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Extreme solutions of system of fuzzy relation equations with triangular fuzzy sets

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In this paper, we observe and compare results concerning solvability of systems of fuzzy relation equations with different types of composition. I try to find maximal and minimal solutions for system of fuzzy relation equations with triangular fuzzy sets. I investigate simple case, where width of fuzzy sets is same and other condition must hold then the maximal solution for system with $inf \rightarrow$ - composition and the minimal solution for system with $sup - *$ - composition look very simple and similarly.

K e y w o r d s: System of fuzzy relation equations; sup-$*$-composition and $inf\rightarrow$ composition; solvability of fuzzy relation equation; maximal and minimal solutions.

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1 INTRODUCTION

Generally two types of composition are known namely $sup - *$ and $inf \rightarrow$ - composition where $*$ is a t-norm and $\rightarrow_*$ is residuation operation with respect to the t-norm $*$. These compositions are used in the fuzzy relation equations. We investigate solvability of these equations and their set of solutions.

The pioneering work was done by Sanchez (1976) [8], but many other researchers dealt with this problem, for instance De Baets [1], Di Nola, Sessa, Pedrycz [2], Klir, Yuan [4], Mamdani, Assilian [5], Perfilieva [7], etc.

This contribution wants to show, how can look maximal or minimal solution for system of fuzzy relation equations. However I deal with only particular case of system of fuzzy relation equations here.

At first I will introduce some basic notions and then I will present solutions of system of fuzzy relation equations.

2 PRELIMINARIES

2.1 Residuated lattice

We choose a complete residuated lattice as the basic algebra of operations.

Definition 1
A residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$ with four binary operations and two constants such that

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice where the ordering $\leq$ defined using operations $\vee, \wedge$ as usual, and $\mathbf{0}, \mathbf{1}$ are the least and the greatest elements;*
- $\langle L, *, \mathbf{1} \rangle$ *is a commutative monoid,*
- *the operation $\rightarrow$ is a residuation operation with respect to $*$, i.e.*

$$a * b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

The well known examples of residuated lattices are boolean algebras, Gödel, Łukasiewicz and product algebras. In particular case $L = [0,1]$, multiplication $*$ is known as a t-norm. I work especially with Łukasiewicz algebra in this contribution. Operations in Łukasiewicz algebra are Łukasiewicz product and implication with respect to the product:

$$a \otimes b = \max\{a+b-1, 0\}; \quad a \rightarrow_{\otimes} b = \min\{1-a+b, 1\}$$

2.2 Fuzzy sets and fuzzy relations

We accept here a mathematical definition of a fuzzy set. In the rest of this paper we suppose that a complete residuated lattice $\mathcal{L}$ with a support L is fixed, and $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary non-empty sets. Then a *fuzzy set* or better, a fuzzy subset A of $\mathbf{X}$, is identified with a function $A : \mathbf{X} \longrightarrow L$. This function is known as *membership function* of the fuzzy set A. The set of all fuzzy subsets of $\mathbf{X}$ is denoted by $\mathcal{F}(\mathbf{X})$, so that

$$\mathcal{F}(\mathbf{X}) = \{A \mid A : \mathbf{X} \longrightarrow L\} = L^{\mathbf{X}}.$$

For two fuzzy subsets A and B of $\mathbf{X}$ we write $A = B$ or $A \leq B$ if $A(x) = B(x)$ or $A(x) \leq B(x)$, holds for all $x \in \mathbf{X}$, respectively. A fuzzy set $A \in \mathcal{F}(\mathbf{X})$ is called *normal* if $A(x_0) = \mathbf{1}$ for some $x_0 \in \mathbf{X}$.

The algebra of operations over fuzzy subsets of $\mathbf{X}$ is introduced as an induced residuated lattice on $L^{\mathbf{X}}$. This means that each operation from $\mathcal{L}$ induces the corresponding operation on $L^{\mathbf{X}}$ taken pointwise. We demonstrate this on the example of $*$-operation between fuzzy sets A and B:

$$(A * B)(x) = A(x) * B(x).$$

Obviously, operations over fuzzy subsets fulfill the same properties as operations in the respective residuated lattice.

