UNIVERSITY OF OSTRAVA

Institute for Research and Applications of Fuzzy Modeling

Fuzzy Transform for Practical Problems

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Research report No. 114

2007

Submitted/to appear:

Inteligentní systémy pro praxi

Supported by:

DAR 1M0572 of the MŠMT ČR

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ABSTRACT:

Fuzzy transform is a method which belongs to fuzzy approximation models. Its fundamentals are studied from different points of view. This approach allows to inherit results from different areas which are closely related to this method. Basically, it is shown to be a powerful approximation tool preserving features typical for fuzzy models. It is an appropriate tool for further numerical methods. Its robustness, computational simplicity, noise removing ability and other properties made possible to successfully applied the method to numerical solution of differential equations or image processing. Finally, the method can significantly influence fuzzy control strategy, especially an identification process. This fact is demonstrated on a real fuzzy control application dealing with an autonomous robot.

1 Introduction

The fuzzy transform method has been proposed in [15]. The method is design as a pair of two transform. The first one maps a continues function to a real vector. The second, inverse one, is naturally a mapping from the space of real vectors to the space of continuous function.

Originally, it has been studied as a transform. Compared to other, especially integral, transforms. Mainly, approximation properties were focused.

Later on, more and more applications has been appearing to be appropriate because of several valuable properties which are typical for the method.

This paper tries to make short overview of these valuable properties, recall, main result, but avoids technical and mathematical details since they are not crucial for such a more or less survey paper.

Its main task is to introduce the method to practitioners, present it in the context of some other methods which are well known, present some advantages and typical features. Especially practical aspects and practical problems which can be solved using the fuzzy transform or algorithms involving the fuzzy transform are addressed.

2 Fuzzy transform

The method belonging to fuzzy modelling and discussed in this paper is the *fuzzy transform* (F-transform) [13]. It is a fuzzy approximation method (approximating a functional dependency i.e. a continuous function $f : X \to Y$) based on two transforms - a direct one and an inverse one. It deals with a fuzzy partition of the domain X given by fuzzy sets called *basic functions* $\mathbf{A}_i \underset{\sim}{\subset} X$ $i = 1, \ldots, n$ fulfilling several conditions including the Ruspini condition [17]

$$\sum_{i=1}^{n} \mathbf{A}_i(x) = 1 \quad \forall x \in X. \tag{1}$$

The technique deals with triangular shaped fuzzy sets or sinusoidal shaped fuzzy sets at most but the shape is not restricted at all so i.e. polynomial basic functions are allowed as well. For details see [12] or [13].

Usually, the uniform fuzzy partition is used i.e. n equidistant nodes $c_i = c_{i-1} + h$ are fixed and basic functions are determined to fulfill:

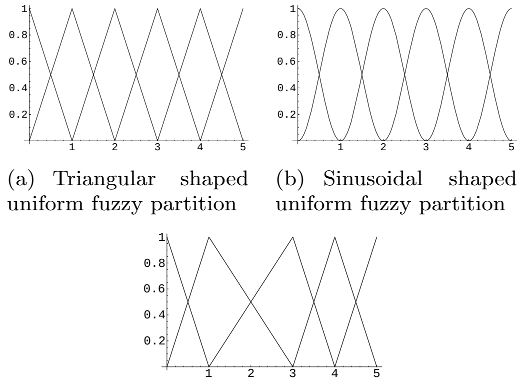
$$\mathbf{A}_i(c_i) = 1 \text{ and } \mathbf{A}_i(x) = 0 \text{ for } x \notin (c_{i-1}, c_{i+1}).$$

