University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

Evaluating Linguistic Expressions and Functional Fuzzy Theories in Fuzzy Logic

Vilém Novák, Irina Perfilieva

Research report No. 10

February 12, 1998

Submitted/to appear:

L. A. Zadeh and J. Kacpryk (eds.): Computing with Words in Systems Analysis. Springer-Verlag, Heidelberg 1998

Supported by:

Grant 201/96/0985 GA ČR, COST Action 15, ZI 104/27-1 (part 436 RUS 113/197/O(R))

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
Bráfova 7, 701 03 Ostrava 1, Czech Republic

tel.: +420-69-622 2808 fax: +420-69-22 28 28
e-mail: novakv@osu.cz, vep@oktava.msk.su

Evaluating Linguistic Expressions and Functional Fuzzy Theories in Fuzzy Logic

Vilém Novák

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
Bráfova 7, 70100 Ostrava 1, Czech Republic

and

Institute of the Theory of Information and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4, 186 02 Praha 8, Czech Republic

Irina Perfilieva

Moscow State Academy of Instrument Making
Stromynka 20, 107846 Moscow, Russia

Abstract

In this paper, we introduce a new mathematical model of the meaning of the basic linguistic trichotomy, which are the canonical words "small", "medium" and "big". The model is based on the concept of *horizon* as elaborated in the Alternative Set Theory. Such a model makes also possible to include naturally the linguistic hedges which form a consistent class of functions. Each linguistic hedge is thus characterized by one number only.

Then it is shown that continuous functional dependencies between x and y can be described (precisely or approximately) by the collections of logical formulas of implicative form with predicates interpreted by fuzzy sets with meaning of the basic linguistic trichotomy. It demonstrates the expressive power of modified by linguistic hedges membership functions of fuzzy sets from the basic triplet.

1 Introduction

One of the most important features of fuzzy set theory which made it very attractive for applications is its potential for modelling of the meaning of natural language expressions. Most works done on this topic either continue directly, or have been inspired by the works of L. A. Zadeh (see e.g., [23, 24, 25, 26]). He focused on some parts of natural language, mostly those which correspond to the so called *evaluating linguistic expressions*, i.e. the expressions which characterize position on an ordered scale.

In this paper, we deal with two problems. First, we propose the model of the meaning of simple *linguistic evaluating expressions* of the form

$$\langle \text{linguistic modifier} \rangle \langle \text{atomic term} \rangle, \tag{1}$$

which is based on modification of the corresponding fuzzy sets.

The second problem elaborated in this paper is formalization of the linguistic characterization of functions in fuzzy logic and their approximation properties.

The *atomic terms* in (1) characterize various properties of objects such as "small, medium, big, slim, stout, sour, sweet", etc. The *linguistic modifiers* (often called also *linguistic hedges*) specify various nuances of properties. In general, they modify the meaning of the atomic terms before which they stand. The leading role among them is played by words such as "very, extremely, roughly ", etc.

[1]This work has been supported by the grant 201/96/0985 of the GA ČR, partially by COST Action 15 and also partially by ZI 104/27-1 (part 436 RUS 113/197/O(R)).

It is easy to see that these properties form ordered linguistic scales. Inherent vagueness of the above considered terms then makes us possible to characterize the set of objects using three basic atomic terms, namely ⟨small, medium, big⟩, ⟨slim, normal, stout⟩, etc. As *canonical*, we take the triple ⟨small, medium, big⟩.

The deliberation leading to our model of the meaning of (1) is based on the concept of *horizon* and its *shift*. Starting from this, we obtain the model of the meaning both of the basic linguistic triple as well as of the linguistic modifiers using specific fuzzy sets. This approach makes possible to give systematic representation of (1) by a unified parametric class of membership functions. The mathematical outcome of this approach is that all the membership functions modeling the meaning of the evaluating expressions (1) are composed of simple functions of two kinds, called Z- and S-functions. We will mention also practical applications of this theory.

2 The meaning of simple evaluating linguistic expressions

2.1 Basic linguistic triple of atomic terms

2.1.1 The concept of horizon and finite numbers

The accepted form of the membership functions modelling the meaning of the basic linguistic triple are Z-, S- and Π-functions (see Fig. 1). Notable on their form is the following: first, there is a kernel consisting of the elements which surely have the property in concern (i.e., elements which are surely "small, medium" or "big"). Second, when getting farther from the kernel (to the right or left), the truth that the elements have this property decreases until the point is reached when it vanishes completely. Let us remark that this course is justified by the intuition and some psychological experiments. The functions used in order to express the described course are mostly of one of the three kinds: linear, quadratic, and exponential.

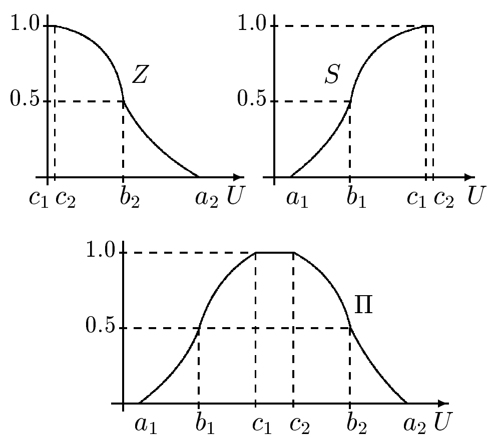

Figure 1: Typical membership functions of the basic linguistic triple.

However, the question raises, whether is it possible to find also other arguments in favour of the above described course and possible, the used functions. To this goal, we will employ the concept of *finite numbers* which seems to be related to the meaning of the atomic term *small*. The working mathematical model of finite and infinite numbers can be found, for example, in the Alternative Set Theory (AST) [22] and in non-standard analysis (cf. [4]), and a simplified one not leaving classical mathematics is constructed also in fuzzy logic.

Let us briefly mention the main idea of finite and infinite numbers in these theories. In classical set theory, infinity is taken as impossibility to reach the end; whatever number is at our disposal but we cannot regard the end of the infinite set of all numbers. On the other hand, the AST begins with the observation that big finite numbers behave as infinite ones (for demonstration, see [22]). We come to a new understanding of infinity and therefore, we may distinguish between finite and infinite numbers.

The infinity of some number n stems from our *inability to verify* whether n is finite. For example, there is no direct way how to verify that the number $1,000,000,005$ is indeed finite[2]. Hence, some (classically finite!) numbers are thus finite and some are infinite. However, there is no clear border between them and, consequently, there is no last finite number. We may distinguish finite numbers from infinite ones but we are not able to say about *each* number whether it is finite or not. Note that the finiteness is there a synonym for transparency, i.e. when seeing a finite number, we are able to verify easily that this is indeed the case. The fundamental concept on the basis of which finite numbers are constructed in AST is that of *horizon*, which represents the border between finite and infinite numbers. The horizon is seen by the *observer* (identified with the number 0). Precise specification of the position of the horizon, however, is not possible[3].

In this paper, we use these ideas for construction of the meaning of the basic linguistic triples. This provides us with a unified way how the membership functions can be constructed. Furthermore, the model of linguistic modifiers is naturally included in a way to extend the unification of membership functions to all (or, at least, a wide subclass of) evaluating expressions of the form (1).

For our purposes, the direct application of the AST is unnecessary and therefore, we use fuzzy logic to obtain a possible mathematical model of the horizon and finite numbers. Its idea follows the principles discussed above, i.e. that finiteness of a number means the *ability to verify* it. This may be imagined, for example, as a necessity to count a number of lines written on a paper. For numbers 0 or 1, we immediately see that this is indeed the case (we immediately see no or one line). However, verification that x is finite does not generally imply that we will be able to verify finiteness of $x+1$ with the same effort. Verification of the finiteness of the latter requires to repeat the same procedure we did for x. For example, if we verify that 1000 is finite by counting one thousand lines then verification that 1001 is also finite means that we have to count one line more, i.e. our effort to verify finiteness of 1001 is a little (imperceptibly) greater than that for 1000. In fuzzy logic, we will measure this effort by the *degree of truth*, i.e. less effort means higher truth degree and vice-versa.

Let $\mathbb{FN}(x)$ denote the proposition "the number x is finite". Then the above discussion justifies the formula $\mathbb{FN}(x) \Rightarrow \mathbb{FN}(x+1)$ to be true only in a degree $1-\varepsilon$ where ε is some small number. Starting from this, we may formally construct the fuzzy set of finite numbers — the numbers laying between the observer identified with 0 and the horizon.

The formal demonstration that such a procedure is reasonable is given by the following theorem.

