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Systems of Fuzzy Relation Equations: New Solvability Criteria Based on the Orthogonality Condition

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Abstract: Inference mechanisms and interpretations of fuzzy rule bases are studied together from the point of view of systems of fuzzy relation equations. A proper use of an inference mechanism connected to a fuzzy relation interpreting a fuzzy rule base is certified by keeping the fundamental interpolation condition. The paper aims at new solutions of systems of fuzzy relation equations which are motivated by several fuzzy approximation methods. The study of this new solutions yields new solvability criteria which are introduced.

Keywords: Fuzzy relation equations, Compositions, Fuzzy inference, Fuzzy rule base, Additive interpretation.

1 Introduction

Fuzzy if-then rule base may be viewed as a partial function from $\mathcal{F}(X)$, the collection of fuzzy sets over X, to $\mathcal{F}(Y)$, the collection of fuzzy sets over Y. Now, building a fuzzy inference module on the base of a rule base means extending this partial function to a total one; in some reasonable manner, we have to associate with an arbitrary $\mathbf{A} \in \mathcal{F}(X)$ some $\mathbf{B} \in \mathcal{F}(Y)$.

So,all information given by a fuzzy rule base consisting of the following n fuzzy rules

$$\textbf{IF } x \text{ is } \mathcal{A}_i \textbf{ THEN } y \text{ is } \mathcal{B}_i \quad i = 1, \dots, n \tag{1}$$

is hidden in n pairs $(\mathbf{A}_i, \mathbf{B}_i)$ of fuzzy sets $\mathbf{A}_i \underset{\sim}{\subset} X, \mathbf{B}_i \underset{\sim}{\subset} Y$ which represent the linguistic expressions $\mathcal{A}_i, \mathcal{B}_i$ on given universes X, Y, respectively.

Fuzzy rule base (1) is modelled by a fuzzy relation, usually one of the following two fuzzy relations

$$\hat{\mathbf{R}}_*(x, y) = \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_* \mathbf{B}_i(y)),$$

$$\check{\mathbf{R}}_*(x, y) = \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y))$$

where $*$ is a left-continuous t-norm. (Any t-norm in the latter will be supposed to be left-continuous.)

By an inference mechanism we understand a mapping which assigns a conclusion $\mathbf{B} \underset{\sim}{\subset} Y$ to a fuzzy set $\mathbf{A} \underset{\sim}{\subset} X$ defined as an image of $\mathbf{A}$ under a fuzzy relation $\mathbf{R} \underset{\sim}{\subset} X \times Y$ interpreting fuzzy rule base (1). In most cases, the image is defined by the sup-$*$ composition

$$\mathbf{B} = \mathbf{A} \circ_* \mathbf{R} \tag{2}$$

which rises from the *compositional rule of inference* introduced by L.A. Zadeh [18] or by the inf-$\rightarrow_*$ composition

$$\mathbf{B} = \mathbf{A} \lhd_* \mathbf{R} \tag{3}$$

which is a particular Bandler-Kohout product [1] and it is given as follows

$$(\mathbf{A} \circ_* \mathbf{R})(y) = \bigvee_{x \in X} (\mathbf{A}(x) * \mathbf{R}(x, y)), \tag{4}$$

$$(\mathbf{A} \lhd_* \mathbf{R})(y) = \bigwedge_{x \in X} (\mathbf{A}(x) \rightarrow_* \mathbf{R}(x, y)) \tag{5}$$

where $\rightarrow_*$ is a residuation operation derived from the t-norm $*$.

As mentioned before, fuzzy rule base (1) defines a partial mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and therefore either we deal with composition (2) or composition (3) while building an inference module extending the mapping to a total one, the fundamental interpolation condition claiming that from $\mathbf{A}_i$ we deduce precisely $\mathbf{B}_i$ should be kept. This leads to the problematic of systems of fuzzy relation equations, see e.g.[6].

Unfortunately, besides the solvability criteria only two solutions $\hat{\mathbf{R}}_*$ and $\check{\mathbf{R}}_*$ have been mainly studied [11] so far. In [16], there have been the so called *additive interpretations* [15] of fuzzy rule bases considered. The additive interpretations have been motivated by Takagi-Sugeno systems [17], some neuro-fuzzy systems [5] or the fuzzy transform [12] and their formalization can be seen in the additive normal forms introduced in [13] and then studied in [2, 3].

