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Abstract

The paper provides a logical theory of a specific class of natural language expressions called intermediate quantifiers (*most, a lot of, many, a few, a great deal of, a large part of, a small part of*), which can be ranked among generalized quantifiers. The formal frame is the fuzzy type theory (FTT).

Our main idea lays in the observation that intermediate quantifiers speak about elements taken from a class that is made "smaller" than the original universe in a specific way. Our theory is based on the formal theory of trichotomous evaluative linguistic expressions. Thus, an intermediate quantifier is obtained as a classical quantifier "for all" or "exists" but taken over a class of elements that is determined using an appropriate evaluative expression. In the paper we will characterize the behavior of intermediate quantifiers and prove many valid syllogisms that generalize classical Aristotle's ones.

Key words: Generalized quantifiers, higher order fuzzy logic, fuzzy type theory, Aristotle's syllogisms, evaluative linguistic expressions, precisiated natural language, fuzzy quantifiers.

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1 Introduction

In fuzzy set theory, the concept of "fuzzy quantifiers" has been introduced by Zadeh [18]. His theory has been further elaborated by several authors, among them let us name Glockner [5]. It turns out that from linguistic point

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of view, fuzzy quantifiers predominantly fall among the, so called intermediate quantifiers that are expressions such as *most, a lot of, many, a few, a great deal of, large part of, small part of.* Intermediate quantifiers have been studied in [15] and [16] †) . The fuzzy quantifiers have been studied especially from the semantic point of view without a clearly distinguished formal logical system.

In this paper, we develop a formal theory of intermediate quantifiers inside the fuzzy type theory (FTT; a higher order fuzzy logic). Our main idea is to introduce them as special formulas constructed in a certain extension of a special formal theory T^{Ev} that is the theory of *trichotomous evaluative linguistic expressions* introduced in [12].

Our approach has the following advantages:

(i) Intermediate quantifiers are only shorts for special formulas. Hence, it is unnecessary to extend syntax of FTT (except for an ordinary definition of a new symbol) and so, its completeness is not harmed, i.e., there is no need to search for a new completeness proof.
(ii) Our theory is sufficiently general to encompass a wide class of generalized quantifiers and provides a unique definition for all of them.
(iii) Properties of the quantifiers can be studied in syntax, thus separating them from their interpretation. The latter can then be defined in many (consistent) ways and so, we can well distinguish various practical situations in which intermediate quantifiers are to be used.

In this paper we deal with syntax only and so, all our proofs are syntactical. We will mention semantics only marginally and postpone this problem to a subsequent paper.

The structure of the paper is the following: Section 2 provides overview of three principal components of our approach: the fuzzy type theory (FTT), the formal theory of trichotomous evaluative linguistic expressions, and fuzzy set theory (taken as formal syntactical theory of FTT). In Section 3 we introduce formally intermediate quantifiers and present some of their basic properties. In Section 4, we introduce several concrete intermediate quantifiers. Their definition, in fact, is a formalization of the quantifiers considered in the book [15]. Soundness of our definitions is demonstrated on many of the generalized syllogisms introduced in the cited book (the syllogisms not presented in this paper have not been formally analyzed up till now).

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†) In [16], the intermediate quantifiers are called *comparative.* According to the literature on generalized quantifiers, however, "comparative quantifiers" are quantifiers such as "more than about", "about three times less than", etc. We will, therefore, keep the term *intermediate* in this paper.

2 Formal tools

2.1 Fuzzy type theory

The detailed explanation of FTT can be found in [10]. The classical type theory is described in details in [1].

2.1.1 Basic syntactic elements

The *Types* is a set of types constructed iteratively from the atomic types ϵ (elements) and o (truth values). Form_α denotes a set of formulas of type $\alpha \in \mathit{Types}$ which is the smallest set satisfying:

(i) Variables $x_\alpha \in \mathit{Form}_\alpha$ and constants $c_\alpha \in \mathit{Form}_\alpha$,
(ii) if $B \in \mathit{Form}_{\beta\alpha}$ and $A \in \mathit{Form}_\alpha$ then $(BA) \in \mathit{Form}_\beta$,
(iii) if $A \in \mathit{Form}_\beta$ then $\lambda x_\alpha\, A \in \mathit{Form}_{\beta\alpha}$,

Each formula has its own type. We will usually write the type of a formula as its subscript, i.e., A_α is a formula of type α. Sometimes, it is more convenient to write $A \in \mathit{Form}_\alpha$ and in this case, we write the formula A without the subscript α. Thus, A_α is equivalent to $A \in \mathit{Form}_\alpha$. However, we will freely write or omit the type when no misunderstanding may occur.

Variables, constants, connectives, and the above defined sequences are also particular formulas. Some of the classical type theories use the term *lambda-term* instead of formula. In this paper, we prefer to stick to the original name introduced by Church [3] and Henkin [6].

It is specific for fuzzy type theory that its basic connective is a fuzzy equality (this is a direct generalization of Henkin introduction of type theory). Therefore, we consider a specific constant $\mathbf{E}_{(o\alpha)\alpha}$ for every $\alpha \in \mathit{Types}$ in the language of FTT. Using it, the fuzzy equality is defined by

$$\equiv \; := \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha.$$

This is a formula of type $(o\alpha)\alpha$. The interpretation of $\equiv$ is introduced below.

If A_α, B_α are formulas then $(A_\alpha \equiv B_\alpha)$ is a formula of type o. Note that, if $\alpha = o$ then $\equiv$ is the logical equivalence, i.e., we read $(A_o \equiv B_o)$ as "the formula A_o (of type truth) is equivalent to B_o". Since we are in fuzzy type theory, the latter formula can be true (in some interpretation) only to a certain degree. Special derived formulas are also logical constants of *truth* $\top$ and *falsity* $\bot$.

Formulas of type o (truth value) can be joined by the following connectives (these are derived formulas): $\vee$ (disjunction), $\wedge$ (conjunction), $\&$ (strong conjunction), ∇ (strong disjunction), $\Rightarrow$ (implication). n-times strong conjunction of A_o will be denoted by A_o^n and similarly, n-times strong disjunction by nA_o. General ($\forall$) and existential ($\exists$) quantifiers are defined as special formulas. For the details about their definition and semantics — see [10].

We will use the following conventions in the sequel: priority of logical connectives:

(1) $\neg, \Delta$.
(2) $\&, \nabla, \wedge, \vee$.
(3) $\equiv$.
(4) $\Rightarrow$.

Sometimes, we will also use the dot convention as follows: the formula $A \cdot B$ is equivalent to $A(B)$. For example, the formula $\lambda x \cdot Ax \wedge \cdot Bx \Rightarrow C$ is equivalent to $\lambda x\, (Ax \wedge (Bx \Rightarrow C))$. As usual, $A \not\equiv B$ is an abbreviation of $\neg(A \equiv B)$, and the like. If type of the formula in concern is clear from the context then we will omit it. The symbol := should be read as "is" and it means a denotation of the expression on the right hand side by a symbol on the left hand side.

If $A \in \mathit{Form}_{o\alpha}$ then A represents a fuzzy set of objects of the type α. Since this fuzzy set can be taken as determined by some property, we will often say "A is a property" (of elements of type α). Similarly, $A_{(o\alpha)\alpha}$ is a fuzzy relation (between elements of type α).

2.1.2 Basic semantic elements

The structure of truth values is generally supposed to form either a complete IMTL$_\Delta$-algebra (see [4]) or a standard Łukasiewicz$_\Delta$ algebra (which can be seen as a special case of IMTL$_\Delta$-algebra). In this paper, we use exclusively the second possibility because FTT based on the Łukasiewicz$_\Delta$ algebra (Ł-FTT) has some nice and useful properties needed for our model.

Recall that the standard Łukasiewicz$_\Delta$ algebra is the algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \oplus, \Delta, \rightarrow, 0, 1 \rangle \tag{1}$$

where

$$\begin{aligned} \wedge &= \text{minimum}, & \vee &= \text{maximum}, \\ a \otimes b &= 0 \vee (a + b - 1), & a \rightarrow b &= 1 \wedge (1 - a + b), \\ \neg a &= a \rightarrow 0 = 1 - a, & a \oplus b &= 1 \wedge (a + b), \end{aligned}$$

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We may consider also a finite Łukasiewicz$_\Delta$ algebra (cf. [13]). The Łukasiewicz biresiduation is the operation $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

A *basic frame* for the language J is a system of sets $(M_\alpha)_{\alpha \in \mathit{Types}}$. A fuzzy set $A \underset{\sim}{\subset} M_\alpha$ is a function $A : M_\alpha \longrightarrow L$. The set of all fuzzy sets over M_α is denoted by $\mathcal{F}(M_\alpha)$.

As noted, important concept in FTT is that of a *fuzzy equality* which is a semantic interpretation of the connective $\equiv$. The fuzzy equality between objects of type $\alpha \in \mathit{Types}$ is a fuzzy relation $=_\alpha$: $M_\alpha \times M_\alpha \longrightarrow L$ having the following properties:

$$[x =_\alpha x] = \mathbf{1}, \qquad \text{(reflexivity)}$$
$$[x =_\alpha y] = [y =_\alpha x], \qquad \text{(symmetry)}$$
$$[x =_\alpha y] \otimes [y =_\alpha z] \leq [x =_\alpha z] \qquad \text{(transitivity)}$$

where $[x =_\alpha y] \in L$ denotes a truth value of the fuzzy equality $x =_\alpha y$ between the objects $x, y \in M_\alpha$.