Theorem 1 *Let T be a fuzzy theory which includes Peano arithmetics and $\varepsilon > 0$. Let $\mathbb{FN}$ be a new predicate. Then the fuzzy theory*

$$T^+ = T \cup \{{}^1/\mathbb{FN}(0), {}^{1-\varepsilon}/(\forall x)(\mathbb{FN}(x) \Rightarrow \mathbb{FN}(x+1), {}^1/(\exists x)\neg\mathbb{FN}(x)\}$$

is a conservative extension of T.

For details and the proof see [13, 15]. The theory T^+ conservatively introduces a predicate $\mathbb{FN}$ which has the properties assumed for finite numbers, namely that 0 is surely finite, that $\mathbb{FN}(x) \Rightarrow \mathbb{FN}(x+1)$ is not surely true and, finally, that there surely exists a number which is not finite.

When interpreting $\mathbb{FN}$ from this theorem, we obtain the following formula for the *fuzzy set* $\mathbb{FN}$ of finite numbers:

$$\mathbb{FN}(x) = \max\{0, (1-\varepsilon)^x\} = \max\{0, 1-\varepsilon x\} \tag{2}$$

where the power is taken with respect to the Łukasiewicz product defined by

$$a \otimes b = \max\{0, a+b-1\}, \qquad a, b \in [0,1].$$

In [15], several important properties of such fuzzy set are also demonstrated.

The *position of the horizon* is given by the least x such that $\mathbb{FN}(x) = 0$. We easily see that the bigger is ε, the sooner we reach the horizon, i.e. $\mathbb{FN}(x) = 0$ for smaller x. Thus, the magnitude of the number ε determines the "distance of the horizon" from 0.

[2] Sitting and counting, say $1,000,000,005$ sticks would be ungrateful work with uncertain ending — could we be sure that we did not make a mistake?

[3] All these concepts and much more have been explained in detail in [22] and from the point of view of natural language discussed also in [15].

Due to formula (2), the horizon is approached linearly by uniform decreasing of the truth of $\mathbb{FN}(x)$ with the increase of x. However, it follows from our discussion that the effort to verify finiteness could increase more rapidly when moving farther from the observer. Consequently, the approach to the horizon might be non-linear. Formula (2) can be made non-linear when considering ε to be dependent on x, i.e. ε is taken as a function $\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$. Then (2) takes the form

$$\mathbb{FN}(x) = \max\{0, 1 - \varepsilon(x) \cdot x\}. \tag{3}$$

The $\varepsilon(x)$ will be called the *horizon approach* function.

2.1.2 Canonical membership functions of "Small" and "Big"

In this section, we apply the above reasoning to the model of the linguistic meaning of the linguistic expressions (1). The existence of the basic linguistic triples demonstrates that people always tend to classify three positions on an ordered scale, namely "the leftmost" (the smallest), "the rightmost" (the biggest), and "in the middle". This observation has also been experimentally verified, e.g. in [10]. Hence, we may find three points on which we base this classification.

Let an ordered universe U be given. Without loss of generality, we put

$$U = [u, v] \subseteq \mathbb{R} \tag{4}$$

where $\mathbb{R}$ is the real line. The above considered three points ("leftmost", "rightmost" and "center") are u, v and $s \in U$, $u < s < v$, respectively. We put $L = [u, s]$ and $R = [s, v]$. Observe that the point s needs not necessarily lay in the exact middle of $[u, v]$. The reason is that big values are less distinguishable than small ones (cf., e.g., the discussion in [6]).

The meaning of *small* is obtained when realizing that the border of small numbers lays "somewhere to the right from u". The *u is the position of the observer* and there is no last small number — we may encounter only horizon of small numbers running "somewhere towards big ones". Thus, small numbers behave similarly as finite ones and it seems natural to identify the former with the latter.

Using (3), we will construct the *canonical fuzzy set $Sm \underset{\sim}{\subset} L$ of small* (numbers) with the membership function given by

$$Sm(x) = 1 - \varepsilon_{Sm}(x, a, c)x \tag{5}$$

where $\varepsilon_{Sm} : L \longrightarrow [0, 1]$ is a non-decreasing parametric function with the parameters $a, c \in L$, $c \leq a$ such that

$$\varepsilon_{Sm}(x, a, c) = \begin{cases} 0 & \text{for } x \leq c, \\ \text{increasing} & \text{for } c < x < a, \\ 1 & \text{for } a \leq x. \end{cases}$$

The interval $[u, c]$ is the set of *numbers being surely small*, i.e. $Sm(x) = 1$ for $x \in [u, c]$. Similarly, the interval $[a, s]$ is the set of numbers surely not being small, i.e. $Sm(x) = 0$ for $x \in [a, s]$.

As mentioned, the horizon approach function $\varepsilon_{Sm}(x, a, c)$ "implements" the idea of approaching the horizon. The bigger $\varepsilon_{Sm}(x, a, c)$ is, the shorter is our way to it. By setting specific formulas for $\varepsilon_{Sm}(x, a, c)$, we obtain various concrete membership functions. Note that in general, they will be Z-functions.

The atomic term "big" is not a complement of "small" but its *antonym*. Therefore, big numbers also behave as finite ones but taken in the reversed ordering from the rightmost side of the universe. To achieve this, we take the ordering of U reversely and change the position of the observer from u to v. The horizon of *big* then lays "somewhere to the left from v". Of course, it may not be in the same distance from the observer as the horizon of small ones. Thus, the corresponding horizon approach function ε_{Bi}, in general, differs from ε_{Sm}.

The *canonical fuzzy set* of *big* (numbers) $Bi \underset{\sim}{\subset} R$ is given by

$$Bi(x) = 1 - \varepsilon_{Bi}(x, a, c)x \tag{6}$$

where $\varepsilon_{Bi} : R \longrightarrow [0,1]$ is a non-increasing parametric function with the parameters $a, c \in R$, $a \leq c$ such that

$$\varepsilon_{Bi}(x,a,c) = \begin{cases} 1 & \text{for } x \leq a \\ \text{decreasing} & \text{for } a < x < c, \\ 0 & \text{for } c \leq x. \end{cases}$$

The interval $[c, v]$ is the set of *numbers being surely big*, i.e. $Bi(x) = 1$ for $x \in [c, v]$ and $[s, a]$ is the set of numbers surely not being big, i.e. $Bi(x) = 0$ for $x \in [s, a]$. By setting specific formulas for $\varepsilon_{Bi}(x, a, c)$, we obtain concrete membership functions, being generally S-functions.

2.1.3 Canonical membership function of "Medium"

In the case of the atomic term "medium", the observer is placed in the center $s \in U$ (cf. (4)). The horizon is spread both to the left as well as to the right from it. Hence, we may find two points $c^1_{Me} \leq s \leq c^2_{Me}$ and analogously as above define the *canonical fuzzy set* $Me \underset{\sim}{\subseteq} U$ of *medium* numbers given by

$$Me(x) = \begin{cases} 1 - \varepsilon^1_{Me}(x, a^1, c^1)x & \text{for } x \in L, \\ 1 - \varepsilon^2_{Me}(x, a^2, c^2)x & \text{for } x \in R \end{cases} \tag{7}$$

where ε^1_{Me} and ε^2_{Me} are parametric functions fulfilling the following conditions:

1. $\varepsilon^1_{Me} : L \longrightarrow [0,1]$ is a non-increasing parametric function with the parameters $a^1, c^1 \in L$, $a^1 \leq c^1$ such that
$$\varepsilon^1_{Me}(x, a^1, c^1) = \begin{cases} 1 & \text{for } x \leq a^1 \\ \text{decreasing} & \text{for } a^1 < x < c^1, \\ 0 & \text{for } c^1 \leq x. \end{cases}$$

2. $\varepsilon^2_{Me} : R \longrightarrow [0,1]$ is a non-decreasing parametric function with the parameters $a^2, c^2 \in R$, $c^2 \leq a^2$ such that
$$\varepsilon^2_{Me}(x, a^2, c^2) = \begin{cases} 0 & \text{for } x \leq c^2, \\ \text{increasing} & \text{for } c^2 < x < a^2, \\ 1 & \text{for } a^2 \leq x. \end{cases}$$

The interval $[c^1_{Me}, c^2_{Me}]$ is the set of *numbers being surely medium*, i.e. $Me(x) = 1$ for $x \in [c^1_{Me}, c^2_{Me}]$.