A (binary) *fuzzy relation* on $\mathbf{X}\times\mathbf{Y}$ is a fuzzy subset of the Cartesian product. $\mathcal{F}(\mathbf{X}\times\mathbf{Y})$ denotes the set of all binary fuzzy relations on $\mathbf{X}\times\mathbf{Y}$. Analogously, an n-ary fuzzy relation can be introduced.

Let $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and $S \in \mathcal{F}(\mathbf{Y} \times \mathbf{Z})$, then the fuzzy relation $T = R \circ S$ on $\mathbf{X} \times \mathbf{Z}$

$$T(x,z) = \bigvee_{y \in \mathbf{Y}} (R(x,y) * S(y,z))$$

is called a $sup - *$ - composition of R and S and the fuzzy relation $Q = R \looparrowright S$ on $\mathbf{X} \times \mathbf{Z}$

$$Q(x,z) = \bigwedge_{y \in \mathbf{Y}} (R(x,y) \rightarrow_{*} S(y,z))$$

is called an $inf \rightarrow_{*}$ - composition of R and S.

In particular, if A is a unary fuzzy relation on $\mathbf{X}$ or simply a fuzzy subset of $\mathbf{X}$ then the $sup - *$ ($inf \rightarrow_{*}$) - composition between A and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ is the fuzzy subset of $\mathbf{Y}$ defined by

$$(A \circ R)(y) = \bigvee_{x \in \mathbf{X}} (A(x) * R(x,y)),$$

and respectively,

$$(A \looparrowright R)(y) = \bigwedge_{x \in \mathbf{X}} (A(x) \rightarrow_{*} R(x,y)).$$

2.3 Systems of fuzzy relation equations

Let $N \geq 1$. A $sup - *$ - system of fuzzy relation equations

$$A_i \circ R = B_i \quad \text{or} \quad \bigvee_{x \in \mathbf{X}} (A_i(x) * R(x,y)) = B_i(y), \quad 1 \leq i \leq N, \tag{1}$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$, is considered with respect to unknown fuzzy relation R. So that fuzzy set B_i is equal to $sup - *$ - composition of fuzzy set A_i and unknown fuzzy relation R, where $*$ is a t-norm. Analogously, we will consider an $inf \rightarrow_*$ - system of fuzzy relation equations

$$A_i \looparrowright R = B_i \quad \text{or} \quad \bigwedge_{x \in \mathbf{X}} (A_i(x) \rightarrow_* R(x,y)) = B_i(y), \quad 1 \leq i \leq N, \tag{2}$$

with the same description of parameters and also with respect to unknown fuzzy relation R, $\rightarrow_*$ is residuated operation with respect to $*$.

Because solutions of (1) and (2) may not exist, in general, the problem to investigate necessary and sufficient, or only sufficient conditions for solvability arises. This problem has been widely studied in the literature and some nice theoretical results have been obtained (e.g. [3, 7, 8]). The second problem is found a solution. The fundamental theorems say us how to find the greatest respective the least solution. But the question is: How do the further solutions look? This problem is presented e.g. in [1] for special fuzzy relation equations.

Two types of fuzzy relations have been always assumed with respect to the problem of solvability of (1) and (2), namely

$$\check{R}(x,y) = \bigvee_{i=1}^{N} (A_i(x) * B_i(y)) \tag{3}$$

considered in Mamdani & Assilian [5], and

$$\hat{R}(x,y) = \bigwedge_{i=1}^{N} (A_i(x) \rightarrow B_i(y)) \tag{4}$$

first considered in Sanchez [8].