There are numerous examples of fuzzy equalities w.r.t. specific structures of truth values. A typical fuzzy equality between real numbers when considering the Łukasiewicz algebra of truth values is

$$[x =_\epsilon y] = 1 - (1 \wedge |x - y|), \qquad x, y \in \mathbb{R}.$$

2.1.3 Syntax

The syntax of FTT consists of definitions of fundamental formulas, axioms and inference rules. The FTT has 17 axioms that may be divided into the following subsets: fundamental equality axioms, truth structure axioms, quantifier axioms, and description axioms. Let us present a selection of them.

The fundamental equality axioms are the following:

(FT$_I$1) $\boldsymbol{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$
(FT$_I$2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$
(FT$_I$2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$
(FT$_I$3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$ where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).
(FT$_I$4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon))$

Further axioms characterize structure of truth values. These axioms (altogether 11) assure that the predicate IMTL (or Łukasiewicz) fuzzy logic with the $\boldsymbol{\Delta}$ connective is included in FTT, i.e. all theorems of the former are provable also in FTT.

Specific axiom important for the provability properties among them is the axiom

$$(\mathrm{FT}_I 6) \quad (A_o \equiv \top) \equiv A_o.$$

The quantifier axiom is

$$(\mathrm{FT}_I 16) \quad (\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o) \qquad \text{where } x_\alpha \text{ is not free in } A_o.$$

Let us remark that the substitution axiom is a provable formula in FTT.

Finally, the descriptions axioms are the following:

$$(\mathrm{FT}_I 17) \quad \iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon$$

where $\iota_{\alpha(o\alpha)}$ is the *description operator* whose interpretation is defuzzification operation which assigns to a fuzzy set an element from its kernel.

There are two inference rules in FTT:

(R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then, from B infer B' where B' comes from B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

(N) *Let $A_o \in Form_o$. Then, from A_o infer $\boldsymbol{\Delta} A_o$.*

A *formal theory* of FTT is a set of formulas of type o (determined by a subset of special axioms, as usual). Provability is defined as usual. The inference rules of modus ponens and generalization are derived rules in FTT.

In the sequel, we will often refer to axioms and various proved facts from the fuzzy type theory. Since it is not possible to list all of them here, we will simply write "by properties of FTT" and refer the reader to [10].

2.1.4 Semantics

A *frame* for the language J is a tuple $\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$ where $(M_\alpha)_{\alpha \in Types}$ is a basic frame, $\mathcal{L}_\Delta$ is the algebra of truth values (in this paper, we consider it to be the Łukasiewicz$_\Delta$ algebra) and each $=_\alpha$, $\alpha \in Types$, is a fuzzy equality on M_α. Recall that if $\beta\alpha$ is a type then the corresponding set $M_{\beta\alpha}$ contains (not necessarily all) functions $f : M_\alpha \longrightarrow M_\beta$. We put $M_o = L$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{L}_\Delta$.

Let p be an assignment of elements from $\mathcal{M}$ to variables. An interpretation $\mathcal{I}^{\mathcal{M}}$ is a function that assigns every formula A_α, $\alpha \in \mathit{Types}$ and every assignment p a corresponding element, that is, a function of the type α. A general model is a frame $\mathcal{M}$ such that $\mathcal{I}_p^{\mathcal{M}}(A_\alpha) \in M_\alpha$ holds true.

A formula A_o is *crisp* if $\vdash A_o \vee \neg A_o$. It can be proved that A_o is crisp iff $\vdash A_o \equiv \mathbf{\Delta} A_o$. Let us remark that in FTT, we can introduce also special crisp formulas representing non-zero truth values, namely $\Upsilon_{oo} := \lambda z_o \cdot \neg\mathbf{\Delta}(\neg z_o)$ (nonzero truth value) and $\hat{\Upsilon}_{oo} := \lambda z_o \cdot \neg\mathbf{\Delta}(z_o \vee \neg z_o)$ (general truth value).

Some of properties of Ł-FTT needed in the sequel are summarized in the following lemma.

Lemma 1
Let $A, B, C \in \mathit{Form}_o$.

(a) If $\vdash (A \Rightarrow B) \,\&\, (A \Rightarrow \neg B)$ *then* $\vdash \mathbf{\Delta} A \Rightarrow B \,\&\, \neg B$,
(b) $\vdash (A \wedge B) \,\&\, (A \wedge C) \Rightarrow A \wedge (B \,\&\, C)$,
(c) $\vdash (\forall x)(Ax \Rightarrow Bx) \Rightarrow (\forall x)Ax \Rightarrow (\forall x)Bx$,
(d) $\vdash (\exists x)(A \,\&\, B) \equiv (\exists x)A \,\&\, B$ *provided that x is not free in B,*
(e) $\vdash (A \wedge (B \Rightarrow C)) \Rightarrow (A \wedge B \Rightarrow A \wedge C)$.
(f) Let $\vdash \mathbf{\Delta} A_o \equiv B_o$. *Then* $\vdash B_o \equiv \mathbf{\Delta} B_o$.

Note that by (f), if B_o is equivalent to a crisp formula then it is crisp itself.

2.2 Evaluative linguistic expressions

Trichotomous evaluative linguistic expressions (TEv-expressions for short) are expressions of natural language, for example, *small, medium, big, about twenty five, roughly one hundred, very short, more or less deep, not very tall, roughly warm or medium hot, quite roughly strong, roughly medium*, and many others. They form a small but very important part of natural language and they are present in its everyday use any time because people very often need to evaluate phenomena around them, make decisions based on them, learn how to control, and use them in many other activities. In this paper, we will consider only simple TEv-expressions, whose syntactic structure is the following:

$$\langle\text{linguistic hedge}\rangle\langle\text{TE-adjective}\rangle \tag{2}$$

where TE-adjective[†)] is one of the adjectives *small, medium, big.* The latter are *canonical* adjectives that can be, in a concrete situation, replaced by more convenient ones such as *short, medium short, long*, etc.

The $\langle$linguistic hedge$\rangle$ is an intensifying adverb making the meaning of the evaluative expressions either more, or less specific:

$$\langle\text{linguistic hedge}\rangle := \textsf{empty hedge} \mid \langle\text{narrowing adverb}\rangle \mid \\ \langle\text{widening adverb}\rangle \mid \langle\text{specifying adverb}\rangle$$

Typical representatives of linguistic hedges are the following: *extremely, significantly, very* (narrowing), *more or less, roughly, quite roughly, very roughly* (widening) or *rather* (specifying).

We will also include *negative evaluative expressions* of the form

$$\textsf{not}\ (\textsf{empty hedge}\langle\text{TE-adjective}\rangle). \tag{3}$$

Let $\mathcal{A}$ be an evaluative expression. The *abstracted evaluative predication* is an expression of the form

$$X \textsf{ is } \mathcal{A}$$

where X is a variable whose values can be arbitrary elements. This predication is synonymous with '$\mathcal{A}X$' (for example, "temperature is very small" and "very small temperature" are taken as synonyms).

In the sequel, we will use the script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ as variables either for evaluative expressions or predications. All the details about formal theory of evaluative linguistic expressions can be found in [11, 12]. As usual, we distinguish *intension* (a property), and *extension* (class of elements) in a given *context* of use (we may speak also about *possible world*).

The formal theory of the meaning of TEv-expressions is based on the following general characteristics:

(i) *Linguistic context* of evaluative expressions and predications is a nonempty, linearly ordered and bounded scale. In each context, three distinguished limit points can be determined: *left bound* (denoted by by v_L), *right bound* (denoted by by v_R), and a *central point* (denoted by by v_S which lays somewhere in between v_L and v_R).

(ii) Intension of an evaluative expression or predication is a function from the set of contexts into a set of fuzzy sets. This means that each context is

[†)] TE stands for *trichotomous evaluative.* These adjectives form a specific subclass of evaluative ajdectives and their typical feature is that they form pairs of antonyms (e.g., small–big) completed by the middle member (medium).

assigned a fuzzy set inside it which forms an *extension* of the evaluative expression or predication in the given context.

(iii) Each of the limit points in (i) is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point, namely the horizons left bound — central point, right bound back — central point, and central point — both left and right bounds. Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox and vanishes beyond the limit point.

(iv) Extension of any evaluative expression is delineated by a specific horizon obtained by shifting (modifying) one of the above horizons. Each element of the context is contained in extensions of several simple evaluative expressions differing only in their hedges. Each limit point due to (i) laying inside extension of an evaluative expression is *typical* for it.

(v) Each scale is vaguely partitioned by the *fundamental evaluative trichotomy* consisting of a pair of antonyms, and a middle member.

A formal logical theory of evaluative linguistic expressions T^{Ev} in Ł-FTT is constructed on the basis of the above characteristics. The language J^{Ev} of T^{Ev} has the following special symbols:

(i) The constants $\dagger \in Form_o$ (middle truth value) and $\sim \in Form_{oo}$ (additional fuzzy equality on truth values).

(ii) Three special formulas $LH, MH, RH \in Form_{oo}$ for the left, middle, and right horizon.

(iii) A special constant $\bar{\boldsymbol{\nu}} \in Form_{oo}$ for the standard (i.e., empty) hedge and further special constants $Ex, Si, Ve, ML, Ro, QR, VR \in Form_{oo}$ for specific hedges "extremely, significantly, very, more or less, roughly, quite roughly, very roughly", respectively.

