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On Fuzzy Normal Forms

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Abstract

This paper provides a brief survey of normal forms and results in this area. Moreover, it defines a new class of normal forms - of the multiplicative ones - which are studied from the approximation point of view in the context of the previous normal forms.

Key words fuzzy rule base; normal forms; fuzzy relation equations

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1 Introduction

Fuzzy normal forms have been originally introduced by I. Perfilieva in [9] as fuzzy relations in a unified form aiming at approximate representation of the ideal one. They have a direct relationship to interpretations of fuzzy rule bases (FRB) [1], hence to solutions of systems of fuzzy relation equations and other fields of applications involving FRB.

We provide a brief survey of normal forms and results relating to their approximating properties [8, 9, 3]. Moreover, we add new normal forms, which have not been studied yet. These normal forms are based on different couple of operations in use and their investigation was motivated by new results [10] obtained in the field of fuzzy relation equations.

2 Normal forms

Throughout the whole text we assume that M is a non-empty set of objects and $F \underset{\sim}{\subseteq} M, E \underset{\sim}{\subseteq} M \times M$, i.e. fuzzy sets on the respective universe. Furthermore, $*$ denotes a left-continuous t-norm and $\rightarrow_*$ its adjoint residuation and $\leftrightarrow_*$ associated biresiduation, which we will call $*$-equivalence. Finally, $\otimes$ denotes the Łukasiewicz t-norm and $\oplus$ denotes the Łukasiewicz t-conorm.

Definition 1 Let $N \subseteq M$. The right hand sides of the following formulas

$$F_{DNF,*}(x) = \bigvee_{c \in N} (E(c,x) * F(c)) \tag{1}$$

$$F_{ANF,*}(x) = \bigoplus_{c \in N} (E(c,x) * F(c)), \tag{2}$$

$$F_{CNF,*}(x) = \bigwedge_{c \in N} (E(x,c) \rightarrow_* F(c)), \tag{3}$$

$$F_{MNF,*}(x) = \bigotimes_{c \in N} (E(x,c) \rightarrow_* F(c)) \tag{4}$$

are the *disjunctive*, the *additive*, the *conjunctive* and the *multiplicative normal forms* of F w.r.t. N, respectively.

Let us recall the definition of the generalized extensionality property which is essential for further results in this area.

Definition 2 We say that F is extensional w.r.t. E and $*$ on M if

$$E(x,y) * F(x) \leq F(y), \quad \forall x, y \in M \tag{5}$$

We recall the orthogonality property which was used to prove main properties of the additive normal forms, see [9, 3].

Definition 3 Let $N \subseteq M$. We say that E fulfils the orthogonality condition (or is orthogonal) w.r.t. N on M if

$$\bigoplus_{c \in N \setminus \{d\}} E(c,x) = 1 - E(d,x), \quad \forall x \in M, \forall d \in N. \tag{6}$$

The orthogonality condition has a close relationship to the Ruspini condition, the property that is often used in the construction process of FRB.

Lemma 1 *Fuzzy relation E is orthogonal w.r.t. N on M if and only if*

$$\sum_{c \in N} E(c,x) = 1, \quad \forall x \in M. \tag{7}$$

3 Comparison of Normal forms

The following inequalities (8) and (9) stem from the results published in [8] and [9], respectively.

Proposition 1

- *[2] If F is extensional w.r.t. E and $*$ then*

$$F_{DNF,*}(x) \leq F(x) \leq F_{CNF,*}(x), \quad \forall x \in M. \tag{8}$$

- *[3] If E is symmetric and orthogonal w.r.t. N on M then*

$$F_{DNF,*}(x) \leq F_{ANF,*}(x) \leq F_{CNF,\otimes}(x), \quad \forall x \in M. \tag{9}$$

Without any assumption on F or E, the inequalities $F_{DNF,*}(x) \leq F_{ANF,*}(x)$ and $F_{MNF,*}(x) \leq F_{CNF,*}(x)$ are valid for each $x \in M$.