The *direct F-transform* is a discrete simplified representation of the function f given by a real vector $[F_1, \ldots, F_n]$ where

$$F_i = \frac{\int_X f(x)\mathbf{A}_i(x)dx}{\int_X \mathbf{A}_i(x)dx}. \tag{2}$$

Formula (2) represents the center of gravity of function values above the subdomain given by the support of the i-th basic functions which is weighted by membership degrees of elements x to the corresponding basic functions. It means that each *component* F_i of the F-transform expresses local information about the original function f. This fact is supported by the following equality

$$F_i = f(c_i) + O(h^2) \tag{3}$$



(a) Triangular shaped uniform fuzzy partition

(b) Sinusoidal shaped uniform fuzzy partition

(c) Triangular shaped non-uniform fuzzy partition with a symmetry

Figure 1: Graphic presentation of distinct fuzzy partitions.

where h is the step between nodes of a uniform fuzzy partition. Let us stress that similar result is obtained even for non-uniform fuzzy partitions [19].

Obviously, we can observe that the F-transform as a mapping from the space of continuous functions to the space of real vectors is linear. It means that if f, g, ψ are continuous functions on X such that $\psi = \alpha f + \beta g$ where α, β are real numbers. Then the following equality holds

$$[\Psi_1, \dots, \Psi_n] = \alpha[F_1 \dots, F_n] + \beta[G_1, \dots, G_n] \tag{4}$$

where $[\Psi_1, \dots, \Psi_n], [F_1, \dots, F_n]$ and $[G_1, \dots, G_n]$ are the F-transforms of ψ, f and g w.r.t. the given fuzzy partition, respectively [12].

If we deal with an approximation of a function it is necessary to distinguish the approximation among other possible ones. This is usually guaranteed by a minimization of a certain criterion. It is provable that the F-transform components F_i minimize the following *piecewise integral least square criterion* [12]

$$\Phi(Q_1, \dots, Q_n) = \int_X \left(\sum_{i=1}^{n} (f(x) - Q_i)^2 \mathbf{A}_i(x) \right) dx. \tag{5}$$

The local closeness of the components to the function values given by (3) makes from the F-transform an appropriate candidate for replacing the original function in complex computations (as usual in many numerical methods). After finishing a numerical algorithm, its approximate result is supposed to be transformed back to the space of continuous functions.

For this purpose, the *inverse F-transform* mapping has been proposed. It is gives a continuous function on X and it is given by a linear combination of the basic functions and the components F_i of the direct F-transform i.e.

$$f_n^F(x) = \sum_{i=1}^{n} F_i \mathbf{A}_i(x). \tag{6}$$

Indeed, the uniform convergence of a sequence of the inverse F-transform to the original function is an essential property.

The convergence, local closeness, integral optimality given by (5) and low computational cost promises the F-transform method to be a useful approximation method for practice.

Finally, let us recall a discrete version of the direct F-transform which is used in such case when no analytical description is at disposal or its usage is from unspecified reason impossible. Then the components are given as follows

$$F_i = \frac{\sum_{j=1}^{N} f(x_j) \mathbf{A}_i(x_j)}{\sum_{j=1}^{N} \mathbf{A}_i(x_j) dx} \tag{7}$$

where $(x_j, f(x_j))$ $j = 1, \ldots, m$ is a set of (measured) samples which is at disposal and where $N >> n$, in principle.

Of course, a more-dimensional case of any approximation method is highly desirable. Here, we briefly recall a direct extension of the method for functions $f : X \times Y \to Z$, for details we refer to [22].

If the domain is given by a Cartesian product of two real intervals $X \times Y$ then we construct two independent fuzzy partitions, $\mathbf{A}_1, \ldots, \mathbf{A}_n \underset{\sim}{\subset} X$ and $\mathbf{B}_1, \ldots, \mathbf{B}_m \underset{\sim}{\subset} Y$ (see Figure 2) and the direct F-transform is then given by a real matrix composed from components F_{ij} given as follows

$$F_{ij} = \frac{\int_Y \int_X f(x,y)\mathbf{A}_i(x)\mathbf{B}_j(y)dxdy}{\int_Y \int_X \mathbf{A}_i(x)\mathbf{B}_j(y)dxdy}. \tag{8}$$

Obviously, all the mentioned properties (convergence, integral optimality, linearity etc.) are preserved.