(iv) Special constants $\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{b}_{\boldsymbol{\nu}}, \mathbf{c}_{\boldsymbol{\nu}}$ associated with each hedge $\boldsymbol{\nu} \in \{Ex, Si, Ve, ML, Ro, QR, VR\}$.

Let us recall that a *hedge*, in general, is a formula $\boldsymbol{\nu} \in Form_{oo}$ with specific properties. Moreover, we can also define a formula $Hedge \in Form_{o(oo)}$ such that $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ means that $\boldsymbol{\nu}$ is a hedge. We refer the reader to [12] for the technical details. We assume that the following is provable: $T^{\mathrm{Ev}} \vdash Hedge\, \bar{\boldsymbol{\nu}}$ as well as $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ for all $\boldsymbol{\nu} \in \{Ex, Si, Ve, ML, Ro, QR, VR\}$ where the latter denotes the hedges *extremely, significantly, very, more or less, roughly, quite roughly, very roughly*, respectively.

Let $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ be two hedges, i.e., $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}_1$ and $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}_2$. We define a relation of partial ordering of hedges by

$$\preceq := \lambda p_{oo}\, \lambda q_{oo} \cdot (\forall z)(p_{oo} z \Rightarrow q_{oo} z). \tag{4}$$

We will write $\boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2$ instead of $(\preceq \boldsymbol{\nu}_1)\boldsymbol{\nu}_2$. If

$$T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2 \tag{5}$$

then we say that the hedge $\boldsymbol{\nu}_1$ has a *narrowing effect* with respect to $\boldsymbol{\nu}_2$, and $\boldsymbol{\nu}_2$ has a *widening effect* with respect to $\boldsymbol{\nu}_1$.

Moreover, the following ordering of the specific hedges is supposed to be provable:

$$Ex \preceq Si \preceq Ve \preceq \bar{\boldsymbol{\nu}} \preceq ML \preceq Ro \preceq QR \preceq VR \tag{6}$$

where we understand in (6) that (5) is provable for each given pair of hedges. By (5), the hedges *Ex*, *Si*, *Ve* have narrowing effect with respect to the empty hedge and *ML*, *Ro*, *QR*, *VR* have widening effect. Finally, we suppose that for each special constant $\mathbf{c}_{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \{Ex, Si, Ve, ML, Ro, QR, VR\}$, the following is provable:

$$T^{\mathrm{Ev}} \vdash \dagger \Rightarrow \mathbf{c}_{\boldsymbol{\nu}}. \tag{7}$$

Our theory enables us to distinguish clearly the abstract evaluative linguistic expressions from the evaluative linguistic predications where the latter require a specific context (e.g., size of a "small star" is very different from the size of a "small dog"). In this paper, however, we can consider only abstract evaluative expressions that are free from the specific context.

Let $\boldsymbol{\nu} \in Form_{oo}$, $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ and $z \in Form_o$ be a variable of type o (truth value). The TEv-formula is one of the following:

(i) *S-formula:* $Sm := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(LH\, z)$,
(ii) *M-formula:* $Me := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(MH\, z)$,
(iii) *B-formula:* $Bi := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(RH\, z)$,

where $LH, MH, RH \in Form_{oo}$ are special formulas representing the three basic horizons discussed above (left, middle and right one, respectively). Clearly, $Sm\,\boldsymbol{\nu}, Me\,\boldsymbol{\nu}, Bi\,\boldsymbol{\nu} \in Form_{oo}$. These formulas represent intensions of pure evaluative expressions (2). For example, $Sm\,Ex$ is a formula representing intension of the evaluative expression "*extremely small*".

A specific role in this theory is played by the formulas $Sm\,\boldsymbol{\Delta}$, $Me\,\boldsymbol{\Delta}$, $Bi\boldsymbol{\Delta}$ where the connective $\boldsymbol{\Delta}$ has been used as a specific hedge that can be taken as "utmost" (or, alternatively a "limit").

Lemma 2
(a) $T^{Ev} \vdash \boldsymbol{\Delta}(z \equiv \bot) \Rightarrow (Sm\,\boldsymbol{\Delta})z$,
(b) $T^{Ev} \vdash \boldsymbol{\Delta}(z \equiv \dagger) \Rightarrow (Me\,\boldsymbol{\Delta})z$
(c) $T^{Ev} \vdash \boldsymbol{\Delta}(z \equiv \top) \Rightarrow (Bi\,\boldsymbol{\Delta})z$

PROOF: (a) can be proved from $T^{\mathrm{Ev}} \cup \{z \equiv \bot\} \vdash \mathbf{\Delta}(\bot \sim z)$ using deduction theorem. (b) and (c) are similar. $\square$

Intension of negative evaluative expressions of the form(3) is either of the formulas $\neg(Sm\,\bar{\nu})$, $\neg(Me\,\bar{\nu})$, $\neg(Bi\,\bar{\nu})$. We will often use a general metavariable $Ev \in Form_{oo}$ (or in more details, $(Ev\,\boldsymbol{\nu})$ if necessary) for either of $(Sm\,\boldsymbol{\nu}), (Me\,\boldsymbol{\nu}), (Bi\,\boldsymbol{\nu})$.

The variables t, z will be often used as variables of type o (truth value). Since the meaning of the evaluative expressions is identified with their intension, we will often say "evaluative expression Ev" instead of the more precise "intension Ev of the evaluative predication $\mathcal{A}$".

Some important properties of evaluative expressions needed in the sequel are summarized in the following theorem.

Theorem 1
Let $\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ be hedges.

(a) $T^{\mathrm{Ev}} \vdash (Sm\,\boldsymbol{\nu})\bot$,
(b) $T^{\mathrm{Ev}} \vdash (Me\,\boldsymbol{\nu})\dagger$,
(c) $T^{\mathrm{Ev}} \vdash (Bi\,\boldsymbol{\nu})\top$.
(d) For all S-formulas $T^{\mathrm{Ev}} \vdash (\forall t)(\forall z)(\mathbf{\Delta}(t \Rightarrow z) \Rightarrow ((Sm\,\boldsymbol{\nu})z \Rightarrow (Sm\,\boldsymbol{\nu})t))$.
(e) For all M-formulas

$$T^{\mathrm{Ev}} \vdash (\forall t)(\forall z)(\mathbf{\Delta}(t \Rightarrow z \,\&\, z \Rightarrow \dagger) \Rightarrow ((Me\,\boldsymbol{\nu})t \Rightarrow (Me\,\boldsymbol{\nu})z)),$$
$$T^{\mathrm{Ev}} \vdash (\forall t)(\forall z)(\mathbf{\Delta}(\dagger \Rightarrow t \,\&\, t \Rightarrow z) \Rightarrow ((Me\,\boldsymbol{\nu})z \Rightarrow (Me\,\boldsymbol{\nu})t)).$$

(f) For all B-formulas, $T^{\mathrm{Ev}} \vdash (\forall t)(\forall z)(\mathbf{\Delta}(t \Rightarrow z) \Rightarrow ((Bi\,\boldsymbol{\nu})t \Rightarrow (Bi\,\boldsymbol{\nu})z))$.
(g) Let $T^{\mathrm{Ev}} \vdash Hedge\,\boldsymbol{\nu}_1 \wedge Hedge\,\boldsymbol{\nu}_2$ *and* $Ev_{\boldsymbol{\nu}_1}, Ev_{\boldsymbol{\nu}_2}$ *differ only in the hedge. Then*

$$T^{\mathrm{Ev}} \vdash (\forall t)(\boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2 \Rightarrow (Ev_{\boldsymbol{\nu}_1}\,t \Rightarrow Ev_{\boldsymbol{\nu}_2}\,t)).$$

(h) $T^{\mathrm{Ev}} \vdash (\forall x)((Sm\,\boldsymbol{\nu})z \Rightarrow \neg(Bi\,\boldsymbol{\nu})z)$ *and* $T^{\mathrm{Ev}} \vdash (\forall z)((Bi\,\boldsymbol{\nu})z \Rightarrow \neg(Sm\,\boldsymbol{\nu})z)$.

The *canonical model* of T^{Ev} is based on the frame

$$\mathcal{M}^0 = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

where $\mathcal{L}_\Delta$ is the standard Łukasiewicz$_\Delta$-algebra and the set $M_\epsilon = [u, v] \subseteq \mathbb{R}$ of elements is assumed to be an interval (not necessarily proper) of real numbers $\mathbb{R}$. Such a model serves for natural interpretation of the semantics of evaluative linguistic expressions including the sophisticated theory of contexts. Interpretation of the constant $\dagger$ in the set M_o is 0.5.

Interpretation of basic items of the language J^{Ev} and special formulas as well as extensions of evaluative expressions in a specific context is schematically depicted on Fig. 1. It is important to note that interpretation of linguistic

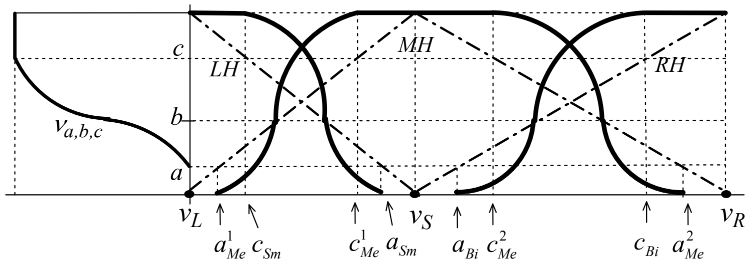

Fig. 1. Scheme of the construction of extensions of evaluative expressions ($\nu_{a,b,c}$ is a function interpreting hedge $\boldsymbol{\nu}$ in $\mathcal{M}^0$; it is turned 90° counterclockwise in the figure). Each extension is a fuzzy set in the context determined by three points $v_L, v_S, v_L \in M_\alpha$ (left bound, central point, and right bound). The extension is constructed from the corresponding horizon LH, MH, RH and the hedge $\nu_{a,b,c}$.

hedges assures that in each context, the kernel of the corresponding fuzzy set is made shorter or longer depending on the kind of the hedge (necessity of such behavior has been linguistically justified already in [7]).

