbol{\Leftrightarrow} B \text{ is } (A \boldsymbol{\Rightarrow} B) \boldsymbol{\&} (B \boldsymbol{\Rightarrow} A).$$

Terms are object variables and object constants. *Atomic formulas* have the form $P(t_1, \ldots, t_n)$ where P is a n-ary predicate and $t_1, \ldots, t_n$ are terms. If A, B are formulas and x is an object variable then $A \boldsymbol{\Rightarrow} B$, $A \boldsymbol{\wedge} B$, $A \boldsymbol{\&} B$, $(\forall x)A$, $(\exists x)A$, $\top$, $\bot$ are formulas.

The set of all the well-formed formulas for the language J is denoted by F_J (we will also speak about J-formulas). The notions of *free and bound variables, substitutable term, closed and open formula*, are the same as in classical logic. As usual, $A(x_1, \ldots, x_n)$ denotes a formula whose all free variables are among $x_1, \ldots, x_n$. We will often write $A(\bar{x})$ instead of $A(x_1, \ldots, x_n)$. If $t_1, \ldots, t_n$ are terms substitutable in A for $x_1, \ldots, x_n$, respectively then $A_{x_1,\ldots,x_n}(t_1, \ldots, t_n)$ denotes *instance* of A in which all free occurrences of $x_1, \ldots, x_n$ are replaced by $t_1, \ldots, t_n$, respectively. If $x_1, \ldots, x_n$ are clear from the context then we will write simply $A(t_1, \ldots, t_n)$.

By $\Sigma(x_1, \ldots, x_n)$ we denote a set of J-formulas such that each formula of it has all its free variables among $x_1, \ldots, x_n$. We will often write $\Sigma(\bar{x})$ instead of $\Sigma(x_1, \ldots, x_n)$. If $x_1, \ldots, x_n$ are known from the context then we will write Σ instead of $\Sigma(x_1, \ldots, x_n)$.

The following schemes of formulas are axioms of predicate MTL∀:

(A1) $(A \boldsymbol{\Rightarrow} B) \boldsymbol{\Rightarrow} ((B \boldsymbol{\Rightarrow} C) \boldsymbol{\Rightarrow} (A \boldsymbol{\Rightarrow} C))$
(A2) $(A \boldsymbol{\&} B) \boldsymbol{\Rightarrow} A$
(A3) $(A \boldsymbol{\&} B) \boldsymbol{\Rightarrow} (B \boldsymbol{\&} A)$

†) The arity n depends on the given symbol.

(A4) $(A \wedge B) \Rightarrow A$
(A5) $(A \wedge B) \Rightarrow (B \wedge A)$
(A6) $(A \,\&\, (A \Rightarrow B)) \Rightarrow (A \wedge B)$
(A7a) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \,\&\, B) \Rightarrow C)$
(A7b) $((A \,\&\, B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$
(A8) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$
(A9) $\bot \Rightarrow A$
($\forall$1) $(\forall x)A(x) \Rightarrow A(t)$ (t substitutable for x in $A(x)$)
($\exists$1) $A(t) \Rightarrow (\exists x)A(x)$ (t substitutable for x in $A(x)$)
($\forall$2) $(\forall x)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall x)B)$ (x not free in A)
($\exists$2) $(\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow B)$ (x not free in B)
($\forall$3) $(\forall x)(A \vee B) \Rightarrow (A \vee (\forall x)B)$ (x not free in A)

Deduction rules of the predicate MTL$\forall$ are *Modus Ponens* (from A and $A \Rightarrow B$ infer B) and *generalization* (from A infer $(\forall x)A$). The basic notions of a *proof* and a *provable formula* in MTL$\forall$ are defined in the classical way.

A *theory* over MTL$\forall$ is a set T of formulas. By $J(T)$ we denote language of a theory T and by $F_{J(T)}$ the set of all the well formed formulas of $J(T)$. A theory T is *consistent* if $T \not\vdash \bot$. We say that T is *complete* if for each pair A, B of closed formulas, $T \vdash A \Rightarrow B$ or $T \vdash B \Rightarrow A$. We say that T is *Henkin* if for each closed formula of the form $(\forall x)A$ such that $T \not\vdash (\forall x)A$ there is a constant c in the language of $J(T)$ such that $T \not\vdash A(c)$.