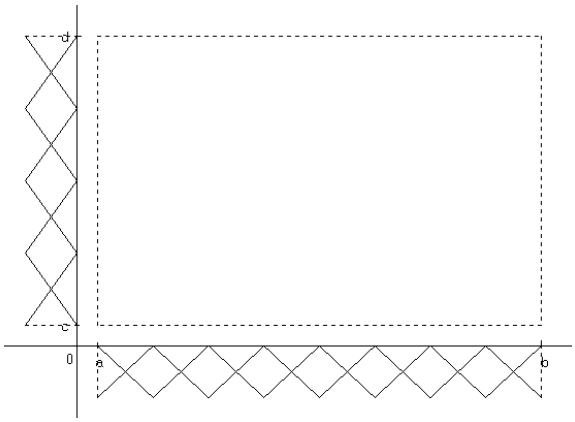


Figure 2: Uniform fuzzy partition of $X \times Y$ comprised from triangular shaped basic functions on both axes.

A generalization to more than 2-dimensional case is straightforward.

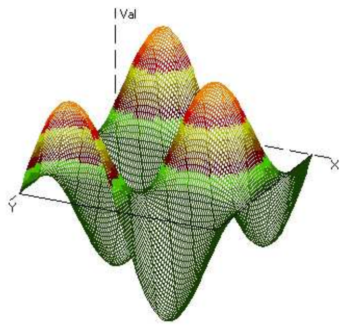
3 Numerical methods

Besides the fact that the methods provide us with a discrete approximate representation (direct F-transform) and a continuous approximation (inverse F-transform) of a given function it can be very useful in further numerical methods.

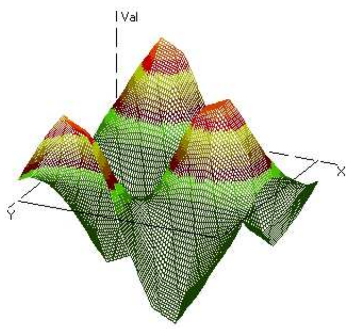
For instance, the integral $\int_X f(x)dx$ can be computed as follows

$$\int_X f(x)dx = h\left(\frac{1}{2}F_1 + F_2 + \cdots + F_{n-1} + \frac{1}{2}F_n\right)$$

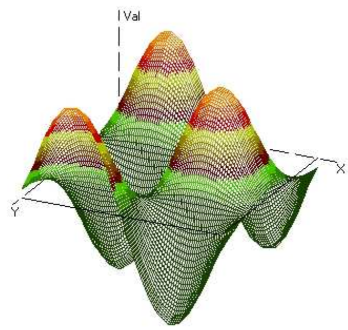
in the case of a uniform fuzzy partition. In the case of a non-uniform fuzzy partition, a similar formula can be obtained. But there is always a question, how to determine a non uniform fuzzy partition. An answer to this questions can be found e.g. in [20] or [19]. These works investigate a neural approach to the F-transform. It means, that the method is visualized in the style of neural network (RBF ϕ-neural networks [9], in particular) and the discrete direct F-transform formula is replaced by an incremental learning. Moreover, an unsupervised learning for a fuzzy partition determination is involved as well. Finally, what we get, is a powerful, computationally simple and fast, incremental tool for an approximation of a function, which moreover incrementally computes a numerical integral of f [19], see Figure 4.



(a) $f(x, y) = \sin(x) \cdot \cos(y)$



(b) The inverse F-transform w.r.t. a uniform fuzzy partition comprised from 10 triangular shaped basic functions on each axis



(c) The inverse F-transform w.r.t. a uniform fuzzy partition comprised from 20 triangular shaped basic functions on each axis

Figure 3: An illustration of the uniform convergence of the inverse F-transform to the original function $f(x, y) = \sin(x) \cdot \cos(y)$ on $\mathcal{D}^2 = [0, 2\pi]^2$.