In this paper, we will need only *standard context* which is the set of truth values M_o with the left bound $v_L = 0$, the central point $v_S = 0.5$ and the right bound v_R.

2.3 Fuzzy sets and measure in FTT

The fuzzy set theory can be introduced in FTT in a straightforward way. Our tools are closely related to those presented in [2].

In this paper, we identify fuzzy sets with their membership functions. Since interpretation of a formula $A_{o\alpha}$ is a function from M_α to truth values, we can represent a fuzzy set in a universe M_α syntactically by $A_{o\alpha}$. We can also use the lambda notation. Let B_o be a formula of type o whose interpretation is a truth value. Furthermore, let the variable x_α occur freely in B_o. Then we can write a fuzzy set explicitly as a formula

$$A_{o\alpha} := \lambda u_\alpha \, B_{o,x_\alpha}[u_\alpha] \tag{8}$$

where $B_{o,x_\alpha}[u_\alpha]$ denotes instance of B_o, in which all free occurrences of x_α are replaced by u_α. Thus, (8) represents a function assigning to each u_α a truth value of $B_{o,x_\alpha}[u_\alpha]$ — we have obtained a fuzzy set.

Let $u \in Form_\alpha$. Operations on fuzzy sets can be introduced as special formulas in the following way:

$$\emptyset_{o\alpha} \equiv \lambda u \, \bot, \tag{9}$$
$$V_{o\alpha} \equiv \lambda u \, \top, \tag{10}$$
$$\epsilon_{o(o\alpha)\alpha} \equiv \lambda u \, \lambda x_{o\alpha} \, x_{o\alpha} u, \tag{11}$$
$$\subseteq_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, (\forall u)(x_{o\alpha} u \Rightarrow y_{o\alpha} u), \tag{12}$$
$$\subset_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, (\forall u)(x_{o\alpha} u \Rightarrow y_{o\alpha} u) \wedge (x_{o\alpha} \not\equiv y_{o\alpha}), \tag{13}$$
$$\cap_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, \lambda u_\alpha \, (x_{o\alpha} u_\alpha \wedge y_{o\alpha} u_\alpha), \tag{14}$$
$$\otimes_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, \lambda u_\alpha \, (x_{o\alpha} u_\alpha \,\&\, y_{o\alpha} u_\alpha), \tag{15}$$
$$\uplus_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, \lambda u_\alpha \, (x_{o\alpha} u_\alpha \nabla y_{o\alpha} u_\alpha), \tag{16}$$
$$\cup_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, \lambda u_\alpha \, (x_{o\alpha} u_\alpha \vee y_{o\alpha} u_\alpha), \tag{17}$$
$$-_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha} \, \lambda y_{o\alpha} \, \lambda u_\alpha \, (x_{o\alpha} u_\alpha \,\&\, \neg y_{o\alpha} u_\alpha), \tag{18}$$
$$Hgt_{o(o\alpha)} \equiv \lambda x_{o\alpha} \, (\exists u)(x_{o\alpha} u). \tag{19}$$

The meaning of these definitions is obvious, namely (9) is empty (fuzzy) set, (10) is the universe (i.e., its interpretation is the whole set M_α), (11) is the (fuzzy) membership relation and (12)–(18) are common fuzzy set operations. The formula (19) is *height* of a fuzzy set.

Clearly, the formulas $\emptyset_{o\alpha} u$ and $V_{o\alpha} u$ are crisp. In the other cases, we obtain fuzzy formulas. This means that, e.g., $A_{o\alpha} \subseteq B_{o\alpha}$ is a formula that may be true only in some general degree (and thus, not provable), i.e., its truth in a model is determined by truth of the formula $(\forall u)(A_{o\alpha} u \Rightarrow B_{o\alpha} u)$. Similarly for the other formulas.

The following follows immediately from the basic properties of FTT.

Lemma 3
(a) $\vdash \emptyset_{o\alpha} \subseteq x_{o\alpha}$,
(b) $\vdash (x_{o\alpha} \subseteq \emptyset_{o\alpha}) \equiv (x_{o\alpha} \equiv \emptyset_{o\alpha})$.
(c) $\vdash x_{o\alpha} - y_{o\alpha} \subseteq x_{o\alpha}$

We will introduce measure as a special formula such that its truth value (in a model) is the measure of the corresponding fuzzy set. Let us stress, however, that our theory is purely formal. Concrete measures can be assigned only in special models and so, the flexibility of our theory is very high.

Definition 1
Let $R \in Form_{o(o\alpha)(o\alpha)}$ *be a formula. Put*

$$\mu := \lambda z_{o\alpha} \, \lambda x_{o\alpha} \, (R z_{o\alpha}) x_{o\alpha}. \tag{20}$$

We say that $\mu \in Form_{o(o\alpha)(o\alpha)}$ *represents* a measure on fuzzy sets *in the universe of type* $\alpha \in Types$ *if it has the following properties:*

(M1) $\mathbf{\Delta}(x_{o\alpha} \equiv z_{o\alpha}) \equiv ((\mu z_{o\alpha})x_{o\alpha} \equiv \top)$,
(M2) $\mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq y_{o\alpha}) \Rightarrow \cdot(\mu z_{o\alpha})x_{o\alpha} \Rightarrow (\mu z_{o\alpha})y_{o\alpha}$
(M3) $\mathbf{\Delta}(z_{o\alpha} \not\equiv \emptyset_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow \cdot(\mu z_{o\alpha})(z_{o\alpha} - x_{o\alpha}) \equiv \neg(\mu z_{o\alpha})x_{o\alpha}$.

The meaning of these axioms is the following: (M1) the measure of the distinguished fuzzy set represented by $z_{o\alpha}$ is unit, (M2) it is a monotonic function, and (M3) the measure of the complement fuzzy set is negation of the measure of the original fuzzy set.

The monotonicity property (M2) is important for the subsequent use of the theory of evaluative expressions. We say that μ is *strictly increasing* if

$$\vdash \mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq y_{o\alpha} \wedge x_{o\alpha} \not\equiv y_{o\alpha}) \\ \Rightarrow \cdot((\mu z_{o\alpha})x_{o\alpha} \Rightarrow (\mu z_{o\alpha})y_{o\alpha}) \wedge ((\mu z_{o\alpha})x_{o\alpha} \not\equiv (\mu z_{o\alpha})y_{o\alpha}).$$

Lemma 4
(a) $\vdash (\forall z_{o\alpha})(\mu z_{o\alpha})z_{o\alpha}$,
(b) $\vdash (\forall z_{o\alpha})((z_{o\alpha} \not\equiv \emptyset_{o\alpha}) \Rightarrow \neg(\mu z_{o\alpha})\,\emptyset_{o\alpha})$.

PROOF: (a) This follows from $\vdash z \equiv z$ and axiom (FT$_I$6) by the properties of FTT.

(b) By (M3) and $\vdash \emptyset_{o\alpha} \subseteq z_{o\alpha}$ we get $\vdash z_{o\alpha} \not\equiv \emptyset_{o\alpha} \Rightarrow \cdot(\mu z_{o\alpha})z_{o\alpha} \equiv \neg(\mu z_{o\alpha})\emptyset_{o\alpha}$ from which (b) follows.

□

Thus, if $z_{o\alpha}$ is empty then $(\mu\emptyset_{o\alpha})$ is a degenerated measure with respect to which measure of the (fuzzy) set $\emptyset_{o\alpha}$ is $\top$.

We may also introduce an additivity property by

$$\vdash \mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \,\&(x_{o\alpha} \otimes y_{o\alpha} \equiv \emptyset_{o\alpha}) \Rightarrow \cdot \\ (\mu z_{o\alpha})(x_{o\alpha} \uplus y_{o\alpha}) \equiv (\mu z_{o\alpha})x_{o\alpha} \,\mathbf{\nabla}\, (\mu z_{o\alpha})y_{o\alpha}. \quad (21)$$

We will not need additivity in the sequel.

Definition 2
Let $\mathcal{S} \subseteq$ Types be a distinguished set of types and $\{R \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$ be a set of new constants. The theory of intermediate quantifiers T^{IQ} *w.r.t. $\mathcal{S}$ is a formal theory of Ł-FTT with the language $J^{\mathrm{Ev}} \cup \{R_{o(o\alpha)(o\alpha)} \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$ which is extension of T^{Ev} such that $\mu \in Form_{o(o\alpha)(o\alpha)}$, $\alpha \in \mathcal{S}$, defined in (20) represents a measure on fuzzy sets in the sense of Definition 1.*

The reasons, why we distinguish the set $\mathcal{S}$ of types are discussed in Section 5 below.

3 The main idea

The main idea for the definition of intermediate quantifiers is FTT-formalization of the concept of FB-quantifiers from [8]. The formula characterizing the meaning of a quantified sentence consists of two parts:

(i) Characterization of a size of a given fuzzy set using specific measure and some evaluative linguistic expression,
(ii) ordinary quantification (general or existential) of the resulting formula.