Let J be a predicate language and let $\mathcal{L}$ be an MTL-chain. An $\mathcal{L}$*-structure*

$$\mathcal{M} = \langle M, \mathcal{L}, \{P_\mathcal{M} \mid P \in \mathrm{Pred}(J)\}, \{v_\mathcal{M} \mid v \in \mathrm{OC}(J)\}\rangle, \tag{1}$$

for J has a non-empty *domain* M, for each n-ary predicate $P \in \mathrm{Pred}(J)$, $P_\mathcal{M}$ is n-ary $\mathcal{L}$-fuzzy relation on M and for each constant $v \in \mathrm{OC}(J)$, $v_\mathcal{M} \in M$. We will often write the structure (1) as $(\mathcal{M}, \mathcal{L})$.

Let J be a predicate language and $\mathcal{M}$ an $\mathcal{L}$-structure for J. A $\mathcal{M}$*-evaluation of object variables* is a mapping e assigning to each object variable x an element $e(x) \in M$. The *degree of satisfaction* $\|A\|^\mathcal{L}_{\mathcal{M},e}$ of a J-formula is a truth value $a \in L$ defined in $\mathcal{M}, e$ in the usual way. We will write

$$\|A\|^\mathcal{L}_{\mathcal{M},e} = a. \tag{2}$$

The connectives $\neg, \wedge, \vee, \&, \Rightarrow, \Leftrightarrow$ are interpreted in $\mathcal{L}$ by the operations $\neg, \wedge, \vee, \otimes, \rightarrow, \leftrightarrow$, respectively. A structure $\mathcal{M}$ is *safe* if (2) is defined for each A and each evaluation e of variables (this means that all the necessary suprema and infima in $\mathcal{L}$ exist).

Let $A(x_1, \dots, x_n)$ be an J-formula and $e(x_1) = v_1, \dots, e(x_n) = v_n$. Then we will write $\|A(v_1/x_1, \dots, v_n/x_n)\|^\mathcal{L}_\mathcal{M}$ or simply $\|A(v_1, \dots, v_n)\|^\mathcal{L}_\mathcal{M}$ instead of $\|A(x_1, \dots, x_n)\|^\mathcal{L}_{\mathcal{M},e}$. To simplify the notation, we will often write $\bar{v} \in M^n$ for the n-tuple of elements $v_1, \dots, v_n \in M$ and $A(\bar{v}/\bar{x})$ instead of $A(v_1/x_1, \dots, v_n/x_n)$, or simply $A(\bar{v})$. Similarly when dealing with the set $\Sigma(x_1, \dots, x_n)$, we will write $\Sigma(v_1/x_1, \dots, v_n/x_n)$, or $\Sigma(v_1, \dots, v_n)$, or only

$\Sigma(\bar{v})$.

Definition 2
Let A be a J-formula and let $\mathcal{M}$ be a safe $\mathcal{L}$-structure for J. The truth value *of A in $\mathcal{M}$ is*

$$\|A\|^{\mathcal{L}}_{\mathcal{M}} = \inf\{\|A\|^{\mathcal{L}}_{\mathcal{M},e} \mid e \textit{ is a } \mathcal{M}\textit{-evaluation}\}.$$

By $(\mathcal{M},\mathcal{L}) \models A$ we denote the fact that $\|A\|^{\mathcal{L}}_{\mathcal{M},e} = \mathbf{1}$ for each $\mathcal{M}$-evaluation e. We say that $(\mathcal{M},\mathcal{L})$ is a model for J instead of saying that $\mathcal{L}$ is a MTL-chain and $\mathcal{M}$ is a safe $\mathcal{L}$-structure for J. Thus if we say "for each model $(\mathcal{M},\mathcal{L})$" we mean "for each MTL-chain $\mathcal{L}$ and each safe $\mathcal{L}$-structure $\mathcal{M}$". Finally, we say that $(\mathcal{M},\mathcal{L})$ is a model of a theory T and write $(\mathcal{M},\mathcal{L}) \models T$ if $(\mathcal{M},\mathcal{L})$ is a model and $(\mathcal{M},\mathcal{L}) \models A$ for all axioms $A \in T$.