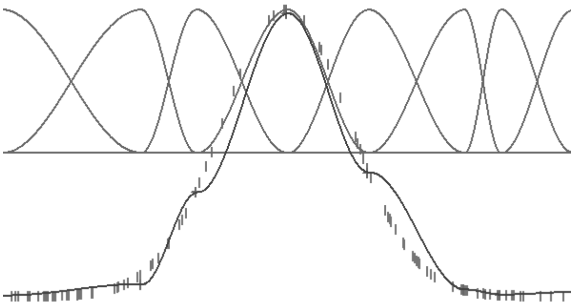


Figure 4: Measured samples of a function and its approximation by the proposed neural improvement of the fuzzy transform, learning coefficient.

(a) First damaged figure

(b) Second damaged figure

(c) Figure created by fusion of both damaged figures

Figure 5: Demonstration of image fusion algorithm proposed in [4] directly using iterations of the F-transforms.

The F-transform has been demonstrated to be appropriate robust method. It has been proved that it can remove a periodical noise upon some conditions [16]. Moreover, it can significantly reduce even random noise. From an engineering point of view it is so called low-pass filter.

This fact has been used in further numerical computations where high impreciseness caused by measurements is expected. An application to numerical solution of a particular ordinary differential equation (with a geological application) has been published in [12]. These results were later extended even for a more-dimensional case i.e. for partial differential equations and their numerical solutions [22].

Some attempts of applications of the method to image processing (rescaling) have been proceeded as well [13]. Especially attempts to involve the method to image fusion are promising [4]. It can be stated, that the algorithm proposed in [4] which directly uses iterations of the direct F-transforms gives comparable results to the well known and for a very long time studied wavelet transform which is a standard method used for this problem.

To provide readers with an idea about results of the image fusion algorithm, we present Figure 5.

4 Fuzziness of the F-transform

Let us discuss the questions what is fuzzy on fuzzy transform. Of course, the fuzzy partition used for partitioning the domain is and old and well known notion. The method obviously disposes of the most typical features for fuzzy systems such as robustness, approximation ability, usage of interpretable elements (basic functions can be easily interpreted by fuzzy numbers).

On the other hand, it fully belongs to regression analysis as well. But if we recall Takagi-Sugeno rules [23] consequents being equal to real functions (polynomials of the k-th order)

$$\textbf{IF } x \text{ is } \mathcal{A}_i \textbf{ THEN } y \text{ is } f_i(x) \quad i = 1, \ldots, n \tag{9}$$

which are evaluated according to the following formula

$$f^A(x) = \frac{\sum_{i=1}^n \mathbf{A}_i(x) \cdot f_i(x)}{\sum_{i=1}^n \mathbf{A}_i(x)} \tag{10}$$

If we recall the inverse F-transform formula and the fact that the method requires to fulfill the Ruspini condition on the antecedent part, we can state that the inverse F-transform is an inference method belonging to singleton models, equal to T-S rules of the 0th order, in particular, while the direct F-transform is an identification (learning) algorithm. In the case of the discrete direct F-transform, it is even equal to the identification of the T-S rules of the 0-th order. So, the F-transform provide us with a continuous (integral) extension of the identification of specific T-S rules.

But still, one could claim that even T-S rules is a a regression method as well and its conditional form is highly questionable since formula (10) has nothing common with logical constraints.

Generally, we can distinguish between two types of fuzzy rules. A set of n fuzzy rules is usually interpreted either by fuzzy relation $\hat{\mathbf{R}}_* \underset{\sim}{\subset} X \times Y$

$$\hat{\mathbf{R}}_*(x, y) = \bigwedge_{i=1}^{n} \left(\mathbf{A}_i(x) \rightarrow_* \mathbf{F}_i(y)\right). \tag{11}$$

or by fuzzy relation $\check{\mathbf{R}}_* \underset{\sim}{\subset} X \times Y$

$$\check{\mathbf{R}}_*(x, y) = \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{F}_i(y)) \tag{12}$$

where $*$ is a left continuous t-norm [8] and $\rightarrow_*$ is its residuation operation.