The following result holds for multiplicative normal form of F with an orthogonal E and a specially chosen t-norm $*$.

Theorem 1 *If E is orthogonal w.r.t. finite N on M and $* \leq \otimes$ then*

$$F_{DNF,\otimes}(x) \leq F_{MNF,*}(x) \leq F_{CNF,*}(x), \quad \forall x \in M \tag{10}$$

and consequently

$$F_{DNF,*}(x) \leq F_{MNF,*}(x) \leq F_{CNF,*}(x), \quad \forall x \in M \tag{11}$$

PROOF: Let $c' \in N$ be such that

$$F_{DNF,\otimes}(x) = \bigvee_{c \in N} (E(c,x) \otimes F(c)) = E(c',x) \otimes F(c') = E(c',x) + F(c') - 1.$$

Using $a \rightarrow_* b \geq a \rightarrow_\otimes b = (1-a) \oplus b \geq (1-a)$, we can estimate

$$F_{MNF,*}(x) = \bigotimes_{c \in N \setminus \{c'\}} (E(x,c)) \rightarrow_* F(c)) \otimes (E(x,c')) \rightarrow_* F(c'))$$

by

$$F_{MNF,*}(x) \geq \bigotimes_{c \in N \setminus \{c'\}} (1 - E(x,c)) \otimes (E(x,c')) \rightarrow_* F(c'))$$

which equals to

$$\left(0 \vee \left(1 - \sum_{c \in N \setminus \{c'\}} E(x,c)\right)\right) \otimes (E(x,c')) \rightarrow_* F(c')).$$

Due to the orthogonality of E and the fact that $* \leq \otimes$:

$$F_{MNF,*}(x) \geq E(x,c') \otimes (E(x,c')) \rightarrow_\otimes F(c')) \geq E(x,c') \otimes (1 - E(x,c') + F(c'))$$

and

$$E(x,c') \otimes (1 - E(x,c') + F(c')) = E(x,c') + E(x,c') + 1 - 1 + F(c') = F(c').$$

So, finally $F_{MNF,*}(x) \geq F(c') \geq E(c',x) + F(c') - 1 = F_{DNF,\otimes}(x)$.

Inequality (11) immediately comes from the fact that $* \leq \otimes$. $\square$

Notice that $N \subseteq M$ can be chosen arbitrarily. Below, we will investigate the extremal case $N = M$.

Proposition 2 *[3] Let $N = M$ and E be reflexive. Then F is extensional w.r.t. E and $*$ if and only if*

$$F_{DNF,*}(x) = F(x) = F_{CNF,*}(x), \quad \forall x \in M. \tag{12}$$

Remark 1 *Note, that if E is reflexive and $N = M$ then the orthogonality property is valid if and only if E is trivial , i.e., $E(x, y) = 1$ for $x = y$, otherwise $E(x, y) = 0$.*

Proposition 3 *[3] Let $N = M$, E be reflexive and $F(x) \in (0, 1)$ for all $x \in M$. Then E fulfils the orthogonality condition w.r.t. N if and only if*

$$F_{ANF,*}(x) = F(x), \quad \forall x \in M. \tag{13}$$

Proposition 4 *Let $N = M$ and E be reflexive and orthogonal w.r.t. N. Then*

$$F_{MNF,*}(x) = F(x), \quad \forall x \in M. \tag{14}$$

PROOF:

$$F_{MNF,*}(x) = (E(x, x) \rightarrow_* F(x)) \otimes \bigotimes_{c \in N \setminus \{x\}} (E(x, c) \rightarrow_* F(c))$$

and because of Remark 1 $F_{MNF,*}(x) = (1 \rightarrow_* F(x)) \otimes 1 = F(x)$. □

4 Approximation properties

Since the $*$-equivalence can be used to estimate a closeness between two objects similarly as standard metrics with the reversed ordering, therefore, we will try to provide lower estimation of the given fuzzy relation and an appropriate normal form.