Let Σ be a set of J-formulas. By $(\mathcal{M},\mathcal{L}) \models \Sigma$ we denote the fact that $(\mathcal{M},\mathcal{L})$ is a model for J such that $(\mathcal{M},\mathcal{L}) \models A$ for each $A \in \Sigma$.

The following theorem summarizes the most important properties used below.

Theorem 1 ([3])
Let T be a theory over MTL$\forall$.

(a) Let $A \in F_{J(T)}$ be a closed formula. Then

$$T \cup A \vdash B \textit{ iff there is } n \textit{ such that } T \vdash A^n \Rightarrow B$$

holds for all $B \in F_{J(T)}$ (deduction theorem).
(b) For each complete Henkin theory T and each closed formula A unprovable in T there is a model $(\mathcal{M},\mathcal{L})$ of T such that $\|A\|^{\mathcal{L}}_{\mathcal{M}} < \mathbf{1}$.
(c) Let A be a J-formula. T proves A iff $\|A\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$ holds for each model $(\mathcal{M},\mathcal{L}) \models T$ (completeness theorem).

2.2 Core predicate fuzzy logic

This term has been introduced by P. Hájek and P. Cintula in [4]. We say that an expansion of MTL$\forall$ is a *core (predicate) fuzzy logic* (with traditional syntax) if the deduction Theorem 1 is provable in it and

$$A \Leftrightarrow B \vdash C(A) \Leftrightarrow C(B).$$

holds for all formulas $A, B, C \in F_J$.

The following theorem is important.

Theorem 2 ([4])
Let T be a theory of a core fuzzy logic and $A, B, C \in F_{J(T)}$ closed formulas.

(a) *If* $T \cup \{A\} \vdash C$ *and* $T \cup \{B\} \vdash C$ *then* $T \cup \{A \vee B\} \vdash C$ *(Proof by Cases).*
(b) *If* $T \cup \{A \Rightarrow B\} \vdash C$ *and* $T \cup \{B \Rightarrow A\} \vdash C$ *then* $T \vdash C$ *(Prelinearity Property).*
(c) *Let* $c \notin J(T)$ *be a new constant. Then* $T \cup \{A(c)\}$ *is a conservative extension of* $T \cup \{(\exists x)A(x)\}$.

The equivalent formulation of the prelinearity property is the following: if $T \nvdash C$ and A, B is a pair of closed formulas then

$$T' = T \cup \{A \Rightarrow B\} \nvdash C \text{ or } T' = T \cup \{B \Rightarrow A\} \nvdash C. \tag{3}$$

Thus, if we need to keep the unprovability of C in the extension T' of T, we put $T' = T \cup (A \Rightarrow B)$ in the former case and $T' = T \cup (B \Rightarrow A)$ in the latter. We will say that T' *decides the formulas* A, B.

We also say that a set of formulas Ψ is *directed* if to every $A, B \in \Psi$ there is a formula $C \in \Psi$ such that $A \Rightarrow C$ as well as $B \Rightarrow C$ are provable.

2.3 Omitting types theory in classical predicate logic

For the reader's convenience, we will briefly overview the omitting types theory in in classical logic (see [1,6]). Let $\Sigma(x_1, \ldots, x_n)$ be a set of J_{CL}-formulas (J_{CL} is a language of classical predicate logic). Let $\mathcal{M}$ be a model for J_{CL}. We say that $\mathcal{M}$ *realizes* $\Sigma(\bar{x})$ if there is $\bar{v} \in M^n$ such that

$$\mathcal{M} \models A(\bar{v})$$

holds for every $A \in \Sigma(\bar{x})$. We say that $\mathcal{M}$ *omits* $\Sigma(\bar{x})$ if $\mathcal{M}$ does not realize Σ.