Obviously, the first fuzzy relation given (11) is an interpretation of the following fuzzy rules

$$\begin{array}{ccc} \textbf{IF } x \text{ is } \mathcal{A}_1 & \textbf{THEN} & y \text{ is } \mathcal{F}_1 \\ & \ldots & \\ & \textbf{AND} & \\ & \ldots & \\ \textbf{IF } x \text{ is } \mathcal{A}_n & \textbf{THEN} & y \text{ is } \mathcal{F}_n \end{array} \tag{13}$$

where $\mathcal{A}_i, \mathcal{F}_i$ are linguistic expression [24] represented by fuzzy sets $\mathbf{A}_i \underset{\sim}{\subset} X$ and $\mathbf{F}_i \underset{\sim}{\subset} Y$, respectively. It means the real IF-THEN fuzzy rules expressing logical constraints.

The second fuzzy relation given by (12) is an interpretation of the following fuzzy rules

$$\begin{array}{ccc} x \text{ is } \mathcal{A}_1 & \textbf{AND} & y \text{ is } \mathcal{F}_1 \\ & \ldots & \\ & \textbf{OR} & \\ & \ldots & \\ x \text{ is } \mathcal{A}_n & \textbf{AND} & y \text{ is } \mathcal{F}_n. \end{array} \tag{14}$$

It is worth mentioning that distinguishing between the conditional (IF-THEN) form of fuzzy rules (13) and the Cartesian product (AND) form of fuzzy rules (14) on a syntactical level is not very common but it can be found e.g. in [2, 7, 10]. Usually only the form given by (13) is because of several (e.g. historical reasons or equivalence of both form sin the classical case) considered and the differences are taken into account only on a semantical level. But the differences can play a crucial role for further implementations and therefore they should be kept in mind. For more detailed study concerning both rule forms we refer to [6, 11, 7] and to an exhaustive investigation in [5].

From the common knowledge from the area of systems of fuzzy relation equations we can state that fuzzy rules (13) predetermine the direct image inference (sup-* composition i.e. *compositional rule of inference*) while fuzzy rules (14) predetermine the subdirect image inference (inf$\rightarrow_*$ composition i.e. Bandler-Kohout subproduct) [18].

So called additive interpretation [21]

$$\mathbf{R}_*^{\oplus}(x,y) = \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) * \mathbf{F}_i(y)) \tag{15}$$

of fuzzy rule base (14) has been proposed to formalize Takagi-Sugeno rules, F-transform and some neuro-fuzzy methods using weighted mean approach rather than logical one. Indeed, in the case of $\mathbf{F}_i$ being singletons and having in mind the definition of the Łukasiewicz t-conorm $\oplus$, it coincides with many mentioned models including the inverse F-transform formula. It has been shown that the additive interpretation is an appropriate fuzzy model. It means, that it is a solution to systems of fuzzy relation equation with subdirect inference and under some special assumptions even to systems with direct inference [18]. The Ruspini condition played a crucial role in the results.

5 F-transform and the additive interpretation

Let us look at the additive interpretations and the F-transform a bit closer. Fuzzy rules (13) interpreted by (11) were from an approximation point of view formalized by the so called *conjunctive normal form*. Fuzzy rules (14) interpreted by (12) were from an approximation point of view formalized by the so called *disjunctive normal form*. Both forms were studied in [14]. Moreover, an *additive normal form* motivated by the F-transform can be found in the cited paper as well. Furthermore, publication [3] continues in investigations of additive normal forms and presents the F-transform as a special additive normal form.

Based on these results, we can define extended fuzzy transform of a fuzzy relation $\mathbf{F} : X \times Y \to [0,1]$ which can be viewed as a fuzzy set-valued function $\mathbf{F} : X \to [0,1]^Y$ i.e. as a mapping which assigns a fuzzy subset of Y to each node $x \in X$. In the latter, we will not distinguish between both points of view since they will be always clear from the context.