Definition 3
Let T^{IQ} be a theory of intermediate quantifiers in the sense of Definition 2 and $Ev \in Form_{oo}$ be intension of some TEv-expression. Furthermore, let $A, B \in Form_{o\alpha}$ be formulas and $z \in Form_{o\alpha}$ and $x \in Form_{\alpha}$ variables where $\alpha \in \mathcal{S}$. Then a type $\langle 1, 1\rangle$ intermediate generalized quantifier interpreting the sentence

"$\langle$Quantifier$\rangle$ B's are A"

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)(\mathbf{\Delta}(z \subseteq B) \wedge Ev((\mu B)z) \wedge (\forall x)(z\, x \Rightarrow Ax)), \quad (22)$$

$$(Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)(\mathbf{\Delta}(z \subseteq B) \wedge Ev((\mu B)z) \wedge (\exists x)(zx \wedge Ax)). \quad (23)$$

Note that $(\mu B)z$ is a formula of type o.

These quantifiers represent formal meaning of natural language expressions, such as

"most (many, a lot of, a few, some, etc.) B's are A".

We can rewrite (22) and (23) using more common symbols as follows:

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)(\mathbf{\Delta}(z \subseteq B) \wedge Ev((\mu B)z) \wedge z \subseteq A), \quad (24)$$
$$(Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)(\mathbf{\Delta}(z \subseteq B) \wedge Ev((\mu B)z) \wedge Hgt(z \cap A)). \quad (25)$$

Theorem 2
(a) $T^{IQ} \vdash (Q^{\forall}_{Bi\mathbf{\Delta}} x)(B, A) \equiv (\forall x)(Bx \Rightarrow Ax)$,
(b) $T^{IQ} \vdash (Q^{\exists}_{Bi\mathbf{\Delta}} x)(B, A) \equiv (\exists x)(Bx \wedge Ax)$,
(c) $T^{IQ} \vdash (Q^{\forall}_{Bi\mathbf{\Delta}} x)(V, A) \equiv (\forall x)Ax$,

(d) $T^{\mathrm{IQ}} \vdash (Q^{\exists}_{Bi\boldsymbol{\Delta}} x)(V, A) \equiv (\exists x)Ax.$

PROOF: By Lemma 4(b) and (M1) we conclude that $T^{\mathrm{IQ}} \vdash (Bi\boldsymbol{\Delta})((\mu B)\, B) \equiv \top$. From this we conclude that $T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(B \subseteq B) \wedge (Bi\boldsymbol{\Delta})((\mu B)\, B)$ and consequently,

$$T^{\mathrm{IQ}} \vdash (\forall x)(B\, x \Rightarrow Ax) \Rightarrow (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\boldsymbol{\Delta})((\mu B)\, z) \wedge (\forall x)(z\, x \Rightarrow Ax)) \tag{26}$$

by the properties of FTT (we used the $\exists$-substitution axiom).

On the other hand, using (M1) and the properties of FTT we obtain

$$T^{\mathrm{IQ}} \vdash (\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\boldsymbol{\Delta})((\mu B)\, z) \wedge (\forall x)(z\, x \Rightarrow Ax)) \Rightarrow (\boldsymbol{\Delta}(z \equiv B) \wedge (\forall x)(z\, x \Rightarrow Ax)). \tag{27}$$

Furthermore, by the equality axiom and further properties of FTT we have $\vdash \boldsymbol{\Delta}(z \equiv B) \Rightarrow ((\forall x)(z\, x \Rightarrow Ax) \Rightarrow (\forall x)(B\, x \Rightarrow Ax))$ which is equivalent with

$$\vdash \boldsymbol{\Delta}(z \equiv B)\, \boldsymbol{\&} (\forall x)(z\, x \Rightarrow Ax) \Rightarrow \cdot(\forall x)(B\, x \Rightarrow Ax)). \tag{28}$$

Because $\vdash \top \wedge A \equiv \top \boldsymbol{\&} A$ and $\vdash \bot \wedge A \equiv \bot \boldsymbol{\&} A$, we can replace $\boldsymbol{\&}$ by $\wedge$ in (28) using the rule of two cases (cf. [10]). Joining the latter and (27), we obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)(\boldsymbol{\Delta}(z \equiv B) \wedge (Bi\boldsymbol{\Delta})((\mu B)\, z) \wedge (\forall x)(z\, x \Rightarrow Ax)) \Rightarrow (\forall x)(B\, x \Rightarrow Ax)$$

which together with (26) implies (a) by the properties of equivalence.

(b) is proved similarly.

(c) follows from (a) by $\vdash (\forall x_\alpha)(V x_\alpha \equiv \top)$ that can be obtained from (10) by lambda conversion.

(d) follows from (b) by the same argument.

□

One can see from this lemma that the hedge "utmost" degenerates intermediate quantifiers to the ordinary classical ones. Do not forget, however, that still truth values of the respective formulas can lay between zero and unit.

4 Specific intermediate quantifiers

4.1 Basic definitions

In this section, we will introduce several specific intermediate quantifiers and prove some valid syllogism based on them. Our definitions below are based on the analysis provided by the Peterson's book [15] as well as the intuition. Each of the following intermediate quantifiers is denoted by a specific letter to which we will refer when dealing with their properties and the syllogism. We suppose that $A, B \in \mathit{Form}_{o\alpha}$.

$$
\begin{aligned}
\textbf{A:}\ \text{All } B \text{ are } A :=\ & Q^{\forall}_{Bi\boldsymbol{\Delta}}(B,A) \equiv (\forall x)(Bx \Rightarrow Ax),\\
\textbf{E:}\ \text{No } B \text{ are } A :=\ & Q^{\forall}_{Bi\boldsymbol{\Delta}}(B,\neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),\\
\textbf{P:}\ \text{Almost all } B \text{ are } A :=\ & Q^{\forall}_{Bi\,Ex}(B,A) \equiv \\
& (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\,Ex)((\mu B)z) \wedge (\forall x)(zx \Rightarrow Ax)),\\
\textbf{B:}\ \text{Few } B \text{ are } A\ (:= \ & \text{Almost all } B \text{ are not } A) := Q^{\forall}_{Bi\,Ex}(B,\neg A) \equiv \\
& (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\,Ex)((\mu B)z) \wedge (\forall x)(zx \Rightarrow \neg Ax)),\\
\textbf{T:}\ \text{Most } B \text{ are } A :=\ & Q^{\forall}_{Bi\,Ve}(B,A) \equiv \\
& (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\,Ve)((\mu B)z) \wedge (\forall x)(zx \Rightarrow Ax)),\\
\textbf{D:}\ \text{Most } B \text{ are not } A :=\ & Q^{\forall}_{Bi\,Ve}(B,\neg A) \equiv \\
& (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\,Ve)((\mu B)z) \wedge (\forall x)(zx \Rightarrow \neg Ax)),\\
\textbf{K:}\ \text{Many } B \text{ are } A :=\ & Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B,A) \equiv \\
& (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z) \wedge (\forall x)(zx \Rightarrow Ax)),\\
\textbf{G:}\ \text{Many } B \text{ are not } A :=\ & Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B,\neg A) \equiv \\
& (\exists z)(\boldsymbol{\Delta}(z \subseteq B) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z) \wedge (\forall x)(zx \Rightarrow \neg Ax)),\\
\textbf{I:}\ \text{Some } B \text{ are } A :=\ & Q^{\exists}_{Bi\boldsymbol{\Delta}}(B,A) \equiv (\exists x)(Bx \wedge Ax),\\
\textbf{O:}\ \text{Some } B \text{ are not } A :=\ & Q^{\exists}_{Bi\boldsymbol{\Delta}}(B,\neg A) \equiv (\exists x)(Bx \wedge \neg Ax).
\end{aligned}
$$

The definition of "Few B are A" in **B** is based on the definition introduced in Peterson [15]. Recall that the formula $(Bi\,Ex)((\mu B)z)$ represents truth value of the linguistic expression "the measure $(\mu B)z$ of the fuzzy set z is extremely big", $(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)$ represents truth value of the linguistic expression "the measure $(\mu B)z$ of the fuzzy set z is small" and similarly the other formulas.

Classical interpretation of the quantifier $\mathit{Most}(B,A)$ in standard theory of generalized quantifiers (cf. [14, 17]) is taken as true if $|B \cap A| > |B - A|$ where the properties A, B are represented by finite sets and $|\cdot|$ denotes a number of their elements.

Lemma 5
Let $A, B \in Form_{o\alpha}$ and assume that "Most B are A" is defined by $Q^{\forall}_{Bi\ Ve}(B, A)$ in the definition $\boldsymbol{T}$ above. Let the measure (μB) be also additive (21). If $T^{IQ} \vdash (Bi\ Ve)((\mu B)(B \cap A))$ then

$$T^{IQ} \vdash Q^{\forall}_{Bi\ Ve}(B, A) \tag{29}$$

and

$$T^{IQ} \vdash ((\mu B)(B - A) \Rightarrow (\mu B)(B \cap A)) \wedge ((\mu B)(B \cap A) \not\equiv (\mu B)(B - A)). \tag{30}$$

PROOF: It is easy to show that $\vdash \boldsymbol{\Delta}(B \cap A \subseteq B)$ as well as $\vdash (\forall x)((B \cap A)x \Rightarrow Ax)$. Then from the assumption and the properties of the existential quantifier we derive (29).