A theory T be a theory. A set $\Sigma(\bar{x})$ is T*-consistent* if there is a model $\mathcal{M}$ of T such that $\mathcal{M}$ realizes $\Sigma(\bar{x})$. More precisely, we may say that set $\Sigma(\bar{x})$ is T*-consistent* if there is $\mathcal{M} \models T$ and $\bar{v} \in M^n$ such that $\mathcal{M} \models T \cup \Sigma(\bar{v})$.

Definition 3
Let J_{CL} *be a language,* T *be a theory,* $\Sigma(\bar{x})$ *be a set of* J_{CL}*-formulas. We say that* Σ *is* locally realized *(isolated) in* T *if there is a* J_{CL}*-formula* $A(\bar{x})$ *such that*

(i) $A(\bar{x})$ *is* T*-consistent,*
(ii) *for every model* $\mathcal{M} \models T$, $\mathcal{M} \models A(\bar{x}) \Rightarrow B(\bar{x})$ *holds for every* $B \in \Sigma$.

This means that if $\mathcal{M} \models A(\bar{v})$ *for some* $\bar{v} \in M^n$ *then* $\mathcal{M} \models B(\bar{v})$ *for all* $B \in \Sigma$.

We say that Σ is *locally omitted* (non-isolated) in T if Σ is not locally realized in T. Note that $\Sigma(\bar{x})$ is *locally omitted* in T if to every J_{CL}-formula $A(\bar{x})$, every model $\mathcal{M} \models T$ and $\bar{v} \in M^n$ such that $\mathcal{M} \models T \cup A(\bar{v})$ (i.e. A is T-consistent on $\bar{v} \in M^n$) there is $B \in \Sigma$ such that

$$\mathcal{M} \models A(\bar{v}) \wedge \neg B(\bar{v}).$$

Theorem 3 (Omitting of $\Sigma(\bar{x})$)
Let T be a consistent theory in a countable language J_{CL} and $\Sigma(\bar{x})$ be a set of locally omitted J_{CL}-formulas in T. Then there exists a countable model of T which omits $\Sigma(\bar{x})$.

3 Omitting types theory in MTL∀

3.1 Basic definitions

Let us now introduce the above concepts into fuzzy logic. Since we deal with fuzzy logic with traditional syntax, the generalization is limited. Therefore, we will mostly confine to the situation when a formula is true in the degree 1 and say that such a *formula is realized*. The case when it is true in a smaller degree means that such a formula is not realized. For more general concepts that can be introduced in the fuzzy logic with evaluated syntax see [7].

Definition 4
Let J be a predicate language, $\Sigma(\bar{x})$ a set of J-formulas, and $(\mathcal{M}, \mathcal{L})$ be a model for J.

(i) We say that $(\mathcal{M}, \mathcal{L})$ realizes *Σ if there is some $\bar{v} \in M^n$ such that*

$$\|B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$$

(or, more precisely, $\|(B(\bar{v}/\bar{x})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$) holds for all $B \in \Sigma$.

(ii) A model $(\mathcal{M}, \mathcal{L})$ omits *Σ if Σ is not realized in a model $(\mathcal{M}, \mathcal{L})$, i.e. to every $\bar{v} \in M^n$ there is a formula $B \in \Sigma$ such that $\|B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} < \mathbf{1}$.*

The following is obvious.

Lemma 1
Let $(\mathcal{M}, \mathcal{L})$ be a model for J and Σ be a set of J-formulas. Then $(\mathcal{M}, \mathcal{L}) \models \Sigma$ iff every $\bar{v} \in M^n$ realizes Σ in $(\mathcal{M}, \mathcal{L})$.

Definition 5
Let J be a predicate language, T a theory, and $\Sigma(\bar{x})$ be a set of J-formulas.

