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Riemann-Stieltjes type integral based on generated pseudo-operations

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Abstract: We shall consider generated pseudo-operations of the following form: $x \oplus y = g^{(-1)}(g(x) + g(y))$, $x \odot y = g^{(-1)}(g(x)g(y))$, *where* g *is a positive strictly monotone generating function and* $g^{(-1)}$ *is its pseudo-inverse. Using this type of pseudo-operations, the Riemann-Stieltjes type integral will be introduced and investigated.*

Keywords: pseudo-operatins, generating function, Riemann-Stieltjes integral.

1 Introduction

It is a well known fact that classical Riemann-Stieltjes integral has great application in several areas of analysis as well as in probability theory and physics (see [3, 4]). The main aim of this paper is to present pseudo-analysis' counterpart of this integral. By the means of generalized generated pseudo-operations and corresponding measure-like set function, pseudo Riemann-Stieltjes integral has been constructed. This new Riemann type of integral belongs to the area of pseudo-analysis, i.e., to the theory that combines approaches from many different fields and is capable of supplying solutions that were not achieved by the classical tools. Some of important results concerning pseudo-analysis, both theory and application, can be found in [1, 2, 5, 6, 7, 10, 13, 16, 17].

Section 2 contains preliminary notions, such as generalized generated pseudo-operations and measure-like set function. Operations in question are generalizations of pseudo-operations that are in the core of g-calculus ([9, 11, 12, 13, 14]). Set function defined in this section is based on generating function g and, under some additional conditions, has properties of fuzzy measure ([13, 17]). In the third section, construction of pseudo Riemann-Stieltjes integral and some of its properties have been presented. Specially, for increasing continuous generating function (we should stress that in this case used operations are operations from g-semiring) strong connection between proposed integral and Lebesque integral has been obtained. Additionally, connections with g-integral ([12, 13]) and another Lebesgue type of integral known as pseudo-Lebegue-Stieltjes integral ([8]) have been given.

2 Preliminary notions

The basic preliminary notions needed in this paper are notions of generated pseudo-operations. More precisely, operations in question are generalizations of so called g-operations, i.e. operations from g-semiring. Before introducing this form of generated pseudo-operations, a short overview of the "classical" g-semiring from [13] is given.

Let $[a, b]$ be closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be total order on $[a, b]$. A semiring is structure $([a, b], \oplus, \odot)$ when following hold:

- $\oplus$ is *pseudo-addition*, i.e., a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by $\mathbf{0}$;
- $\odot$ is *pseudo-multiplication*, i.e., a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing ($x \preceq y$ implies $x \odot z \preceq y \odot z$, $z \in [a, b]_+ = \{x : x \in [a, b], \ \mathbf{0} \preceq x\}$), associative and for which exists a unit element denoted by $\mathbf{1}$;

- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and non idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by monotone and continuous generator function $g : [a, b] \to [0, +\infty]$, i.e. g-semirings, form the second class, and semirings with both idempotent operations belong to the third class. More on this structure as well as on measures and integrals constructed on it can be found in [7, 10, 13, 14, 15].

As already mentioned, of special interest for this paper are generalizations of operations from semirings of the second class. In this case, constrains normally put on generating function $g : [a, b] \to [0, +\infty]$ are weakened.

Definition 1 *Let g be a positive strictly monotone function defined on $[a, b] \subset [-\infty, +\infty]$ such that $0 \in Ran(g)$.* The generalized generated pseudo-addition $\oplus$ *and* the generalized generated pseudo-multiplication $\odot$ *are given by*

$$x \oplus y = g^{(-1)}(g(x) + g(y)), \tag{1}$$

$$x \odot y = g^{(-1)}(g(x)g(y)), \tag{2}$$

where $g^{(-1)}$ is pseudo-inverse function for function g.

Remark 2 For non-decreasing function $f : [a, b] \to [a_1, b_1]$, where $[a, b]$ and $[a_1, b_1]$ are closed subintervals of extended real line $[-\infty, +\infty]$, pseudo-inverse is

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) < y\}.$$

If f is non-increasing function, its pseudo-inverse can be obtained by following formula:

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) > y\}.$$

More on this subject can be found in [6].