Furthermore, by the properties of T^{IQ} we have $T^{\mathrm{IQ}} \vdash (Bi\ Ve)\dagger \equiv \bot$. From this, (7) and the assumption we obtain

$$T^{\mathrm{IQ}} \vdash (\dagger \Rightarrow (\mu B)(B \cap A)) \wedge ((\mu B)(B \cap A) \not\equiv \dagger). \tag{31}$$

From the properties of Ł-FTT we conclude the following:

$$\begin{aligned} &\vdash (B \cap A) \otimes (B - A) \equiv \emptyset, \\ &\vdash (B \cap A) \uplus (B - A) \equiv B, \\ &\vdash B - (B \cap A) \equiv B - A, \\ T^{\mathrm{IQ}} &\vdash (\mu B)(B \cap A) \,\boldsymbol{\nabla} (\mu B)(B - A) \equiv \top, \\ T^{\mathrm{IQ}} &\vdash (\mu B)(B - A) \equiv \neg(\mu B)(B \cap A). \end{aligned}$$

From this and the assumed properties of (μB) we obtain $T^{\mathrm{IQ}} \vdash (\mu B)(B - A) \Rightarrow \dagger$. When joining the latter with (31), we obtain (30).

□

Corollary 1
Let $\mathcal{M}$ be a model such that $\mathcal{M}(B_{o\alpha}) = B$, $\mathcal{M}(A_{o\alpha}) = A$ are crisp finite sets, and let $\mathcal{M}((\mu B)z_{o\alpha}) = \frac{|\mathcal{M}(z_{o\alpha})|}{|B|}$ if B is nonempty and $\mathcal{M}((\mu B)z_{o\alpha}) = \mathbf{1}$ otherwise. If B is nonempty then

$$\mathcal{M}(Q^{\forall}_{Bi\ Ve}(B_{o\alpha}, A_{o\alpha})) = \mathbf{1} \quad \textit{implies} \quad |B - A| < |B \cap A|.$$

It follows from this corollary that our definition of "most" encompasses its standard definition in the classical theory of generalized quantifiers (cf. [14, 17]; they do not consider the degenerated case when B is empty).

4.2 Valid implications

We will show the basic monotonicity property of our theory that states that "wider" quantifiers imply "narrower" ones, i.e. if, for example, we learn that "All S are P" then we know that also "Almost all S are P", etc. The precise statement is in the following theorem.

Theorem 3
Let $\boldsymbol{A}, \ldots, \boldsymbol{G}$ be the basic intermediate quantifiers defined above. Then the following sets of implications are provable in T^{IQ}:

(a) $T^{IQ} \vdash (\mathbf{A}) \Rightarrow (\mathbf{P})$, $\quad T^{IQ} \vdash (\mathbf{P}) \Rightarrow (\mathbf{T})$, $\quad T^{IQ} \vdash (\mathbf{T}) \Rightarrow (\mathbf{K})$.
(b) $T^{IQ} \vdash (\mathbf{E}) \Rightarrow (\mathbf{B})$, $\quad T^{IQ} \vdash (\mathbf{B}) \Rightarrow (\mathbf{D})$, $\quad T^{IQ} \vdash (\mathbf{D}) \Rightarrow (\mathbf{G})$.

PROOF: If $T^{\mathrm{IQ}} \vdash \boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2$ then $T^{\mathrm{IQ}} \vdash (Bi\,\boldsymbol{\nu}_1)((\mu B)z) \Rightarrow (Bi\,\boldsymbol{\nu}_2)((\mu B)z)$ by Theorem 1(g). From it follows that

$$\boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\,\boldsymbol{\nu}_1)((\mu B)z) \wedge (\forall x)(zx \Rightarrow Ax) \Rightarrow \\ \boldsymbol{\Delta}(z \subseteq B) \wedge (Bi\,\boldsymbol{\nu}_2)((\mu B)z) \wedge (\forall x)(zx \Rightarrow Ax)$$

and we conclude that both parts of this theorem follow from the ordering (6) using the properties of FTT. The implications $(\mathbf{T}) \Rightarrow (\mathbf{K})$ and $(\mathbf{D}) \Rightarrow (\mathbf{G})$ follow from Theorem 1(h).

□

4.3 Valid syllogisms

By a syllogism, we will understand a triple of formulas $\langle P_1, P_2, C\rangle$ where P_1 is a *major premise*, P_2 is a *minor premise* and C is a *conclusion*. A syllogism is *formally valid* if $\vdash P_1$ and $\vdash P_2$ imply $\vdash C$. This holds if $\vdash P_1 \mathbin{\&} P_2 \Rightarrow C$, or equivalently, if $\vdash P_1 \Rightarrow (P_2 \Rightarrow C)$.

In this subsection, we will show that many of the syllogisms presented in [15] are formally valid in our theory. These syllogisms are generalizations of the classical Aristotle's syllogisms.

As usual, the syllogism are put into four figures where Q_1, Q_2, Q_3 are intermediate quantifiers and $X, Y, M \in Form_{o\alpha}$ are formulas representing properties *) .

Figure I	Figure II	Figure III	Figure IV
$\dfrac{Q_1\ M \text{ is } Y \quad Q_2\ X \text{ is } M}{Q_3\ X \text{ is } Y}$	$\dfrac{Q_1\ Y \text{ is } M \quad Q_2\ X \text{ is } M}{Q_3\ X \text{ is } Y}$	$\dfrac{Q_1\ M \text{ is } Y \quad Q_2\ M \text{ is } X}{Q_3\ X \text{ is } Y}$	$\dfrac{Q_1\ Y \text{ is } M \quad Q_2\ M \text{ is } X}{Q_3\ X \text{ is } Y}$

In this subsection, we will present some of the formally valid syllogisms for each figure. In accordance with [15], each syllogism is denoted by a sequence of three letters corresponding to the definitions of specific intermediate quantifiers introduced above. The first two are premises, the third one is a conclusion.

Figure I.

AAA-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{All } X \text{ are } M}{\text{All } X \text{ are } Y}$

AAP-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{All } X \text{ are } M}{\text{Almost all } X \text{ are } Y}$

AAT-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{All } X \text{ are } M}{\text{Most } X \text{ are } Y}$

AAK-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{All } X \text{ are } M}{\text{Many } X \text{ are } Y}$

APP-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{Almost all } X \text{ are } M}{\text{Almost all } X \text{ are } Y}$

APT-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{Almost all } X \text{ are } M}{\text{Most } X \text{ are } Y}$

APK-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{Almost all } X \text{ are } M}{\text{Many } X \text{ are } Y}$

ATT-I: $\dfrac{\text{All } M \text{ are } Y \qquad \text{Most } X \text{ are } M}{\text{Most } X \text{ are } Y}$

*) We can easily extend these syllogisms to manipulation with properties as considered in the intensional logic, i.e., formulas of type $(o\alpha)\omega$ where ω is a type for possible worlds. This is, however, unnecessary at this stage.

$$\textbf{ATK-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Most } X \text{ are } M\end{array}}{\text{Many } X \text{ are } Y} \qquad \textbf{AKK-I:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Many } X \text{ are } M\end{array}}{\text{Many } X \text{ are } Y}$$

Theorem 4
All the above syllogisms are formally valid.

PROOF: It is sufficient to prove **AAA-I**, **APP-I**, **ATT-I** and **AKK-I**. Formal validity of the other syllogisms follows from the latter and from Theorem 3 by the properties of FTT.

AAA-I: This follows from

$$\vdash (Mx \Rightarrow Yx) \,\&\, (Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow Yx)$$

by Lemma 1(c) and the properties of FTT.

APP-I: Let us denote $Ez := \boldsymbol{\Delta}(z \subseteq X) \wedge (Bi\ Ex)((\mu X)z)$. Then, using Lemma 1(c) and the properties of FTT we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow Yx) \,\&\, (Ez \wedge (\forall x)(zx \Rightarrow Mx)) \Rightarrow (Ez \wedge (\forall x)(zx \Rightarrow Yx))$$

From this using Lemma 1(d) and the properties of FTT we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow Yx) \,\&\, (\exists z)(Ez \wedge (\forall x)(zx \Rightarrow Mx)) \Rightarrow \\ (\exists z)(Ez \wedge (\forall x)(zx \Rightarrow Yx))$$

which is just validity of the syllogism **APP-I** when replacing Ez by $\boldsymbol{\Delta}(z \subseteq X) \wedge (Bi\ Ex)((\mu X)z)$.

In a similar way, if we put $Ez := \boldsymbol{\Delta}(z \subseteq X) \wedge (Bi\ Ve)((\mu X)z)$ then we obtain **ATT-I** and $Ez := \boldsymbol{\Delta}(z \subseteq X) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu X)z)$ leads to **APP-I**.

□

Figure II.

$$\textbf{EAE-II:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{All } X \text{ are } M \\ \hline \text{No } X \text{ are } Y \end{array}$$

$$\textbf{EAB-II:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{All } X \text{ are } M \\ \hline \text{Almost all } X \text{ are not } Y \\ \textit{(Few } X \textit{ are } Y\textit{)} \end{array}$$

$$\textbf{EAD-II:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{All } X \text{ are } M \\ \hline \text{Most } X \text{ are not } Y \end{array}$$

$$\textbf{EAG-II:}\quad \begin{array}{l} \text{No } Y \text{ are } M \\ \text{All } X \text{ are } M \\ \hline \text{Many } X \text{ are not } Y \end{array}$$

Theorem 5
All the above syllogisms are formally valid.