(i) We say that Σ is consistent *with T (T-consistent) if there is a model $(\mathcal{M}, \mathcal{L})$ of T and $\bar{v} \in M^n$ such that $\|B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$ for all $B \in \Sigma$.*

(ii) We say that Σ is locally realized *in T if there is a formula $A(\bar{x}) \in F_{J(T)}$ such that*
  (a) $A(\bar{x})$ is T-consistent,
  (b) for every model $(\mathcal{M}, \mathcal{L}) \models T$ and for every $\bar{v} \in M^n$, $\|A(\bar{v} \Rightarrow B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$.

(iii) We say that Σ is locally omitted *in T if to every T-consistent formula $A(\bar{x}) \in F_{J(T)}$ there is a formula $B \in \Sigma$ and $\bar{v} \in M^n$ such that $\|A(\bar{v}) \Rightarrow B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} < \mathbf{1}$.*

Obviously, if Σ is locally omitted in T then $T \not\vdash A(\bar{x}) \Rightarrow B(\bar{x})$. As a special case, if $T \vdash A(\bar{x})$ then $T \not\vdash B(\bar{x})$, i.e., if $\|A(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$ then $\|B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} < \mathbf{1}$.

From the previous definitions we immediately get the following lemma.

Lemma 2
If Σ is locally realized in the theory T then Σ is T-consistent.

3.2 Omitting types theorem in MTL∀

The scheme of the proof of the following theorem is analogous to the classical proof, i.e. we construct a complete Henkin extension that, moreover, fulfils the omitting condition. However, the concrete steps are different and must be elaborated carefully. Note that the part concerning Henkin completion is taken from the proof of Lemma 5.2.7 from [3].

Theorem 4
Let J be a countable language of MTL∀ and T be a theory. Let $\Sigma(\bar{x})$ be a set of J-formulas non-isolated in T. Then there exists a countable model $(\mathcal{M}, \mathbf{L})$ which omits $\Sigma(\bar{x})$.

PROOF: Without lack of generality, we will consider a set $\Sigma(x)$ with one free variable only. Let $K = \{c_0, c_1, \ldots\} \not\subset J$ be a countable set of new constant symbols and put $J_K = J \cup K$. Let $A_0, A_1, \ldots, A_m, \ldots$ be a list of closed formulas of the language J_K. To avoid the use of unnecessary subscripts, we will denote formulas from this list by the letters E, F. We will suppose that also the constants from J_K are put on an ordered list.

By induction we will construct an increasing sequence of consistent theories $T = T_0 \subset T_1 \subset \cdots \subset T_m \cdots$ in the language J_K such that for each m, the following is fulfilled:

(i) Each T_m is a consistent theory and it is obtained from $T = T_0$ by extending it by finitely many axioms which contain finite number of new variables (free or bound).
(ii) For each pair of closed formulas $E, F \in \{A_0, \ldots, A_m\}$, either $T_m \vdash E \Rightarrow F$ or $T_m \vdash F \Rightarrow E$.
(iii) If $T_m \not\vdash (\forall y)E(y)$ then $T_m \not\vdash E_y(c_p)$ for the first constant c_p (from the list) not occurring in T_m (the Henkin property).
(iv) There exists a formula $B \in \Sigma$ such that $T_m \not\vdash B(c_m)$ for a corresponding constant c_m from the list.

Let us put $T_0 = T$ and assume that we already have the theory $T_m = T \cup \{C_1, \ldots, C_r\}$ being a consistent extension of T fulfilling (i)–(iv) and such that $T_m \not\vdash D_m$ for a certain formula D_m which must be kept unprovable to assure the properties (iii) and (iv).

We construct T_{m+1} as follows. Put

$$C = C_1 \wedge \cdots \wedge C_r$$

and let the list $c_m, c_1, \ldots, c_n$ contain all constants from K occurring in the formula C. We replace these constants by new variables $x, y_1, \ldots, y_n$, respectively and obtain a new formula which we denote by H'. Finally, we form a new formula $H(x)$ from H' by adding existential quantifiers $(\exists y_1), \ldots, (\exists y_n)$. Since T_m is a consistent extension of T, $H(x)$ is consistent with T. Because Σ is locally omitted in T there is a formula $B(x) \in \Sigma$ such that $T \not\vdash H(x) \Rightarrow B(x)$. Moreover, we have $T \vdash H(c_m)$ and so, $T \not\vdash B(c_m)$ (otherwise we would obtain $T \vdash H(c_m) \Rightarrow B(c_m)$, i.e. a contradiction).