Remark 3 For continuous generating function, operations (1) and (2) are operations from g-semiring, i.e., $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x) + g(y))$.

As in the case of g-semiring, monotonicity of generating function g is closely connected with order $\preceq$ on $[a, b]$:

$$x \preceq y \Leftrightarrow g(x) \leq g(y).$$

Additionally, $x \prec y$ if and only if $g(x) \leq g(y)$ and $x \neq y$.

Further on, by pseudo-addition and pseudo-multiplication, if not stated differently, operations (1) and (2) will be considered.

It is obvious that operations (1) and (2) are commutative, however they need not be associative. Some basic properties of pseudo-operations in question are given by the following proposition.

Proposition 4 *Let $\oplus$ and $\odot$ be pseudo-operations from Definition 1.*

(a) If $g(x) + g(y), g(z)g(x), g(z)g(y) \in Ran(g)$, $\odot$ is distributive over $\oplus$, i.e.,

$$z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y).$$

(b) Neutral element for $\oplus$ is $g^{(-1)}(0)$.

(c) If $1 \in Ran(g)$, the neutral element for $\odot$ is $g^{(-1)}(1)$.

(d) $g^{(-1)}(0) \odot x = x \odot g^{(-1)}(0) = g^{(-1)}(0)$ for all $x \in [a, b]$.

(e) $\oplus$ *is non-decreasing function, i.e., for* $x \preceq y$ *we have* $x \oplus z \preceq y \oplus z, \quad x, y, z \in [a, b]$.

(f) $\odot$ *is non-decreasing function, i.e., for* $x \preceq y$ *we have* $x \odot z \preceq y \odot z, \quad x, y, z \in [a, b]$.

(g) In the general case, associativity does not hold for $\oplus$.

(h) In the general case, the cancellation law does not hold for $\oplus$.

Proofs for properties (a)-(f) follow directly from the Definition 1 and properties of pseudo-inverse function (see [6]). Claims (g) and (h) are illustrated by the following example.

Example 5 Let $g : [0, +\infty] \to [0, +\infty]$ given by

$$g(x) = \begin{cases} \ln(x+1), & x \in [0, 2], \\ e^x, & x \in (2, +\infty] \end{cases}$$

be a generating function for pseudo-addition $\oplus$. Its pseudo-inverse is

$$g^{(-1)}(x) = \begin{cases} e^x - 1, & x \in [0, \ln 3], \\ \ln x, & x \in (e^2, +\infty], \\ 2, & x \in (\ln 3, e^2]. \end{cases}$$

This pseudo-inverse function is continuous and strictly increasing on $Ran(g)$.

Now, for this choice of generating function and corresponding pseudo-operation it can be easily shown that the following holds:

$$\left(\frac{3}{2} \oplus \frac{1}{2}\right) \oplus 3 = \ln\left(\ln 3 + e^3\right) \neq \ln\left(\ln \frac{5}{2} + e^3\right) = \frac{3}{2} \oplus \left(\frac{1}{2} \oplus 3\right),$$

therefore, associativity, in the general case, does not hold.

Also, it can be easily shown that $\frac{3}{2} \oplus \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{3}$, i.e., the cancellation law does not hold either.

Remark 6 Due to the strict monotonicity of generating function, left neutral element for $\oplus$ is a, if the generating function is increasing, or b for g decreasing. This implies that $[a, b]_+ = \{x \mid x \in [a, b],\ g^{(-1)} \preceq x\}$ is whole interval $[a, b]$ and positive non-decreasingness for $\odot$ from the "classical" case is equivalent to non-decreasing property of $\odot$ from generalized g-semiring.

Since $\oplus$ is not necessarily associative operation, further on the following notation has been used:

$$\bigoplus_{i=1}^{n} \alpha_i = (\ldots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \ldots) \oplus \alpha_n,$$

where $\alpha_i \in [a, b]\ \ i \in \{1, 2, \ldots n\}$.

Also, by the means of generating function g is possible to introduce a metric. Let $d : [a, b]^2 \to [0, +\infty]$ be a function given by

$$d(x, y) = |g(x) - g(y)|, \tag{3}$$

where $x, y \in [a, b]$ and g is a generating function for $\oplus$. It is easy to check that d fulfills all conditions for being a metric.