The canonical membership functions (5), (6) and (7) serve as a basis for derivation of the mentioned accepted form of the membership functions Z, S and Π-curves, respectively. It has always been accepted that and Z and S functions are special cases of the Π-one. However, the opposite understanding — that the latter is composed of the formers — is also possible. Therefore, "medium" can be understood as a composed term of the form

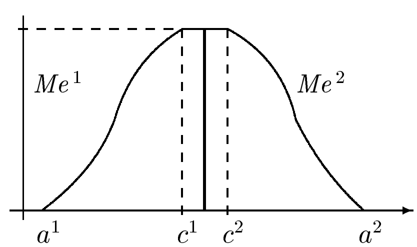

Figure 2: Membership function of "medium = medium^1 OR medium^2" composed of two adjacent S and Z curves.

$$medium = medium^1 \ OR \ medium^2$$

where "medium^1" and "medium^2" are atomic terms with the meaning analogous to the meaning of the terms "big" and "small" respectively (cf. Fig. 2). The fuzzy set Me in (7) is then the union of two fuzzy sets

$$Me = Me^1 \cup Me^2$$

defined by the corresponding two cases in (7).

Note 1 Let us remark that *fuzzy numbers* are generally accepted to be triangular fuzzy sets where the peak corresponds to the given number. For example "about 25" is represented by a triangular number with the peak in 25. Due to our approach, also fuzzy numbers can be explained in the same way as the term "medium". Thus, we obtain a unique representation for all the basic atomic terms as well as for a wider class of evaluating linguistic expressions which includes also fuzzy numbers.

2.1.4 Concrete membership functions of basic atomic terms

Concrete shapes of the membership functions depend on the form of the horizon approach function $\varepsilon(x, c)$. We will consider three basic kinds of membership functions, namely linear, quadratic and exponential ones which lead to the well known membership function widely used in applications.

For the term "small", the corresponding membership functions are the following.

(a) *linear*

$$Sm(x) = \begin{cases} 1 & \text{if } 0 \le x \le c \\ \frac{a-x}{a-c} & \text{if } c \le x \le a \\ 0 & \text{if } a < x \end{cases}$$

(b) *quadratic*

$$Sm(x) = \begin{cases} 1 & \text{if } 0 \le x \le c \\ 1 - \frac{1}{2}\left(\frac{x-c}{b-c}\right)^2 & \text{if } c \le x \le b \\ \frac{1}{2}\left(\frac{a-x}{a-b}\right)^2 & \text{if } b \le x \le a \\ 0 & \text{if } a < x \end{cases}$$

(c) *exponential*

$$Sm(x) = \begin{cases} 1 & \text{if } 0 \le x \le c \\ e^{-k(x-c)^2} & \text{if } c < x \end{cases}$$

where $u \le a \le b \le c \le v$ are parameters specifying position of the corresponding curve in the universe $U = [u, v]$. The parameter b is auxiliary and may be missing (cf. Fig. 1 and 3).

These functions can be obtained from (5) by putting:

$$\varepsilon_{Sm}(x, a, c) = \begin{cases} 0 & \text{for } x \le c, \\ \frac{x-c}{x(a-c)} & \text{for } c < x < a, \\ 1 & \text{for } a \le x, \end{cases} \qquad (Sm \text{ linear}) \tag{8}$$

$$\varepsilon_{Sm}(x, a, b, c) = \begin{cases} 0 & \text{for } x \le c, \\ \frac{(x-c)^2}{2x(b-c)^2} & \text{for } c < x < b, \\ \frac{2(a-b)^2-(a-x)^2}{2x(a-b)^2} & \text{for } b < x < a \\ 1 & \text{for } a \le x, \end{cases} \qquad (Sm \text{ quadratic}) \tag{9}$$

$$\varepsilon_{Sm}(x, a, c) = \begin{cases} 0 & \text{for } x \le c, \\ \frac{1-e^{-k(x-c)^2}}{x} & \text{for } c < x < a, \\ 1 & \text{for } a \le x. \end{cases} \qquad (Sm \text{ exponential}) \tag{10}$$

Similarly, the corresponding membership functions for *big* have the horizon approach functions as follows:

$$\varepsilon_{Bi}(x,a,c) = \begin{cases} 0 & \text{for } c \le x, \\ \frac{c-x}{x(c-a)} & \text{for } a < x < c, \\ 1 & \text{for } x \le a, \end{cases} \qquad (Bi \text{ linear}) \tag{11}$$

$$\varepsilon_{Bi}(x,a,b,c) = \begin{cases} 0 & \text{for } c \le x, \\ \frac{(c-x)^2}{2x(c-b)^2} & \text{for } b < x < c, \\ \frac{2(b-a)^2-(x-a)^2}{2x(b-a)^2} & \text{for } a < x < b, \\ 1 & \text{for } x \le a, \end{cases} \qquad (Bi \text{ quadratic}) \tag{12}$$

$$\varepsilon_{Bi}(x,a,c) = \begin{cases} 0 & \text{for } c \le x, \\ \frac{1-e^{-k(c-x)^2}}{x} & \text{for } a < x < c, \\ 1 & \text{for } x \le a. \end{cases} \qquad (Bi \text{ exponential}) \tag{13}$$

The membership functions for *medium* are obtained analogously.

The linear membership functions of the basic triple of atomic terms have trapezoidal shape depicted on Fig. 3. The other two have the shape depicted on Fig. 1 (exponential, however, has the disadvantage of asymptotical approaching of the x axis, which is not favourable behavior).

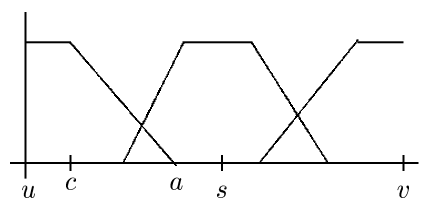

Figure 3: The meaning of the basic triple of atomic terms if the horizon approach function $\varepsilon(x,a,c)$ gives linear membership functions (trapezoidal shape).

2.2 Linguistic hedges

2.2.1 Existing models of linguistic hedges

The model of the meaning of natural language in fuzzy set theory works especially for adjectives and nouns. Verbs are omitted due to their extreme complexity[4].

L. A. Zadeh proposed in his works the model of the meaning of the linguistic hedges using certain transformations of the membership functions. More exactly, if $A \underset{\sim}{\subset} U$ is a fuzzy set assigned to the word $\mathcal{A}$ and $\mathcal{M}$ is a linguistic hedge so that $\mathcal{MA}$ is a linguistic expression then its meaning is a fuzzy set

$$M(\mathcal{MA}) = m(A) \tag{14}$$

where m is a transformation assigned to $\mathcal{M}$ which acts on the membership function A. The first studied linguistic hedge was the hedge *very.* The transformation assigned to it is the operation $CON(a)$ defined by $CON(a) = a^2$, $a \in [0,1]$. For example, the meaning of *very small* is the membership function A^2_{Sm} where A_{Sm} is the fuzzy set assigned to the word *small.* The effect of such modifier is depicted on Fig. 4.

Later on, L. A. Zadeh proposed the model of some other hedges [23]. He used several classes of functions which can be combined in various ways. Modifications of his theory have been proposed, e.g., in [1, 11, 14].

[4]The only elaborated mathematical model known to the author, which includes also the meaning of verbs, is presented in his book [15].

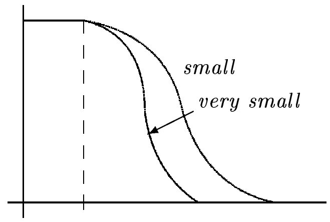

Figure 4:

G. Lakoff in his detailed linguistic analysis [11] demonstrated that the above model does not fit well the real linguistic effect. The hedge *very* and other similar ones should steepen the membership function as well as *shift it.* Furthermore, the shifting depends on the kind of adjective so that there is a difference between the modification of the adjectives *small* and *big* (see Fig. 5).

Modification of Zadeh's proposal aiming at fulfilling the above requirements was proposed in [14]. Due to it, the meaning of $\mathcal{MA}$ above is computed using the composition of functions

$$M(\mathcal{MA}) = \nu_{\mathcal{M}} \circ A \circ \zeta_{\mathcal{M}} \tag{15}$$

where $\nu_{\mathcal{M}} : [0,1] \longrightarrow [0,1]$ is a modification function and $\zeta_{\mathcal{M}} : U \longrightarrow U$ is a shifting function. For example, the hedge *very* is determined using the functions

$$\nu_{very}(a) = CON(a) = a^2, \qquad a \in [0,1],$$

and

$$\zeta_{very} = x + (-1)^k d \,\|\,\mathrm{Ker}(A)\|$$

where $k = 1$ for $\mathcal{A} :=$ '*small*', $k = 2$ for $\mathcal{A} :=$ '*big*', d is a shifting factor (experimentally it has been found that $d \in [0.25, 0.4]$) and $\|\,\mathrm{Ker}(a)\|$ is the width of the kernel of A, i.e., $\mathrm{Ker}(A) = \{x \mid Ax = 1\}$ (for details and other hedges, see [14]). This model behaves in accordance with Lakoff's discussion. However, it is fairly complicated and thus, most authors incorrectly stick to the original Zadeh's proposal.