The main goal of the paper is to search for solutions in the following (additive) form

$$\mathbf{R}_*^{\oplus}(x,y) = \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)) \tag{6}$$

where $\oplus$ is the Łukasiewicz t-conorm. It means that we search for specific conditions related to operations in inference mechanisms and fuzzy sets in a rule base which ensure keeping the interpolation condition and so a proper use of the *additive interpretations* (6). Moreover, we will show that such study can yield even new solvability criteria.

2 Systems of Fuzzy Relation Equations - Preliminaries

This section focuses on systems of fuzzy relation equations and recalls basic results which can be found e.g. in [4, 7, 9]. Interpolation condition leads to the following systems of fuzzy relation equations

$$\mathbf{A_i} \circ_* \mathbf{R} = \mathbf{B}_i \quad i = 1,\ldots,n \tag{7}$$

and

$$\mathbf{A_i} \lhd_* \mathbf{R} = \mathbf{B}_i \quad i = 1,\ldots,n. \tag{8}$$

Now, we recall necessary and sufficient conditions of the solvability of the given systems of fuzzy relation equations.

Theorem 1 *The system*

$$\mathbf{A_i} \circ_* \mathbf{R} = \mathbf{B}_i \quad i = 1,\ldots,n \tag{9}$$

is solvable if and only if $\hat{\mathbf{R}}_*$ *is its solution and then* $\hat{\mathbf{R}}_*$ *is its greatest solution.*

Theorem 2 *The system*

$$\mathbf{A_i} \lhd_* \mathbf{R} = \mathbf{B}_i \quad i = 1,\ldots,n \tag{10}$$

is solvable if and only if $\check{\mathbf{R}}_*$ *is its solution and then* $\check{\mathbf{R}}_*$ *is its least solution.*

The following theorem presents a sufficient condition of solvability w.r.t. system (7) from [7].

Theorem 3 *Let* $\mathbf{A}_i$*,* $i = 1,\ldots,n$*, be normal. Then* $\check{\mathbf{R}}_*$ *is a solution to (7) if and only if the following condition*

$$\bigvee_{x\in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \le \bigwedge_{y\in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y)) \tag{11}$$

holds for arbitrary $i, j \in \{1,\ldots,n\}$*.*

We present a sufficient condition for the solvability of system (8) published in [10]. This criterion is somehow inverse to Theorem 3.

Theorem 4 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal. Then $\hat{\mathbf{R}}_*$ is a solution to (8) if and only if the following condition*

$$\bigvee_{x \in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \leq \bigwedge_{y \in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y))$$

holds for arbitrary $i, j \in \{1, \ldots, n\}$.

3 New Solutions and Criteria

Let us recall a definition of the generalized orthogonality [13] which will be crucial in the latter.

Definition 1 We say that $\mathbf{A}_i \underset{\sim}{\subset} X,\ i = 1, \ldots, n$ keep the orthogonality condition if

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^{n} \mathbf{A}_i(x) = 1 - \mathbf{A}_j(x). \tag{12}$$

Since the investigation of the additive solutions was motivated by searching for condition under which $\mathbf{R}_*^{\oplus}$ is a solution to either (7) or (8) the solvability of such systems was always assumed. Let us recall the following theorem from [16].

Theorem 5 *Let (8) be solvable, $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and keep the orthogonality condition. Then $\mathbf{R}_*^{\oplus}$ is a solution to (8).*

And a direct corollary of Theorem 5 and Theorem 4.

Corollary 1 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and condition (11) holds. If $\mathbf{A}_i$ keep the orthogonality condition then $\mathbf{R}_*^{\oplus}$ is a solution to (8).*

Such results clarify the proper use of the relation $\mathbf{R}_*^{\oplus}$ as an interpretation of fuzzy rules connected to $\lhd_*$ inference technique. But they do not claim anything about solvability since it is assumed already. On the other hand, the proofs in [16] can be modified and we can state the following solvability criterion (sufficient condition).