So the direct F-transform is then defined analogously i.e. by

$$\mathbf{F}_i(y) = \frac{\int_a^b \mathbf{F}(x,y)\mathbf{A}_i(x)dx}{\int_a^b \mathbf{A}_i(x)dx},\ i = 1,\dots,n \tag{16}$$

as well as the inverse one

$$\mathbf{F}_n^F(x,y) = \sum_{i=1}^{n}(\mathbf{A}_i(x) \cdot \mathbf{F}_i(y)). \tag{17}$$

Obviously, the inverse F-transform $\mathbf{F}_n^F$ given by (17) is equal to the additive interpretation $\mathbf{R}_{\odot}^{\oplus}$ of fuzzy rule base (14) where $\odot$ is the product t-norm. Which presents again the F-transform in the light of fuzzy models.

In the identification process, we expect knowledge of the approximated fuzzy relation viewed a fuzzy set valued function $\mathbf{F} : X \to [0,1]^Y$ at nodes $p_1,\dots,p_N \in X$. So, let us be given data $(p_k, \mathbf{F}(p_k,\cdot))$ where $\mathbf{F}(p_k,\cdot) \underset{\sim}{\subseteq} Y$ for $j = 1,\dots,N$. Then the vector $[\mathbf{F}_1,\dots,\mathbf{F}_n]$ of fuzzy sets on Y is the *discrete direct F-transform* of the fuzzy relation $\mathbf{F}$ w.r.t. $\mathbf{A}_1,\dots,\mathbf{A}_n$ if

$$\mathbf{F}_i(y) = \frac{\sum_{k=1}^{N} \mathbf{F}(p_k,y)\mathbf{A}_i(p_k)}{\sum_{k=1}^{N} \mathbf{A}_i(p_k)},\ i = 1,\dots,n. \tag{18}$$

Let only shortly stress, that having a set of $(p_k, \mathbf{F}(p_k, \cdot))$ is not unnatural. It can be either obtained by asking an expert, how to control a given process at node p_k while $\mathbf{F}(p_k, \cdot)$ is a fuzzy sets representing a linguistic expression the expert used in his answer. (For instance: what to do when the right hand side wall is 5cm close? Answer: turn the wheel to the right very much. Then $p_k = 5$ and $\mathbf{F}(p_k, \cdot)$ is an extension of expression *very much* in the context Y).

The second possibility is to (during a manual control) collect nodes $(p_k, f(p_k))$ and to "fuzzify" control actions $f(p_k)$. This increases the robustness since manual control actions are typical but never precise. Further reasons for this approach will be discussed later on.

6 Identification strategy

There are two main approaches for an identification of a fuzzy rule base controlling a given process. Fuzzy rule based (FRB) systems were motivated by dealing with an expert knowledge which is more or less always expressed by a natural language where the phenomenon of vagueness is essential. However, for some systems an expert knowledge acquisition is not a trivial task or transformation of such knowledge into an FRB is technically hardly feasible [1]. For these cases a data-driven approach has to be used either to *adapt* some initial very rough FRB or to *generate* a new one if no initial FRB is attainable. In general, data-driven approaches (neural learning, heuristic algorithms, adaptation optimizing a cost function etc.) deal with some training data obtained by experiments.

In the case of a generation of an FRB we just approximate given data but it might be done by any classical approach and thus the usage of *fuzzy* is questionable although it surely provides undoubtable advantages e.g. interpretability, transparency or robustness. In the second case we assume that we are given some initial FBR which is to be adapted. But an adaptation algorithm can lead to something completely different from the initial FRB and therefore it might be set randomly. So, we either do not have a prior expert knowledge or we lose it, at least partially.

Finally, in any type of learning, there is a problem that the system must *learn* all possible situation otherwise the system will not be able to behave correctly. This can lead to a huge mass, of experiments and even this need not be sufficient.