2.4 Triangular fuzzy sets

We use a triangular fuzzy sets. These fuzzy sets are normal, but they have only one element in their kernel, we denote it x_i for fuzzy set A_i on universe $\mathbf{X}$ ($A_i(x_i) = 1$) and y_i for fuzzy set B_i on universe $\mathbf{Y}$ ($B_i(y_i) = 1$). Further triangular fuzzy sets are symmetrical and convex. They have following membership functions:

$$A_i(x) = \max\left\{ \left(1 - \frac{|x - x_i|}{d_i}\right), 0 \right\}, \quad B_i(y) = \max\left\{ \left(1 - \frac{|y - y_i|}{h_i}\right), 0 \right\}.$$

Where x_i $[y_i]$ is element from the kernel of fuzzy set A_i $[B_i]$ and d_i $[h_i]$ is mean length of the arm of the fuzzy set A_i $[B_i]$, so for instance $d_i = x_i - \tilde{x}$, $\tilde{x} = \sup\{x \in \mathbf{X} | A_i(x) = 0, x < x_i\}$ $[h_i = y_i - \tilde{y}$, $\tilde{y} = \sup\{y \in \mathbf{Y} | B_i(y) = 0, y < y_i\}]$.

3 COMPLETE SET OF SOLUTIONS

In this section we will recall fundamental criteria of solvability. At first we present criterion for $sup - *$ - system of fuzzy relation equations. The proof of the theorem is e.g. in [8].

Theorem 1
The system (1) is solvable if and only if the fuzzy relation $\hat{R}$ is its solution. If the system (1) is solvable then $\hat{R}$ is its greatest solution.

Further we show analogous criterion for $inf \rightarrow_*$ - system of fuzzy relation equations. The proof of this theorem is e.g. [6].

Theorem 2
The system (2) is solvable if and only if the fuzzy relation $\check{R}$ is its solution. If the system (2) is solvable then $\check{R}$ is its smallest solution.

It is easy to see that if system (1)[(2)] is solvable then the set of solutions forms a $\vee$-semi-lattice [$\wedge$-semi-lattice], i.e. a fuzzy relation $R_1 \vee R_2$ [$R_1 \wedge R_2$] is a solution to (1) [(2)] whenever R_1 and R_2 are its solutions.

Further we will present when the fuzzy relation $\check{R}$ will be solution of $sup - *$ - system of fuzzy relation equations and when the fuzzy relation $\hat{R}$ will be solution of $inf \rightarrow_*$ - system of fuzzy relation equations. If system (1) (respective (2)) is solvable then the relation $\check{R}$ (respective $\hat{R}$) does not need to be its solution. In [3], the necessary and sufficient condition of the solvability of (1) with respect to $\check{R}$ is given and for system (2) was given in [6].

Theorem 3
Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ be normal and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \le i \le n$. Then the fuzzy relation $\hat{R}$ [$\check{R}$] is a solution to (2) [(1)] if and only if for all $i, j = 1, \ldots, n$ the following inequality

$$\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \le \bigwedge_{y \in \mathbf{Y}} (B_i(y) \leftrightarrow B_j(y)) \tag{5}$$

holds.

4 SPECIAL CASE

In this section we will try to find some minimal respective maximal solution. We will consider only simplified system of fuzzy relation equations with triangular fuzzy sets here. For this special case with conditions mentioned bellow, we will be able to say theorems about maximal and minimal solutions.

Let the system of fuzzy relation equations with triangular fuzzy sets satisfy:

1. Triangular fuzzy sets satisfy inequality

$$\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \le \bigwedge_{y \in \mathbf{Y}} (B_i(y) \leftrightarrow B_j(y))$$

(i.e. both fuzzy relations $\hat{R}$ and $\check{R}$ are solutions of the system of fuzzy relation equations with $sup - *$ and $inf \rightarrow$ composition).

2. All fuzzy sets A_i have identical width ($d_i = d_j = d$ for all $i, j = 1, \ldots, N$). And all fuzzy sets B_i have identical width ($h_i = h_j = h$ for all $i, j = 1, \ldots, N$).

3. t-norm is Łukasiewicz product.

4. If $A_i(x_j) > 0$ then $A_i(x_j) < B_i(y_j)$ for all $j \neq i$.

Theorem 4
Let the system of fuzzy relation equations with triangular fuzzy sets satisfy above mentioned conditions then

$$R_{min}(x, y) = \begin{cases} B_i(y) & \textit{if } x = x_i; \\ 0 & \textit{otherwise.} \end{cases} \tag{6}$$

is a minimal solution for $sup - *$ - system and

$$R_{max}(x, y) = \begin{cases} B_i(y) & \textit{if } x = x_i; \\ 1 & \textit{otherwise.} \end{cases} \tag{7}$$

is a maximal solution for $inf \ \rightarrow_*$ - system of fuzzy relation equations.