Shifting of the membership function as a method for realization of the linguistic hedges is proposed also by B. Bouchon in [1]. Her approach considers only linear membership functions.

2.2.2 A horizon shifting model of linguistic hedges

What does it mean "very small"? The hedge *very* makes the meaning of *small numbers* more accurate. *Very small* numbers are small but there are small numbers which are not very small. A very small number is closer to 0 and, therefore, it is more transparent. Of course, the set (interval) of "surely very small numbers" is smaller than that of "surely small ones". In other words, the *horizon of very small numbers is closer to the observer than that of small ones.* This suggests to consider the meaning of the expression *very small* in the same way as small simply by changing the horizon approach function $\varepsilon_{Sm}(x, a, c)$.

We will introduce a new special parameter corresponding to the given hedge. Hence, the fuzzy set *VeSm* modeling the meaning of *very small numbers* is given by

$$VeSm(x) = 1 - \varepsilon'_{Sm}(x, a, c, \sigma_V)x \tag{16}$$

where $0 < \sigma_V < 1$ is some parameter such that $\varepsilon'_{Sm}(x, a, c, \sigma_V) > \varepsilon_{Sm}(x, a, c)$. Since $\varepsilon_{Sm}(x, a, c)$ can be understood as a special case of $\varepsilon'_{Sm}(x, a, c, \sigma_V)$ with the parameter $\sigma_V = 1$, we will omit the apostrophe in the sequel.

The function (16) realizes both the required actions: it shifts the horizon closer and steepens the membership function (cf. Fig. 5). Note that $[u, \sigma_V c] \subset [u, c]$, i.e., the interval of surely very small numbers is shorter than that of surely small numbers.

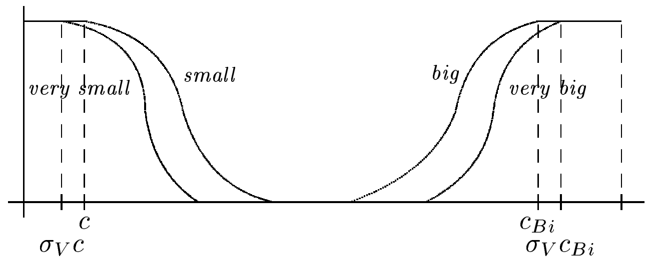

Figure 5: The effect of the horizon shifting model of the hedge "very". Note that the membership function of *big* is shifted to the right.

Opposite effect, i.e. shifting the horizon farther from the observer, is encountered for the hedges such as *more or less, roughly*, etc. In general, we speak about hedges with *narrowing effect* (very, highly, etc.) and those with *widening effect* (more or less, roughly, etc.).

Let $\mathcal{M}$ denote a linguistic hedge. Then, applying it to the atomic term *small* in (5), we generally get

$$\mathcal{M}Sm(x) = 1 - \varepsilon_{Sm}(x, a, c, \sigma_{\mathcal{M}})x \tag{17}$$

where $a, c \in L$ and $\varepsilon_{Sm}(x, a, c, \sigma_{\mathcal{M}}) : L \longrightarrow [0,1]$ is a parametric function with the parameters $c \leq a$ such that

$$\varepsilon_{Sm}(x, a, c, \sigma_{\mathcal{M}}) = \begin{cases} 0 & \text{for } x \leq \min\{\sigma_{\mathcal{M}}c, s\}, \\ \text{increasing} & \text{for } \min\{\sigma_{\mathcal{M}}c < x < \min\{\sigma_{\mathcal{M}}a, s\}, \\ 1 & \text{for } \min\{\sigma_{\mathcal{M}}a, s\} \leq x \end{cases}$$

where $x \in [u, s]$ and moreover,

$$\varepsilon_{Sm}(x, a, c, \sigma_{\mathcal{M}}) \geq \varepsilon_{Sm}(x, a, c, \sigma'_{\mathcal{M}})$$

for $\sigma_{\mathcal{M}} \leq \sigma'_{\mathcal{M}}$ and $\varepsilon_{Sm}(x, a, c, 1) = \varepsilon_{Sm}(x, a, c)$.

The parameter $\sigma_{\mathcal{M}}$ works as follows: $\sigma_{\mathcal{M}} = 1$ when no hedge in (1) is present (we may also speak about *empty hedge*). Furthermore, $0 < \sigma_{\mathcal{M}} < 1$ for $\mathcal{M}$ being the hedge with narrowing effect and $1 < \sigma_{\mathcal{M}}$ for $\mathcal{M}$ being the hedge with widening effect.

The interval $[u, \min\{\sigma_{\mathcal{M}}c, s\}]$ is the set of *numbers being surely $\mathcal{M}$ small*, i.e. $\mathcal{M}Sm(x) = 1$ for $x \in [u, \min\{\sigma_{\mathcal{M}}c, s\}]$. Similarly, the interval $[\min\{\sigma_{\mathcal{M}}a, s\}, s]$ is the set of numbers surely not being $\mathcal{M}$ small, i.e. $\mathcal{M}Sm(x) = 0$ for $x \in [\min\{\sigma_{\mathcal{M}}a, s\}, s]$. Each linguistic hedge is thus characterized by one parameter σ only.

Similarly we proceed for the linguistic hedges of *big* and *medium*, where we have

$$\mathcal{M}Me = \mathcal{M}Me^1 \cup \mathcal{M}Me^2.$$

Note that there is no linguistic hedge with narrowing effect applied to the atomic term *medium*.

2.2.3 Linguistic hedges for specific types of membership functions

The horizon approach functions for the atomic term "small" when linguistic modifiers are included have the following form:

Sm linear:

$$\varepsilon_{Sm}(x, a, c, \sigma_{\mathcal{M}}) = \begin{cases} 0 & \text{for } x \leq \sigma_{\mathcal{M}}c, \\ \frac{x - \sigma_{\mathcal{M}}c}{x\sigma_{\mathcal{M}}(a-c)} & \text{for } \sigma_{\mathcal{M}}c < x < \sigma_{\mathcal{M}}a, \\ 1 & \text{for } \sigma_{\mathcal{M}}a \leq x, \end{cases} \tag{18}$$

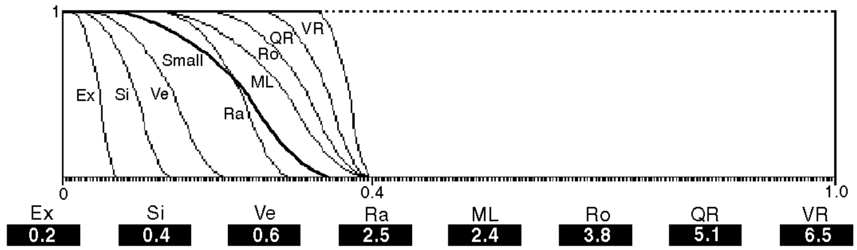

Figure 6: The effect of the hedges *extremely*, *significantly*, *very*, *more or less roughly*, *quite roughly* and *very roughly* for the atomic term "small". Included is also the hedge *rather*, but this is exceptional and does not belong to the class of the previous ones.