2.1 Measure-like set function given by generator g

Another notion essential for the construction of pseudo Riemann-Stieltjes integral is the notion of measure-type set function. Therefore, we shall consider a set function that is given by the means of generating function g and defined on family of subintervals of the real line.

Let $\mathcal{C}$ be a family of semiclosed subintervals $(c, d]$ of $\mathbb{R}$ where $c \leq d$, then $\mathcal{C}$ is semiring of sets, i.e.,

1. $\emptyset \in \mathcal{C}$,
2. $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$,
3. for all $A, B \in \mathcal{C}$ there exists $C_1, ..., C_n$ from $\mathcal{C}$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$ and $A \backslash B = \cup_{k=1}^{n} C_k$.

Definition 7 *Let g be generating function from Definition 1 and let ϕ be a bounded function defined on the real line. A mapping $m : \mathcal{C} \to [a, b]$ is called* g_ϕ-set-function *if*

$$m((c, d]) = g^{(-1)} (\phi(d) - \phi(c)) . \tag{4}$$

Some of properties of set function given by the previous definition are:

- $m(\emptyset) = g^{(-1)}(0)$,
- if functions g and ϕ are of same monotonicity, m is monotone set function, i.e., fuzzy measure.

Additionally, if $\mathcal{P} = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$ is a n-partition of some interval $(c, d] \in \mathcal{C}$ such that $c = x_0 \leq x_1 \leq \ldots \leq x_n = d$, following hold:

- if g is continuous function, m is pseudo-additive on $\mathcal{P}$, i.e.,

$$m((c, d]) = m(\cup_{i=0}^{n-1} (x_i, x_{i+1}]) = \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]),$$

- if g is either strictly increasing left-continuous or strictly decreasing right-continuous, m is pseudo-superadditive on $\mathcal{P}$, i.e.,

$$m((c, d]) = m(\cup_{i=0}^{n-1} (x_i, x_{i+1}]) \succeq \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]),$$

- if g is either strictly decreasing left-continuous or strictly increasing right-continuous function m is pseudo-subadditive $\mathcal{P}$, i.e.,

$$m((c, d]) = m(\cup_{i=0}^{n-1} (x_i, x_{i+1}]) \preceq \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]).$$

3 Pseudo Riemann-Stieltjes integral

The main aim of this section is to introduce pseudo-analysis' counterpart of the well known Riemann-Stieltjes integral (see [4]). For this construction, operations, metric and measure presented in the previous section will be used.

Let g be generating function from Definition 1 defined on interval $[a, b]$ and $\oplus$ and $\odot$ pseudo-operations given by (1) and (2), respectively. Let ϕ be a bounded function defined on the real line. If $\mathcal{P} = \{(\omega_i, (x_{i-1}, x_i])\}_{i=1}^{n}$ is a tagged partition of $[c, d]$, i.e., $c = x_0 \leq x_1 \leq \ldots \leq x_n = d$ and $\omega_i \in (x_{i-1}, x_i]$, *the Riemann-Stieltjes pseudo-sum of f with respect to ϕ for the tagged partition $\mathcal{P}$* is

$$\bigoplus_{\mathcal{P}} f = \bigoplus_{i=1}^{n} f(\omega_i) \odot m((x_{i-1}, x_i]),$$

where $f : [c, d] \to [a, b]$.

Definition 8 *Function $f : [c, d] \to [a, b]$ is pseudo Riemann-Stieltjes integrable with respect to ϕ on $[c, d]$ whenever there is a real number PI satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d\left(\bigoplus_{\mathcal{P}} f, PI\right) < \varepsilon,$$

for all tagged partitions $\mathcal{P}$ of $[c, d]$ that fulfills $\max\{x_i - x_{i-1} \,|\, 1 \leq i \leq n\} < \delta$.

It is easy to check that number PI from previous definition is uniquely determined. This number PI is pseudo Riemann-Stieltjes integral of f on $[c,d]$ and it will be denoted by $(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\,d\phi$.

Remark 9 Specialy, for $g(x) = x$ previos definition will give us classical Riemann-Stieltjes integral $(RS)\int_c^d f\,d\phi$. More on this integral can be found in [4].