Theorem 2 *[3] If F is extensional w.r.t. E and $*$ then*

$$C_*(x) \leq F_{DNF,*}(x) \leftrightarrow_* F(x), \quad x \in M \tag{15}$$

$$C_*(x) \leq F_{CNF,*}(x) \leftrightarrow_* F(x), \quad x \in M \tag{16}$$

where

$$C_*(x) = \bigvee_{c \in N} E(x, c) * E(c, x), \quad x \in M \tag{17}$$

The equivalence between F and $F_{ANF,*}$ can be estimated as follows:

Theorem 3 *[3] Let E be symmetric and orthogonal w.r.t. N. Let F be extensional w.r.t. E and $*$ and moreover, let F be extensional w.r.t. E and $\otimes$.*

(1) *If $*$ is weaker than $\otimes$ then for all $x \in M$*

$$C_*(x) \leq F_{ANF,*}(x) \leftrightarrow_* F(x), \quad x \in M. \tag{18}$$

(2) *If $*$ is stronger than $\otimes$ then for all $x \in M$*

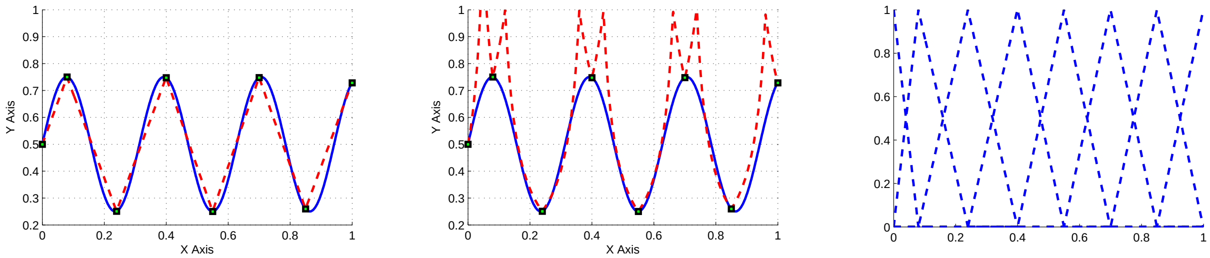
$$C_\otimes(x) \leq F_{ANF,*}(x) \leftrightarrow_\otimes F(x), \quad x \in M. \tag{19}$$

Corollary 1 *Let E be symmetric and orthogonal w.r.t. N and let F be extensional w.r.t. E and $\otimes$. Then*

$$C_\otimes(x) \leq F_{ANF,\otimes}(x) \leftrightarrow_\otimes F(x), \quad x \in M. \tag{20}$$

Theorem 4 *Let N be finite and F be extensional w.r.t. E and $*$ where $* \leq \otimes$. Then*

$$C_*(x) \leq F_{MNF,*}(x) \leftrightarrow_* F(x), \quad x \in M. \tag{21}$$



(a) $F_{ANF,\cdot,}$ (b) $F_{MNF,\cdot,}$ (c) Final fuzzy numbers

Figure 1: Approximations of F from Example 1 using normal forms.

PROOF: Using Proposition 1 and Theorem 2. □

Corollary 2 *Let N be finite and let F be extensional w.r.t. E and $\otimes$ then*

$$C_{\otimes}(x) \leq F_{MNF,\otimes}(x) \leftrightarrow_{\otimes} F(x), \quad x \in M. \tag{22}$$

We see that the precision of approximation of the original fuzzy relation measured by $*$-equivalence depends only on the choice of E and the distribution of the nodes $N \subseteq M$.

Example 1 *Let us consider $*$ to be the product t-norm, $F(x) = 0.25\sin(20x)+0.5$, $N = [0, 0.08, 0.24, 0.4, 0.55, 0.7, 0.85, 1]$, and $E(x, y) = 1 - k|x - y|$, where k is chosen dependently on N such that the orthogonality property remains valid. Then we can construct all kinds of normal forms. First let us consider $F_{MNF,\cdot}$ and $F_{ANF,\cdot}$, which are depicted on Figure 1(a) and 1(b), respectively.*

In the case of $F_{DNF,\cdot}$ and $F_{CNF,\cdot}$, we do not consider the orthogonality, hence, the choice of k is now limited only by the extensionality property. The resulting approximations are depicted on Figure 2, where the nodes of $F_{DNF,\cdot}$ and $F_{CNF,\cdot}$ differ, and they are marked by the squared box.