Sm quadratic:

$$\varepsilon_{Sm}(x,a,c,\sigma_{\mathcal{M}}) = \begin{cases} 0 & \text{for } x \le \sigma_{\mathcal{M}} c, \\ \frac{(x-\sigma_{\mathcal{M}} c)^2}{2x\sigma_{\mathcal{M}}^2 (b-c)^2} & \text{for } \sigma_{\mathcal{M}} c < x \le \sigma_{\mathcal{M}} b, \\ \frac{2\sigma_{\mathcal{M}}^2 (a-b)^2 - (\sigma_{\mathcal{M}} a - x)^2}{2x\sigma_{\mathcal{M}}^2 (a-b)^2} & \text{for } \sigma_{\mathcal{M}} b \le x < \sigma_{\mathcal{M}} a, \\ 1 & \text{for } \sigma_{\mathcal{M}} a \le x, \end{cases} \tag{19}$$

Sm exponential:

$$\varepsilon_{Sm}(x,a,c,\sigma_{\mathcal{M}}) = \begin{cases} 0 & \text{for } x \le \sigma_{\mathcal{M}} c, \\ \frac{1-e^{-k(x-\sigma_{\mathcal{M}} c)^2}}{x} & \text{for } \sigma_{\mathcal{M}} c < x < \sigma_{\mathcal{M}} a, \\ 1 & \text{for } \sigma_{\mathcal{M}} a \le x. \end{cases} \tag{20}$$

This model has been implemented in the software system called Linguistic Fuzzy Logic Controller (LFLC). Its theory is presented, for example in [18]. The behaviour of a few kinds of hedges used in it including the values of the corresponding parameters $\sigma_{\mathcal{M}}$ is depicted on Fig. 6. The membership functions are quadratic, which seem to fit best all the linguistic requirements. The universe is $[u,v] = [0,1]$ and the center s is set to $s = 0.4$. There are also few successful applications of LFLC in the control of real processes.

The horizon approach functions (18), (19) and (20) for the terms "big" and "medium" are defined analogously. Let us stress, however, that the values of $\sigma_{\mathcal{M}}$ are not unique. They depend on the definition of the modified membership function as well as on the kind of the atomic term in concern, i.e. they should differ for the atomic terms "small", "medium" and "big". This is a necessary tax for the more realistic behaviour[5]. Let us also mention that this theory opens a promising way for finding effective learning procedures which could concentrate on finding the optimal values of the parameters $\sigma_{\mathcal{M}}$.

2.3 Class of functions modeling the meaning of evaluating terms

It follows from the previous discussion that simple evaluating expressions (1) have the meaning which can be modeled using fuzzy sets with membership functions taken from the class of functions described in this section.

Let a set (interval) $[p,q] = \{x \mid x \in U, p \le x \le q\}$ be given where U is, by assumption, an ordered set of elements. Then the following class contains all the functions using which membership functions of the meaning of the simple evaluating expressions (1) can be constructed:

$$\mathcal{Z}([p,q]) = \{Z, S \mid Z, S : [p,q] \longrightarrow [0,1]\} \tag{21}$$

[5]All these problems are already solved in LFLC.

where Z, S are functions given by

$$Z(x, a_Z, c_Z, \sigma_Z) = 1 - \varepsilon_Z(x, a_Z, c_Z, \sigma_Z)x, \tag{22}$$
$$S(x, a_S, c_S, \sigma_S) = 1 - \varepsilon_S(x, a_S, c_S, \sigma_S)x \tag{23}$$

where $a_Z, c_Z, a_S, c_S \in [p, q]$, $\sigma_Z, \sigma_S > 0$ and

$$\varepsilon_Z(x, a_Z, c_Z, \sigma_Z) : [p, q] \longrightarrow [0, 1],$$
$$\varepsilon_S(x, a_S, c_S, \sigma_S) : [p, q] \longrightarrow [0, 1]$$

are functions with the following properties:

(a) Denote $u_Z = \min\{\sigma_Z c_Z, q\}$ and $v_Z = \min\{\sigma_Z a_Z, q\}$. Then $\varepsilon_Z(x, a_Z, c_Z, \sigma_Z)$ is a parametric function such that

$$\varepsilon_Z(x, a_Z, c_Z, \sigma_Z) = \begin{cases} 0 & \text{for } x \le u_Z, \\ \text{increasing} & \text{for } u_Z < x < v_Z, \\ 1 & \text{for } v_Z \le x, \end{cases}$$

(b) Denote $u_S = \min\{\sigma_S c_S, q\}$ and $v_S = \min\{\sigma_S a_S, p\}$. Then $\varepsilon_S(x, a_S, c_S, \sigma_S)$ is a parametric function such that

$$\varepsilon_S(x, a_S, c_S, \sigma_S) = \begin{cases} 1 & \text{for } x \le v_S, \\ \text{decreasing} & \text{for } v_S < x < u_S, \\ 0 & \text{for } u_S \le x. \end{cases}$$

Both functions are, moreover, decreasing with respect to σ, i.e.

$$\varepsilon(x, a, c, \sigma) > \varepsilon(x, a, c, \sigma')$$

for $\sigma < \sigma'$. The functions from the class $\mathcal{Z}([p, q])$ are depicted in Fig. 7.

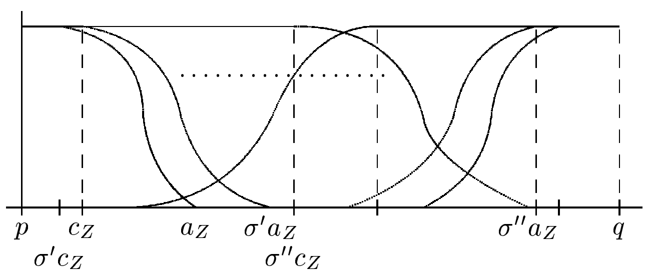

Figure 7: Scheme of the class of functions from $\mathcal{Z}([p, q])$. The σ' has narrowing and σ'' widening effect (this is marked for Z-functions only).

The $\varepsilon_Z, \varepsilon_S$ are special parametric functions (horizon approach functions) providing concrete shapes of the membership functions of the the meaning of simple evaluating linguistic expressions (1).

The outcome of this approach is that these membership functions are formed using only two forms of functions from the class $\mathcal{Z}([p, q])$. Furthermore, the use of them is well justified to fit both the results of the linguistic analysis as well as the intuition.

3 Representation of continuous functions using formulas containing linguistic terms

3.1 Expressional ability of the 'computing with words' methodology

Amazing ability of fuzzy logic formulas to represent functional dependencies does not surprize specialists thanks to the results on fuzzy logic controllers to be universal approximators (see [3, 2, 7, 9]). This technique is based on the representation of continuous functions using fuzzy logic formulas of special type. It is notable that this kind of representation is a result of formalization of expert rules used for the control of simple dynamic processes, such as steam engine, etc. For more complicated controlled objects, however, one kind of expert knowledge is insufficient.

As often happens, the experience has been generalized and a mechanism of elaboration of dependencies of various kind (expert data and knowledge, algebraic expressions) has been expressed in a form of logical formulas comprizing sets implications between antecedent and succedent. The fundamental part of this mechanism is based on making use of properties of logical operations being general for all algebraic systems interpreting fuzzy logic. It has been demonstrated that the class of algebraic systems based on any of the three basic t-norms (minimum, Łuksiewicz product and ordinary product) has a sufficient expressing power for approximation of continuous functions.

The secondary part of this mechanism is based on the choice of basic membership functions interpreting fuzzy logic predicates in antecedents and succedents. In the existing literature on the approximate description of functions by fuzzy logic formulas, these membership functions are generalized characteristic functions of certain areas in the universe without any other specificity. A consequent of this is unlimited extension of the number of the used logical formulas (in the form of implications) necessary for approximation of the given dependence with a given precision. In some sense, this makes impossible a direct use of the approximation result in the practice. Hence, we face a problem of effective representation of dependencies using logical formulas, i.e. how to find a minimal number of them.

We solve this problem in this paper by choosing basic membership functions reflecting the specificity of the given dependence. These basic functions serve also as interpretations of the meaning of words forming the above introduced canonical linguistic triple ("small, medium, high").

To fit the concrete dependence, it is necessary to modify the membership functions of the basic triple. We will show that using linguistic modifiers introduced in the previous section, it is possible to represent effectively an arbitrary continuous function, monotonous on finite number of intervals, using logical formulas formed from the basic functions of the type mentioned above. Consequently, this representation is possible using the simple evaluating linguistic expressions (1).

The obtained result means that the 'computing with words' methodology has an expressing power comparable, for example, with polynomial or other representations. However, the possibility to use linguistic expressions in the description of the dependence makes this methodology significantly attractive for the applications.

3.2 Normal forms for fuzzy logic formulas

To achieve the results outlined in the previous section, we will work with formulas in the form of implications $A(x) \Rightarrow B(y)$, where $A(x)$ is an atomic formula or equals to a conjunction of atomic formulas and $B(y)$ is an atomic formula. Let us have a collection of such kind of formulas

$$\{A_i(x) \Rightarrow B_i(y) \mid 1 \leq i \leq N\}. \tag{24}$$

We aim at proving that continuous functional dependencies between x and y can be logically described (precisely or approximately) by those collections. We obtain this result by successive solution of the following tasks:

(i) interpretation of (24) by a fuzzy relation;

(ii) transformation of a fuzzy relation to a function;

(iii) reverse representation of a function by a collection of fuzzy logic formulas;

(iv) generalization of (iii) to some special class of functions.