PROOF: We make the proof in two steps. At the first step we proof that fuzzy relation R_{min} $[R_{max}]$ is solution of system of fuzzy relation equations with $sup - *$ $[inf \ \rightarrow]$ - composition.

For system with $sup - *$ - composition:

$$\begin{aligned} \bigvee_{x \in \mathbf{X}} A_i(x) * R_{min}(x, y) &= (1 * B_i(y)) \vee \left(\bigvee_{j \neq i} A_i(x_j) * B_j(y) \right) \vee 0 = \\ &= B_i(y) \vee \left(\bigvee_{j \neq i} A_i(x_j) * B_j(y) \right) = B_i(y). \end{aligned}$$

The last step is from the first request to system of fuzzy relation equations:

$$\begin{aligned} A_i(x_j) * A_j(x_j) &\leq B_j(y) \rightarrow B_i(y) \\ A_i(x_j) * B_j(y) &\leq B_i(y) \end{aligned}$$

For system with $inf \ \rightarrow$ - composition:

$$\begin{aligned} \bigwedge_{x \in X} A_i(x) \rightarrow R_{max}(x, y) &= (A_i(x_i) \rightarrow B_i(y)) \wedge \left(\bigwedge_{j \neq i} A_i(x_j) \rightarrow B_j(y) \right) \wedge 1 = \\ &= B_i(y) \wedge \left(\bigwedge_{j \neq i} A_i(x_j) \rightarrow B_j(y) \right) = B_i(y). \end{aligned}$$

The last step is from the first request to system of fuzzy relation equations again:

$$\begin{aligned} A_i(x_j) * A_j(x_j) &\leq B_i(y) \to B_j(y) \\ A_i(x_j) &\leq B_i(y) \to B_j(y) \\ B_i(y) &\leq A_i(x_j) \to B_j(y) \end{aligned}$$

Fuzzy relation $\mathbf{R}_{min}$ [$\mathbf{R}_{max}$] is solution of system of fuzzy relation equations with $sup - *$ [$inf \ \rightarrow$] - composition. Now we proceed to the second step, that $\mathbf{R}_{min}$ [$\mathbf{R}_{max}$] is minimal [maximal] solution. For the system with respect to $sup - *$ - composition, we will proof that fuzzy relation $\mathbf{R}_{min}$ is a minimal solution, so that for all solutions $\mathbf{R}$ of this system such that $\mathbf{R}_{min} \geq \mathbf{R}$ then $\mathbf{R}_{min} = \mathbf{R}$. I perform the proof by contradiction. Let fuzzy relation $\mathbf{R}$ be solution of the system ($\bigvee_{x \in \mathbf{X}} A_i(x) * R(x,y) = B_i(y)$) and $\mathbf{R}_{min} > \mathbf{R}$, so that there is point $(\tilde{x}, \tilde{y}) \in \mathbf{X} \times \mathbf{Y}$:

$$R(\tilde{x},\tilde{y}) < R_{min}(\tilde{x},\tilde{y}) = \begin{cases} 0 \\ B_i(\tilde{y}) \end{cases}$$

nothing is less than zero, so $\tilde{x}$ is a node ($\tilde{x} = x_k$). Hence $R(x_k,\tilde{y}) < R_{min}(x_k,\tilde{y}) = B_i(\tilde{y})$. All points different from $(x_k,\tilde{y})$ have following relation with $\mathbf{R}_{min}$: $R(x,y) = R_{min}(x,y)$.