Theorem 6 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and keep the orthogonality condition. Then system (8) is solvable and $\mathbf{R}_*^{\oplus}$ is its solution.*

PROOF: Let $j \in \{1, \ldots, n\}$ be an arbitrary fixed subindex and let

$$\mathbf{B}(y) = \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)))$$

then

$$\mathbf{B}(y) \geq \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)))$$

$$\mathbf{B}(y) \geq \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* (\mathbf{A}_j(x) * \mathbf{B}_j(y)))$$

which yields $\mathbf{B} \subseteq \mathbf{B}_j$.

On the other hand, because of the isotonicity of residuation operations the following inequality

$$\mathbf{B}(y) \leq \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigoplus_{\substack{i=1 \\ i \neq j}}^{n} (\mathbf{A}_i(x)) \oplus (\mathbf{B}_j(y)))$$

holds. The orthogonality (see [13]) yields

$$\mathbf{B}(y) \leq \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* ((1 - \mathbf{A}_j(x)) \oplus \mathbf{B}_j(y)))$$

Let $x' \in X$ such that $\mathbf{A}_j(x') = 1$ then

$$\mathbf{B}(y) \leq (\mathbf{A}_j(x') \rightarrow_* ((1 - \mathbf{A}_j(x')) \oplus \mathbf{B}_j(y)))$$

which yields $\mathbf{B} \supseteq \mathbf{B}_j$.

□

Corollary 2 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and keep the orthogonality condition. Then $\hat{\mathbf{R}}_*$ is a solution to system (8).*

Now, let us recall the definition of a stronger t-norm, see [8].

Definition 2 Let $*$ and $\blacktriangle$ be t-norms. Then we say that $\blacktriangle$ is a stronger t-norm than $*$ ($* \leq \blacktriangle$) if $\forall x_1, x_2 \in [0, 1] : \ x_1 * x_2 \leq x_1 \blacktriangle x_2$.

We directly obtain another corollary of Theorem 6.

Corollary 3 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and keep the orthogonality condition. If $* \leq \blacktriangle$ then the fuzzy relation $\mathbf{R}_{\blacktriangle}^{\oplus}$ is a solution to (8).*

A possibility of taking an additive interpretation of a fuzzy rule base together with the inf$\rightarrow_*$ composition as an inference method was partially clarified by the previous theorems and corollaries. Moreover, the investigation lead to a new solvability criterion. In the latter, we focus on the sup-$*$ composition, which is a bit more complicated case.

Theorem 7 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and let $\mathbf{A}_i$ keep the orthogonality condition. If $* \leq \otimes$ where $\otimes$ is the Łukasiewicz t-norm then $\mathbf{R}_*^{\oplus}$ is a solution to (7).*

PROOF: Let $j \in \{1, \ldots, n\}$ be an arbitrary fixed subindex and let

$$\mathbf{B}(y) = \bigvee_{x \in X} (\mathbf{A}_j(x) * \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)))$$

then by direct assignments one gets $\mathbf{B} \supseteq \mathbf{B}_j$. On the other hand, because of the orthogonality conditions the following inequality

$$\mathbf{B}(y) \leq \bigvee_{x \in X} (\mathbf{A}_j(x) * \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_{\otimes} \mathbf{B}_i(y)))$$

holds (used technique is adopted from [13] and has been already used in the proof of Theorem 6). Since $* \leq \otimes$ we obtain

$$\mathbf{B}(y) \leq \bigvee_{x\in X} (\mathbf{A}_j(x) \otimes (\mathbf{A}_j(x) \rightarrow_{\otimes} \mathbf{B}_j(y)))$$

and because $a * (a \rightarrow_* b) \leq b$ for any $*$, we immediately get $\mathbf{B} \subseteq \mathbf{B}_j$. □

Theorem 7 requires to use a t-norm which is even weaker than the Łukasiewicz one which is already a very weak t-norm, see [8]. So, for practical applications, perhaps only the case when $* = \otimes$ worths mentioning. In this case, the Łukasiewicz t-norm is used for both, the sup-$\otimes$ composition as an inference method and for the interpretation of a fuzzy rule base by the fuzzy relation $\mathbf{R}^{\oplus}_{\otimes}$.

The result is strengthened by the following one.