This observation enables us to continue in construction of T_{m+1} in such a way that the properties (i)–(iv) are fulfilled.

(a) First, we assure (ii) and (iv). Since $T_m \not\vdash D_m \vee B(c_m)$, we will now consider a theory T'_{m+1} which is a consistent extension of T_m deciding a couple of formulas $E, F \in \{A_0, \ldots, A_{m+1}\}$ (in the sense of (3)). Thus, $T'_{m+1} \not\vdash D_m \vee B(c_m)$ and we put $D'_{m+1} = D_m \vee B(c_m)$. This step can be repeated if necessary (finitely many times). We conclude that T'_{m+1} fulfils (ii) and (iv).

(b) Now we assure (iii). Let a formula A_m be of the form $A_m = (\forall y)E(y)$ and let c_p be the first constant not occurring in T_m. We must distinguish two cases:

Case 1: Let $T'_{m+1} \not\vdash D'_{m+1} \vee E(c_p)$. Then also $T'_{m+1} \not\vdash (\forall y)E(y)$. We put $T_{m+1} = T'_{m+1}$ and $D_{m+1} = D'_{m+1} \vee E(c_p)$.

Case 2: Let $T'_{m+1} \vdash D'_{m+1} \vee E(c_p)$. Thus $T'_{m+1} \vdash D'_{m+1} \vee E(y)$ for a suitable y. Hence $T'_{m+1} \vdash (\forall y)(D'_{m+1} \vee E(y))$ and using the axiom ($\forall$3), $T'_{m+1} \vdash (D'_{m+1} \vee (\forall y)E(y))$. From the definition of $\vee$ we have

$$T'_{m+1} \vdash ((D'_{m+1} \Rightarrow (\forall y)E(y)) \Rightarrow (\forall y)E(y)) \wedge (((\forall y)E(y) \Rightarrow D'_{m+1}) \Rightarrow D'_{m+1}).$$

By properties of MTL$\forall$ we get

$$T'_{m+1} \cup \{(\forall y)E(y) \Rightarrow D'_{m+1}\} \vdash D'_{m+1}.$$

Therefore, we put $T_{m+1} = T'_{m+1} \cup \{D'_{m+1} \Rightarrow (\forall y)E(y)\}$ and $D_{m+1} = D'_{m+1}$.

Thus $T_{m+1} \not\vdash D_{m+1}$, and either $T_{m+1} \not\vdash (\forall y)E(y)$ implies $T_{m+1} \not\vdash E(c_p)$ (Case 1) or this situation does not happen. Consequently, we verify that all properties (i)–(iv) are fulfilled by T_{m+1}.

Let us now put $T^+ = \bigcup_{m \in \mathbb{N}} T_m$. It follows for the above construction that T^+ is a complete Henkin theory. The rest is analogous to the classical proof. Namely, we consider a canonical model $(\mathcal{W}^+, \mathcal{L})$ of T^+. Due to item (iv), each element of W is an interpretation of some constant c_p and so, restriction of $(\mathcal{W}^+, \mathbf{L})$ to J provides a countable model $(\mathcal{M}, \mathcal{L}) \models T$ which omits Σ because to every element $v \in M$ which interprets the constant c_p we can find $B \in \Sigma$ such that $T \not\vdash B(c_p)$. Therefore,

$$\|B(v)\|^{\mathcal{L}}_{\mathcal{M}} < \mathbf{1}.$$

□

4 Omitting types theory core predicate fuzzy logic

In this section we will generalize the omitting types theorem to arbitrary core fuzzy logics and to languages with arbitrary cardinality α. Our theorem is a generalization of the α-omitting types theorem from [1].