We take only problematic case namely k-th equation in element $\tilde{y}$. So we have equation:

$$\bigvee_{x \in \mathbf{X}} A_k(x) * R(x,\tilde{y}) = B_k(\tilde{y}).$$

We have:

$$\begin{aligned} \bigvee_{x \in \mathbf{X}} A_k(x) * R(x,\tilde{y}) &= (A_k(x_k) * R(x_k,\tilde{y})) \vee \bigvee_{x \in \mathbf{X}, x \neq x_k} (A_k(x) * R_{min}(x,\tilde{y})) = \\ &= R(x_k,\tilde{y}) \vee \bigvee_{i \neq k} A_k(x_i) * B_i(\tilde{y}) = \begin{cases} R(x_k,\tilde{y}) \\ \bigvee_{i \neq k} A_k(x_i) * B_i(\tilde{y}) \end{cases} \end{aligned}$$

In the first case it is clear $R(x_k,\tilde{y}) < B_i(\tilde{y})$ so $R(x_k,\tilde{y})$ is not solution of the system.

If $R(x_k,\tilde{y}) < \bigvee_{i \neq k} A_k(x_i) * B_i(\tilde{y})$ then it arises the second case. We need to prove: $\bigvee_{i \neq k} A_k(x_i) * B_i(\tilde{y}) < B_k(\tilde{y})$. This problem we can solve by contradiction again. So let $\bigvee_{i \neq k} A_k(x_i) * B_i(\tilde{y}) \geq B_k(\tilde{y})$.

$$\bigvee_{i \neq k} A_k(x_i) \otimes B_i(\tilde{y}) = A_k(x_j) \otimes B_j(\tilde{y}) \geq B_k(\tilde{y}).$$

Without loss of generality, let $j < k$. We will use the fourth assumption: $A_k(x_j) > 0$ so $A_k(x_j) < B_k(y_j)$, it means that

$$\frac{y_k - y_j}{h} < \frac{x_k - x_j}{d}.$$

Hence

$$\begin{aligned}
A_k(x_j) \otimes B_j(\tilde{y}) &\geq B_k(\tilde{y}) \\
A_k(x_j) + B_j(\tilde{y}) - 1 &\geq B_k(\tilde{y}) \\
1 - \frac{x_k - x_j}{d} + 1 - \frac{|y_j - \tilde{y}|}{h} - 1 &\geq 1 - \frac{|y_k - \tilde{y}|}{h} \\
\frac{|y_k - \tilde{y}|}{h} - \frac{|y_j - \tilde{y}|}{h} &\geq \frac{x_k - x_j}{d} > \frac{y_k - y_j}{h} \\
|y_k - \tilde{y}| - |y_j - \tilde{y}| &> y_k - y_j.
\end{aligned}$$

We have three possibilities:

- $\tilde{y} \leq y_j$: $y_k - \tilde{y} - y_j + \tilde{y} = y_k - y_j > y_k - y_j$,
- $y_j < \tilde{y} \leq y_k$: $y_k - \tilde{y} + y_j - \tilde{y} = y_k + y_j - 2\tilde{y} > y_k - y_j$ hence $y_j > \tilde{y}$,
- $y_k < \tilde{y}$: $-y_k + \tilde{y} + y_j - \tilde{y} = y_j - y_k = -(y_k - y_j) > y_k - y_j$.

All possibilities make for contradiction. Hence $\bigvee_{i \neq k} A_k(x_i) \otimes B_i(\tilde{y}) < B_k(\tilde{y})$.

The proof for $inf\rightarrow$-system of fuzzy relation equations is performed analogous. □

5 Conclusion

I wanted to find some minimal respective maximal solution for the system of fuzzy relation equations with respect to the triangular fuzzy sets. However during investigation it was shown that it is not so easy. So I presented one example of minimal respective maximal solution for Łukasiewicz t-norm. Every system has different maximal respective minimal solutions but some points are same, so we can say that every minimal solution looks as following fuzzy relation: $R_{min}(x, y)$ is equal to $\hat{R}(x, y)$ in some points (these points depend on the form of the system) and 0 otherwise. Maximal solutions for the system with $inf\rightarrow$ composition look like: $R_{max}(x, y)$ is equal to $\check{R}(x, y)$ in some points and 1 otherwise. Where $R_{max}(x, y)$ is equal to $\check{R}(x, y)$ is again depended on specific of the system.

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