Theorem 8 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal and $\mathbf{A}_i$ keep the orthogonality condition. If $* \leq \otimes$ and $* \leq \blacktriangle$ then $\mathbf{R}^{\oplus}_{\blacktriangle}$ is a solution to (7).*

PROOF: Let $j \in \{1, \ldots, n\}$ be an arbitrary fixed subindex and let

$$\mathbf{B}(y) = \bigvee_{x\in X} (\mathbf{A}_j(x) * \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) \blacktriangle \mathbf{B}_i(y)))$$

then the following inequality

$$\mathbf{B}(y) \geq \bigvee_{x\in X} (\mathbf{A}_j(x) * \bigvee_{i=1}^{n}(\mathbf{A}_i(x) * \mathbf{B}_i(y)))$$

holds. Let $x' \in X$ such that $\mathbf{A}_j(x') = 1$ then

$$\mathbf{B}(y) \geq (\mathbf{A}_j(x') * (\mathbf{A}_j(x') * \mathbf{B}_j(y)))$$

and therefore $\mathbf{B} \supseteq \mathbf{B}_j$.

The other side is proved analogously to the proof of Theorem 7 just using the fact that $* \leq \otimes$. □

Theorem 8 allows us to deal with a t-norm weaker or equal to the Łukasiewicz one only in the sup-$*$ composition as an inference method but the interpretation of a fuzzy rule base can be build w.r.t. a t-norm $\blacktriangle$ which is stronger.

To demonstrate the results introduced above, let us consider the following simple example.

Example 1 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ keep the Ruspini condition [14]*

$$\sum_{i=1}^{n} \mathbf{A}_i(x) = 1, \quad \forall x \in X \tag{13}$$

which is very often required for antecedent fuzzy sets when constructing a fuzzy rule based systems. Then the fuzzy sets obviously keep the orthogonality condition. Let us add the normality of the antecedent fuzzy sets.

Then due to Theorem 6 the fuzzy relation

$$\mathbf{R}^{\oplus}_{\otimes}(x, y) = \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) \otimes \mathbf{B}_i(y)) \tag{14}$$

is a solution to the following system of fuzzy relation equations

$$\mathbf{A_i} \lhd_{\otimes} \mathbf{R} = \mathbf{B}_i \quad i = 1, \ldots, n,$$

as well as fuzzy relations

$$\hat{\mathbf{R}}_{\otimes}(x, y) = \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_{\otimes} \mathbf{B}_i(y)) \quad \textit{and}$$

$$\mathbf{R}_{\odot}^{\oplus}(x, y) = \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) \odot \mathbf{B}_i(y)) = \sum_{i=1}^{n} \mathbf{A}_i(x)\mathbf{B}_i(y)$$

due to Corollary 2 and Corollary 3, respectively.

Moreover, fuzzy relation (14) is also a solution to the following system of fuzzy relations

$$\mathbf{A_i} \circ_{\otimes} \mathbf{R} = \mathbf{B}_i \quad i = 1, \ldots, n, \tag{15}$$

and due to Theorem 8, the solution to system (15) can be also found in the form of $\mathbf{R}_{\odot}^{\oplus}$.

4 Conclusion

The paper dealt with systems of fuzzy relation equations and a retrieval of new solutions while up to now, usually only two solutions $\hat{\mathbf{R}}$ and $\check{\mathbf{R}}$ have been studied. Besides providing readers with new possible solutions, the investigation clarified possible combinations of fuzzy inference techniques and the additive interpretations of fuzzy rule bases which are formalizations of various fuzzy approximation techniques. This results we consider to be necessary for a proper use of the additive interpretations.

Finally, the investigation lead even to new solvability criteria. The given criteria can be considered to be very useful since they put assumptions only on antecedents fuzzy sets $\mathbf{A}_i$ conversely to the original criteria. Theorem 1 and Theorem 2 require knowledge whether special relations composed from $\mathbf{A}_i$ and $\mathbf{B}_i$ are solutions. Theorem 3 and Theorem 4 required knowledge whether $\mathbf{A}_i$ and $\mathbf{B}_i$ are in a special predefined relationship. New criteria put only clear assumption on antecedent fuzzy sets $\mathbf{A}_i$ - normality and orthogonality (which can be easily fulfilled e.g. by keeping the Ruspini condition). Then the systems will be solvable and the paper introduces possible solutions. It means, that the consequent fuzzy sets $\mathbf{B}_i$ can be determined arbitrarily e.g. by some learning and such fuzzy inference system will keep the interpolation condition regardless of $\mathbf{B}_i$.

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