In this section, we consider task (i) and show how the collection (24) can be interpreted by a fuzzy relation in the universe $X \times Y$.

Let $A_i(x), B_i(y)$ be atomic formulas with one variable being interpreted by the corresponding fuzzy sets on some universes X and Y, respectively. As the strict distinction between formulas and their interpretations is not necessary in this paper, we will employ the same symbols also for fuzzy sets. Hence, $A_i \underset{\sim}{\subset} X$ and $B_i \underset{\sim}{\subset} Y$ express that $A_i(x), B_i(y)$ are interpreted by the respective fuzzy sets. We will work with these representations only, in the sequel. Using this agreement, (24) is represented by a fuzzy relation in the universe $X \times Y$.

There are two basic general forms how (24) can be realized. They generalize the common disjunctive and conjunctive normal forms (cf. [20], and also [7]), namely

$$\bigvee_{i=1}^{N} (A_i(x) * B_i(y)) \tag{25}$$

and

$$\bigwedge_{i=1}^{N} (A_i(x) \rightarrow_* B_i(y)) \tag{26}$$

where $*$ denotes some t-norm and $\rightarrow_*$ the implication operator adjoint to $*$.

When viewed as functions of independent variable x and dependent variable y, expressions (25) and (26) define fuzzy relations on $X \times Y$. Both these relations are used as interpretations of the set of logical formulas (24).

Due to results in the theory of t-norms (cf., for example, [8]), the following t-norms are sufficient for representation of all continuous t-norms.

$$a * b = a \wedge b = \min(a, b), \qquad a \rightarrow_G b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } b < a. \end{cases}$$

$$a * b = a \otimes b = \max\{0, a + b - 1\}, \qquad a \rightarrow_L b = 1 \wedge (1 - a + b).$$

$$a * b = a \cdot b \qquad a \rightarrow_P b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{if } b < a. \end{cases}$$

where $\rightarrow_G, \rightarrow_L, \rightarrow_P$ are the corresponding adjoints.

3.3 From fuzzy relations to functions

Let $X, Y \subseteq \mathbb{R}$ be subsets of the set of real numbers and $R \underset{\sim}{\subset} X \times Y$ be a fuzzy relation corresponding to normal form (25) or (26) for some fixed t-norm $*$. We show the way how the relation $R(x, y)$ can be transformed to a function $f_R : X \longrightarrow Y$ (and it will be a solution of the task (ii)).

Let $x^0 \in X$ be an arbitrary but fixed element in X. Consider $R(x^0, y)$ and note that it defines a membership function of some fuzzy set in Y. To define a function f_R it is sufficient to choose one element from $R(x^0, y)$ and correspond it to x^0. We will do it following the procedure of defuzzification. Recall that $DF : \mathcal{F}(Y) \longrightarrow Y$ is a defuzzification function where $\mathcal{F}(Y)$ is the set of all fuzzy sets on Y, when it assigns an element y^0 to the fuzzy set $R(x^0, y)$ by a certain procedure so that

$$y^0 = DF(R(x^0, y)).$$

The function f_R we are looking for can be defined by

$$f_R : x^0 \mapsto DF(R(x^0, y)).$$

We call this function as *function realized by fuzzy relation* R or, we can say also that *fuzzy relation* R *defines the function* f_R.

In the sequel we will restrict ourselves to the defuzzification function such that

$$DF(B(y)) = \inf\{y \mid B(y) = \sup_{u \in Y} B(u)\}.$$

3.4 Representation of continuous and piecewise monotone functions

In the previous section we showed that a collection of fuzzy logic formulas can represent a function. Precisely this fact makes fuzzy logic applicable in many areas. But choosing a fuzzy logic representation of a function, one should know how to obtain its algorithm and how to evaluate a complexity of such kind of representation. To answer these questions, we associate the complexity of a fuzzy logic representation of a function with a number of formulas in the representing collection (24).

In this section we show, how is it possible to obtain a precise representation of a function by non-excessive collections of fuzzy logic formulas. It will be a solution of the task (iii). For the simplicity, we consider the case of functions with one variable.

Theorem 2 *To any continuous and strictly monotonous real-valued real function $f(x) : [a, b] \longrightarrow \mathbb{R}$ there are fuzzy sets $A_1, A_2 \underset{\sim}{\subset} [a, b]$ and $B_1, B_2 \underset{\sim}{\subset} f([a, b])$ such that the collection of fuzzy logic formulas*

$$\{A_i(x) \Rightarrow B_i(y) \mid 1 \leq i \leq 2\},$$

describes a function $f_R(x) : [a, b] \longrightarrow \mathbb{R}$ and the fuzzy relation in the conjunctive normal form (see (26))

$$R(x, y) = \bigwedge_{i=1}^{2} (A_i(x) \rightarrow B_i(y))$$

defines $f_R(x)$ in such a way that $f_R(x) = f(x)$ for every $x \in [a, b]$.

PROOF: The proof is constructive and can be used as a basis for an algorithm. Fix a t-norm $\mathbf{t}$ and let $\rightarrow$ be its adjoint implication. Since $f(x)$ is continuous and monotonous on $[a, b]$ it defines a one-to-one correspondence between $[a, b]$ and $[f(a), f(b)]$. Therefore, the inverse function $f^{-1}(y)$ exists.

Assume that $f(x)$ monotonously increases. The proof consists in construction of membership functions of the fuzzy sets A_1, A_2 and B_1, B_2 in such a way that the equality $f_R(x) = f(x)$ is fulfilled for all $x \in [a, b]$. We suggest two possible solutions.

1. Membership functions for the fuzzy sets A_1, A_2 and B_1, B_2 are the following:

$$A_1(x) = 1 - \frac{x - a}{b - a}, \qquad A_2(x) = \frac{x - a}{b - a}, \qquad x \in [a, b], \tag{27}$$

$$B_1(y) = 1 - \frac{f^{-1}(y) - a}{b - a}, \quad B_2(y) = \frac{f^{-1}(y) - a}{b - a}, \quad y \in [f(a), f(b)]. \tag{28}$$

Let $x^0 \in [a, b]$ be an arbitrary, but fixed element. We show that $f_R(x^0) = f(x^0)$. Indeed,

$$\begin{aligned} A_1(x^0) \rightarrow B_1(y) = 1 \quad & \text{iff} \quad A_1(x^0) \leq B_1(y) \quad \text{iff} \quad 1 - \frac{x^0 - a}{b - a} \leq \frac{b - f^{-1}(y)}{b - a} \\ & \text{iff} \quad y \leq f(x^0); \\ A_2(x^0) \rightarrow B_2(y) = 1 \quad & \text{iff} \quad A_2(x^0) \leq B_2(y) \quad \text{iff} \quad \frac{x^0 - a}{b - a} \leq \frac{f^{-1}(y) - a}{b - a} \\ & \text{iff} \quad f(x^0) \leq y. \end{aligned}$$

Thus,

$$R(x^0, y) = 1 \quad \text{iff} \quad f(x^0) = y,$$

and then

$$\sup_y R(x^0, y) = R(x^0, f(x^0)) = 1.$$

By the definition,

$$f_R(x^0) = DF(R(x^0, y)) = f(x^0).$$

2. The second variant of membership functions of the fuzzy sets A_1, A_2 and B_1, B_2 is the following:

$$A_1(x) = 1 - \frac{f(x) - f(a)}{f(b) - f(a)}, \qquad A_2(x) = \frac{f(x) - f(a)}{f(b) - f(a)}, \qquad x \in [a, b],$$

$$B_1(y) = 1 - \frac{y - f(a)}{f(b) - f(a)}, \qquad B_2(y) = \frac{y - f(a)}{f(b) - f(a)}, \qquad y \in [f(a), f(b)].$$

Analogously as above, let $x^0 \in [a,b]$ be an arbitrary, but fixed element. We show that $f(R(x^0)) = f(x^0)$. Indeed,

$$\begin{aligned}
A_1(x^0) \to B_1(y) = 1 \quad &\text{iff} \quad A_1(x^0) \le B_1(y) \\
&\text{iff} \quad 1 - \frac{f(x^0) - f(a)}{f(b) - f(a)} \le 1 - \frac{y - f(a)}{f(b) - f(a)} \quad \text{iff} \quad y \le f(x^0); \\
A_2(x^0) \to B_2(y) = 1 \quad &\text{iff} \quad A_2(x^0) \le B_2(y) \\
&\text{iff} \quad \frac{f(x^0) - f(a)}{f(b) - f(a)} \le \frac{y - f(a)}{f(b) - f(a)} \quad \text{iff} \quad f(x^0) \le y.
\end{aligned}$$

The rest follows the same way as above. □

To extend the class of functions which are non-excessively realized by fuzzy relation R we introduce the following definition.