5 Discussion

We have summarized all the known results relating to the normal forms. Moreover, we have introduced the multiplicative normal form. This normal form can be viewed as a dual one to the additive normal form although it does not preserve the same properties – compare Theorem 1 and Proposition 1.

Besides the survey and new results related to the normal forms, one more issue has to be discussed. It is the terminological issue. The usage of the notion "normal form" might be questionable because of several reasons. Usually, an object is in a normal form if it is given in a certain predefined form. In our case, four types of special forms are given by right hand sides of formulas (1), (2), (3) and (4) from Definition 1.

Furthermore, it is assumed that if a particular object can be given in a normal form then the equivalence between this normal form and the object itself can be proved. In our case, a fuzzy relation is not in general equivalent to its normal form. What can be shown is the so called *conditional equivalence* of a fuzzy relation and normal form of this relation (see Theorems 2, 3, 4 and Corollaries 1, 2), i.e. the degree of the equivalence is bounded from below by a degree of fulfillment of some condition (the left sides of inequalities in the above listed propositions). Hence, it can be find out when normal forms are fully equivalent to the original fuzzy relation.

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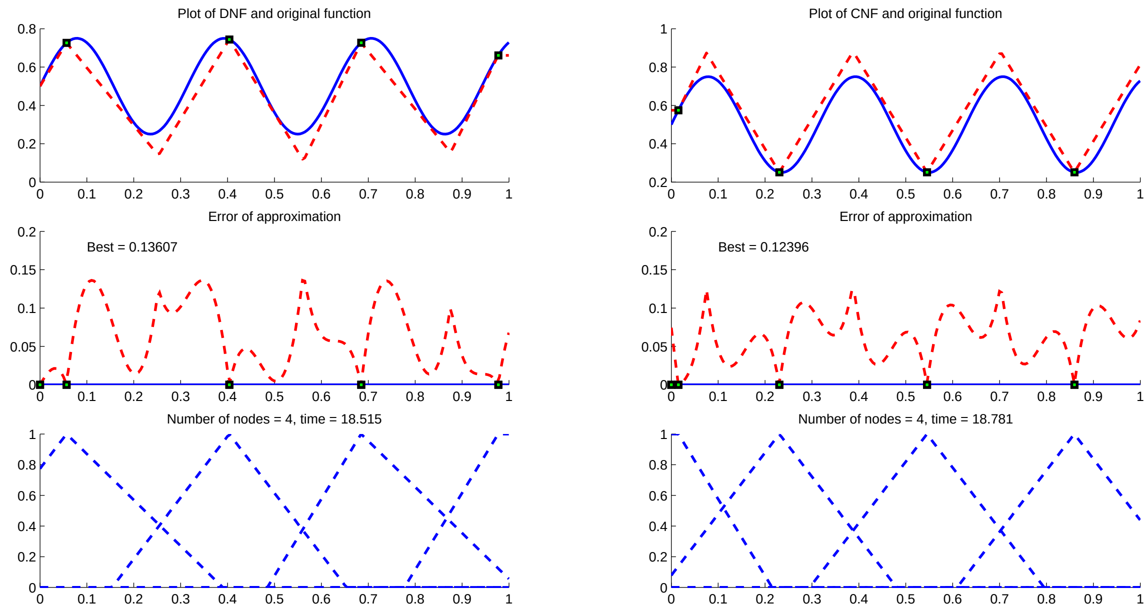

(a) $F_{DNF,\cdot}$,

(b) $F_{CNF,\cdot}$,

Figure 2: Approximations of F from Example 1 using normal forms.

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