Let us consider a core (predicate) fuzzy logic with the language J whose set of formulas F_J has the cardinality $|F_J| = \alpha$.

Definition 6
Let $\Sigma(\bar{x})$ be a set of J-formulas having the cardinality α.

(i) We say that Σ is α-realized *in T if there is a set of formulas $\Psi(\bar{x}) \subset F_{J(T)}$ with the cardinality smaller than α, such that:*
  (a) $\Psi(\bar{x})$ is T-consistent, i.e. there is a model $(\mathcal{M}, \mathcal{L}) \models T$, and $\bar{v} \in M^n$ such that $\|A(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$ holds for every $A \in \Psi$,
  (b) for every model $(\mathcal{M}, \mathcal{L}) \models T$, every $\bar{v} \in M^n$ and every $A \in \Psi$, if $\|A(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$ then $\|B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} = \mathbf{1}$ holds for all $B \in \Sigma$.

(ii) We say that Σ is α-omitted *in T if to every T-consistent set of formulas $\Psi(\bar{x}) \subset F_{J(T)}$ with the cardinality smaller than α there is a model $(\mathcal{M}, \mathcal{L}) \models T$, $\bar{v} \in M^n$, and $B \in \Sigma$ such that for all $A \in \Psi$ $\|A(\bar{v}) \Rightarrow B(\bar{v})\|^{\mathcal{L}}_{\mathcal{M}} < \mathbf{1}$. This means that $T \not\vdash A(x) \Rightarrow B(x)$ holds for all $A \in \Psi$.*

Theorem 5
Let us consider a core predicate fuzzy logic with the language J and the cardinality of $|F_J| = \alpha$. Let T be a theory of this logic and $\Sigma(\bar{x}) \subset F_J$ a set of formulas α-omitted in T. Then T has a model $(\mathcal{M}, \mathbf{L})$ of the cardinality α which omits $\Sigma(\bar{x})$.

PROOF: Analogously as in the proof of Theorem 4, we will construct a complete Henkin theory which fulfils the omitting condition. Therefore, we will also adopt the proof of Lemma 2 from [4]. Without loss of generality we consider formulas with one free variable x only.

First, we put closed formulas $A_0, A_1, \ldots, A_\gamma, \ldots$ from F_J onto a list and enumerate them by ordinal numbers. Furthermore, let $K = \{c_0, c_1, \ldots, c_\alpha\} \not\subset J$ be a set of new constant symbols and put $J_K = J \cup K$. By transfinite recursion we will construct a sequence of theories $T_0 \subseteq T_1 \subseteq \cdots T_\gamma \subseteq \cdots$ such that for each γ the following conditions are fulfilled:

(i) Each T_γ is a consistent extension of T so that $T_\gamma = T \cup \Psi_\gamma$ where $\Psi_\gamma \subset F_J$ is constructed by a transfinite recursion.
(ii) For each couple of formulas $E, F \in \{A_0, \ldots, A_\gamma\}$ either $T_\gamma \vdash E \Rightarrow F$ or $T_\gamma \vdash F \Rightarrow E$.
(iii) If $T_\gamma \nvdash (\forall y)E(y)$ for some formula $E(y)$ then there is a constant c_μ not occurring in T_γ such that $T_\gamma \nvdash E(c_\mu)$.
(iv) There is a formula $B \in \Sigma(x)$ such that $T_\gamma \nvdash B(c_\gamma)$ for a corresponding constant from K.

Let us suppose that we are already in a step $\gamma \leq \alpha$. We will construct a theory T_γ and a directed set of unprovable formulas Λ_γ which assures conditions (iii) and (iv). We will use the following notation: $\Psi_{<\gamma} = \bigcup_{\delta<\gamma} \Psi_\delta$, $T_{<\gamma} = T \cup \Psi_{<\gamma}$, and $\Lambda_{<\gamma} = \bigcup_{\delta<\gamma} \Lambda_\delta$.

(a) We will construct a T-consistent set $\overline{\Psi}_{<\gamma}(x)$ which has a cardinality smaller than α.