Definition 1 *By a piecewise monotonous function f on a compact U we mean a function, for which there exists a finite partition of U such that the restriction of f to each set from the partition is strictly monotonous.*

We again consider only the case of functions with one variable.

Theorem 3 *For any continuous and piecewise monotonous real-valued real function $f(x) : [a,b] \longrightarrow \mathbb{R}$ there exist a number N and fuzzy sets $A_{i1}, A_{i2} \underset{\sim}{\subsetneq} [a,b]$ and $B_{i1}, B_{i2} \underset{\sim}{\subsetneq} f([a,b])$, $1 \le i \le N$, such that the collection of fuzzy logic formulas*

$$\{A_{ij}(x) \Rightarrow B_{ij}(y) \mid 1 \le i \le N,\ 1 \le j \le 2\},$$

describes a function $f_R(x) : [a,b] \longrightarrow \mathbb{R}$ and the fuzzy relation in the conjunctive normal form (see (26))

$$R(x,y) = \bigwedge_{i=1}^{N} \bigwedge_{j=1}^{2} (A_{ij}(x) \to B_{ij}(y))$$

defines $f_R(x)$ in such a way that $f_R(x) = f(x)$ for every $x \in [a,b]$.

PROOF: Since $f(x)$ is continuous and piecewise monotonous on $[a,b]$, there exists a finite partition of $[a,b]$ into a finite number of subintervals $J_1, \ldots, J_N$ such that the restriction $f|J_i$, $1 \le i \le N$, to each set from the partition is strictly monotonous. In accordance with Theorem 2, for each such restriction there exist fuzzy sets $A_{i1}, A_{i2} \underset{\sim}{\subsetneq} J_i$ and $B_{i1}, B_{i2} \underset{\sim}{\subsetneq} f(J_i)$ such that the fuzzy relation

$$R_i(x,y) = \bigwedge_{j=1}^{2} (A_{ij}(x) \to B_{ij}(y))$$

defines a function $f_{R_i}(x)$ such that $f(x) = f_{R_i}(x)$ for all $x \in J_i$.

We extend the definitions of the membership functions $A_{ij}(x), B_{ij}(y)$, $1 \le i \le N$, $1 \le j \le 2$, to $[a,b]$ and $f([a,b])$, by setting them equal to 0 outside of J_i and $f(J_i)$, respectively. Consider the fuzzy relation

$$R(x,y) = \bigwedge_{i=1}^{N} \bigwedge_{j=1}^{2} (A_{ij}(x) \to B_{ij}(y))$$

where $x \in [a,b]$, $y \in f([a,b])$ and $A_{ij}(x), B_{ij}(y)$ are the extensions. Since any arbitrary $x^0 \in [a,b]$ belongs to exactly one subinterval J_i (except for the bounds of subintervals) then not more than two membership functions $A_{i1}(x), A_{i2}(x)$ ($A_{i-1,2}(x), A_{i1}(x)$ or $A_{i2}(x)$, $A_{i+1,1}(x)$, $2 \le i \le N-1$, for internal bounds) can differ from 0 in x^0. Thus,

$$R(x^0, y) = R_i(x^0, y),\ y \in f(J_i),$$

and then

$$f_R(x^0) = f_{R_i}(x^0) = f(x^0).$$

□

Corollary 1 *Let conditions of Theorem 3 be satisfied and in addition, let $f(x)$ be monotonously increasing or decreasing on the whole interval $[a,b]$. Then there exist fuzzy sets $A_1, \dots, A_{N+1} \subsetneqq [a,b]$ and $B_1, \dots, B_{N+1} \subsetneqq f([a,b]$ such that*

$$R(x,y) = \bigwedge_{i=1}^{N+1} (A_i(x) \to B_i(y))$$

defines $f_R(x)$ in such a way that $f_R(x) = f(x)$ for every $x \in [a,b]$.

PROOF: We show only the case when $f(x)$ is monotonously increasing. Consider the membership functions $A_{ij}(x), B_{ij}(y)$, $1 \le i \le N$, $1 \le j \le 2$, constructed in Theorem 3 and define new ones as follows:

$$\begin{aligned} A_1(x) &= A_{11}(x), \\ A_i(x) &= A_{i-1,2}(x) \vee A_{i,1}(x), \\ A_{N+1}(x) &= A_{N2}(x), \end{aligned}$$

and analogously for $B_1(y), B_i(y), B_{N+1}(y)$, $2 \le i \le N-1$. It is easy to verify that

$$\bigwedge_{i=1}^{N+1} (A_i(x) \to B_i(y)) = \bigwedge_{i=1}^{N} \bigwedge_{j=1}^{2} (A_{ij}(x) \to B_{ij}(y)).$$

□

3.5 Representation of functions and computing with words

In this subsection we jointly consider linguistic and functional points of view on fuzzy sets and combine results from the previous sections. We show that each continuous function defined on a compact set can be realized or approximated by fuzzy relation with membership functions similar to those from the basic triplet *small, medium, big*. This fact precisely means that having at disposal three given linguistic labels and the set of linguistic hedges, it is possible to construct a collection of logical formulas (based only on their combinations) and a fuzzy relation using which we can construct a function which either fits precisely or approximately each continuous function.

Similarly as above, we start with a continuous piecewise monotonously increasing function $f(x)$ defined on interval $[a,b] \subseteq \mathbb{R}$. From Corollary 1 we see that to realize such a function by a fuzzy relation, we need membership functions of three types in the universe X divided into N subintervals $[a_1,b_1], \dots, [a_N,b_N]$, namely

$$\begin{aligned} A_1(x) &= 1 - \frac{f(x)-f(a_1)}{f(b_1)-f(a_1)}, \qquad x \in [a_1,b_1], \\ A_i(x) &= \begin{cases} \frac{f(x)-f(a_{i-1})}{f(b_{i-1})-f(a_{i-1})}, & \text{for } x \in [a_{i-1},b_{i-1}], \\ 1 - \frac{f(x)-f(a_i)}{f(b_i)-f(a_i)}, & \text{for } x \in [a_i,b_i], \end{cases} \qquad 2 \le i \le N, \\ A_{N+1}(x) &= \frac{f(x)-f(a_N)}{f(b_N)-f(a_N)}, \qquad x \in [a_N,b_N]. \end{aligned}$$

Similarly, we divide the universe Y into N subintervals

$$[f(a_1), f(b_1)], \dots, [f(a_N), f(b_N)]$$

and obtain the membership functions

$$B_1(y) = 1 - \frac{y - f(a_1)}{f(b_1) - f(a_1)}, \qquad y \in [f(a_1), f(b_1)],$$
$$B_i(y) = \begin{cases} \frac{y - f(a_{i-1})}{f(b_{i-1}) - f(a_{i-1})}, & \text{for } y \in [f(a_{i-1}), f(b_{i-1})], \\ 1 - \frac{y - f(a_i)}{f(b_i) - f(a_i)}, & \text{for } y \in [f(a_i), f(b_i)], \end{cases} \qquad 2 \le i \le N,$$
$$B_{N+1}(y) = \frac{y - f(a_N)}{f(b_N) - f(a_N)}, \qquad y \in [f(a_N), f(b_N)].$$

Note, that the character and form of these membership functions depend on the character and form of the function $f(x)$. Consider different cases for $f(x)$.

1. The function $f(x)$ is piecewise linear and monotonously increasing. In this case it is not difficult to see that:

   - the membership functions $A_1(x), B_1(y)$ are precisely described and expressed by a modified linguistic term "small",
   - the membership functions $A_i(x), B_i(y)$, $2 \le i \le N$, are precisely described and expressed by a modified linguistic term "medium",
   - the membership functions $A_N(x), B_N(y)$ are precisely described and expressed by modified linguistic term "big".

   We verify this on the example of $A_1(x), B_1(y)$. Let $f|[a_1, b_1](x) = kx + l$. Then

   $$A_1(x) = 1 - \frac{k(x - a_1)}{k(b_1 - a_1)} = 1 - \frac{x - a_1}{b_1 - a_1} = \frac{b_1 - x}{b_1 - a_1},$$
   $$B_1(y) = 1 - \frac{f^{-1}(y) - a_1}{b_1 - a_1} = 1 - \frac{\frac{y-l}{k} - a_1}{b_1 - a_1} = \frac{(kb_1 + l) - y}{(kb_1 + l) - (ka_1 + l)},$$

   i.e. we obtain the linear membership functions of simple evaluating linguistic expressions as discussed in Sections 2.1.4 and 2.2.3.