This section recalls [21] an identification strategy which tries to deal with the mentioned problems. Briefly, it can be described as follows:

- manually control a given process and collect input-output pairs of data $(p_k, f(p_k))$
- fuzzify the collected control actions $f(p_k)$ to get pairs $(p_k, \mathbf{F}(p_k, \cdot))$
- construct a fuzzy partition of X
- compute components of the extended direct F-transform
- proceed experiments with an automatic control by fuzzy rules (14) interpreted by fuzzy relation $\mathbf{R}^{\oplus}_{\odot}$ i.e. by the extended inverse F-transform (with an appropriate defuzzification e.g. COG)

At this moment an expert only observe behaviour of the process i.e. correctness of the model controlling the process. For those situation which were not learned sufficiently (or were not learned at all), the expert specifies an appropriate control action by linguistic expression. It means that originally collected data $(p_k, f(p_k))$ for $k = 1, \ldots, N$ were modified to $(p_k, \mathbf{F}(p_k, \cdot))$ and finally enriched by data $(p_{N+l}, \mathbf{F}(p_{N+l}, \cdot))$ given linguistically by an expert where $l = 1, \ldots, M$.

Then the F-transform components, serving us as consequent fuzzy sets, are recomputed from all data $(p_k, \mathbf{F}(p_k, \cdot))$ where $k = 1, \ldots, M + N$. This approach has been successfully applied to an approximation based control [21] of an autonomous dynamic robot, see Figure 6.

Remark 1 *The autonomous robot control was double-input-single-output problem. The input variables were the distance to the middle of a given corridor e and its derivation Δe. This 2-dimensional case of the extended F-transform was not recalled in this paper since it is straightforward and based on the 2-dimensional case of the original F-transform, see formula (8).*

This approach has brought some advantages which worth mentioning. It required extremely low time (in minutes) for initial experiments since only 112 input-output pairs were collected (so $N = 112$). Fuzzy partition was constructed (9 basic function on each axis which implied 49 fuzzy rules) and the extended direct F-transform computed. Automatic control was immediately able to be tested. After few experiments, an expert (actually for this control process anyone who is able to drive a car) specified 6 expression for 6 critical positions (obtained from PC observing the behaviour).

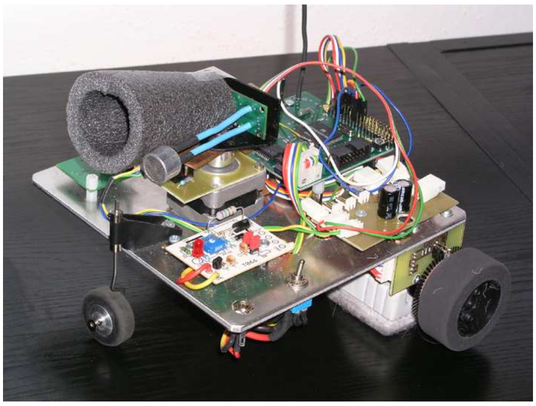


Figure 6: The dynamic robot controlled by the F-transform.

The F-transform was recomputed which did not cause any increase of number of rules since the number is predetermined by the chosen fuzzy partition already.

For more details concerning this fuzzy control application we refer to [21].

7 Conclusions

The F-transform method has been recalled in the paper. As mentioned in the Introduction, the main task was to introduce the method to practitioners without deeper details. Its impact to practical problems was accented more.

Numerical techniques, noise removing abilities, connection to differential equations has been at least shortly recalled or hinted. The correlations of the method and neural networks has been remarked as well. A considerable part of the paper was devoted to a relationship between the method and fuzzy rule based systems, especially interpretations of fuzzy rules and their identification.

From real practical applications, function approximation, image fusion, numerical solution of differential equations and especially fuzzy control of an autonomous robot were discussed or cited. The fuzzy control of the robot is of course just a demonstrative “benchmark” and the suggested approach should be understood generally.

Acknowledgement

This investigation has been supported by project DAR 1M0572 of the MŠMT ČR.

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