Let $c_1, \ldots, c_\beta \in K$ be a set of all constants occurring in formulas from Ψ_δ and let x be a variable not occurring in them. First, we take all formulas containing a constant c_1. In each of these formulas we replace c_1 by x, the remaining constants by some substitutable variables $y, \ldots$ and put $(\exists y), \ldots$ before it. Then take the remaining formulas containing the constant c_2, replace it by x, replace the remaining constants by $y, \ldots$ and add $(\exists y), \ldots$, etc. The result is a set $\overline{\Psi}_\delta(x)$ and we put $\overline{\Psi}_{<\gamma}(x) = \bigcup_{\delta<\gamma} \overline{\Psi}_\delta(x)$. Since $T_{<\gamma}$ is a consistent extension of T, $\overline{\Psi}_{<\gamma}(x)$ is T-consistent and by the construction below it has a cardinality smaller than α.

(b) By the assumption, there is a formula $B \in \Sigma(x)$ such that $T \nvdash A(x) \Rightarrow B(x)$ holds for all $A \in \overline{\Psi}_{<\gamma}(x)$. Let c_γ be an unused constant. Then $T_{<\gamma} \nvdash A(c_\gamma) \Rightarrow B(c_\gamma)$ and since $T_{<\gamma} \vdash A(c_\gamma)$, we have $T_{<\gamma} \nvdash B(c_\gamma)$ [†)].

We put

$$\Lambda'_\gamma = \{D \vee B(c_\gamma) \mid D \in \Lambda_{<\gamma}\}. \tag{4}$$

This assures condition (iv).

(c) we will put $\Psi'_\gamma = \Psi_{<\gamma} \cup \Phi$ where Φ is a set of all couples of formulas $E, F \in \{A_0, \ldots, A_\gamma\}$ such that $T_{<\gamma}$ decides E, F. This assures condition (ii).

(d) Let us now consider a formula $(\forall y)E(y)$ and assure condition (iii) (Henkin property). For this, we consider cases (H1) and (H2) as in the proof of Lemma 2 from [4]:

(H1) $T_{<\gamma} \vdash D \vee (\forall y)E(y)$ for some $D \in \Lambda'_\gamma$. We put $\Psi_\gamma = \Psi_{<\gamma} \cup \{(\forall y)E(y)\}$ and $\Lambda_\gamma = \Lambda_{<\gamma}$.
(H2) $T_{<\gamma} \nvdash D \vee (\forall y)E(y)$ for some $D \in \Lambda'_\gamma$. We put $\Psi_\gamma = \Psi'_\gamma$ and $\Lambda_\gamma = \Lambda_{<\gamma} \cup \{D \vee E(c_\mu) \mid D \in \Lambda'_\gamma\}$ for some unused constant c_μ.

The proof that the theory T_γ fulfils condition (iii) (and also (iv)) is the same as the proof of cases (H1) and (H2) in Lemma 2 from [4]. We must, of course, assure that Λ_γ is directed. But this is again the same as in the cited lemma.

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†) In fact, $T_{<\gamma}$ is obtained as a conservative extension by adding the constant c_γ to the language $J(T_{<\gamma})$

We finish with the theory $T_\alpha = T \cup \Psi_\alpha$ which is a complete Henkin theory with the cardinality α. Then, analogously as in the proof of Theorem 4, we consider a canonical model of T_α which has the cardinality α and from it we obtain a model of T α-omitting the set Σ.

□

5 Conclusion

This paper is a continuation of the development of model theory of fuzzy logic. We focused on MTL∀-fuzzy logic and core predicate fuzzy logics. The main result of this paper are theorems generalizing classical omitting types theorems in both kinds of logics. Since these fuzzy logics have traditional syntax, the omitting types theory has many points common with classical logic. One of the consequences is that only the truth degree 1 in the model is interesting. Perhaps some deeper generalizations might be possible, for example, when introducing logical (truth) constants in the syntax. We leave this question to some of the subsequent papers.

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