2. The function $f(x)$ is continuous and monotonously increasing on $[a, b]$, but not necessarily piecewise linear. It is well known that for the given function and each $\epsilon > 0$ there exist a partition of $[a, b]$ into a finite number of subintervals $J_1, \ldots, J_N$ and a piecewise linear function $f^L(x)$ such that $f^L|J_i$ is linear and

   $$|f(x) - f^L(x)| \le \epsilon$$

   holds for for all x. It follows that $f^L(x)$ admits a representation by some fuzzy relation R^L with the membership functions expressed by (linear) fuzzy sets representing the meanings of simple evaluating terms (1). Thence, the original function $f(x)$ admits an approximation by R^L.

3. The function $f(x)$ is continuous on $[a, b]$ but the character of monotonicity can be different. Then for the given function and each $\epsilon > 0$ there exists a piecewise linear function $f^L(x)$ which approximates $f(x)$ on the whole interval $[a, b]$ with the precision ϵ. In this case, a fuzzy relation R^L defining function $f^L(x)$ can be constructed on the basis of Theorem 3 rather than on its corollary. It means that membership functions characterized by a modified term "medium" can be missing. With this restriction, the fuzzy relation R^L approximately defines $f(x)$.

So far, we have exploited only linear membership functions among those modeling the meaning of the simple evaluating linguistic expressions (1). The non-linear (quadratic or exponential) membership functions are used mainly for approximation. It is necessary to point out that the approximation of continuous functions based on fuzzy relation with non-linear membership functions is more preferable because it enables to reduce the number of formulas in (24) and fits better the linguistic environment. A precise formulation of this problem and its solution will be done in the future paper.

On the other hand, if we stick on the precise description of the given function $f(x)$, have have to employ the membership functions constructed in accordance with Theorem 3. However, these may not necessarily have the form suggested for the basic linguistic terms *small, medium, big*, even if they are modified due to the theory presented in Section 2.

4 Discussion

In this paper, we have proposed a horizon shifting method for computation of the linguistic hedges which technically leads to parametric classes of membership functions. The method meets the requirements of linguists, i.e., that the membership function should be shifted as well as modified its steepness. Furthermore, it has a potential for further development because it affects the basic parameters of the defined membership functions in a unified way and using one parameter only. Thence, the potential for effective learning is open.

The expressing power of basic linguistic terms together with modifiers has been demonstrated on the example of the representation of an arbitrary continuous piecewise monotonous function. The obtained result means that the methodology of 'computing with words' is comparable, for example, with polynomial or other representations. However, the possibility to use linguistic expressions in the description of functions makes this methodology significantly attractive for the applications.

To conclude, let us remark the following. In this paper, we have elaborated the model of the meaning of linguistic expressions in a traditional way, which is purely extensional understanding of the semantics of the linguistic expressions. This means that their semantics is explained only on the basis of the groupings (i.e., fuzzy sets) of elements falling into the meaning of the expressions in concern. This is correct and quite well working in various applications of fuzzy set theory. However, the groupings vary in various contexts and in time and therefore, the extensional approach is often criticized by linguists. A more realistic theoretically well founded interpretation of the linguistic semantics must take the intension (property) into consideration (cf., e.g. [21]). However, pure fuzzy set theory is unable to fit this requirement. It is a challenge to find a formal apparatus capable to model both the intension as well as extension of the linguistic semantics including its vagueness (fuzziness). Promising in this respect seems to be *fuzzy logic in broader sense* (see [17, 19]) which is an extension of the fuzzy logic in narrow sense[6]. Fuzzy logic in broader sense attempts to elaborate methods of approximate reasoning including the formal model of linguistic meaning. A more elaborated theory of the meaning of the evaluated linguistic expressions developed in the frame of fuzzy logic in broader sense is a task of future work.

References

[1] Bouchon-Meunier, B.: *Fuzzy logic and knowledge representation using linguistic modifiers.* In: Zadeh, L.A. and J. Kacprzyk: **Fuzzy Logic for the Management of Uncertainty**, J. Wiley, New York 1992.

[2] Buckley, J. J. and Y. Hayashi, *Fuzzy input-output controllers are universal approximators*, Fuzzy Sets and Systems **58**(1993), 273–278.

[3] Castro, J., *Fuzzy Logic Controllers are Universal Approximators*, IEEE Trans. on Systems, Man, and Cybernetics **25**(1995), 629–635.

[4] Davis, M.: **Applied nonstandard analysis**. J. Wiley, New York 1977.

[5] Hájek P.: **Metamathematics of fuzzy logic**. Kluwer, Dordrecht (to appear).

[6] Klawonn, F., *Fuzzy sets and vague environments*, Fuzzy Sets and Systems **66**(1994), 207–221.

[7] Klawonn, F. and V. Novák: *The Relation between Inference and Interpolation in the Framework of Fuzzy Systems.* Fuzzy Sets and Systems **81**(1996), 331–354.

[6] Presentation of fuzzy logic in narrow sense can be found, for example, in [5, 17] and elsewhere.

[8] Klement, E. P., Mesiar, R. and E. Pap. **Triangular Norms**. Monograph in preparation.

[9] Kosko, B., *Fuzzy Systems as Universal Approximators*, Proc. IEEE Conf. on Fuzzy Systems, San Diego 1992, 1153–1162.

[10] Kuzmin, V. B.: *About semantical structure of linguistic hedges: an experimental hypothesis.* BUSEFAL 24, 1986, pp. 117–125.

[11] Lakoff, G.: *Hedges: A study in meaning criteria and logic of fuzzy concepts*, J. Philos. Logic **2**(1973), 458–508.

[12] Lee, J. and Chae, S.: *Analysis on function duplicating capabilities of fuzzy controllers.* Fuzzy Sets and Systems **56**(1993), 127–143.

[13] Novák, V.: *Sorites-like First–order Fuzzy Theories.* Proc. of IIIrd IFSA Congress, Seattle 1989.

[14] Novák, V., **Fuzzy Sets and Their Applications**, Adam–Hilger, Bristol, 1989.

[15] Novák, V., **The Alternative Mathematical Model of Linguistic Semantics and Pragmatics**, Plenum, New York, 1992.

[16] Novák, V.: *Towards Formalized Integrated Theory of Fuzzy Logic.* In: Bien, Z, and K. Min (eds.), **Fuzzy Logic and Its Applications to Engineering, Information Sciences, and Intelligent Systems**, Kluwer, Dordrecht 1995, 353–363.

[17] Novák, V.: *Paradigm, Formal Properties and Limits of Fuzzy Logic.* Int. J. of General Systems **24**, 1996.

[18] Novák, V.: *Linguistically Oriented Fuzzy Logic Controller and Its Design.* Int. J. of Approximate Reasoning 1995, **12**(1995), 263–277.

[19] Novák, V. and I. Perfilieva: *On Logical and Algebraic Foundations of Approximate Reasoning.* Proc. FUZZ-IEEE'97, Barcelona, 693–698.

[20] Perfilieva, I.: *Fuzzy logic normal forms for control law representation* In: Fuzzy Algorithms for Control Engineering — from misunderstanding to benefits and challenges. Eds. H. B. Verbruggen, H. J. Zimmermann, 1998 (to appear).

[21] Sgall, P., Hajičová, E., Panevová J., **The meaning of the sentence in its semantic and pragmatic aspects**, D. Reidel, Dordecht 1986.

[22] Vopěnka, P.: **Mathematics in the Alternative Set Theory**. Teubner, Leipzig 1979.

[23] Zadeh, L.A.: *A fuzzy-set-theoretic interpretation of linguistic hedges.* J. Cyber. **2**(1972), 2–34.

[24] Zadeh, L.A.: *Quantitative Fuzzy Semantics*, Inf. Sci.,**3**(1973), 159–176.

[25] Zadeh, L.A.: *The concept of a linguistic variable and its application to approximate reasoning I, II, III*, Inf. Sci., **8**(1975), 199–257, 301–357; **9**(1975), 43–80.

[26] Zadeh, L. A. *Test-score semantics for natural languages and meaning representation via PRUF.* In: Rieger, Burghard, **Empirical Semantics I**. Dr. N. Brockmayer, Bochum, 1981.