PROOF: From

$$\vdash (Yx \Rightarrow \neg Mx)\,\&(zx \Rightarrow Mx)) \Rightarrow (zx \Rightarrow \neg Yx).$$

we obtain **EAE-II** by contraposition and the properties of quantifiers in FTT. The formal validity of the syllogism **EAB-II** is obtained from this in a similar way as in the proof of Theorem 4 by putting $Ez := \mathbf{\Delta}(z \subseteq X) \wedge (Bi\ Ex)((\mu X)z)$, and the like also the other syllogisms. □

Figure III.

The following syllogisms require presupposition which means that we will assume $T^{\mathrm{IQ}} \vdash (\exists x)Mx$. Since the existential quantifier is interpreted by supremum in fuzzy logic, in the model this does not necessarily mean existence of some specific object having the property M.

$$\textbf{EAO-III:}\quad \begin{array}{l} \text{No } M \text{ are } Y \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

$$\textbf{DAO-III:}\quad \begin{array}{l} \text{Most } M \text{ are } Y \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y \end{array}$$

$$\textbf{BAO-III:}\quad \frac{\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y} \qquad \textbf{BKO-III:}\quad \frac{\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Many } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

Theorem 6
If $T^{\mathrm{IQ}} \vdash (\exists x)Mx$ then all the above syllogisms are formally valid.

PROOF: The syllogism **EAO-III** is classical and its formal validity follows from the provable formula

$$\vdash (Mx \Rightarrow \neg Yx) \mathbin{\&} (Mx \Rightarrow Xx) \Rightarrow (Mx \Rightarrow (Xx \wedge \neg Yx)).$$

DAO-III: Let us put $Ez := \boldsymbol{\Delta}(z \subseteq M) \wedge (Bi\ Ve)((\mu M)z)$ and $E'z := \boldsymbol{\Delta}(z \subseteq M) \wedge (Bi\boldsymbol{\Delta})((\mu M)z)$. Note that

$$T^{\mathrm{IQ}} \vdash E'z \Rightarrow Ez. \tag{32}$$

Furthermore, since $T^{\mathrm{IQ}} \vdash Bi\ Ve\top$, we have $T^{\mathrm{IQ}} \vdash (Bi\ Ve)((\mu M)M)$ and we conclude from the properties of the existential quantifier that

$$T^{\mathrm{IQ}} \vdash (\exists z)Ez. \tag{33}$$

By the properties of FTT and T^{IQ}, we can prove that

$$T^{\mathrm{IQ}} \vdash Ez \wedge ((zx \Rightarrow \neg Yx) \mathbin{\&} (zx \Rightarrow Xx)) \Rightarrow (Ez \wedge (zx \Rightarrow (Xx \wedge \neg Yx)))$$

and from this by Lemma 1(b), properties of quantifiers and (32)

$$T^{\mathrm{IQ}} \vdash ((Ez \wedge (\forall x)(zx \Rightarrow \neg Yx)) \mathbin{\&} (E'z \wedge (\forall x)(zx \Rightarrow Xx))) \Rightarrow \\ (Ez \wedge ((\exists x)zx \Rightarrow (\exists x)(Xx \wedge \neg Yx)))$$

and further, by Lemma 1(e) we derive

$$T^{\mathrm{IQ}} \vdash ((Ez \wedge (\forall x)(zx \Rightarrow \neg Yx)) \mathbin{\&} (E'z \wedge (\forall x)(zx \Rightarrow Xx))) \Rightarrow \\ (Ez \wedge (\exists x)zx) \Rightarrow (Ez \wedge (\exists x)(Xx \wedge \neg Yx)). \tag{34}$$

Now using the rule of generalization with respect to $\forall z$ and using the properties of quantifiers and Theorem 2(a) we realize that $T^{\mathrm{IQ}} \vdash (\exists z)(E'z \wedge (\forall x)(zx \Rightarrow Xx)) \equiv (\forall x)(Mx \Rightarrow Xx)$. Using the premises of the syllogism we finally obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)(Ez \wedge (\exists x)zx) \Rightarrow (\exists z)(Ez \wedge (\exists x)(Xx \wedge \neg Yx)).$$

When rewriting the formula Ez and using the properties of T^{IQ} we conclude that the implication $T^{\mathrm{IQ}} \vdash (EM \wedge (\exists x)Mx) \Rightarrow (\exists z)(Ez \wedge (\exists x)zx)$ is provable and using the presupposition we conclude that $T^{\mathrm{IQ}} \vdash (\exists z)(Ez \wedge (\exists x)zx)$. Finally, using the properties of quantifiers and (32) we obtain

$$T^{\mathrm{IQ}} \vdash (\exists x)(Xx \wedge \neg Yx)$$

which is the expected conclusion of the examined syllogism.

Quite similarly we can prove also formal validity of syllogism **BAO-III**.

□

Figure IV.

The following syllogisms require even stronger presupposition than above, and namely, $T^{\mathrm{IQ}} \vdash (\exists x)\mathbf{\Delta} Mx$. This means that we may introduce a new special constant $\mathbf{u}_\alpha$ *representing some object* of type α so that $T^{\mathrm{IQ}} \vdash M\mathbf{u}_\alpha$ is provable.

AAI-IV:

$$\begin{array}{l} \text{All } Y \text{ are } M \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are } Y \end{array}$$

PAI-IV:

$$\begin{array}{l} \text{Almost all } Y \text{ are } M \\ \text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are } Y \end{array}$$

Theorem 7
*If $T^{IQ} \vdash (\exists x)\mathbf{\Delta} Mx$ then both syllogisms **AAI-IV**and **PAI-IV** are formally valid.*

PROOF: The syllogism **AAI-IV** is classical and it leads to $\vdash (\forall x)(Yx \Rightarrow Xx)$ from which, using the presupposition, we prove $\vdash (\exists x)(Xx \wedge Yx)$. Quite similarly we prove also validity of **PAI-IV** where the use of the quantifiers *Almost all* plays no significant role. □

AEE-IV:

$$\begin{array}{l} \text{All } Y \text{ are } M \\ \text{No } M \text{ are } X \\ \hline \text{No } X \text{ are } Y \end{array}$$

AEB-IV:

$$\begin{array}{l} \text{All } Y \text{ are } M \\ \text{No } M \text{ are } X \\ \hline \text{Almost all } X \text{ are not } Y \\ \textit{(Few } X \textit{ are } Y\textit{)} \end{array}$$

$$\textbf{AED-IV:}\quad \begin{array}{l} \text{All } Y \text{ are } M \\ \text{No } M \text{ are } X \\ \hline \text{Most } X \text{ are not } Y \end{array} \qquad\qquad \textbf{AEG-IV:}\quad \begin{array}{l} \text{All } Y \text{ are } M \\ \text{No } M \text{ are } X \\ \hline \text{Many } X \text{ are not } Y \end{array}$$

Theorem 8
All the above syllogisms are formally valid.

PROOF: Formal validity of the classical syllogism **AEE-IV** follows immediately from the provable formula $\vdash (\neg Mx \Rightarrow \neg Yx)\,\&(Xx \Rightarrow \neg Mx) \Rightarrow (Xx \Rightarrow \neg Yx)$. Using it and Theorem 3(b), we may prove also validity of the other three syllogisms. □

We are convinced that we have sufficiently demonstrated the power of our theory. Though there are much more formally valid generalized syllogisms than we demonstrated above (the book [15] presents 105 of them), we expect that all of them will be verified.

Let us also remark that, e.g., the syllogism **TAT-III**

$$\begin{array}{l} \text{Most } M \text{ are } Y \\ \text{All } M \text{ are } X \\ \hline \text{Most } X \text{ are } Y \end{array}$$

presented in [15] as invalid is indeed invalid also in our theory. Namely, its analysis leads to the requirement that $T^{\mathrm{IQ}} \vdash (Bi\ Ve)((\mu X)X \cap Y)$. It is not difficult to construct a model in which interpretation of the latter formula is smaller than $\mathbf{1}$.

5 Interpretation

In this section we will outline the way how our theory can be interpreted in a model. Since we need to introduce a measure in our theory, it can happen that for a more complex model the interpretation can be a quite non-trivial problem. In this section, however, we will construct only a very simple model with finite set M_ϵ of elements.

Let us consider a frame

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \tag{35}$$

where $M_o = [0,1]$ is the support of the standard Łukasiewicz$_\Delta$ algebra. The fuzzy equality $=_o$ is the Łukasiewicz biresiduation $\leftrightarrow$. Furthermore, $M_\epsilon = \{u_1, \dots, u_r\}$ is a finite set with fixed numbering of its elements and $=_\epsilon$ is defined by

$$[u_i =_\epsilon u_j] = (1 - \left(1 \wedge \frac{|i-j|}{s}\right))$$

for some fixed natural number $s \leq r$. This a separating fuzzy equality.

The distinguished set $\mathcal{S} \subset$ *Types* is defined by: $\alpha \in \mathcal{S}$ iff α is a type not containing the type o of truth values.