Theorem 10 *Let $g : [a,b] \to [0,\infty]$ be strictly increasing left-continuous or strictly decreasing right-continuous generating function and $f : [c,d] \to [a,b]$ pseudo Riemann-Stieltjes integrable function on $[c,d]$ with respect to a bounded function ϕ. Then*

$$g\left((pRS)\int_{[c,d]}^{(\oplus,\odot)} f\,d\phi\right) \leq (RS)\int_c^d g\circ f\,d\phi, \tag{5}$$

if the right side of inequality exists.

For g being continuous function some stronger connection between pseudo Riemann-Stieltjes integral and Riemann-Stieltjes integral can be proved: if f is pseudo Riemann-Stieltjes integrable on $[c,d]$ then $g\circ f$ is Riemann-Stieltjes integrable function on $[c,d]$ and

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\,d\phi = g^{-1}\left((RS)\int_a^b g\circ f\,d\phi\right). \tag{6}$$

Also, if f is Riemann-Stieltjes integrable on $[c,d]$ it can be proved that $g^{-1}\circ f$ is pseudo Riemann-Stieltjes integrable function on $[c,d]$.

Additionally, based on (6), some pseudo-linear properties of the pseudo Riemann-Stieltjes with continuous generating function can be proved, i.e., if f and h are pseudo Riemann-Stieltjes integrable functions on $[c,d]$ with respect to ϕ, then:

- for some $\alpha \in \mathbb{R}$, αf is pseudo Riemann-Stieltjes integrable on $[c,d]$ with respect to ϕ, f is pseudo Riemann-Stieltjes integrable on $[c,d]$ with respect to $g(\alpha)\phi$ and

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} \alpha\odot f\,d\phi = \alpha\odot(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\,d\phi = (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\,d\left(g(\alpha)\phi\right);$$

- $f\oplus h$ is pseudo Riemann-Stieltjes integrable on $[c,d]$ with respect to ϕ, and

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\oplus h\,d\phi = (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\,d\phi \oplus (pRS)\int_{[c,d]}^{(\oplus,\odot)} h\,d\phi.$$

There is a useful relationship between the Lebesgue integral and the pseudo Riemann-Stieltjes integral given by the following theorem.

Theorem 11 *Let $g : [a,b] \to [0,\infty]$ be a bounded strictly increasing continuous generating function and $f : [c,d] \to [a,b]$ a measurable function. Let μ be Lebesgue measure and ϕ_f distribution function of f given by $\phi_f(x) = \mu\left(\{t\in[c,d] \mid f(t) > x\}\right)$. Then*

$$(L)\int_c^d g\circ f\,d\mu = -g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\,d\phi_f\right). \tag{7}$$

Remark 12 Some additional constrains put on genrator g will ensure us following connection with g-integral ([12, 13]):

- if g^{-1} is even function, then

$$\int_{[c,d]} f\odot\,d\nu = (pRS)\int_{[a,b]}^{(\oplus,\odot)} x\,d\phi_f;$$

- if g^{-1} is odd function, then

$$\int_{[c,d]} f \odot \, d\nu = -(pRS) \int_{[a,b]}^{(\oplus,\odot)} x \, d\phi_f;$$

where a g-integral of function f is denoted by $\int_{[c,d]} f \odot \, d\nu$.

Remark 13 Problem of pseudo-integartion, similar to one presented in this paper, that has been focused on pseudo-probability space, continuous generating function and Lebesgue type of integral has been investigated in [8]. Some additional conditions will insure us following connection between pseudo Lebesgue-Stieltjes integral from [8] and pseudo Riemann-Stieltjes integral: if (Ω, S, P) is a pseudo-probability space, $\xi : \Omega \to [a, b]$ a random variable, F_g pseudo-distribution function of random variable ξ and $f : [a, b] \to [a, b]$ a measurable function, then

$$g\left((pLS) \int_{[a,b]}^{\oplus} f \, dF_g\right) = -g\left((pRS) \int_{[a,b]} x \, d\phi_{f\circ\xi}\right),$$

where integral on the left is pseudo Lebesgue-Stieltjes integral ([8]).

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