Let $A \underset{\sim}{\subset} M_\alpha$, $\alpha \in \mathcal{S}$ be a fuzzy set. We will put

$$|A| = \sum_{u \in \mathrm{Supp}(A)} A(u), \qquad u \in M_\alpha. \tag{36}$$

Furthermore, for fuzzy sets $A, B \underset{\sim}{\subset} M_\alpha$, $\alpha \in \mathcal{S}$ we define

$$F_R(B)(A) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = B, \\ \frac{|A|}{|B|} & \text{if } B \neq \emptyset \text{ and } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

The interpretation $\mathcal{I}^\mathcal{M}$ of the constants $R \in Form_{o(o\alpha)(o\alpha)}$, $\alpha \in S$ is defined by $\mathcal{I}^\mathcal{M}(R) = F_R$ where $F_R : \mathcal{F}(M_\alpha) \times \mathcal{F}(M_\alpha) \longrightarrow L$ is the function (37). It can be verified that axioms (M1)–(M3) are true in the degree 1 in $\mathcal{M}$.

The interpretation of the evaluative linguistic expressions is obtained using simple quadratic functions that can be found in [9]. Using them, extensions of several selected TEv-expressions in the context [0,1] are obtained. Their graphs are depicted in Figure 2.

Let, for example, the set M_ϵ be a set of people. Let $\mathrm{Memb}_{o\epsilon}$ be a formula *Members of a society* with the interpretation $\mathcal{I}^\mathcal{M}(\mathrm{Memb}_{o\epsilon}) = B$ where $B \subset M_\epsilon$ is a (classical) set of 500 people. Furthermore, let $\mathrm{Assemb}_{o\epsilon}$ be a formula *Members came to the assembly* with the interpretation $\mathcal{I}^\mathcal{M}(\mathrm{Assemb}_{o\epsilon}) = A$ where $A \subset M_\epsilon$ is a set of 450 people. Then the statement *Most members came to the assembly* is represented by a formula $Q^\forall_{Bi\ Ve}(\mathrm{Memb}, \mathrm{Assemb})$ whose truth value can be computed as follows.

Apparently, the set $\mathcal{I}^\mathcal{M}(z) = A$ is the biggest set such that $\mathcal{I}^\mathcal{M}(\mathbf{\Delta}(z \subseteq \mathrm{Memb})) = 1$ as well as $\mathcal{I}^\mathcal{M}((\forall x)(zx \Rightarrow \mathrm{Assemb}\, x)) = 1$. Thus, it remains to evaluate $\mathcal{I}^\mathcal{M}(Bi\ Ve)((\mu\,\mathrm{Memb})z) = 0.99$ so that we obtain

$$\mathcal{I}^\mathcal{M}(Q^\forall_{Bi\ Ve}(\mathrm{Memb}, \mathrm{Assemb})) = 0.99. \tag{38}$$

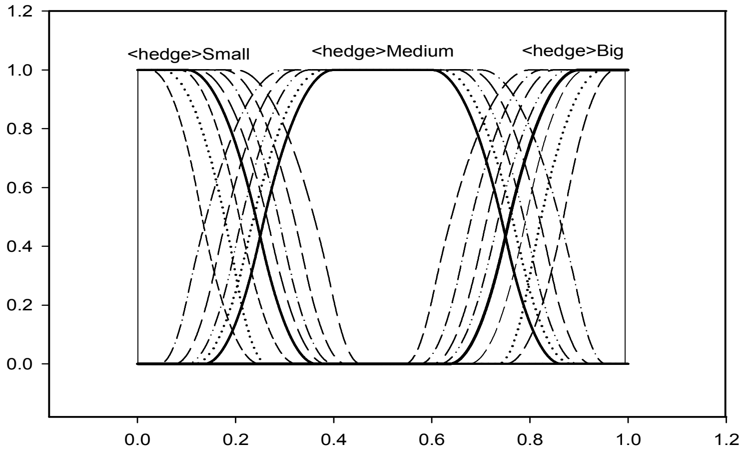

Fig. 2. Shapes of extensions of some evaluative expressions in the context [0,1]. The hedges are {Extremely, Significantly, Very, empty hedge} for "small" and "big" and {More-or-Less, Roughly, Quite Roughly, Very Roughly} for "small", "medium", and "big".

For other cardinalities of A, for example $|A| = 460, 400, 370, 300$ we obtain the truth values 1, 0.7, 0.3, 0, respectively. Thus, the number of more than about 450 people out of 500 can be taken quite convincingly as representing the quantity of "most". For comparison, "almost all" would mean $|A| = 460$ with the degree 0.9 and $|A| = 475$ with the degree 1. The results would be, of course, different when defining A, B as fuzzy sets so that $\mathcal{I}^{\mathcal{M}}((\forall x)(zx \Rightarrow \mathrm{Assemb}\, x))$ might be smaller than 1. This task is left to the reader.

One can see that our concept heavily depends on the definition of the measure μ. It can be quite difficult to define in models with $M_\epsilon = \mathbb{R}$ a reasonable measure for slightly more complex types, such as $o(\epsilon\epsilon)$. Therefore, we have limited our considerations only to a certain subset $\mathcal{S}$ of types. Another question is raised in connection with linguistic. Does it have a sense to use the intermediate quantifiers such as "most, many", etc. to characterize quantities of elements taken from very complex sets with very high cardinalities? The task to define such complex models of our theory is the topic of further research.

6 Conclusion

In this paper, we have introduced a special formal theory T^{IQ} of intermediate generalized quantifiers. This theory is developed as an extension of the formal theory T^{Ev} of evaluative linguistic expressions. Both theories are special theories of Ł-FTT.

The intermediate quantifiers are introduced as specific formulas of T^{IQ} based on the idea that they are, in fact, classical general or existential quantifiers but limited to smaller (fuzzy) sets (for example, fuzzy sets with smaller support) whose size is measured using some measure represented by a specific formula of FTT. Since our theory is syntactical, we use only basic properties of the measure without considering more specific measures which can be introduced in specific models. The measured size of the fuzzy set is evaluated by a certain evaluative linguistic expression. From it also follows that our definition is independent on the elements of sets over which the quantification is defined, i.e., the ISOM property of the theory of generalized quantifiers is fulfilled (cf. [14, 17]). On the other hand, since our concept depends on the measure, we cannot introduce intermediate quantifiers in arbitrary frame. We do not see this as imperfection since our motivation is to model the meaning of (specific) natural language expressions and people hardly quantify in common language, e.g., over non-measurable sets.

We have proved validity of 24 of over 100 generalizations of classical Aristotlean syllogisms presented in [15]. Though this work is not yet finished, we are convinced that all intuitively valid generalized syllogisms can be proved to be formally valid also in our theory.

To conclude, let us remark that our theory has a wider potential. For example, the description operator $\iota_{\alpha(o\alpha)}$ can be understood as the quantifier *indefinite article* ("a dog", etc.) since it selects some prototypical element from $x_{o\alpha}$. Also the comparative quantifiers ("much more than about half of", etc.) can be introduced in T^{IQ}. This will be the topic of some of subsequent papers.

References

[1] P. Andrews, An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof, Kluwer, Dordrecht, 2002.

[2] L. Běhounek, P. Cintula, Fuzzy class theory, Fuzzy Sets and Systems 154 (2005) 34–55.

[3] A. Church, A formulation of the simple theory of types, J. Symb. Logic 5 (1940) 56–68.

[4] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems 124 (2001) 271–288.

[5] I. Glockner, Fuzzy Quantifiers: A Computational Theory, Springer, Berlin, 2006.

[6] L. Henkin, Completeness in the theory of types, J. Symb. Logic 15 (1950) 81–91.

[7] G. Lakoff, Hedges: A study in meaning criteria and logic of fuzzy concepts, Journal of Philosophical Logic 2 (1973) 458–508.

[8] V. Novák, Antonyms and linguistic quantifiers in fuzzy logic, Fuzzy Sets and Systems 124 (2001) 335–351.

[9] V. Novák, Fuzzy logic deduction with words applied to ancient sea level

estimation, in: R. Demicco, G. Klir (Eds.), Fuzzy logic in geology, Academic Press, Amsterdam, 2003, pp. 301–336.

[10] V. Novák, On fuzzy type theory, Fuzzy Sets and Systems 149 (2005) 235–273.

[11] V. Novák, Fuzzy logic theory of evaluating expressions and comparative quantifiers, in: Proc. 11^{th} Int. Conf. IPMU, Paris, July 2006, Vol. 2, Éditions EDK, Les Cordeliers, Paris, 2006, pp. 1572–1579.

[12] V. Novák, A comprehensive theory of trichotomous evaluative linguistic expressions, in: S. Gottwald, H. P., U. Höhle, E. P. Klement (Eds.), Proc. of 26th Linz Seminar on Fuzzy Set Theory, (to appear), research Report No. 71, http://irafm.osu.cz/.

[13] V. Novák, I. Perfilieva, J. Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer, Boston, 1999.

[14] S. Peters, D. Westerståhl, Quantifiers in Language and Logic, Claredon Press, Oxford, 2006.

[15] P. Peterson, Intermediate Quantifiers. Logic, linguistics, and Aristotelian semantics, Ashgate, Aldershot, 2000.

[16] N. Pfeifer, Contemporary syllogistics: Comparative and quantitative syllogisms, in: G. Kreuzbauer, G. Dorn (Eds.), Argumentation in Theorie und Praxis: Philosophie und Didaktik des Argumentierens, LIT., Wien, 2006, pp. 57–71.

[17] D. Westerståhl, Quantifiers in formal and natural languages, in: D. Gabbay, F. Guenthner (Eds.), Handbook of Philosophical Logic vol. IV, D. Reidel, Dordrecht, 1989, pp. 1–131.

[18] L. A. Zadeh, A computational approach to fuzzy quantifiers. in natural languages, Computers and Mathematics 9 (1983) 149–184.